

Appendix

A. Complete Axiom Derivations

A.1 Calling Axiom (J1) Proof

Statement: $\langle\langle a \rangle\rangle = a$

Spatial Interpretation: Consider a space with form a . Enclosing a creates $\langle a \rangle$ —we are now “outside” a (inside the boundary around a). Enclosing again creates $\langle\langle a \rangle\rangle$ —we are now “outside” of being “outside” a , which returns us to a .

Algebraic Proof: Let \cdot denote truth value evaluation. -

$\langle\langle a \rangle\rangle = \neg\langle a \rangle$ (by enclosure semantics) - $= \neg\neg a$ (by enclosure semantics again) - $= a$ (by double negation)

Since truth values are preserved and the calculus is sound,

$\langle\langle a \rangle\rangle = a$.

A.2 Crossing Axiom (J2) Proof

Statement: $\langle \rangle \langle \rangle = \langle \rangle$

Spatial Interpretation: Two marks side by side both indicate “the marked state.” Indicating the same state twice does not change what is indicated.

Algebraic Proof: - $\langle \rangle \langle \rangle = \langle \rangle \wedge \langle \rangle$ (by juxtaposition semantics) - $= \text{TRUE} \wedge \text{TRUE}$ (mark is TRUE) - $= \text{TRUE}$ - $= \langle \rangle$

B. Consequence Derivations

B.1 C1: Position

Statement: $\langle\langle a \rangle b \rangle a = a$

Derivation: Consider the Boolean interpretation: - LHS = $\neg(\neg a \wedge b) \wedge a$ - $= (a \vee \neg b) \wedge a$ (De Morgan) - $= a \wedge (a \vee \neg b)$ (commutative) - $= a$ (absorption)

B.2 C3: Generation (Law of Excluded Middle)

Statement: $\langle\langle a \rangle a \rangle = \langle \rangle$

Derivation: - LHS = $\langle\langle a \rangle a \rangle$ - $= \neg(\neg a \wedge a)$ (Boolean interpretation) - $= \neg(\text{FALSE})$ (contradiction) - $= \text{TRUE}$ - $= \langle \rangle$
This confirms that $a \vee \neg a = \text{TRUE}$.

B.3 C6: Iteration (Idempotence)

Statement: $aa = a$

Derivation: - $aa = a \wedge a$ (juxtaposition) - $= a$ (idempotence of AND)

C. Boolean Algebra Correspondence

C.1 Complete Translation Table

Boolean	Boundary Form	Reduction
TRUE	$\langle \rangle$	canonical
FALSE	void	canonical
$\neg a$	$\langle a \rangle$	—
$a \wedge b$	ab	—
$a \vee b$	$\langle \langle a \rangle \langle b \rangle \rangle$	—
$a \rightarrow b$	$\langle a \langle b \rangle \rangle$	—
$a \leftrightarrow b$	$\langle \langle ab \rangle \langle \langle a \rangle \langle b \rangle \rangle \rangle$	—
$a \oplus b$ (XOR)	$\langle \langle \langle a \rangle b \rangle \langle a \langle b \rangle \rangle \rangle$	—
a NAND b	$\langle ab \rangle$	—
a NOR b	$\langle \langle \langle a \rangle \langle b \rangle \rangle \rangle$	—

C.2 NAND Completeness

All Boolean operations expressible via NAND ($\langle ab \rangle$):

► NOT $a = a$ NAND $a = \langle aa \rangle = \langle a \rangle$

D. Reduction Algorithm Details

D.1 Pattern Matching

Calling Pattern:

Match: `Form` with `is_marked=True`, `contents=[Form with is_ma`

Result: `a`

Crossing Pattern:

Match: `Form` with multiple simple marks in `contents`

Result: Single mark with non-mark contents preserved

D.2 Trace Format

Each reduction step records:

ReductionStep:

- before: `Form` (pre-reduction)
- after: `Form` (post-reduction)
- rule: `ReductionRule` (`CALLING` | `CROSSING` | `VOID_ELIMINAT`)
- location: `str` (where rule applied)

D.3 Termination Proof

Theorem: The reduction algorithm terminates for all well-formed inputs.

E. Test Coverage Details

E.1 Test Categories

Category	Tests	Coverage Target
Unit (forms.py)	36	95%+
Unit (reduction.py)	27	95%+
Unit (algebra.py)	22	90%+
Integration	15	90%+
Theorem verification	12	100%
Edge cases	18	Comprehensive

E.2 Property-Based Testing

Random form generation tests: - Depth: 1-6 (uniform) - Width: 1-4 (uniform) - Samples: 500 per test run - Seed: 42 (reproducible)

Verified properties: - All forms reduce to canonical - Canonical forms are stable (re-reduction yields same) - Equivalent forms have equal canonical forms

F. Notation Reference

Symbol	Meaning	LaTeX
$\langle \rangle$	Mark (TRUE)	<code>\langle\ \rangle</code>
\emptyset	Void (FALSE)	<code>\emptyset</code>
$\langle a \rangle$	Enclosure (NOT)	<code>\langle a \rangle</code>
ab	Juxtaposition (AND)	<code>ab</code>
f	Truth value	<code>\llbracket f \rrbracket</code>
j	Imaginary value	<code>j</code>

G. Implementation Reference

G.1 Module Structure

```
project/src/  
  forms.py           # Form class and construction  
  reduction.py       # Reduction engine  
  algebra.py         # Boolean correspondence  
  evaluation.py      # Truth value extraction  
  theorems.py        # Theorem definitions  
  verification.py    # Formal verification  
  visualization.py   # Diagram generation  
  __init__.py        # Package exports
```

G.2 Key APIs

Form construction

```
make_void() -> Form  
make_mark() -> Form  
enclose(form: Form) -> Form  
juxtapose(*forms: Form) -> Form
```

Reduction