Understanding the classical monad-theory correspondence

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Objective

• Prove the equivalence:

 $Law \simeq Mnd_{fin}(Set)$

Objective

• ProveUnderstand the equivalence:

 $Law \simeq Mnd_{fin}(Set)$

Plan

Take three steps:

$$\begin{aligned} \operatorname{Law} &\simeq \operatorname{ProMnd}_{\times}(\mathbb{F}^{\operatorname{op}}) \\ &\simeq \operatorname{RMnd}(\mathbb{F} \hookrightarrow \operatorname{Set}) \\ &\simeq \operatorname{Mnd}_{fin}(\operatorname{Set}) \end{aligned}$$

- Although not necessary to write down a proof, hopefully these all feel very natural by the end!
- Arkor's thesis takes this approach in extreme generality

Outline

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Part I
       Lawvere theories
       Cartesian promonads
       Law \simeq ProMnd_{\times}(\mathbb{F}^{op})
Part II
       Finitary monads
       Relative monads
       \operatorname{RMnd}(\mathbb{F} \hookrightarrow \operatorname{Set}) \simeq \operatorname{Mnd}_{fin}(\operatorname{Set})
Part III
       \operatorname{ProMnd}_{\times}(\mathbb{F}^{\operatorname{op}}) \simeq \operatorname{RMnd}(\mathbb{F} \hookrightarrow \operatorname{Set})
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Part I

- Presentation invariant descriptions of algebraic theories
- Can be quite tricky to wrap your head around

• Consider a presentation of the theory of monoids:

$$u:0$$
 $\oplus:2$

$$u \oplus x = x$$
$$x \oplus u = x$$
$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

We get lots of derivable operations—e.g.,

$$x \oplus (y \oplus (z \oplus w))$$

$$x \oplus x$$

$$(u \oplus x) \oplus (x \oplus u)$$

$$(x \oplus y) \oplus (z \oplus w)$$

• Some of these are provably equal:

$$\mathbf{x} \oplus \mathbf{x} = (\mathbf{u} \oplus \mathbf{x}) \oplus (\mathbf{x} \oplus \mathbf{u})$$
$$(\mathbf{x} \oplus \mathbf{y}) \oplus (\mathbf{z} \oplus \mathbf{w}) = \mathbf{x} \oplus (\mathbf{y} \oplus (\mathbf{z} \oplus \mathbf{w}))$$

Important: Copying and discarding of variables is allowed

- Lawvere's idea: no matter how you present the theory, the same operations should be derivable and satisfy the same equations
- A Lawvere theory bundles derivable operations and their equations into a category

- What does this look like?
- For each $n \in \mathbb{N}$, write down the set of derivable operations in at most n variables, modulo provable equality:

• Extend this to all $m \in \mathbb{N}$ taking

$$T(n,0) = \{\star\}$$

$$T(n,m+1) = T(n,m) \times T(n,1)$$

In other words, T(n, m) consists of tuples of m operations each in (at most) n variables

- **Idea**: T(-,=) describes the hom-sets of a category.
- Composition is substitution!

$$T(n,m)$$
 \times $T(m,1)$
 m operations in n variables 1 operation in m variables

 $T(n,1)$
substitute the m variables for the m derived operations

• We get more than just a category, we get a *cartesian* category:

$$n \stackrel{\langle \mathbf{x}_i \rangle_{i \in \underline{n}}}{\longleftarrow} n + m \stackrel{\langle \mathbf{x}_{i+n} \rangle_{i \in \underline{m}}}{\longrightarrow} m \qquad \qquad n \stackrel{\langle \rangle}{\longrightarrow} 0$$

 This is intimately connected with the fact that we've allowed variables to be copied and discarded

Relationship to \mathbb{F}^{op}

- Call $\mathbb F$ the category with objects $n\in\mathbb N$ and morphisms $\mathbb F(n,m)=\underline n o\underline m$
- Claim: \mathbb{F}^{op} corresponds to the Lawvere theory determined by the empty presentation:
 - The only derivable operations are the variables:

$$T_{\varnothing}(n,1) = \underline{n}$$

 $\cong \mathbb{F}(1,n)$

Extending,

$$T_{\varnothing}(n,0) = \{\star\} \cong \mathbb{F}(0,n)$$
 $T_{\varnothing}(n,m+1) = T_{\varnothing}(n,m) \times T_{\varnothing}(n,1)$
 $\cong \mathbb{F}(m,n) \times F(1,n)$
 $\cong \mathbb{F}(m+1,n)$

Relationship to \mathbb{F}^{op}

- If we have two presentations $\Theta \subseteq \Theta'$, we get a (unique) corresponding identity-on-objects functor $T_{\Theta} \to T_{\Theta'}$
 - Hand-waving: All the operations derivable in the old theory are still derivable in the new theory, but might be identified via the new equations
- In particular, we always have a (unique) identity-on-objects functor

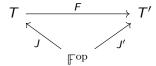
$$\mathbb{F}^{op} \to \mathcal{T}_\Theta$$

- Moreover, this functor will always preserve products strictly
- We can use this to define Lawvere theories semantically!

Semantic definition

 Definition: a Lawvere theory is a category T equipped with a strictly product-preserving identity-on-objects functor
 J: F^{op} → T

• We obtain a category Law of Lawvere theories and triangles:



• **Definition**: A promonad is a monoid in the category of endoprofunctors on a category $\mathcal C$

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• **Definition**: A promonad is a profunctor $P: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ equipped with maps

$$\mu: \int^{c:\mathcal{C}} P(-,c) \times P(c,=) \to P(-,=)$$
$$\eta: \mathcal{C}(-,=) \to P(-,=)$$

subject to...

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subject to...

- Promonads are an extremely useful way to build new categories from old ones
- Promonads show up everywhere but aren't given the credit they deserve

- ullet Say we have some category ${\mathcal C}$
- ullet We love the objects of ${\mathcal C}$
- But the morphisms are a bit of a disappointment:
 - Missing some maps
 - Not enough equations
 - Still important though!
- Promonads are a technical tool for describing the morphisms we wish we had and how they relate to the morphisms we've got right now
- If we set things up properly, we get a new category $\mathcal D$ with the same objects as $\mathcal C$ and an identity-on-objects functor $\mathcal C \to \mathcal D$

- How do we do this?
- Start with a profunctor $P(-,=): \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$
 - For each $x, y \in C$, P(x, y) is the hom-set you wish you had
 - The functorial actions tell you how to compose your dream maps with your disappointments
- Ask for a natural transformation

$$\eta: \mathcal{C}(-,=) \rightarrow P(-,=)$$

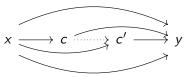
- Every disappointment has something to live up to
- Not necessarily injective

- We've also got to explain how to compose our ideal maps
- For each $c \in \mathcal{C}$ we *want* to say we have a natural transformation:

$$\mu_c: P(-,c) \times P(c,=) \rightarrow P(-,=)$$

• But it's a bit more subtle!

 We need all our composition operations to line up with each other:

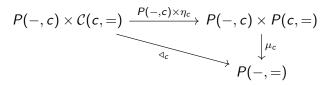


 The fancy way to say this is that we have a single natural transformation

$$\mu:\int^{c:\mathcal{C}}P(-,c)\times P(c,=)\to P(-,=)$$

Jumping straight to coends is missing the point

- Finally, we need a couple of laws to hold:
 - Composition should be associative
 - Lifting and composing should agree with acting



• We get a category $\operatorname{ProMnd}(\mathcal{C})$ whose objects are promonads on \mathcal{C} and morphisms are natural transformations which respect the composition and lifting

- What's this bought us?
- We certainly get a functor:

$$\operatorname{ProMnd}(\mathcal{C}) \to (\mathcal{C}/\operatorname{Cat})_{\mathsf{ioo}}$$

• In fact, we get an equivalence:

$$\operatorname{ProMnd}(\mathcal{C}) \simeq (\mathcal{C}/\operatorname{Cat})_{\mathsf{ioo}}$$

Cartesian promonads

You may have spotted, we've got the following:

$$\mathrm{Law} = (\mathbb{F}^\mathrm{op}/\mathrm{Cat})_{\mathsf{ioo},\times} \hookrightarrow (\mathbb{F}^\mathrm{op}/\mathrm{Cat})_{\mathsf{ioo}} \simeq \mathrm{ProMnd}(\mathbb{F}^\mathrm{op})$$

• Can we restrict the right-hand side to get an equivalence?

Cartesian promonads

- We only want to consider promonads which induce cartesian functors
- Consider $(P : \mathbb{F} \times \mathbb{F}^{\mathrm{op}} \to \mathrm{Set}, \mu, \eta)$ a promonad on \mathbb{F}^{op}
- Because the induced functor is identity-on-objects, it will strictly preserve products iff our dream maps still validate the universal properties
- In other words,

$$P(-,0) \cong \top$$

 $P(-,n+m) \cong P(-,n) \times P(-,m)$

naturally in m and n

Cartesian promonads

$$P(-,0) \cong \top$$

 $P(-,n+m) \cong P(-,n) \times P(-,m)$

Key point: this is the same as asking that the curried functor

$$P: \mathbb{F} \to [\mathbb{F}^{\mathrm{op}}, \mathrm{Set}]$$

lands in cartesian functors

We want cartesian profunctors:

$$P: \mathbb{F} \to [\mathbb{F}^{\mathrm{op}}, \mathrm{Set}]_{\times}$$

The first equivalence

 $\mathrm{Law} \simeq \mathrm{ProMnd}_{\times}(\mathbb{F}^{\mathrm{op}})$

Part II

 \bullet A monad is a monoid in the category of endofunctors on a category ${\cal C}$

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ullet A monad is a lax 2-functor ${f 1}
ightarrow {
m Cat}$

 \bullet A monad is a lax 2-functor $1 \to \mathrm{Cat}$

- Monads are a technical tool for describing algebraic structures internal to general categories
- Rather than take a syntactic approach, monads are fundamentally semantically motivated
- The monad-theory correspondence for Set essentially says that the semantic approach and the syntactic approach are secretly the same
- Mumble mumble technicalities...

- Fundamental observation: an algebra is an object equipped with some operations we can somehow evaluate
- Take an object $x \in \mathcal{C}$, what does it mean to evaluate in x?
- Choose another object $Tx \in \mathcal{C}$ of 'computations' and a map

$$a: Tx \rightarrow x$$

- How do we make sure we've chosen a sensible notion of computation?
- First, we make $T: \mathcal{C} \to \mathcal{C}$ an endofunctor:
 - The notion of computation should be independent of the specific x I've chosen
 - \circ Functoriality says that T can't 'see' x

 If we've already got a (generalised) element of x, we should have a 'do nothing' computation:

$$\eta_x: x \to Tx$$

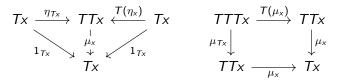
 Similarly, if I have a computation that computes a computation, this should reduce to a single computation that works out what it needs to do and does it:

$$\mu_{\mathsf{x}}: TT\mathsf{x} \to T\mathsf{x}$$

These should be natural (again, we shouldn't look at x):

$$\eta: 1_{\mathcal{C}} \to T \qquad \qquad \mu: TT \to T$$

 The monad laws express three more sensible properties of computation when you think in these terms!



• We get a category $\mathrm{Mnd}(\mathcal{C})$ whose objects are monads on \mathcal{C} and whose morphisms are natural transformations preserving all the structure

Finitary monads

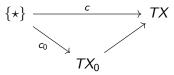
 Disclaimer: this part is quite technical, so I'm going to brush over a lot of details, but hopefully the picture still comes out!

Finitary monads

- We're not always interested in notions of computation in the most general sense
- Sometimes we want to ensure that our computations are somehow 'finitely describable':
 - o If we have a monad T on Set , we might hope that a computation $c \in TX$ can be described using at most finitely many elements of X
 - For example, computations might be formal sums of at most finitely many elements
- If we're looking for a connection with universal algebra, this is certainly a sensible restriction!

Finitary monads

- Category theory gives us some very technical, but very useful, notions of finiteness in general categories
- As we're only really interested in monads on Set today:
 - A monad T on Set is finitary iff for every set X and element $c \in TX$, there is a finite subset $X_0 \subseteq X$ through which c factors:



- We'll call the full subcategory of $\operatorname{Mnd}(\operatorname{Set})$ spanned by finitary monads $\operatorname{Mnd}_{\mathit{fin}}(\operatorname{Set})$
- It only really matters what T does to finite sets!

• A monad relative to a functor $J:\mathcal{C}\to\mathcal{E}$ is a monoid in the skew-monoidal category $([\mathcal{C},\mathcal{E}],\circ^J,J)$

• A monad relative to a functor $J: \mathcal{C} \to \mathcal{E}$ is a monoid in the skew-monoidal category $([\mathcal{C}, \mathcal{E}], \circ^J, J)$

- Relative monads are a technical tool for describing notions of computation constrained to some particular diagram in a category
- The computations might form objects in a much larger category, but are only described for a (potentially) smaller system of objects

- What does this look like?
- First, pick your system of objects $J: \mathcal{C} \to \mathcal{E}$
- For each object $x \in \mathcal{C}$, define the computations $Tx \in \mathcal{E}$
- Again, we want 'do nothing' computations

$$\eta_{\mathsf{X}}: \mathsf{J}\mathsf{X} \to \mathsf{T}\mathsf{X}$$

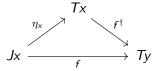
- This time, we can't necessarily build computations that compute computations
- However, we can introduce a mechanism for *sequencing* computations

$$(-)^{\dagger}: \mathcal{E}(Jx, Ty) \to \mathcal{E}(Tx, Ty)$$

I find it helps to think of maps $Jx \to Ty$ as computations with a 'parameter'

 Similar to monads, we have some sensible properties we expect to hold

$$(\eta_x)^\dagger = 1_x$$



$$Jx \xrightarrow{f} Ty \xrightarrow{g^{\dagger}} Tz$$

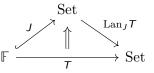
$$Tx \xrightarrow{f^{\dagger}} Ty \xrightarrow{g^{\dagger}} Tz$$

- These laws automatically guarantee functoriality of T and naturality of η and $(-)^{\dagger}!$
- For each $J: \mathcal{C} \to \mathcal{E}$, we get a category $\mathrm{RMnd}(J)$ of monads relative to J and natural transformations preserving the structure

The second equivalence

 $\mathrm{RMnd}(\mathbb{F} \hookrightarrow \mathrm{Set}) \simeq \mathrm{Mnd}_{\mathit{fin}}(\mathrm{Set})$

The picture



Part III

The third equivalence

 $\operatorname{ProMnd}_{\times}(\mathbb{F}^{\operatorname{op}}) \simeq \operatorname{RMnd}(\mathbb{F} \hookrightarrow \operatorname{Set})$

What we've got

- P: F → [F^{op}, Set]_×
 η: F^{op}(-,=) → P
 μ: ∫^{c:C} P(-,c) × P(c,=) → P
- ...

- $T : \mathbb{F} \to Set$
- $\eta: J \to T$
- $(-)^{\dagger} : \operatorname{Set}(J(-), T(=)) \to \operatorname{Set}(T(-), T(=))$
- ...

Key ingredient

 $[\mathbb{F}^{\mathrm{op}}, \mathrm{Set}]_\times \simeq \mathrm{Set}$

What we've got

• $P: \mathbb{F} \to [\mathbb{F}^{op}, \operatorname{Set}]_{\times}$

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• \eta : \mathbb{F}^{\mathrm{op}}(-, =) \to P

• \mu : \int^{c:\mathcal{C}} P(-, c) \times P(c, =) \to P

• ...

• T : \mathbb{F} \to \mathrm{Set} \simeq [\mathbb{F}, \mathrm{Set}]_{\times}

• \eta : J \to T \eta : \mathbb{F}^{\mathrm{op}}(-, =) \to T

• (-)^{\dagger} : \mathrm{Set}(J(-), T(=)) \to \mathrm{Set}(T(-), T(=))
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 $T(m,n) \cong [\mathbb{F}^{op}, Set]_{\times}(\mathbb{F}^{op}(n,-), Tm) \to [\mathbb{F}^{op}, Set]_{\times}(Tn, Tm)$

Summary

- We've covered a lot of ground:
 - Lawvere theories
 - Finitary monads
 - Relative monads
 - Promonads
 - Ways these all link up!
- Hopefully you've got some intuitions for some of these and can go away and look at the details

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