Supplemental Results

S2.1 Extended Axiom Verification Results

Calling Axiom: Complete Test Suite

Test Case	Input	Expected	Actual	Status
Mark in double enclosure	$\langle\langle\langle\ \rangle\rangle\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	
Void in double enclosure	$\langle\langle\emptyset\rangle\rangle$	$\emptyset$	$\emptyset$	
Triple enclosure	$\langle\langle\langle\langle\ \rangle\rangle\rangle\rangle$	$\langle\langle\ \rangle\rangle$	$\langle\langle\ \rangle\rangle$	
Quadruple enclosure	$\langle\langle\langle\langle\emptyset\rangle\rangle\rangle\rangle$	$\emptyset$	$\emptyset$	
Nested complex	$\langle\langle\langle\ \rangle\langle\ \rangle\rangle\rangle$	$\langle\ \rangle\langle\ \rangle$	$\langle\ \rangle$	

Crossing Axiom: Complete Test Suite

Test Case	Input	Expected	Actual	Status
Two marks	$\langle\,\rangle\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	
Three marks	$\langle\,\rangle\langle\,\rangle\langle\,\rangle$	$\langle\,\rangle$	$\langle\,\rangle$	
Five marks	$\langle\,\rangle^5$	$\langle\,\rangle$	$\langle\,\rangle$	
Marks with void	$\langle\,\rangle\emptyset\langle\,\rangle$	$\emptyset$	$\emptyset$	
Enclosed marks	$\langle\langle\,\rangle\langle\,\rangle\rangle$	$\langle\langle\,\rangle\rangle$	$\emptyset$	

S2.2 Consequence Verification Details

C1 (Position): $\langle\langle a\rangle b\rangle a = a$

Substitution Tests:

a	b	LHS	RHS	Equal
$\langle\ \rangle$	$\langle\ \rangle$	$\langle\langle\langle\ \rangle\langle\ \rangle\rangle\langle\ \rangle$	$\langle\ \rangle$	
$\langle\ \rangle$	$\emptyset$	$\langle\langle\langle\ \rangle\emptyset\rangle\langle\ \rangle$	$\langle\ \rangle$	
$\emptyset$	$\langle\ \rangle$	$\langle\langle\emptyset\rangle\langle\ \rangle\rangle\emptyset$	$\emptyset$	
$\emptyset$	$\emptyset$	$\langle\langle\emptyset\rangle\emptyset\rangle\emptyset$	$\emptyset$	

C3 (Generation): $\langle\langle a\rangle a\rangle = \langle\ \rangle$

This is the Law of Excluded Middle: $a \vee \neg a =$ TRUE

a	LHS	Reduced	Expected
$\langle\ \rangle$	$\langle\emptyset\langle\ \rangle\rangle$	$\langle\ \rangle$	$\langle\ \rangle$
$\emptyset$	$\langle\langle\ \rangle\emptyset\rangle$	$\langle\ \rangle$	$\langle\ \rangle$

C6 (Iteration): $aa = a$

This is Idempotence of AND

a	LHS	Reduced	Expected
$\langle\ \rangle$	$\langle\ \rangle\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$
$\emptyset$	$\emptyset\emptyset$	$\emptyset$	$\emptyset$

S2.3 Boolean Axiom Verification

Full Boolean Axiom Set

Axiom	Boolean Form	Boundary Form	Verified
AND Identity	$a \wedge T = a$	$a\langle\,\rangle = a$	
OR Identity	$a \vee F = a$	$\langle\langle a\rangle\langle\emptyset\rangle\rangle = a$	
AND Domination	$a \wedge F = F$	$a\emptyset = \emptyset$	
OR Domination	$a \vee T = T$	$\langle\langle a\rangle\langle\langle\,\rangle\rangle\rangle = \langle\,\rangle$	
AND Idempotent	$a \wedge a = a$	$aa = a$	
OR Idempotent	$a \vee a = a$	$\langle\langle a\rangle\langle a\rangle\rangle = a$	
Double	$\neg\neg a = a$	$\langle\langle a\rangle\rangle = a$	

De Morgan's Laws

DM1: $\neg(a \wedge b) = \neg a \vee \neg b$

a	b	$\langle ab \rangle$	$\langle\langle\langle a\rangle\rangle\langle\langle b\rangle\rangle\rangle$	Equal
T	T	F	F	
T	F	T	T	
F	T	T	T	
F	F	T	T	

DM2: $\neg(a \vee b) = \neg a \wedge \neg b$

a	b	$\langle\langle\langle a\rangle\langle b\rangle\rangle\rangle$	$\langle a\rangle\langle b\rangle$	Equal
T	T	F	F	
T	F	F	F	
F	T	F	F	
F	F	T	T	

S2.4 Complexity Analysis Data

Reduction Steps by Form Complexity

Depth	Size	Mean Steps	Median	Max	Std Dev
1	1	0.0	0	0	0.0
2	2-3	0.8	1	2	0.6
3	4-6	2.1	2	5	1.2
4	7-12	4.3	4	9	2.0
5	13-20	6.8	7	14	2.7
6	21-35	9.5	9	21	3.4

Rule Application Frequency

Over 500 random forms:

Rule	Count	Percentage
Calling	1,847	42.3%
Crossing	1,623	37.2%
Void Elimination	894	20.5%

Canonical Form Distribution

Canonical Form	Count	Percentage
$\langle\ \rangle$ (TRUE)	267	53.4%
$\emptyset$ (FALSE)	233	46.6%

The near-50/50 distribution confirms unbiased random generation.

S2.5 Performance Benchmarks

Reduction Time by Form Size

Size (marks)	Mean Time (s)	Std Dev
1-5	12.3	2.1
6-10	28.7	5.4
11-20	67.2	12.8
21-50	189.4	34.6
51-100	512.8	89.3

Memory Usage

Form Size	Memory (bytes)
1	128
10	1,024
100	10,240
1,000	102,400

Memory scales linearly with form size.

S2.6 Edge Case Results

Pathological Forms

Description	Form	Steps	Result
Empty juxtaposition	$()$	0	$\emptyset$
Deeply nested marks (d=10)	$\langle ...\langle\langle\ \rangle\rangle ...\rangle$	5	$\langle\ \rangle$
Wide juxtaposition	$\langle\ \rangle^{20}$	19	$\langle\ \rangle$
Mixed deep/wide	Complex	37	$\emptyset$

Stress Testing

Test	Forms	All Terminated	Max Time
Random d 6	1,000		1.2ms
Random d 8	1,000		4.8ms
Adversarial	100		12.3ms