

Equations and Math Supplement

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1 Equations and Math Supplement (Appendix)

1.1 Volume of a Tetrahedron (Lattice)

$$V = \frac{1}{6} |\det [P_1 - P_0, P_2 - P_0, P_3 - P_0]| \quad (1)$$

Notes.

- P_0, \dots, P_3 are vertex coordinates; the determinant computes the volume of the parallelepiped spanned by edge vectors, with the $1/6$ factor converting to tetra volume.

Tom Ace 5×5 tetravolume (IVM units):

$$V_{ivm} = \frac{1}{4} \left| \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 1 \\ b_0 & b_1 & b_2 & b_3 & 1 \\ c_0 & c_1 & c_2 & c_3 & 1 \\ d_0 & d_1 & d_2 & d_3 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \right| \quad (2)$$

Notes.

- Rows correspond to Quadray 4-tuples of the vertices; the last row encodes the affine constraint. Division by 4 returns IVM tetravolume.

XYZ determinant volume and S3 conversion:

$$V_{xyz} = \frac{1}{6} \left| \det \begin{pmatrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1 \\ x_d & y_d & z_d & 1 \end{pmatrix} \right|, \quad V_{ivm} = S3 V_{xyz}, \quad S3 = \sqrt{\frac{9}{8}} \quad (3)$$

Notes.

- Homogeneous determinant in Cartesian coordinates for tetra volume; conversion to IVM units uses $S3 = \sqrt{9/8}$ as used throughout.

See code: `tetra_volume_cayley_menger`. For tetrahedron volume background, see [Tetrahedron - volume](#). Exact integer determinants in code use the [Bareiss algorithm](#). External validation: these formulas align with implementations in the 4dsolutions ecosystem. See the [Resources](#) section for comprehensive details.

1.2 Fisher Information Matrix (FIM)

Background: [Fisher information](#).

$$F_{i,j} = \mathbb{E} \left[\frac{\partial \log p(x; \theta)}{\partial \theta_i} \frac{\partial \log p(x; \theta)}{\partial \theta_j} \right] \quad (4)$$

Notes.

- Defines the Fisher information matrix as the expected outer product of score functions; see [Fisher information](#).

Figure: empirical estimate shown in the FIM heatmap figure. See code: `fisher_information_matrix`.

See `src/information.py` — empirical outer-product estimator (`fisher_information_matrix`).

1.3 Natural Gradient

Background: [Natural gradient](#) (Amari).

$$\theta \leftarrow \theta - \eta F(\theta)^{-1} \nabla_{\theta} L(\theta) \quad (5)$$

Explanation.

- Natural gradient update: right-precondition the gradient by the inverse of the Fisher metric (Amari); see [Natural gradient](#).

See code: `natural_gradient_step`.

See `src/information.py` — damped inverse-Fisher step (`natural_gradient_step`).

1.4 Free Energy (Active Inference)

$$\mathcal{F} = -\log P(o \mid s) + \text{KL}[Q(s) \parallel P(s)] \quad (6)$$

Explanation.

- **Partition:** variational free energy decomposes into expected negative log-likelihood and KL between approximate posterior and prior; see [Free energy principle](#).

See code: `free_energy`.

See `src/information.py` — discrete-state variational free energy (`free_energy`).

Note: The main figures demonstrating natural gradient trajectories and free energy landscapes are shown in [Section 4: Optimization in 4D](#). The appendix focuses on unique figures specific to mathematical formulations and validation.

1.5 Quadray Normalization (Fuller.4D)

Given $q = (a, b, c, d)$, choose $k = \min(a, b, c, d)$ and set $q' = q - (k, k, k, k)$ to enforce at least one zero with non-negative entries.

1.6 Distance (Embedding Sketch; Coxeter.4D slice)

Choose linear map M from quadray to \mathbb{R}^3 (or \mathbb{R}^4) consistent with tetrahedral axes; then $d(q_1, q_2) = \|M(q_1) - M(q_2)\|_2$.

1.7 Minkowski Line Element (Einstein.4D analogy)

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \quad (7)$$

Background: [Minkowski space](#).

1.8 High-Precision Arithmetic Note

When evaluating determinants, FIMs, or geodesic distances for sensitive problems, use quad precision (binary128) via GCC's `libquadmath` (`__float128`, functions like `expq`, `sqrtq`, and `quadmath_snprintf`). See [GCC libquadmath](#). Where possible, it is useful to use symbolic math libraries like SymPy to compute exact values.

1.8.1 Reproducibility artifacts and external validation

- **This manuscript's artifacts:** Raw data in `quadmath/output/` for reproducibility and downstream analysis:
 - `fisher_information_matrix.csv` / `.npz`: empirical Fisher matrix and inputs
 - `fisher_information_eigenvalues.csv` / `fisher_information_eigensystem.npz`: eigenspectrum and eigenvectors
 - `natural_gradient_path.png` with `natural_gradient_path.csv` / `.npz`: projected trajectory and raw coordinates
 - `ivm_neighbors_data.csv` / `ivm_neighbors_edges_data.npz`: neighbor coordinates (Quadray and XYZ)
 - `polyhedra_quadray_constructions.png`: synergetics volume relationships schematic
- **External validation resources:** The [4dsolutions ecosystem](#) provides extensive cross-validation. See the [Resources](#) section for comprehensive details on computational implementations and validation.

1.9 Namespaces summary (notation)

- Coxeter.4D: Euclidean E^4 ; regular polytopes; not spacetime (cf. Coxeter, *Regular Polytopes*, Dover ed., p. 119). Connections to higher-dimensional lattices and packings as in Conway & Sloane.
- Einstein.4D: Minkowski spacetime; indefinite metric; used here only as a metric analogy when discussing geodesics and information geometry.
- Fuller.4D: Quadrays/IVM; tetrahedral lattice with integer tetravolume; unit regular tetrahedron has volume 1; synergetics scale relations (e.g., S3).