Equations and Math Supplement

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1 Equations and Math Supplement (Appendix)

1.1 Volume of a Tetrahedron (Lattice)

$$V = \tfrac{1}{6}\, \left|\det\left[\, P_1 - P_0,\ P_2 - P_0,\ P_3 - P_0 \,\right]\right| \tag{1}$$

Notes.

- $P_0, \ldots, P_3$ are vertex coordinates; the determinant computes the volume of the parallelepiped spanned by edge vectors, with the $1/6$ factor converting to tetra volume.

Tom Ace 5×5 tetravolume (IVM units):

$$V_{ivm} = \tfrac{1}{4} \left| \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 1 \\ b_0 & b_1 & b_2 & b_3 & 1 \\ c_0 & c_1 & c_2 & c_3 & 1 \\ d_0 & d_1 & d_2 & d_3 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \right| \tag{2}$$

Notes.

- Rows correspond to Quadray 4-tuples of the vertices; the last row encodes the affine constraint. Division by 4 returns IVM tetravolume.

XYZ determinant volume and S3 conversion:

$$V_{xyz} = \frac{1}{6} \left| \det \begin{pmatrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1 \\ x_d & y_d & z_d & 1 \end{pmatrix} \right|, \qquad V_{ivm} = S3\, V_{xyz}, \quad S3 = \sqrt{\frac{9}{8}} \tag{3}$$

Notes.

- Homogeneous determinant in Cartesian coordinates for tetra volume; conversion to IVM units uses $S3 = \sqrt{9/8}$ as used throughout.

See code: `tetra_volume_cayley_menger`. For tetrahedron volume background, see Tetrahedron – volume. Exact integer determinants in code use the Bareiss algorithm. External validation: these formulas align with implementations in the 4dsolutions ecosystem. See the Resources section for comprehensive details.

1.2 Fisher Information Matrix (FIM)

Background: Fisher information.

$$F_{i,j} = \mathbb{E}\left[\frac{\partial \log p(x;\theta)}{\partial \theta_i} \frac{\partial \log p(x;\theta)}{\partial \theta_j} \right] \tag{4}$$

Notes.

- Defines the Fisher information matrix as the expected outer product of score functions; see Fisher information.

Figure: empirical estimate shown in the FIM heatmap figure. See code: `fisher_information_matrix`.

See `src/information.py` — empirical outer-product estimator (`fisher_information_matrix`).

1.3 Natural Gradient

Background: Natural gradient (Amari).

$$\theta \leftarrow \theta - \eta\, F(\theta)^{-1}\, \nabla_\theta L(\theta) \tag{5}$$

Explanation.

- Natural gradient update: right-precondition the gradient by the inverse of the Fisher metric (Amari); see Natural gradient.

See code: `natural_gradient_step`.

See `src/information.py` — damped inverse-Fisher step (`natural_gradient_step`).

1.4 Free Energy (Active Inference)

$$\mathcal{F} = -\log P(o \mid s) + \mathrm{KL}\big[Q(s) \parallel P(s)\big] \tag{6}$$

Explanation.

- **Partition**: variational free energy decomposes into expected negative log-likelihood and KL between approximate posterior and prior; see Free energy principle.

See code: `free_energy`.

See `src/information.py` — discrete-state variational free energy (`free_energy`).

Note: The main figures demonstrating natural gradient trajectories and free energy landscapes are shown in Section 4: Optimization in 4D. The appendix focuses on unique figures specific to mathematical formulations and validation.

1.4.1 Figures

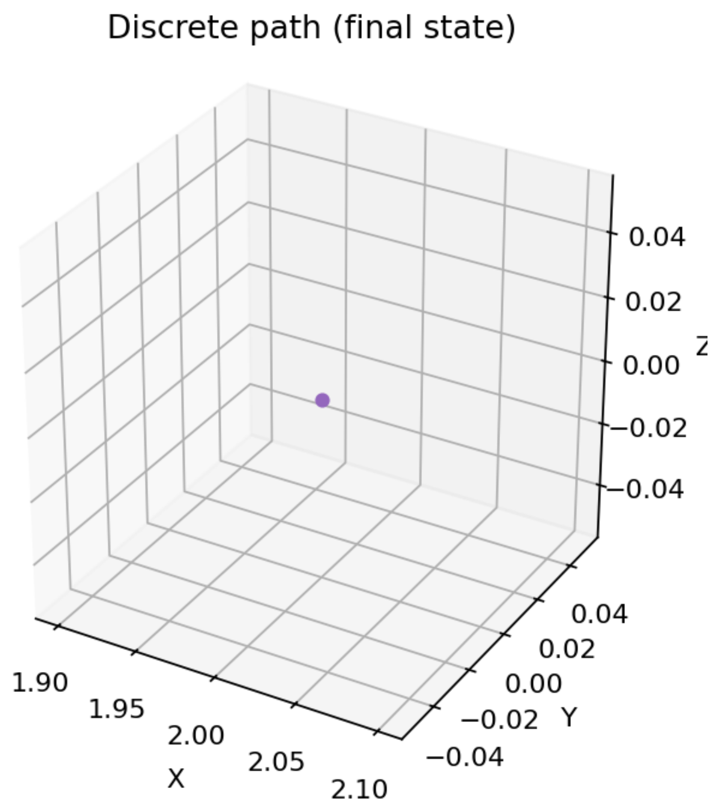

Figure 1: **Discrete IVM descent optimization path (final converged state)**. This static frame shows the final position of a discrete variational descent algorithm operating on the integer Quadray lattice. **Points**: Colored spheres representing the final optimization state, each positioned at integer Quadray coordinates projected to 3D space via the default embedding matrix. **Colors**: Each point has a distinct color for easy identification of different optimization components. **Optimization context**: These points represent the final state of the discrete IVM descent algorithm after converging to a local optimum on the integer lattice. The tight clustering of points indicates successful convergence, with the algorithm having found a stable configuration. **Lattice constraints**: All point positions correspond to integer Quadray coordinates, demonstrating the discrete nature of the optimization. The final configuration represents a stable "energy level" where further discrete moves do not improve the objective function. This visualization complements the time-series trajectory data and demonstrates the effectiveness of discrete optimization on the integer Quadray lattice.

1.5 Quadray Normalization (Fuller.4D)

Given $q = (a, b, c, d)$, choose $k = \min(a, b, c, d)$ and set $q' = q - (k, k, k, k)$ to enforce at least one zero with non-negative entries.

1.6 Distance (Embedding Sketch; Coxeter.4D slice)

Choose linear map M from quadray to $\mathbb{R}^3$ (or $\mathbb{R}^4$) consistent with tetrahedral axes; then $d(q_1, q_2) = \|M(q_1) - M(q_2)\|_2$.

1.7 Minkowski Line Element (Einstein.4D analogy)

$$ds^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2 \tag{7}$$

Background: Minkowski space.

1.8 High-Precision Arithmetic Note

When evaluating determinants, FIMs, or geodesic distances for sensitive problems, use quad precision (binary128) via GCC's `libquadmath` (`__float128`, functions like `expq`, `sqrtq`, and `quadmath_snprintf`). See GCC libquadmath. Where possible, it is useful to use symbolic math libraries like SymPy to compute exact values.

1.8.1 Reproducibility artifacts and external validation

- **This manuscript's artifacts**: Raw data in `quadmath/output/` for reproducibility and downstream analysis:
  - `fisher_information_matrix.csv` / `.npz`: empirical Fisher matrix and inputs
  - `fisher_information_eigenvalues.csv` / `fisher_information_eigensystem.npz`: eigenspectrum and eigenvectors
  - `natural_gradient_path.png` with `natural_gradient_path.csv` / `.npz`: projected trajectory and raw coordinates
  - `ivm_neighbors_data.csv` / `ivm_neighbors_edges_data.npz`: neighbor coordinates (Quadray and XYZ)
  - `polyhedra_quadray_constructions.png`: synergetics volume relationships schematic
- **External validation resources**: The 4dsolutions ecosystem provides extensive cross-validation. See the Resources section for comprehensive details on computational implementations and validation.

1.9 Namespaces summary (notation)

- Coxeter.4D: Euclidean E^4; regular polytopes; not spacetime (cf. Coxeter, Regular Polytopes, Dover ed., p. 119). Connections to higher-dimensional lattices and packings as in Conway & Sloane.
- Einstein.4D: Minkowski spacetime; indefinite metric; used here only as a metric analogy when discussing geodesics and information geometry.
- Fuller.4D: Quadrays/IVM; tetrahedral lattice with integer tetravolume; unit regular tetrahedron has volume 1; synergetics scale relations (e.g., S3).