Generalised free extensions

NATHAN CORBYN*, University of Oxford, UK

The notion of a *free extension* of an algebra serves as the foundation for an expressive framework, abstractly characterising algebras of terms over an algebraic structure — e.g., the ring of polynomials over a ring. This framework finds applications in the optimisation of algebraic computations in meta-programming [7]; in the theory of normalisation by evaluation for first-order term languages; and in proof-synthesis in dependent-type theories [1, 3]. However, the current theoretical treatment of free extensions is limited to universal algebraic structures, whilst many structures of interest in programming language theory cannot be captured universal algebraically.

I present my ongoing work to generalise free extensions to broader classes of equational structures. Here, I focus on *generalised algebraic theories* (GATs) à la Cartmell [2]. Observing that the dependent sorting of GATs precludes a notion of free extension by a *set* of variables, I instead develop a notion of free extension by a *context*. This requires a strict generalisation of the universal property of free extensions, which I show specialises to the usual notion of free extension when applied to a universal algebraic theory (UAT). I conclude by summarising ongoing work to construct effective descriptions of generalised free extensions of structured categories.

1 BACKGROUND

Consider a universal algebraic theory (UAT) Θ — i.e., a collection of finitary operator symbols and equations, such as the theory of commutative monoids. Let $\mathcal{A} \in \mathrm{Alg}(\Theta)$ be a Θ-model and V be a set of variables. Modulo provable equivalence in Θ and reduction in $\mathcal{A}$, the first-order terms over $\mathcal{A}$ containing variables drawn from V form a Θ-model, $\mathcal{A}[V]$, dubbed *the free extension of* $\mathcal{A}$ *by* V. Free extensions enjoy a defining universal property, similar to that of *free models*.

The *free* Θ*-model* on a set X, FX, is the set of Θ-terms generated by X, modulo equivalence in Θ. However, FX also holds a defining universal property: given an *environment* $\theta : X \to \mathcal{W}$ — i.e., a mapping of elements of X into the carrier of a model $\mathcal{W} \in \mathrm{Alg}(\Theta)$, there exists a unique extension of θ to a homomorphism $\tilde{\theta} : FX \to \mathcal{W}$ that substitutes according to θ and reduces in $\mathcal{W}$. This is illustrated in figure 1a, where $|-| : \mathrm{Alg}(\Theta) \to \mathrm{Set}$ denotes the forgetful functor. This amounts to asserting that the F extends to a functor, $F : \mathrm{Set} \to \mathrm{Alg}(\Theta)$, left-adjoint to $|-|$.

Similarly, for free extensions, given an environment $\theta : V \to |\mathcal{W}|$, there is a unique way to extend a homomorphism $h : \mathcal{A} \to \mathcal{W}$ to a homomorphism $\mathcal{A}[V] \to \mathcal{W}$ that structurally evaluates terms by applying h to constants, θ to variables and reducing in $\mathcal{W}$. Figure 1b depicts this arrangement, where $i_{\mathcal{A}}$ and i_V denote the homomorphic insertions of $\mathcal{A}$ and FV into the free extension. This universal property is precisely that of the coproduct of $\mathcal{A}$ with the free algebra FV.

Definition 1.1. The *free extension* of $\mathcal{A} \in \mathrm{Alg}(\Theta)$ by $V \in \mathrm{Set}$, $\mathcal{A}[V]$, is the coproduct, $\mathcal{A} + FV$.

2 MOTIVATION

The definitions presented in §1 are grounded in universal algebra, meaning that there are many structures that do not fit this framework. For example, varieties of structured category — e.g., monoidal categories, cartiesan-closed categories — are core to contemporary programming language theory, but are not classes of models of UATs. However, many such structures do possess

*Author's email: `nathan.corbyn@cs.ox.ac.uk`.
ACM student member №: 2565044.
Student research competition category: Graduate.
Advisors: Max Schäfer (GitHub), Mario Álvarez Picallo (Huawei), & C.-H. Ong (University of Oxford).

$$\begin{tikzcd} FX \arrow[r, dashed, "\exists!\tilde{\theta}"] & \mathcal{W} \end{tikzcd}$$

$$\begin{tikzcd} {|FX|} \arrow[r, "|\tilde{\theta}|"] & {|\mathcal{W}|} \\ X \arrow[u, hook] \arrow[ur, "\theta"'] & \end{tikzcd}$$

(a) Free models as adjoint functors.

$$\begin{tikzcd} & FV \arrow[d, "i_V"] \arrow[ddr, bend left, "\tilde{\theta}"] & \\ \mathcal{A} \arrow[r, "i_{\mathcal{A}}"'] \arrow[drr, bend right, "h"'] & \mathcal{A}[V] \arrow[dr, dashed, "\exists!"] & \\ & & \mathcal{W} \end{tikzcd}$$

(b) Free extensions as coproducts with free models.

$$\begin{tikzcd} F\langle\Gamma\rangle \arrow[r, "F\pi"] \arrow[d, "\tilde{\sigma}"'] & F\langle\Gamma;\Delta\rangle \arrow[d, "i_\Delta"] \arrow[ddr, bend left, "\tilde{\theta}"] & \\ \mathcal{A} \arrow[r, "i_{\mathcal{A}}"'] \arrow[drr, bend right, "h"'] & \mathcal{A}[\sigma|\Delta] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"'] & \\ & & \mathcal{W} \end{tikzcd}$$

(c) Generalised free extensions as push-outs along weakenings.

Fig. 1. Universal properties.

characterisations in terms of the *generalised algebraic theories* (GATs) of Cartmell [2], extending UATs by equipping terms with *dependent sorts* akin to those of Martin-Löf type theory [5].

GATs support similar notions of model and free model to UATs, making it tempting to assume the characterisation of free extensions given in §1 still applies. [1] However, the following argument shows that this naïve generalistion excludes important examples, even for simple GATs.

Ordinary category theory has a straightforward presentation as a GAT, Θ_{Cat} — see appendix A, figures 2 and 3 — with (small) categories as models. Fixing a model $\mathcal{A} \in \mathrm{Alg}(\Theta_{\mathrm{Cat}}) \simeq \mathrm{Cat}$, we want to support extension by a free object $\mathtt{X}$ and a free morphism $\mathtt{f} : A \to \mathtt{X}$, for a fixed $A \in \mathcal{A}$. Assuming that such an extension is given by the coproduct of $\mathcal{A}$ with a free category, as $\mathtt{X} \notin \mathcal{A}$, we should expect to find our freely added morphism in the hom-set $\mathcal{A}[\mathtt{X}, \mathtt{f} : A \to \mathtt{X}](\iota_1 A, \iota_2 \mathtt{X})$. However, as coproducts of categories are given by disjoint pastings of their underlying quivers, every such hom-set is necessarily empty. Hence, a contradiction.

3 CONTEXTS & CONTEXTUAL CATEGORIES

The fundamental distinction between UATs and GATs is the introduction of dependent sorting. Whilst a context for a UAT is always a finite set of variables $\{\mathtt{x}_1, \cdots, \mathtt{x}_n\}$, the structure on sorts means that, for a GAT Θ, a Θ-context is a list of typed variables $\langle \mathtt{x}_1 : \tau_1; \cdots ; \mathtt{x}_n : \tau_n \rangle$, where, for $1 \leq i \leq n$, $\langle x_1 : \tau_1, \cdots, x_{i-1}\tau_{i-1} \rangle \vdash \tau_i$ *type* is derivable in Θ.

The contexts of a GAT Θ form a richly structured category known as a *contextual category* [2, 6], with the contextual category determined by Θ denoted $\mathrm{Ctx}\Theta$. $\mathrm{Ctx}\Theta$ has, as objects, Θ-contexts and, as morphisms, *realisations* — well-typed substitutions of terms in one context for variables of another. Contextual categories possess tree-like inductive structure, arising from the nature of context extension.

As well-formed first-order terms are defined with respect to contexts, it ceases to be meaningful to discuss free extensions by (finite) sets of variables — i.e., universal algebraic contexts, and instead we must consider free extensions by generalised algebraic contexts. Similarly, it is no longer meaningful to talk about the free model on a set, as the generators of a free model now require typing information. Typing information is provided by, again, considering contexts over sets. However, notice that, by the nature of substitution, the free functor $F : \mathrm{Ctx}\Theta^{\mathrm{op}} \to \mathrm{Alg}(\Theta)$ is contravariant. [2]

4 GENERALISED FREE EXTENSIONS

Before defining generalised free extensions, we must first identify the class of objects by which we intend to extend. For a GAT Θ, free extension by a *pure* Θ-context — i.e., objects of $\mathrm{Ctx}\Theta$ — is too

[1] This is something I mistakenly assumed as an undergraduate — see [3] ch. 3.

[2] This is also true in the universal algebraic case, but is masked by the fact that contexts are taken to be $\mathbb{F}^{\mathrm{op}}$ and $(C^{\mathrm{op}})^{\mathrm{op}} = C$ for any C.

limited, as it leaves us unable to refer to constants of a particular model when constructing types. In practice, this is a natural thing to do: e.g., when extending a category by a free map $\mathsf{f} : A \to \mathsf{X}$, from a fixed object A to a free object X, we should be able to refer to A. Thus, we instead consider freely extending by *partial environments*.

Given a context $\Gamma \in \mathrm{Ctx}\Theta$ and a larger context $\Gamma; \Delta \in \mathrm{Ctx}\Theta$, tracing the tree structure of $\mathrm{Ctx}\Theta$ towards its root, we obtain a canonical *weakening* $\pi : \Gamma; \Delta \to \Gamma$, which forgets the suffix Δ. Pairing a weakening $\pi : \Gamma; \Delta \to \Gamma$ with a Γ-environment yields an object which has the flavour of Δ, after substituting the variables of Γ for constants of $\mathcal{A}$ chosen by σ.

Definition 4.1. A *partial $\mathcal{A}$-environment* consists of a weakening $\pi : \Gamma; \Delta \to \Gamma$ and an environment $\sigma : \Gamma \to |\mathcal{A}|$, denoted $(\sigma|\pi)$. When π is clear, we will identify it with Δ.

The *generalised free extension* of $\mathcal{A} \in \mathrm{Alg}(\Theta)$ by a partial $\mathcal{A}$-environment $(\sigma|\Delta)$ is the push-out of the mate $\tilde{\sigma} : F\langle\Gamma\rangle \to \mathcal{A}$ along the weakening $F\pi : F\langle\Gamma\rangle \to F\langle\Gamma; \Delta\rangle$.

Let $(\sigma|\Delta)$ be a partial $\mathcal{A}$-environment and consider the prescribed push-out, pictured in figure 1c. The resulting object is a model $\mathcal{A}[\sigma|\Delta] \in \mathrm{Alg}(\Theta)$, equipped with homomorphic insertions $i_{\mathcal{A}}$ and i_{Δ}. Critically, commutativity ensures that the structure induced by Γ is not present in $\mathcal{A}[\sigma|\Delta]$, and is correctly identified with structure of $\mathcal{A}$, determined by σ. Moreover, given any model $\mathcal{W} \in \mathrm{Alg}(\Theta)$, a homomorphism $h : \mathcal{A} \to \mathcal{W}$ and an environment $\theta : \Gamma; \Delta \to |W|$, so long as θ and h agree on the mapping of Γ's induced structure into $\mathcal{W}$, there is a unique extension of h to a homomorphism $\mathcal{A}[\sigma|\Delta] \to \mathcal{W}$ that structurally evaluates elements of $\mathcal{A}[\sigma|\Delta]$ in a way compatible with h and θ.

For example, consider the free extension of a category $\mathcal{A} \in \mathrm{Alg}(\Theta_{\mathrm{Cat}})$ by a free object X and a free morphism $\mathsf{f} : A \to \mathsf{X}$ for a fixed $A \in \mathcal{A}$. We first encode this data as a partial $\mathcal{A}$-environment, taking $\Gamma = \langle \mathsf{Y} : \mathrm{Obj}\rangle$, $\Delta = \langle \mathsf{X} : \mathrm{Obj}; \mathsf{f} : \mathrm{Hom}(\mathsf{X}, \mathsf{Y})\rangle$ and σ as the map $\mathsf{Y} \mapsto A$. Computing the push-out described, we obtain a category $\mathcal{A}[\sigma|\Delta]$ into which $\mathcal{A}$ and $F\langle\Gamma; \Delta\rangle = \mathsf{X} \to \mathsf{Y}$ embed. By commutativity, the embedding of Y is identified with that of A. Thus, the free morphism f belongs to the hom-set $\mathcal{A}[\sigma|\Delta](i_{\mathcal{A}}A, i_{\Delta}\mathsf{X})$ as required.

Further, observe that when Θ determines a UAT — i.e., Θ has a single sort $*$, this push-out specialises to a push-out over $F\langle\rangle$, as variables of Γ cannot appear in Δ. As $F\langle\rangle$ is initial in $\mathrm{Alg}(\Theta)$, the push-out amounts to a coproduct, recovering the universal algebraic notion of free extension as claimed.

5 CONCLUSION & FUTURE WORK

I have instantiated the framework proposed in §4 to construct and verify, both by hand and mechanically, an effective description of the free extension of a category. Appendix B gives the inductive definitions underpinning this construction in figure 4, alongside the definition of composition in figures 5 & 6. Whilst similar in flavour to the inductive construction of the free extension of a monoid given by Yallop et al. [7], this is a key first step, supporting the correctness of the universal property given in §4.

The ultimate aim of this research is to provide a unified theoretical framework for reasoning about normalisation problems using free extensions, whether this be for simple universal algebraic structures, or feature-rich λ-calculi. With this in mind, the immediate next steps left by this work are to continue expanding the catalogue of examples of effective free extensions for GATs. Key examples of structures of interest are monoidal categories, cartesian categories and cartesian-closed categories. Beyond this, there is the question of further generalising the account of free extensions given in §4 to the second-order algebraic theories of Fiore and Mahmoud [4]. Such a generalisation would provide another lens through which to understand the evidently deep connection between free extensions of algebras and programming language theory.

A CATEGORY THEORY AS A GAT

$$\frac{}{\Gamma \vdash \mathrm{Obj}\ \mathit{type}}\ (\mathrm{Obj}) \qquad \frac{\Gamma \vdash x : \mathrm{Obj}}{\Gamma \vdash \mathrm{id}(x) : \mathrm{Hom}(x, x)}\ (\mathrm{id})$$

$$\frac{\Gamma \vdash x : \mathrm{Obj} \qquad \Gamma \vdash y : \mathrm{Obj}}{\Gamma \vdash \mathrm{Hom}(x, y)\ \mathit{type}}\ (\mathrm{Hom}) \qquad \frac{\Gamma \vdash f : \mathrm{Hom}(x, y) \qquad \Gamma \vdash g : \mathrm{Hom}(y, z)}{\Gamma \vdash g \circ f : \mathrm{Hom}(x, z)}\ (\circ)$$

Fig. 2. Generating rules for the types and terms of Θ_{Cat}.

$$\frac{\Gamma \vdash x : \mathrm{Obj} \qquad \Gamma \vdash f : \mathrm{Hom}(x, y)}{\Gamma \vdash f \circ \mathrm{id}(x) = f : \mathrm{Hom}(x, y)}\ (\circ\text{-}\textsc{unit}_R) \qquad \frac{\Gamma \vdash y : \mathrm{Obj} \qquad \Gamma \vdash f : \mathrm{Hom}(x, y)}{\Gamma \vdash \mathrm{id}(y) \circ f = f : \mathrm{Hom}(x, y)}\ (\circ\text{-}\textsc{unit}_L)$$

$$\frac{\Gamma \vdash f : \mathrm{Hom}(x, y) \qquad \Gamma \vdash g : \mathrm{Hom}(y, z) \qquad \Gamma \vdash h : \mathrm{Hom}(z, w)}{\Gamma \vdash (h \circ g) \circ f = h \circ (g \circ f) : \mathrm{Hom}(x, w)}\ (\circ\text{-}\textsc{assoc})$$

Fig. 3. Generating rules for the equations of Θ_{Cat}.

B FREE EXTENSIONS OF CATEGORIES

$$\frac{}{[] \in N(x, x)} \qquad \frac{g \in (\sigma|\Delta)(y, z) \qquad p \in M(x, y)}{g :: p \in N(x, z)}$$

(a) Neutral forms.

$$\frac{f \in \mathcal{A}(y, z) \qquad p \in N(x, \iota_1 y)}{f :: p \in M(x, \iota_1 z)} \qquad \frac{p \in N(x, \iota_2 y)}{\mathrm{lift}(p) \in M(x, \iota_2 y)}$$

(b) Normal forms.

Fig. 4. Inductive construction of the free extension of a category.

$$\widehat{(-)} : N(x, y) \longrightarrow M(x, y)$$
$$p \mapsto \begin{cases} 1_{y'} :: p & y = \iota_1 y' \\ \mathrm{lift}(p) & y = \iota_2 y' \end{cases}$$

(a) Lift neutral forms to normal forms.

$$\delta_g : M(x, y) \longrightarrow M(x, z)$$
$$p \mapsto \widehat{g :: p}$$

(b) Post-compose a free morphism $g : y \longrightarrow z$.

$$\sigma_h : M(x, \iota_1 y) \longrightarrow M(x, \iota_1 z)$$
$$f :: p \mapsto (h \circ f) :: p$$

(c) Post-compose a concrete morphism $h : y \longrightarrow z$.

Fig. 5. Smart-constructors for normal forms.

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$$\begin{aligned} \cdot : N(y,z) \times M(x,y) &\longrightarrow M(x,z) \\ [] \cdot p &\mapsto p \\ (g :: q) \cdot p &\mapsto \delta_g(q \circ p) \end{aligned}$$

(a) Action of neutral forms on normal forms.

$$\begin{aligned} \circ : M(y,z) \times M(x,y) &\longrightarrow M(x,z) \\ \mathrm{lift}(q) \circ p &\mapsto q \cdot p \\ (f :: q) \circ p &\mapsto \sigma_f(q \cdot p) \end{aligned}$$

(b) Composition of normal forms.

Fig. 6. Mutually inductive definition of composition of normal forms.

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