18.100B Problem Set 3

Due Friday September 29, 2006 by 3 PM

Problems:

1) (10 pts) In vector spaces, metrics are usually defined in terms of *norms* which measure the length of a vector. If V is a vector space defined over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $\|\cdot\|$ satisfying:
   i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$,
   ii) For any $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \|x\|$,
   iii) $\|x + y\| \leq \|x\| + \|y\|$.
   Prove that every norm defines a metric.

2) (10 pts) Let M be a metric space with metric d. Show that d_1 defined by
$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$
is also a metric on M. Observe that M itself is bounded in this metric.

3) (10 pts) Let A and B be two subsets of a metric space M. Recall that A°, the interior of A, is the set of interior points of A. Prove the following:
$$a) A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, \qquad b) A^\circ \cap B^\circ = (A \cap B)^\circ$$
Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

4) (10 pts) Let A be a subset of a metric space M. Recall that $\overline{A}$, the closure of A, is the union of A and its limit points. Recall that a point x belongs to the boundary of A, ∂A, if every open ball centered at x contains points of A and points of A^c, the complement of A. Prove that:
   a) $\partial A = \overline{A} \cap \overline{A^c}$,
   b) $p \in \partial A \iff p$ is in $\overline{A}$, but not in A° (symbolically, $\partial A = \overline{A} \setminus A^\circ$),‘
   c) ∂A is a closed set,
   d) A is closed $\iff \partial A \subseteq A$.

5) (10 pts) Show that, in $\mathbb{R}^n$ with the usual (Euclidean) metric, the closure of the open ball $B_R(p)$, $R > 0$, is the closed ball
$$\{q \in \mathbb{R}^n : d(p, q) \leq R\}.$$
Given an example of a metric space for which the corresponding statement is false.

6) (10 pts) Prove directly form the definition that the set $K \subseteq \mathbb{R}$ given by
$$K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$$
is compact.

7) (10 pts) Let K be a compact subset of a metric space M, and let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in A$ with

$$B_\delta(x) \subseteq \mathcal{U}_\alpha.$$

Extra problems:

1) Let M be a non-empty set, and let d be a real-valued function of ordered pairs of elements of M satisfying
   i) $d(x, y) = 0 \iff x = y$,
   ii) $d(x, y) \leq d(x, z) + d(y, z)$.
   Show that d is a metric on M.

2) Determine the boundaries of the following sets, $A \subseteq X$:
   i) $A = \mathbb{Q}$, $X = \mathbb{R}$
   ii) $A = \mathbb{R} \setminus \mathbb{Q}$, $X = \mathbb{R}$
   iii) $A = (\mathbb{Q} \times \mathbb{Q}) \cap B_R(0)$, $X = \mathbb{R}^2$

3) Describe the interior of the Cantor set.

4) Let M be a metric space with metric d, and let d_1 be the metric defined above (in problem 2). Show that the two metric spaces (M, d), (M, d_1) have the same open sets.