18.100B Problem Set 10

Due Friday December 8, 2006 by 3 PM

Problems:

1) Let (f_n) be the sequence of functions on $\mathbb{R}$ defined as follows.
$$f_0(t) = \sin t \quad \text{and} \quad f_{n+1}(t) = \frac{2}{3} f_n(t) + 1 \quad \text{for } n \in \mathbb{N}.$$
Show that $f_n \to 3$ uniformly on $\mathbb{R}$. What can you say if we choose $f_0(t) = t^2$?
Hint: Consider first the map $T(x) = \frac{2}{3}x + 1$ on $\mathbb{R}$.

2) Suppose $\varphi : [0, \infty) \to \mathbb{R}$ is continuous and satisfies
$$0 \leq \varphi(t) \leq \frac{t}{2+t} \qquad (t \geq 0).$$
Define the sequence (f_n) by setting $f_0(t) = \varphi(t)$ and $f_{n+1}(t) = \varphi(f_n(t))$ for $t \geq 0$ and $n \in \mathbb{N}$. Prove that the series $F(t) = \sum_{n=0}^{\infty} f_n(t)$ converges for every $t \geq 0$ and that F is continuous on $[0, \infty)$.

3) Does $f(t) = \sum_{k=1}^{\infty} \sin^2(t/k)$ define a differentiable function on $\mathbb{R}$?

4) Suppose (f_n) is a sequence of continuous functions such that $f_n \to f$ uniformly on a set E. Prove that
$$\lim_{n\to\infty} f_n(x_n) = f(x)$$
for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

5) Suppose (f_n) is a sequence of real-valued functions that are Riemann-integrable on all compact subintervals of $[0, \infty)$. Assume further that:
   a) $f_n \to 0$ uniformly on every compact subset of $[0, \infty)$;
   b) $0 \leq f_n(t) \leq e^{-t}$ for all $t \geq 0$ and $n \in \mathbb{N}$.

   Prove that
$$\lim_{n\to\infty} \int_0^{\infty} f_n(t)\, dt = 0,$$
where the improper integral $\int_0^{\infty} f_n(t)\, dt$ is defined as $\lim_{b\to\infty} \int_0^b f_n(t)\, dt$. Moreover, give an explicit example for a sequence (f_n), so that condition b) does not hold and the conclusion above fails.

   Remark: In fact, one can relax condition a) to "$f_n \to 0$ pointwise on $[0, \infty)$." But then the proof (of this "dominated convergence theorem") becomes by far more involved when using Riemann's theory of integration.

6) Suppose (f_n) is an equicontinuous sequence of functions on a compact set K, and $f_n \to f$ pointwise on K. Prove that $f_n \to f$ uniformly on K.

7) Show that any uniformly bounded sequence of differentiable functions on a compact interval with uniformly bounded derivatives has a convergent subsequence.
(*Hint:* To apply the Arzela-Ascoli theorem (Thm 7.25) from the book, show that any family $\mathcal{F}$ of real-valued, differentiable functions f defined on $[a, b]$, satisfying $|f'(x)| \leq M$ for some M and all $x \in [a, b]$ and $f \in \mathcal{F}$, must be equicontinuous.)

Extra problems:

1) In class, we have seen that uniform convergence of a sequence of bounded functions on a set E can be equivalently formulated in terms of the metric $d(f, g) = \sup_{x \in E} |f(x) - g(x)|$. That is, we have $d(f_n, f) \to 0$ if and only if $f_n \to f$ uniformly on E.

Having this in mind, we could ask whether an analogous statement holds with respect to pointwise convergence. More specifically, is there a metric $d(f, g)$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise on E? Surprisingly, it turns out that the answer is NO when, for example, $E = [0, 1]$. We therefore cordially invite you to prove the following theorem.

Theorem 1. *There is no metric $d(f, g)$ defined on $C([0, 1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise on $[0, 1]$.*

Before proving this theorem, you may first show the following weaker statement whose proof requires less effort.

Theorem 2. *There is no norm $\|\cdot\|$ defined on $C([0, 1])$ such that $\|f_n\| \to 0$ if and only if $f_n \to 0$ pointwise on $[0, 1]$.*

Hint (for proof of Theorem 2). Consider $f_n \in C([0, 1])$, with $n = 1, 2, 3, \ldots$, such that

i) For every $x \in [0, 1]$, there exists $n_0 = n_0(x)$ such that $f_n(x) = 0$ if $n \geq n_0$.
ii) $f_n \not\equiv 0$ for every $n \geq 1$.

(A possible choice is, for instance, given by $f_n(x) = \sin(n\pi x)$ if $x \in [0, 1/n]$, and $f_n(x) = 0$ if $x \in [1/n, 1]$.) By clever choice of a sequence of real-valued numbers (c_n), prove the claim by considering the sequence $(c_n f_n)$.

Hint (for proof of Theorem 1). Assume there is such a metric $d(f, g)$ on $C([0, 1])$. Then $f_n \to 0$ if and only if, for every $k \in \mathbb{N}$, we have that $f_n \in N_{1/k}(0) = \{y \in C([0, 1]) : d(y, 0) < 1/k\}$, except for finitely many f_n. Use this fact and the specific choice $g_n(x) = e^{-n|x - x_0|}$ for suitable $x_0 \in [0, 1]$ to show that $g_n \to 0$ pointwise on $[0, 1]$, which is false! (Since $g(x_0) = 1$ for all n.)

2) Assume that (f_n) is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence (n_k) such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}$. (This result is usually called *Helly's selection theorem.*)

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

Hint: (i) Some subsequence (f_{n_i}) converges at all rational points r, say, to $f(r)$. (ii) Define $f(x) = \sup_{r \leq x} f(r)$ for any $x \in \mathbb{R}$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. (This is where montonicity is strongly used.) (iv) A subsequence of (f_{n_i}) converges at every point of discontinuity of f, since there are at most countably many such points. This outlines the proof of (a). To prove (b), modify your proof of (iii) appropriately.