

# Discussion

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## 1 Discussion

Quadray geometry (Fuller.4D) offers an interpretable, quantized view of geometry, topology, information, and optimization. Integer volumes enforce discrete dynamics, acting as a structural prior that can regularize optimization, reduce overfitting, prevent numerical fragility, and enable integer-based accelerated methods. Information geometry provides a right language for optimization in the synergetic tradition: optimization proceeds not through arbitrary parameter-space moves in continuous space, but along geodesics defined by information content (see Eq. (??) and Eq. (??) in the equations appendix; overview: [Natural gradient](#)).

Limitations and considerations:

- **Embeddings and distances:** Mapping between quadray and Euclidean coordinates must be selected carefully for distance calculations.
- **Hybrid strategies:** Some problems may require hybrid strategies (continuous steps with periodic lattice projection).
- **Benchmarking:** Empirical benchmarking remains important to quantify benefits across domains.

In practical analysis and simulation, numerical precision matters. Integer-volume reasoning is exact in theory, but empirical evaluation (e.g., determinants, Fisher Information, geodesics) can benefit from high-precision arithmetic. When double precision is insufficient, quad-precision arithmetic (binary128) via GCC’s `libquadmath` provides the `__float128` type and a rich math API for robust computation. See the official documentation for details on functions and I/O: [GCC libquadmath](#).

### 1.1 Fisher Information and Curvature

The Fisher Information Matrix (FIM) defines a Riemannian metric on parameter space and quantifies local curvature of the statistical manifold. High curvature directions (large eigenvalues of  $\mathbb{F}$ ) indicate parameters

to which the model is most sensitive; small eigenvalues indicate sloppy directions. Our eigenspectrum visualization (see the Fisher Information Matrix eigenspectrum figure above) highlights these scales. Background: [Fisher information](#).

Implication: curvature-aware steps using Eq. (??) in the equations appendix adaptively scale updates by the inverse metric, improving conditioning relative to vanilla gradient descent.

A curious connection unites geodesics in information geometry, the physical principle of least action, and Buckminster Fuller’s tensegrity geodesic domes (Fuller.4D). On statistical manifolds, geodesics are shortest paths under the Fisher metric, and natural-gradient flows approximate least-action trajectories by minimizing an information-length functional constrained by curvature (Eqs. (??), (??) in the equations appendix). In tensegrity domes, geodesic lines on triangulated spherical shells distribute stress nearly uniformly while the network balances continuous tension with discontinuous compression, attaining maximal stiffness with minimal material. Both systems exemplify constraint-balanced minimalism: an extremal path emerges by trading off cost (action or information length) against structure (metric curvature or tensegrity compatibility). The shared economy—optimal routing through low-cost directions—links geodesic shells in architecture to geodesic flows in parameter spaces; see background on tensegrity/geodesic domes @Web.

## 1.2 Quadray Coordinates and 4D Structure (Fuller.4D vs Coxeter.4D vs Einstein.4D)

Quadray coordinates provide a tetrahedral basis with projective normalization, aligning with close-packed sphere centers (IVM). Symmetries common in quadray parameterizations often yield near block-diagonal structure in  $F$ , simplifying inversion and preconditioning. Overview: [Quadray coordinates](#) and synergetics background. We stress the namespace boundaries: (i) Fuller.4D for lattice and integer volumes, (ii) Coxeter.4D for Euclidean embeddings, lengths, and simplex families, (iii) Einstein.4D for metric analogies only — not for interpreting synergetic tetravolumes.

## 1.3 Integrating FIM with Quadray Models

Applying the FIM within quadray-parameterized models ties statistical curvature to tetrahedral structure. Practical takeaways:

- Use `fisher_information_matrix` to estimate  $F$  from per-sample gradients; inspect principal directions via `fim_eigenspectrum`.
- Exploit block patterns induced by quadray symmetries to stabilize metric inverses and reduce compute.
- Combine integer-lattice projection with natural-gradient steps to balance discrete robustness and curvature-aware efficiency.
- Purely discrete alternatives (e.g., `discrete_ivm_descent`) provide monotone integer-valued descent when gradients are unreliable; hybrid schemes can interleave discrete steps with curvature-aware continuous proposals.

## 1.4 Implications for Optimization and Estimation

### 1.4.1 Clarifications on “frequency/time” dimensions

- Fuller’s discussions often treat frequency/energy as an additional organizing dimension distinct from Euclidean coordinates. In our manuscript, we keep the shape/angle relations (Fuller.4D) separate from time/energy bookkeeping; when temporal evolution is needed, we use explicit trajectories and metric analogies (Einstein.4D) without conflating with Euclidean 4D objects (Coxeter.4D). This separation avoids category errors while preserving the intended interpretability.

### 1.4.2 On distance-based tetravolume formulas (clarification)

- When volumes are computed from edge lengths, PdF and Cayley–Menger operate in Euclidean length space and are converted to IVM tetravolumes via the S3 factor. In contrast, the Gerald de Jong formula computes IVM tetravolumes natively, agreeing numerically with PdF/CM after S3 without explicit XYZ

intermediates. Tom Ace’s  $5 \times 5$  determinant sits in the same native camp as de Jong’s method. See references under the methods section for links to Urner’s code notebooks and discussion.

### 1.4.3 Symbolic analysis (bridging vs native) (Results linkage)

- Exact (SymPy) comparisons confirm that CM+S3 and Ace  $5 \times 5$  produce identical IVM tetravolumes on canonical small integer-quadray examples. See the bridging vs native comparison figure above and the manifest `sympy_symbolics.txt` alongside `bridging_vs_native.csv` in `quadmath/output/`.
- Curvature-aware optimizers: Kronecker-factored approximations (K-FAC) leverage structure in  $\mathbb{F}$  to accelerate training and improve stability; see [K-FAC \(arXiv:1503.05671\)](#). Similar ideas apply when quadray structure induces separable blocks.
- Model selection: eigenvalue spread of  $\mathbb{F}$  provides a lens on parameter identifiability; near-zero modes suggest redundancies or over-parameterization.
- Robust computation: lattice normalization in quadray space yields discrete plateaus that complement FIM-based scaling for numerically stable trajectories.

## 1.5 Community Ecosystem and Validation

The extensive computational ecosystem around Quadrays and synergetic geometry provides validation, pedagogical context, and practical implementations that complement and extend the methods developed in this manuscript. Cross-language implementations serve as independent verification of algorithmic correctness while educational materials demonstrate practical applications across diverse computational environments. See the Resources section for comprehensive details on the 4dsolutions organization, cross-language implementations, educational frameworks, and community platforms.