18.100B Problem Set 6

Due Friday October 27, 2006 by 3 PM

Problems:

1) Prove that if $\sum |a_n|$ is converges, then $\sum |a_n|^2$ also converges.

2) Prove that
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}\,.$$
Hint: Use a "telescope trick", i. e. an argument of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1$.

3) Investigate the behavior (convergence and divergence) of $\sum_{n=1}^{\infty} a_n$ if

   a) $a_n = \sqrt{n+1} - \sqrt{n}$

   b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

   c) $a_n = \dfrac{1}{1+\alpha^n}$ where $\alpha \geq 0$ is some fixed number.

4) Show that convergence of $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ implies convergence of
$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}\,.$$
Hint: Cauchy–Schwarz inequality.

5) Assume $a_0 \geq a_1 \geq a_2 \geq \cdots$ and suppose that $\sum a_n$ converges. Prove that
$$\lim_{n\to\infty} (n a_n) = 0.$$
Hint: Use and show the inequality $n a_{2n} \leq \sum_{k=n+1}^{2n} a_k$.

6) If X and Y are metric spaces and $f : X \to Y$ is a mapping between them, show that the following statements are equivalent:

   a) $f^{-1}(B)$ is open in X whenever B is open in Y.

   b) $f^{-1}(B)$ is closed in X whenever B is closed in Y.

   c) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

7) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$\lim_{h\to 0} [f(x+h) - f(x-h)] = 0$$
for every $x \in \mathbb{R}$. Does this imply that f is continuous?

Extra problems:

1) Consider the non-absolutetly convergent series

$$\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Find a rearrangement $\sum_{n=1}^{\infty} a_{k_n}$ that diverges to $+\infty$.

2) Let $(p_k) = (2, 3, 5, 7, 11, \ldots)$ be the sequence of prime numbers. Prove that the series

$$\sum_{k=1}^{\infty} \frac{1}{p_k}$$

diverges, for instance, by working out the details of the following argument.

Suppose $\sum \frac{1}{p_k}$ converges. Then, there is an integer ℓ such that

$$\sum_{k=\ell+1}^{\infty} \frac{1}{p_k} = \sum_{k=1}^{\infty} \frac{1}{p_{k+\ell}} < \frac{1}{2}.$$

Next, for any given $x \geq 0$, we let $N(x)$ be the number of positive integers $n \leq x$ that are not divisible by any prime p_k with $k > \ell$. By writing $n = km^2$ with k square-free (which can be done with any integer), prove that

$$N(x) \leq 2^{\ell}\sqrt{x} \quad \text{and conclude that} \quad \frac{x}{2} < N(x) < 2^{\ell}\sqrt{x}.$$

This is a contradiction whenever $x \geq 2^{2\ell+2}$ holds.

Trivia: As an amusing fact, we remark that the series

$$(*) \qquad \sum_{k=1}^{\infty} \Big(\frac{1}{p_k} + \frac{1}{q_k}\Big) = \Big(\frac{1}{3} + \frac{1}{5}\Big) + \Big(\frac{1}{5} + \frac{1}{7}\Big) + \Big(\frac{1}{11} + \frac{1}{13}\Big) + \Big(\frac{1}{17} + \frac{1}{19}\Big) + \Big(\frac{1}{29} + \frac{1}{31}\Big) + \cdots$$

is known to converge. Here (p_k, q_k) runs over the pairs of *twin primes*; that is, p_k and q_k are both prime and $q_k = p_k + 2$. This result was proved by V. Brun in 1919, and it is known today (by numerics) that the value of the series is ≈ 1.902.

Had this series $(*)$ diverged, we would have a proof of the *twin prime conjecture* which means that there are infinitely many twin primes (this is still not known!). Also, if we knew that the value of $(*)$ were an irrational number, then the twin prime conjecture would follow immediately.

3) Let (p_k) be the sequence of prime numbers, and let J_N denote the set of natural numbers whose factorization into primes only involves the primes $\{p_k : 1 \leq k \leq N\}$. Prove the following identity

$$\sum_{n \in J_N} \frac{1}{n^s} = \prod_{k=1}^{N} \frac{1}{1 - p_k^{-s}}$$

for any $N \in \mathbb{N}$ and any $s \in \mathbb{Q} \cap (1, \infty)$. From this deduce *Euler's formula*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{1}{1 - p_k^{-s}} \Big(=: \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}} \Big)$$