18.100B Problem Set 4

Due Friday October 6, 2006 by 3 PM

Problems:

1) Give an example of an open cover of the set $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \subseteq \mathbb{R}^2$ which has no finite subcover. (As usual, $\mathbb{R}^2$ is equipped with standard Euclidean metric.)

2) Consider $\mathbb{R}^k$ and let $\|\mathbf{x}\| = (x_1^2 + \cdots + x_k^2)^{1/2}$ be the Euclidean norm. Show that if $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\| + \|\mathbf{y}\|$ when $\mathbf{x} \neq \mathbf{y}$ and $d(\mathbf{x}, \mathbf{x}) = 0$, then $(\mathbb{R}^k, d)$ is a metric space.

   Are there open sets in $\mathbb{R}^k$ with this new metric $d(x, y)$ that are not open with respect to the Euclidean metric $d_{\text{Euclid}}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ on $\mathbb{R}^k$? Or vice versa?

3) Let X be an infinite set and consider the metric function on X given by $d(x, y) = 1$ when $x \neq y$ and $d(x, x) = 0$. Which sets in X are compact?

4) Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with $d(x, y) = |x - y|$. Define the set $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

5) Prove (e. g., by using results from class or Rudin's book) that the union of two compact sets is always compact. Does this assertion also hold for their intersection?

6) The terms *limit* and *limit point* are often a source of confusion for people not thoroughly accustomed to them. For instance, the constant sequence $\{1, 1, \ldots, 1, \ldots\}$ is convergent with limit 1; but as a subset of the real line its values are just equal to the set $\{1\}$, which cannot have a limit point (why?).

   To clarify the notions of limit and limit point, prove the following statement: If a convergent sequence in a metric space has infinitely many distinct points, then its limit is a limit point of the set of points of the sequence.

7) Find a sequence $\{x_n\}$ with values in $[0, 1]$ that has the following property. For every $x \in [0, 1]$, we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. *Hint:* Think about the rational numbers between 0 and 1.

8) Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

Extra problems:

1) Prove that every open set in $\mathbb{R}$ (with its usual metric) is the union of an at most countable collection of disjoint open intervals $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

   Hint: First show that $\mathbb{R}$ is *separable.* By this we mean that $\mathbb{R}$ contains a countable dense subset. (A subset $E \subseteq X$ of a metric space X is called dense if the closure of E is equal to X, i. e., we have that $\overline{E} = X$.)

2) Consider the following variation of problem 7) above. Is it possible to a find a sequence $\{x_n\}$ with values in $[0, 1]$ such that, for any $x \in [0, 1]$ with $x \neq 1/2$, we can find a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x$ as $k \to \infty$, but we cannot find a subsequence $\{x_{m_k}\}$ such that $x_{m_k} \to 1/2$ as $k \to \infty$? Justify your answer.

3) Let $\{x_n\}$ be a real-valued sequence. Show that at least one of the following statements must be true.
   a) There exists a subsequence $\{x_{m_k}\}$ such that $x_{m_k} \geq x_{m_{k+1}}$ holds for all $k \geq 1$.
   b) There exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq x_{n_{k+1}}$ holds for all $k \geq 1$.

   Hint: Is is useful to consider the set $A = \{m \in \mathbb{N} : x_m \geq x_n \text{ for all } n \geq m\}$.