Chapter 6. Stochastic Process.

6.1. Definitions and classifications.

A stochastic process is a random variable that also depends on time.

$X(t, w)$ → state

$w \in \Omega$, an outcome of an experiment.

$t \in T$, time

If we fix time x, $X_t(w)$ is a function of a random outcome.
if we fix ω, $X_w(t)$ is a realization, a sample path, or a trajectory of a process $X(t, w)$.

If $X_t(w)$ is discrete, then $X(t, w)$ is discrete-state.
If $X_t(w)$ is continuous, then $X(t, w)$ is continuous-state.

If the set of Times, T, is discrete, $X(t, w)$ is a discrete-time process
if T is a connected, possibly unbounded interval, then $X(t, w)$ is a continuous-time process.

6.3. Counting Processes.

A stochastic process X is counting if $X(t)$ is the number of item counted by the time t.

All counting processes are discrete-state.

Binomial Process (discrete-time, discrete-space)

Binomial process $X(n)$ is the number of successes in the first n independent Bernoulli trials, where $n = 0, 1, 2, \cdots$

Arrival rate $\lambda = \frac{p}{\Delta}$ is the average number of successes per one unit of time. The time interval Δ of each Bernoulli trial is called a frame.

$p \to$ probability of arrival (success) during one frame.

The interarrival time (T) is the time between successes.

☆ The key assumption in such models is that no more than 1 arrival is allowed during each Δ-second frame, so each frame is a Bernoulli trial.

$X(\frac{t}{\Delta})$ is the number of arrivals by the time t.

The interarrival period consists of a Geometric number of frames Y, each frame taking Δ seconds.

$T = Y \cdot \Delta$. $\to$ waitting time.

$Y = \text{Geometric}(p)$. $\to$ number of frames.

$X(n) = \text{Binomial}(n, p)$

$n = \frac{t}{\Delta}$

$$\boxed{\begin{aligned} E(T) &= E(Y) \cdot \Delta = \frac{1}{p} \cdot \Delta = \frac{1}{\lambda}. \\ Var(T) &= \frac{1-p}{\lambda^2}. \end{aligned}}$$

Poisson Process.

Poisson process is a continuous-time counting stochastic process obtained from a binomial counting process when its frame size Δ decreases to 0 while the arrival rate λ remains constant.

$\Delta \downarrow 0$, smaller frames.

$n \uparrow \infty$, more and more frames

λ frequency stays constant.

$p = \Delta \cdot \lambda \to 0$, small, rare event.

$\boxed{\sigma = std(T_k) = \frac{\sqrt{k}}{\lambda}}$ Time $\uparrow$ $E(T_k)$

Therefore $\begin{cases} \boxed{\mu = E(T_k) = \frac{k}{\lambda}} \cdots k = \text{total numbers}, \lambda \text{ is frequency} \\ E(X(t)) = np = \frac{tp}{\Delta} = \lambda t. \end{cases}$

$X(t) = Binomial(n, p) \to Poisson(\lambda)$.

$F_T(t) = P\{T \leq t\} = P\{Y \leq n\}$. .... $T = Y \cdot \Delta$, $t = n \cdot \Delta$

$= 1 - (1-p)^n$

$\approx 1 - e^{-\lambda t}$

$P\{T_k \leq t\} = P\{k\text{-th arrival before time } t\}$

$= P\{X(t) \geq k\}$.

$P\{T_k > t\} = P\{X(t) < k\}$. --- possion list.

wait time until the kth arrival

$\begin{cases} E(T) = \frac{1}{\lambda} \to Var(T) = \frac{1}{\lambda^2}, \sigma = \frac{1}{\lambda} \\ E(T_k) = \frac{k}{\lambda} \to Var(T_k) = \frac{k}{\lambda^2}, \sigma = \frac{\sqrt{k}}{\lambda}. \end{cases}$

$P\{\underbrace{X(t)=x}_{\text{x arrivals in time t.}}\} = e^{-\lambda t}\frac{(\lambda t)^x}{x!}$ ... Poisson distribution.

λ: frequency.

T has exponential distribution.

$$\begin{cases} P\{T \le t\} = 1 - e^{-\lambda t} & \text{--- Next success arrives ~~at~~ in next t time.} \\ P\{T > t\} = e^{-\lambda t} & \text{--- Not arrive in next t time.} \end{cases}$$

6.2. Markov processes and Markov chains.

Stochastic process $X(t)$ is Markov if for any $\underbrace{t_1 < \cdots <}_{\text{past}} \underset{\text{present}}{t_n} < \underset{\text{future}}{t}$.

and any sets A; $A_1, A_2, \cdots, A_n$.

$$\underset{\text{future}}{P\{X(t)\in A} \overset{\text{given.}}{|} \underbrace{X(t_1)\in A_1, \cdots,}_{\text{past}} \underbrace{X(t_n)\in A_n}_{\text{present.}}\} = P\{X(t)\in A \mid X(t_n)\in A_n\}$$

future.

That is, if Markov

$P\{\text{future} \mid \text{past}, \text{present}\} = \underline{P\{\text{future} \mid \text{present}\}}$.

if Markov, only its present state is important.

A Markov chain is a stochastic process that is

i). discrete time. → use $\{0, 1, 2, \cdots\}$. t.h.

ii). discrete state. → use $\{0, 1, 2, \cdots, n\}$. from $0 \to \infty$. i, j, n, x.

iii). Markov.

$$P\{X(t+1)=i \mid X(0)=a, X(1)=b, X(2)=c, \cdots, X(t)=j\}$$

$$= P\{X(t+1)=i \mid X(t)=j\}.$$

From $t \to t+1$ one step "transition Probability" denote $P_{ij}(t)$. (lower case) h-step transition probability

$$P_{ij}^{(h)}(t) = P\{X(t+h)=j \mid X(t)=i\}.$$

h steps. of time.

If all its transition probabilities are independent of t, a Markov chain is homogeneous, that means transition from i to j has the same probability at any time.

By the Markov property, each next state should be predicted from the previous state only.

Therefore, the distribution of a Markov chain is completely determined by the initial distribution P_0 and one-step transition probabilities P_{ij}

$$P_0(x) = P\{X(0)=x\} \quad \text{for } x \in \{1, 2, \cdots, n\}.$$

time. 0.

Given that, we find:

i) h-step transition probability $P_{ij}^{(h)}$

ii). P_h, the distribution of states at time h, the forecast for $X(h)$.

iii). The limit of $P_{ij}^{(h)}$ and P_h as $h \to \infty$.

Notation:

$P_{ij} = P\{X(t+1)=j \mid X(t)=i\}$, transition probability.

$P_{ij}^{(h)} = P\{X(t+h)=j \mid X(t)=i\}$, h-step transition probability.

$P_t(x) = P\{X(t)=x\}$, distribution of states at time t.

$P_0(x) = P\{X(0)=x\}$, initial distribution.

Matrix approach:

All one-step transition probabilities P_{ij} can be written in an $n \times n$ transition probability matrix.

$$\begin{array}{cc} & \text{From state:} \\ P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \\ \text{to state: } 1 \quad 2 \quad \cdots \quad n & \end{array}$$

Then h-step transition probability matrix $P^{(h)} = \underbrace{P \cdot P \cdots P}_{h \text{ times.}} = P^h$

Note: Matrix multiplication: $n \times p$

A is $n \times m$ matrix, B is $m \times p$ matrix.

Then $A \times B = (AB)_{ij}$, where $1 \leq i \leq n$, $1 \leq j \leq p$

and $(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$.

Distribution of $X(h)$

The distribution of states after h transitions, or the probability mass functi of $X(h)$, can be written in a $1 \times n$ matrix:

$$P_h = (P_h(1), P_h(2), \cdots, P_h(n)).$$

$$= P_0 \cdot P^h$$

P_0: $h=0$ $[1 \times n]$ matrix

P^h: $[n \times n]$ h-step transition probability.

The initial distribution of X.

Steady-state distribution:

$$\pi(x) = \lim_{h \to \infty} P_h(x) = (\pi_1, \pi_2, \cdots, \pi_n)$$

and $\pi_1 + \pi_2 + \dots + \pi_n = 1$.

In this case. $\pi \cdot P = \pi$. when $n \to \infty$

A Markov chain is regular if

$P_{ij}^{(h)} > 0$ for some h and all i, j.

A regular Markov chain has a steady state.

for some h, if all P_{ij} are positive, then it is regular.

Homework 7.

Discrete time process: view values of variables as occurring at distinct, separate "points in time", or equivalently as being unchanged throughout each non-zero region of time (time period), that is, time is viewed as a discrete variable

Continuous time process: view variables as having a particular value for potentially only an infinitesimally short amount of time.

Between any two points in time there are an infinite number of other points in time. The variable "time" ranges over the entire real number line.

Counting process: is a stochastic process $\{N(t), t \geq 0\}$ with values that are positive, integer; and increasing.

#4. Messages arrive at an interactive message center according to a counting process with the average inter-arrival time of 15 seconds, choosing a frame size of 5 seconds, compute the probability that during 200 minutes of operation, no more than 750 messages arrive.

Solution:

$$E(T) = 15\,s = \frac{1}{\lambda} = \frac{1}{4}\,\text{min} \Rightarrow \lambda = 4 \text{ arrivals/min}.$$

$$\Delta = 5\,s. \quad \cancel{t} = 200\,\text{min}, \quad n = \frac{t}{\Delta} = \frac{200 \times 60}{5} = 2400.$$

$$p = \lambda \cdot \Delta = 4 \cdot \frac{1}{12} = \frac{1}{3}, \quad q = 1 - p = \frac{2}{3}.$$

Therefore. $E(X(t=200\,\text{min})) = n \cdot p = 2400 \times \frac{1}{3} = 800.$

$$\sigma = \sqrt{npq} = \sqrt{2400 \cdot \frac{1}{3} \cdot \frac{2}{3}} = 23.09.$$

$$P\{X(t=200) \leq 750\} = P\{-0.5 < X(t=200) < 750.5\}.$$

$$= \Phi\left(\frac{-0.5-800}{\text{[illegible]}} < Z < \frac{750.5-800}{\text{[illegible]}}\right) = 0.0162.$$