# Equations and Math Supplement

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## Contents



# 1 Equations and Math Supplement (Appendix)

## 1.1 Volume of a Tetrahedron (Lattice)

$$V = \frac{1}{6}\,\left|\det\left[\,P_1 - P_0,\ P_2 - P_0,\ P_3 - P_0\,\right]\right| \tag{1}$$

Notes.

- $P_0, \ldots, P_3$ are vertex coordinates; the determinant computes the volume of the parallelepiped spanned by edge vectors, with the $1/6$ factor converting to tetra volume.

Tom Ace 5×5 tetravolume (IVM units):

$$V_{ivm} = \frac{1}{4}\left|\det\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 1 \\ b_0 & b_1 & b_2 & b_3 & 1 \\ c_0 & c_1 & c_2 & c_3 & 1 \\ d_0 & d_1 & d_2 & d_3 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}\right| \tag{2}$$

Notes.

- Rows correspond to Quadray 4-tuples of the vertices; the last row encodes the affine constraint. Division by 4 returns IVM tetravolume.

## 1.2 Expanded Ace 5×5 Matrix

The expanded form of the Ace 5×5 matrix with explicit Quadray coordinates:

$$M(q_0, q_1, q_2, q_3) = \begin{bmatrix} q_{01} & q_{02} & q_{03} & q_{04} & 1 \\ q_{11} & q_{12} & q_{13} & q_{14} & 1 \\ q_{21} & q_{22} & q_{23} & q_{24} & 1 \\ q_{31} & q_{32} & q_{33} & q_{34} & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad V_{ivm} = \tfrac{1}{4} \left| \det M(q_0, q_1, q_2, q_3) \right| \tag{3}$$

Notes.

- **Matrix structure**: Each row represents a vertex with its Quadray coordinates plus affine coordinate 1.
- **Last row**: Enforces projective normalization constraint.
- **Volume computation**: Determinant divided by 4 gives IVM tetravolume.

XYZ determinant volume and S3 conversion:

$$V_{xyz} = \tfrac{1}{6} \left| \det \begin{pmatrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1 \\ x_d & y_d & z_d & 1 \end{pmatrix} \right|, \qquad V_{ivm} = S3\, V_{xyz}, \quad S3 = \sqrt{\tfrac{9}{8}} \tag{4}$$

Notes.

- Homogeneous determinant in Cartesian coordinates for tetra volume; conversion to IVM units uses $S3 = \sqrt{9/8}$ as used throughout.

## 1.3 Cayley-Menger Determinant (Coxeter.4D)

For tetrahedron volume from edge lengths (Coxeter.4D approach):

$$288\, V^2 = \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 \\ 1 & d_{10}^2 & 0 & d_{12}^2 & d_{13}^2 \\ 1 & d_{20}^2 & d_{21}^2 & 0 & d_{23}^2 \\ 1 & d_{30}^2 & d_{31}^2 & d_{32}^2 & 0 \end{pmatrix} \tag{5}$$

Notes.

- **Pairwise distances**: $d_{ij}$ are Euclidean distances between vertices $P_i$ and $P_j$.
- **Length-only formulation**: Cayley–Menger provides a length-only formula for simplex volumes, here specialized to tetrahedra.
- **Conversion to IVM**: Use $V_{ivm} = S3 \cdot V_{xyz}$ with $S3 = \sqrt{9/8}$ to convert to IVM units.

## 1.4 Piero della Francesca Formula (PDF)

For tetrahedron volume from edge lengths meeting at a vertex:

$$144\, V_{xyz}^2 = 4a^2b^2c^2 - a^2\,(b^2+c^2-f^2)^2 - b^2\,(c^2+a^2-e^2)^2 - c^2\,(a^2+b^2-d^2)^2 + (b^2+c^2-f^2)(c^2+a^2-e^2)(a^2+b^2-d^2) \tag{6}$$

Notes.

- **Edge lengths**: $a, b, c$ are edges meeting at a vertex, $d, e, f$ are opposite edges.
- **Conversion to IVM**: Use $V_{ivm} = S3 \cdot V_{xyz}$ with $S3 = \sqrt{9/8}$.

## 1.5 Gerald de Jong Formula (GdJ)

Native Quadray formula for tetrahedron volume:

$$V_{ivm} = \frac{1}{4} \left| \det \begin{pmatrix} a_1 - a_0 & a_2 - a_0 & a_3 - a_0 \\ b_1 - b_0 & b_2 - b_0 & b_3 - b_0 \\ c_1 - c_0 & c_2 - c_0 & c_3 - c_0 \end{pmatrix} \right| \tag{7}$$

Notes.

- **Quadray differences**: Each column represents edge vectors $P_1 - P_0$, $P_2 - P_0$, $P_3 - P_0$ in Quadray coordinates.
- **Native IVM**: Returns tetravolume directly in IVM units without conversion.
- **Integer arithmetic**: Exact for integer Quadray coordinates.

See code: `tetra_volume_cayley_menger`. For tetrahedron volume background, see Tetrahedron – volume. Exact integer determinants in code use the Bareiss algorithm. External validation: these formulas align with implementations in the 4dsolutions ecosystem. See the Resources section for comprehensive details.

## 1.6 Fisher Information Matrix (FIM)

Background: Fisher information.

$$F_{i,j} = \mathbb{E}\left[ \frac{\partial \log p(x;\theta)}{\partial \theta_i} \frac{\partial \log p(x;\theta)}{\partial \theta_j} \right] \tag{8}$$

Notes.

- Defines the Fisher information matrix as the expected outer product of score functions; see Fisher information.

Figure: empirical estimate shown in the FIM heatmap figure. See code: `fisher_information_matrix`.

See `src/information.py` — empirical outer-product estimator (`fisher_information_matrix`).

## 1.7 Empirical Fisher Information Matrix

For empirical estimation from data, the Fisher Information Matrix is computed as:

$$F_{i,j} = \frac{1}{N} \sum_{n=1}^{N} \frac{\partial \log p(x_n;\theta)}{\partial \theta_i} \frac{\partial \log p(x_n;\theta)}{\partial \theta_j} \tag{9}$$

Notes.

- Empirical estimate of the FIM from $N$ data samples; converges to the theoretical FIM as $N \to \infty$.
- Used in natural gradient descent and information geometry applications.

## 1.8 Natural Gradient

Background: Natural gradient (Amari).

$$\theta \leftarrow \theta - \eta\, F(\theta)^{-1}\, \nabla_\theta L(\theta) \tag{10}$$

Explanation.

- Natural gradient update: right-precondition the gradient by the inverse of the Fisher metric (Amari); see Natural gradient.

See code: `natural_gradient_step`.

See `src/information.py` — damped inverse-Fisher step (`natural_gradient_step`).

## 1.9 Free Energy (Active Inference)

$$\mathcal{F} = -\log P(o \mid s) + \mathrm{KL}\big[Q(s) \parallel P(s)\big] \tag{11}$$

Explanation.

- **Partition**: variational free energy decomposes into expected negative log-likelihood and KL between approximate posterior and prior; see Free energy principle.

See code: `free_energy`.

See `src/information.py` — discrete-state variational free energy (`free_energy`).

**Note**: The main figures demonstrating natural gradient trajectories and free energy landscapes are shown in Section 4: Optimization in 4D. The appendix focuses on unique figures specific to mathematical formulations and validation.

## 1.10 Quadray Normalization (Fuller.4D)

Given $q = (a, b, c, d)$, choose $k = \min(a, b, c, d)$ and set $q' = q - (k, k, k, k)$ to enforce at least one zero with non-negative entries.

## 1.11 Distance (Embedding Sketch; Coxeter.4D slice)

Choose linear map $M$ from quadray to $\mathbb{R}^3$ (or $\mathbb{R}^4$) consistent with tetrahedral axes; then $d(q_1, q_2) = \|M(q_1) - M(q_2)\|_2$.

## 1.12 Minkowski Line Element (Einstein.4D analogy)

$$ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2 \tag{12}$$

Background: Minkowski space.

## 1.13 High-Precision Arithmetic Note

When evaluating determinants, FIMs, or geodesic distances for sensitive problems, use quad precision (binary128) via GCC's `libquadmath` (`__float128`, functions like `expq`, `sqrtq`, and `quadmath_snprintf`). See GCC libquadmath. Where possible, it is useful to use symbolic math libraries like SymPy to compute exact values.

### 1.13.1 Reproducibility artifacts and external validation

- **This manuscript's artifacts**: Raw data in `quadmath/output/` for reproducibility and downstream analysis:
  - `fisher_information_matrix.csv` / `.npz`: empirical Fisher matrix and inputs
  - `fisher_information_eigenvalues.csv` / `fisher_information_eigensystem.npz`: eigenspectrum and eigenvectors
  - `natural_gradient_path.png` with `natural_gradient_path.csv` / `.npz`: projected trajectory and raw coordinates
  - `ivm_neighbors_data.csv` / `ivm_neighbors_edges_data.npz`: neighbor coordinates (Quadray and XYZ)
  - `polyhedra_quadray_constructions.png`: synergetics volume relationships schematic
- **External validation resources**: The 4dsolutions ecosystem provides extensive cross-validation. See the Resources section for comprehensive details on computational implementations and validation.

## 1.14 Namespaces summary (notation)

- Coxeter.4D: Euclidean $E^4$; regular polytopes; not spacetime (cf. Coxeter, Regular Polytopes, Dover ed., p. 119). Connections to higher-dimensional lattices and packings as in Conway & Sloane.
- Einstein.4D: Minkowski spacetime; indefinite metric; used here only as a metric analogy when discussing geodesics and information geometry.
- Fuller.4D: Quadrays/IVM; tetrahedral lattice with integer tetravolume; unit regular tetrahedron has volume 1; synergetics scale relations (e.g., S3).