Non-regular language (Pumping Lemma)

The Pumping Lemma:

Given an infinite regular language L, there exists an integer m (critical length), for any string $w \in L$ with length $|w| \geq m$.

We can write $w = xyz$, with $|xy| \leq m$, $|y| \geq 1$.

such that $xy^iz \in L$, $i = 0, 1, 2, \cdots$

Every non-regular language has to be of infinite size, and every language of finite size has to be regular since we can easily construct an NFA that accepts every string in the language.

How to prove that an infinite language L is not regular?

1. Assume L is regular, then the pumping lemma should hold for L.
2. Use the pumping lemma to obtain a contradiction:
   i). Let m be the critical length for L.
   ii). choose a particular string $w \in L$ which satisfies the length condition $|w| \geq m$.
   iii). Write $w = xyz$.
   iv). Now try to show $w' = xy^iz \notin L$ for some i.
   This gives a contradiction since from pumping lemma we know $w' = xy^iz \in L$.
   v). Therefore, L is not regular.

More Applications of the Pumping Lemma.

1. The language $L = \{vv^R : v \in \Sigma^*\}$ $\Sigma = \{a, b\}$, is not regular.

Proof: pick a string w such that: $w \in L$ and length $|w| \geq m$.

$$\text{Let } w = a^m b^m b^m a^m = xyz \Rightarrow y = a^k, \ 1 \leq k \leq m$$

From the pumping Lemma: $xy^iz \in L$, $i = 0, 1, 2, \cdots$

$$\Rightarrow xy^2z \in L \Rightarrow a^{m+k} b^m b^m a^m \in L \Rightarrow \text{contradiction}.$$

2. The language $L = \{a^n b^l c^{n+l} : n, l \geq 0\}$ is not regular.

Proof: Let m be the critical length of L.

Pick a string w such that: $w \in L$ and length $|w| \geq m$.

We pick $w = a^m b^m c^{2m} = xyz$. $\Rightarrow y = a^k, \ 1 \leq k \leq m$.

From the pumping Lemma: $xy^iz \in L$, $i = 0, 1, 2, \cdots$

$$\Rightarrow xy^0z = xz \in L \Rightarrow a^{m-k} b^m c^{2m} \in L. \ 1 \leq k \leq m \Rightarrow \text{contradiction}.$$

3. The language $L = \{a^{n!} : n \geq 0\}$ is not regular. $n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$.

Proof: Pick $w = a^{m!} = xyz$ with $|xy| \leq m$, $|y| \geq 1$

$$\Rightarrow y = a^k, \ 1 \leq k \leq m$$

From the pumping Lemma: $xy^iz \in L$, $i = 0, 1, 2, \cdots$

Thus. $xy^2z \in L$. $\Rightarrow a^{m!+k} \in L \Rightarrow$

There must exist p such that $m! + k = p!$

However, $m! + k \leq m! + m$

$\leq m! + m!$

$< m \cdot m! + m! = (m+1) \cdot m! = (m+1)!$

$\Rightarrow m! + k < (m+1)! \Rightarrow m! + k \neq p!$ for any p.

$\Rightarrow$ contradiction.