

Quasi-Bayesian Inference for Production Frontiers*

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Abstract

This paper proposes to estimate and infer the production frontier by combining multiple first-stage extreme quantile estimates via the quasi-Bayesian method. We show the asymptotic properties of the proposed estimator and the validity of the inference procedure. The finite sample performance of our method is illustrated through simulations and an empirical application.

Keywords: Approximate Bayesian Computation, Extreme Value Theory, Fixed-k Asymptotics

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1 Introduction

The concept of production frontier (or data envelope) arises naturally in and applies to many fields such as manufacturing, health care, transportation, education, banking, public services, and portfolio management. [Gattoifi, Oral, and Reisman \(2004\)](#) provide a comprehensive survey on the topic. However, the estimation and inference of the production frontier are complicated by the fact that the parameter of interest is on the boundary.

In this article, we combine multiple extreme quantile estimates and construct a point estimate and confidence interval for the production frontier via the quasi-Bayesian method. We treat the first-stage extreme quantile estimates and their joint asymptotic distribution as observations and the corresponding likelihood, respectively. Then, we put a prior on the production frontier, draw from the posterior distribution by Markov Chain Monte Carlo (MCMC) method, and construct the point estimator and confidence interval.

The quasi-Bayesian inference is first considered by [Bickel and Yahav \(1969\)](#) and [Ibragimov and Has'minskii \(1981\)](#). Recently, [Chernozhukov and Hong \(2003\)](#), [Müller \(2013\)](#), [Jun, Pinkse, and Wan \(2015\)](#), [Yu \(2015\)](#), [Forneron and Ng \(2018\)](#), and [Chen, Christensen, and Tamer \(2018\)](#) apply the method in the context of M-estimations, misspecified MLE, Maximum-score type estimations, threshold regressions, GMM, and partially identified models, respectively. [Creel, Gao, Hong, and Kristensen \(2015\)](#) justify the use of kernel regression instead of the MCMC method to make inference in the GMM framework. We differ from the previous literature by applying the method to the *first-stage estimates* rather than the original observations. We treat a finite number of (properly scaled) first-stage estimates as new observations and conduct quasi-Bayesian estimation and inference. We mainly treat our method as an estimator-combination device. First, it is robust to certain amount of outliers as it combines extreme quantiles, rather than using the sample maximum of feasible outputs. Second, it can simultaneously produce point estimates and confidence intervals. Since extreme quantile estimators are not asymptotically normal, the standard bootstrap inference does not control size. The quasi-Bayesian approach provides an asymptotically valid alternative. Third, our method can automatically correct the downward bias between the extreme quantiles and the production frontier. It has good finite-sample performance even in samples with small and moderate sizes, as illustrated in our simulation study.

There is a vast literature on the estimation and inference of production frontiers. [Deprins, Simar, and Tulkens \(1984\)](#) first introduce the free-disposal hull (FDH) estimator. Its asymptotic properties have been studied by [Park, Simar, and Weiner \(2000\)](#), [Daouia, Florens, and Simar \(2010\)](#), and [Daouia, Simar, and Wilson \(2017\)](#). Assuming convexity of the production frontier, [Kneip, Park, and Simar \(1998\)](#) consider the data envelopment analysis (DEA) estimator. The asymptotic properties of DEA estimator have been investigated by [Kneip et al. \(1998\)](#), [Gijbels, Mammen, Park, and Simar \(1999\)](#), [Jeong \(2004\)](#), [Jeong and Park \(2006\)](#), [Kneip, Simar, and Wilson \(2008\)](#), [Park, Jeong, and Simar \(2010\)](#), and [Kneip, Simar, and Wilson \(2015\)](#). However, neither the FDH nor

DEA estimator is robust to any outliers. In addition, the inference of the FDH estimator requires estimating the normalizing rate, while a valid inference for the DEA estimator is still lacking, to the best of our knowledge. Recognizing those drawbacks, Cazals, Florens, and Simar (2002) and Aragon, Daouia, and Thomas-Agnan (2005) suggest estimating an expected frontier, which does not envelope the data. Daouia et al. (2010), Daouia, Florens, and Simar (2012), and Daouia, Girard, and Guillou (2014) propose to first estimate intermediate quantiles, and then extrapolate them to the boundary. Recently, Jirak, Meister, and Reiß (2014) consider nonparametric estimations of data boundary by adaptive kernel smoothing and obtain the optimal rate of convergence. Daouia, Noh, and Park (2016) study the global fit of boundary by constrained polynomial splines and obtain the asymptotic rate of global convergence. Although we only consider the point-wise estimation as in Jirak et al. (2014), we complement both Jirak et al. (2014) and Daouia et al. (2016) by establishing the distributional theory and valid inference procedure for our frontier estimator. Overall, Daouia, Laurent, and Noh (2017) provide an excellent and up-to-date literature review on the estimation and inference of the production frontier.

Bertail, Haefke, Politis, and White (2004), Chernozhukov and Fernández-Val (2011), and Zhang (2018) study the inference of extreme quantiles in the contexts of percentiles, linear quantile regressions, and quantile treatment effects, respectively. Recently, Müller and Wang (2017) study the inference of extreme quantiles by what they refer to as fixed- k asymptotics. Our approach takes inspiration from their idea of treating the first-stage estimates as new observations. Wang and Xiao (2019) further study the estimation of tail properties for censored or truncated data. We differ from the above papers by estimating the data boundary and adopting the quasi-Bayesian inference. Wang and Wang (2016) study the optimal way to combine intermediate quantile estimates in the linear tail quantile regression. Since intermediate quantile estimates are asymptotically normal, the linear combination is optimal. Then, Wang and Wang (2016) derive the optimal weights. On the contrary, we aim to combine extreme quantile estimates, which are not asymptotically normal. The optimal combination may be nonlinear. We propose to use the quasi-Bayesian method to combine these estimates.

The rest of the article is organized as follows. Section 2 sets up the model. Section 3 establishes the asymptotic properties of extreme quantile estimators. Section 4 investigates the asymptotic properties of our quasi-Bayesian method. Section 5 examines the inference procedure on the simulated data. Section 6 applies the approach to an empirical application. We conclude with Section 7. All proofs are collected in the Appendix.

Throughout this article, capital letters, such as A , X , and Y , denote random elements while their corresponding lower cases denote realizations. C denotes an arbitrary positive constant that may not be the same in different contexts. For a sequence of random variables $\{U_n\}_{n=1}^\infty$ and a random variable U , $U_n \rightsquigarrow U$ indicates weak convergence in the sense of van der Vaart and Wellner (1996). Convergence in probability is denoted as $U_n \xrightarrow{p} U$.

2 Setup

Following the definition in [Daouia et al. \(2016\)](#), we suppose that the n pairs of observations (X_i, Y_i) are independently drawn from a joint density function $f(x, y)$. We can interpret $X_i \in \mathbb{R}_+^p$ and $Y_i \in \mathbb{R}_+$ as vectors of production factors (inputs) and a scalar output, respectively. The support \mathbb{T} of the joint density $g(\cdot, \cdot)$ is assumed to be of the form

$$\mathbb{T} = \{(x, y) | y \leq \psi(x)\} \supset \{(x, y) | g(x, y) > 0\} \quad \text{and} \quad \{(x, y) | y > \psi(x)\} \subset \{(x, y) | g(x, y) = 0\},$$

where $\psi(\cdot)$ corresponds to the locus of the curve above which the density f is zero. Intuitively, we can view \mathbb{T} as technology that

$$\mathbb{T} = \{(x, y) \in \mathbb{R}_+^{d_x} \times \mathbb{R}_+ | x \text{ can produce } y\}.$$

Researchers observe a random sample of $\{X_i, Y_i\}_{i=1}^n$ such that for each $i = 1, \dots, n$, $(X_i, Y_i) \in \mathbb{T}$. The parameter of interest is $\psi(x)$, the maximal achievable output for a given level of inputs, i.e.,

$$\psi(x) = \sup\{y | (x, y) \in \mathbb{T}\}.$$

Assumption 1. $\{Y_i, X_i\}_{i=1}^n$ is i.i.d. $p_0 = \mathbb{P}(X_i \leq x) > 0$, where \leq inside the probability operator is pointwise.

In addition, we follow the literature and assume the free disposability.

Assumption 2. If $(x, y) \in \mathbb{T}$, then $(x', y') \in \mathbb{T}$ for any (x', y') such that $x' \geq x$ (component-wise) and $y' \leq y$.

Let $F(y/x) = \mathbb{P}(Y \leq y | X \leq x)$ be the ‘‘non-standard conditional distribution’’ in the production frontiers literature. Then under Assumption 2, [Cazals et al. \(2002\)](#) propose that

$$\psi(x) = \sup\{y \geq 0 | F(y/x) < 1\}. \tag{2.1}$$

Following [Aragon et al. \(2005\)](#) and [Daouia et al. \(2010\)](#), we estimate the production frontier at x by $\hat{q}_n(\hat{\tau}_n)$, where

$$\hat{q}_n(\tau) = \arg \min_q \sum_{i=1}^n \rho_\tau(Y_i - q) \mathbb{1}\{X_i \leq x\}, \tag{2.2}$$

$\rho_\tau(u) = (\tau - \mathbb{1}\{u \leq 0\})u$ is [Koenker and Bassett's \(1978\)](#) check function, and $\hat{\tau}_n$ is some random sequence that is smaller than but converges to 1. Later, following [Daouia et al. \(2010\)](#), we define $\hat{\tau}_n$ that depends on $\hat{p} := \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_i \leq x\}$, which is a consistent estimator of $p_0 = \mathbb{P}(X_i \leq x)$. The deterministic counterpart of $\hat{\tau}_n$ is denoted as τ_n . We further denote $q(\tau_n) = F^{-1}(\tau_n/x)$ where $F^{-1}(\tau_n/x) = \inf\{y : F(y/x) \geq \tau_n\}$. We omit the dependence of $q(\tau_n)$ and $\hat{q}_n(\hat{\tau}_n)$ on x for brevity

as we focus on the point estimation throughout the paper. Based on this notation, $\psi(x)$, the production frontier at x , is just $q(1)$. Let $\hat{\tau}_n = 1 - \frac{k}{np}$ for some $k \in (0, \infty)$.

Assumption 3. *Suppose k is not an integer.*

The population counterpart of $\hat{\tau}_n$ is $\tau_n = 1 - \frac{k}{np_0}$. In the literature, τ_n is referred to as the extreme quantile index by Chernozhukov (2005) and Daouia et al. (2010), and as fixed-k asymptotics by Müller and Wang (2017). For comparison, the quantile index τ'_n is intermediate if

$$n(1 - \tau'_n) = k_n \rightarrow \infty \quad \text{and} \quad k_n/n \rightarrow 0. \quad (2.3)$$

Compared with (2.3), $np_0(1 - \tau_n) = k$, which does not diverge to infinite as the sample size increases. However, since k can be greater than 1, we still use interior data points, rather than the maximum of the feasible outputs, for estimation and inference. Therefore, our method is robust to $\lceil k - 1 \rceil^1$ largest outliers, although it is indeed less robust than the existing inference based on the intermediate quantile estimations. The second part of Assumption 3 is to guarantee that the limiting objective function of our minimization problem in (2.2) has a unique minimizer. This assumption is mild because we have the freedom to choose k and the integers are sparse on the real line.

3 Asymptotic Properties

Before stating the regularity condition for our asymptotic results, we first introduce some definitions. We say the cumulative distribution function (CDF) F belongs to the domain of attraction of type III generalized extreme value (EV) distributions if as $z \rightarrow 0$ and any $v > 0$,

$$\frac{1 - F(z_1 - vz)}{1 - F(z_1 - z)} \rightarrow v^{-1/\xi},$$

where $z_1 = \sup\{z | F(z) < 1\}$ and $\xi < 0$ is the EV index.

Assumption 4. *The conditional CDF of Y_i given $X_i \leq x$ belongs to the domain of attraction of type III generalized EV distributions with the EV index $\xi_0 < 0$.*

Assumption 4 states that $1 - F(y/x)$ decays polynomially (up to some slowing varying function, e.g., $\log(\cdot)$) as y approaching $q(1)$ or equivalently, $F(y/x)$ has a Pareto-type upper tail. This assumption is common in the literature on the inference of extreme quantiles and production frontiers, e.g., Chernozhukov and Fernández-Val (2011), Daouia et al. (2010), Daouia et al. (2012), Daouia et al. (2014), Park et al. (2000), Zhang (2018).

¹ $\lceil u \rceil$ denotes the smallest integer that is greater than or equal to u .

Assumption 5. Let $k_0 > 0$ and $m > 1$ be two constants. Then, $\lceil mk_0 \rceil > \lceil k_0 \rceil$, where for a non-integer k , $\lceil k \rceil$ is the unique integer that satisfies $k \leq \lceil k \rceil \leq k + 1$.

Later, we will propose a random normalization factor $(\hat{\alpha}_n)$ for our first-stage extreme quantile estimates. Assumption 5 guarantees that the normalizing factor is well-defined. This condition is innocuous as researchers have the freedom to choose k_0 and m . We discuss the choice of k_0 , m , and other tuning parameters in practice in Section 5.2.

Now we are ready to describe the limiting distribution of our extreme quantile estimators. For a generic k that satisfies Assumption 3, let

$$Z_\infty(k) = -\left(\sum_{i=1}^{\lceil k \rceil} \mathcal{E}_i\right)^{-\xi_0}, \quad Z_\infty^c(k) = Z_\infty(k) + \eta(k), \quad \text{and} \quad \tilde{Z}_\infty(k) = Z_\infty(k)/(Z_\infty(k_0) - Z_\infty(mk_0)),$$

where $\{\mathcal{E}_i\}_{i \geq 1}$ is a sequence of i.i.d. standard exponential random variables and $\eta(\cdot) = (\cdot)^{-\xi_0}$.

Theorem 3.1. Let $\hat{\alpha}_n = (\hat{q}_n(1 - \frac{k_0}{n\hat{\rho}}) - \hat{q}_n(1 - \frac{mk_0}{n\hat{\rho}}))^{-1}$, $\hat{\tau}_{nl} = 1 - \frac{k_l}{n\hat{\rho}}$ for $l = 1, \dots, L$. If Assumptions 1, 2, and 4 hold, and Assumption 3 holds for $k = k_0, mk_0, k_1, \dots, k_L$, then

$$\begin{pmatrix} \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - q(1)) \\ \vdots \\ \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - q(1)) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \tilde{Z}_\infty(k_1) \\ \vdots \\ \tilde{Z}_\infty(k_L) \end{pmatrix}. \quad (3.1)$$

Several comments are in order. First, Theorem 3.1 establishes the joint asymptotic distribution of $(\hat{q}_n(\hat{\tau}_{n1}), \dots, \hat{q}_n(\hat{\tau}_{nL}))$ which extends the univariate result established in Daouia et al. (2010, Theorem 2.2). Second, we follow Bertail et al. (2004) and Chernozhukov and Fernández-Val (2011) and use a feasible convergence rate $\hat{\alpha}_n$ that is valid without any additional assumption on the tail distribution of the feasible output. Third, (3.1) and the fact that $\hat{\alpha}_n \rightarrow \infty$ imply that $(\hat{q}_n(\hat{\tau}_{n1}), \dots, \hat{q}_n(\hat{\tau}_{nL}))$ are all consistent estimates for the production frontier $q(1)$. The remaining question is how to combine these L estimates to construct a valid point estimate and confidence interval for $q(1)$. In the next section, we achieve this goal by the quasi-Bayesian method.

4 Inference

As pointed out by Bickel and Freedman (1981) and Zarepour and Knight (1999), the standard bootstrap inference for the extreme quantile estimators is inconsistent. Instead, we combine L extreme quantile estimators via a second-stage quasi-Bayesian method to infer the production frontier. Such method is the optimal way to combine these L estimates, as shown in Theorem 4.2 below. It is also possible to just use one extreme quantile estimator and its asymptotic distribution

to make inference, which is a special case of our proposed method when $L = 1$. However, this may lose information.

Denote $\tilde{Z}_n(k_l) = \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nl}) - q(1))$ for $\hat{\tau}_{nl} = 1 - k_l/(np)$, $l = 1, \dots, L$. Then, Theorem 3.1 shows

$$\begin{pmatrix} \tilde{Z}_n(k_1) \\ \vdots \\ \tilde{Z}_n(k_L) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \tilde{Z}_\infty(k_1) \\ \vdots \\ \tilde{Z}_\infty(k_L) \end{pmatrix}.$$

We view $(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L))$ as new observations, whose joint density is parameterized by $q(1)$ and converges to the joint PDF of $(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$, which is denoted as $f(\cdot; \xi_0)$. Although we cannot calculate the exact finite sample likelihood of $(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L))$, we can approximate it by $f(\cdot; \xi_0)$. Then, by putting a prior on $q(1)$, we can write down the posterior distribution and conduct quasi-Bayesian inference.²

The quasi-Bayesian estimator \hat{q}^{BE} of $q(1)$ minimizes the average risk, i.e.,

$$\hat{q}^{BE} = \arg \min_q \int_{\Omega} \ell_n(q - \bar{q}) f(\hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - \bar{q}), \dots, \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - \bar{q}); \xi) \pi(\bar{q}) \phi\left(\frac{\xi - \hat{\xi}}{\hat{\sigma}}\right) 1\{\xi \in \Gamma\} d\bar{q} d\xi, \quad (4.1)$$

where $\ell_n(u) = \ell(\hat{\alpha}_n u)$ is a loss function, $\pi(\cdot)$ is the prior of $q(1)$, Ω is the support of $\pi(\cdot)$ that has $q(1)$ as its interior point, $\phi(\cdot)$ is the standard normal PDF, $\hat{\sigma}$ is some (potentially random) bandwidth, and Γ is an interval that contains ξ_0 as an interior point.

In (4.1), we set the prior for ξ as a normal distribution that has mean $\hat{\xi}$ and standard error $\hat{\sigma}$, and is truncated by support Γ . As the standard error decreases to zero, the effect of this prior will vanish asymptotically. We use this prior to capture the finite sample uncertainty (randomness) of $\hat{\xi}$. In practice, we compute $\hat{\xi}$ by the default Pickands-type method, using function `dfs_pick` in the R package `npbr`. We refer readers to [Daouia et al. \(2017\)](#) for more details about `npbr`. The asymptotic normality of Pickands-type estimator has already been established in the literature (e.g., [Dekkers and De Haan \(1989\)](#)) under some extra conditions. This motivates us the use the Gaussian kernel. In addition, [Chernozhukov \(2000\)](#) and [D'Haultfoeuille, Maurel, and Zhang \(2018\)](#) have already established the validity of bootstrap inference under intermediate quantile index asymptotics, which is the same asymptotic scheme that the Pickands estimator is based on. Therefore, it is natural to construct $\hat{\sigma}$ based on the bootstrap standard error of $\hat{\xi}$. The support restriction Γ is imposed to further regularize the finite sample behaviour of ξ . However, its effect is asymptotically negligible. Although we motive the prior from the asymptotic normality of $\hat{\xi}$, we require only that $\hat{\xi}$ is consistent and $\hat{\sigma} = o_p(1)$ when deriving all the results in this section. Theoretically speaking, it is also valid to just plug in the consistent estimate $\hat{\xi}$ without using the

²We call the method “quasi-Bayesian” because we do not use the true finite-sample likelihood.

Gaussian prior. We recommend using the prior as it can improve the finite-sample coverage. We provide more details about the estimation of $\hat{\xi}$ and $\hat{\sigma}$ in Section 5.

It is also possible to consider the finite-sample maximum likelihood estimator, i.e.,

$$\hat{q}^{MLE} = \arg \max_q f(\hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - q), \dots, \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - q); \hat{\xi}),$$

which corresponds to the mode of the posterior distribution with uninformative priors, i.e., $\hat{\sigma} = 0$, $\Gamma = \Re$, and $\pi(\cdot) = 1$. We prefer the Bayesian estimator to the MLE for three reasons: (1) the Bayesian estimation does not require optimization, (2) it is natural to use prior of ξ to capture the randomness of the estimator $\hat{\xi}$, and (3) the Bayesian estimator can produce point estimate and confidence intervals simultaneously.

Let $v = \hat{\alpha}_n(\bar{q} - q(1))$, $z = \hat{\alpha}_n(q - q(1))$, and $\tilde{Z}_n^{BE} = \hat{\alpha}_n(\hat{q}^{BE} - q(1))$. Then

$$\tilde{Z}_n^{BE} = \theta_n^{BE}(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L); \hat{\xi}),$$

where

$$\theta_n^{BE}(z_1, \dots, z_L; \bar{\xi}) = \arg \min_z Q_n(z, z_1, \dots, z_L; \bar{\xi}), \quad (4.2)$$

$$\begin{aligned} & Q_n(z, z_1, \dots, z_L; \bar{\xi}) \\ &= \hat{\sigma}^{-1} \int \int_{\Omega_n} \ell(z - v) f(z_1 - v, \dots, z_L - v; \xi) \pi(q(1) + v/\hat{\alpha}_n) \phi\left(\frac{\xi - \bar{\xi}}{\hat{\sigma}}\right) 1\{\xi \in \Gamma\} dv d\xi, \end{aligned} \quad (4.3)$$

and $\Omega_n = \hat{\alpha}_n(\Omega - q(1))$. As $n \rightarrow \infty$, it is expected that the RHS of (4.3) converges (up to some constant) to

$$Q_\infty(z, z_1, \dots, z_L) = \int_{\Re} \ell(z - v) f(z_1 - v, \dots, z_L - v; \xi_0) dv. \quad (4.4)$$

Further denote $Z_\infty^{BE} = \theta_\infty^{BE}(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$,

$$\theta_\infty^{BE}(z_1, \dots, z_L) = \arg \min_z Q_\infty(z, z_1, \dots, z_L), \quad (4.5)$$

and

$$\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \bar{\xi}) = \arg \min_\gamma \int_{K_t} \ell(\gamma - v) f(z_1 - v, \dots, z_L - v; \bar{\xi}) dv, \quad (4.6)$$

where $K_t = [-t, t]$ for $t \geq 1$.

Assumption 6. 1. $\ell(u)$ is convex and $\ell(u) \leq C|u|^{d_1}$ for some constants C and $d_1 > 0$.

2. $([k_0], [mk_0], [k_1], \dots, [k_L])$ are distinct from each other.

3. $\hat{\xi} \xrightarrow{p} \xi_0$ and $\hat{\sigma} = o_p(1)$.

4. Let Γ be some compact subset of $(-\infty, 0)$ such that ξ_0 is in the interior of Γ , and \mathcal{N}_0 be some open neighborhood of ξ_0 . Then $f(z_1, \dots, z_L; \xi)$ is continuous in ξ at ξ_0 , for any fixed $M > 0$,

$$\sup_{(z_1, \dots, z_L) \in [-M, M]^L, \xi \in \Gamma} f(z_1 - v, \dots, z_L - v; \xi) \leq H_{1M}(v)$$

and

$$\sup_{(z_1, \dots, z_L) \in [-M, M]^L, (\xi_1, \xi_2) \in \mathcal{N}_0} |f(z_1 - v, \dots, z_L - v; \xi_1) - f(z_1 - v, \dots, z_L - v; \xi_2)| \leq H_{2M}(v)|\xi_1 - \xi_2|,$$

where for any fixed z ,

$$\int |\ell(z - v)|(H_{1M}(v) + H_{2M}(v))dv < \infty.$$

5. For some constant $d_2 > 0$, $\sup_{\xi \in \mathcal{N}_0} |\theta_n^{BE}(z_1, \dots, z_L; \bar{\xi})| \leq C \sum_{l=1}^L |z_l^{d_2}|$, $\sup_{\xi \in \mathcal{N}_0} |\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \bar{\xi})| \leq C \sum_{l=1}^L |z_l^{d_2}|$, and

$$\sup_{v \in [-t, t]} f(z_1 - v, \dots, z_L - v, \xi_0) \leq H_{3t}(z_1, \dots, z_L)$$

such that, for any $t \geq 0$

$$\int_{\mathbb{R}^L} \left(\sum_{l=1}^L |z_l^{d_2}| + t \right)^{d_1} H_{3t}(z_1, \dots, z_L) dz_1 \cdots dz_L < \infty.$$

6. $\pi(\cdot)$ is bounded and continuous at $q(1)$.

7. $Q_\infty(z, \tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$ is finite over a non-empty open set \mathcal{Z}_0 and uniquely minimized at some random variable Z_∞^{BE} w.p.1.

Several comments are in order. First, Assumption 6.1 is common in quasi-Bayesian estimations, e.g., Chernozhukov and Hong (2003) and Chernozhukov and Hong (2004). Both l_1 and l_2 loss functions satisfy this assumption. Second, Assumption 6.2 ensures the limiting likelihood is well-defined. Third, the consistency requirement for $\hat{\xi}$ is mild. The bandwidth $\hat{\sigma}$ will converge to zero, which is the standard requirement for the kernel type estimation. Fourth, Assumptions 6.4 and 6.5 can be verified directly because it is possible to write down $f(z_1, \dots, z_L; \xi)$ analytically. We provide one example in Proposition 4.1. In that example, $f(\cdot; \xi)$ depends on the gamma density function, which only takes values on the positive half of the real line and has an exponential tail at $+\infty$. Fifth, unlike the standard quasi-Bayesian estimation, here we only deal with a finite sample

with L observations. Following the example after Theorem 4.1, if $L = 1$ and $\pi(\cdot) = 1$, then

$$\theta_n^{BE}(z; \xi) = z - c(\xi)$$

in which the $c(\xi)$'s under l_1 and l_2 loss functions are just the median and mean of the random variable with density

$$\frac{\int f(w; \xi + u\hat{\sigma})\phi(u)1\{u \in \Gamma_n\}du}{\int \phi(u)1\{u \in \Gamma_n\}du},$$

respectively. The same comment applies to $\tilde{\theta}_t^{BE}(z; \xi)$ with density

$$\frac{f(u; \xi)1\{u \in [z-t, z+t]\}}{\int_{z-t}^{z+t} f(u; \xi)du}.$$

In these cases, Assumption 6.5 holds. Sixth, Assumptions 6.1, 6.4, and 6.5 induce various integrability conditions which are necessary for applying the dominated convergence theorem. Last, Assumption 6.7 implies the limiting objective function has a unique minimizer, which is necessary for applying the argmin theorem in [van der Vaart and Wellner \(1996\)](#). This type of assumption is common in the literature of quasi-Bayesian estimations, e.g., [Chernozhukov and Hong \(2003\)](#) and [Chernozhukov and Hong \(2004\)](#).

Theorem 4.1. *If Assumptions 1, 2, 4, 5, and 6 hold, and Assumption 3 holds for $k = k_0, mk_0, k_1, \dots, k_L$, then $\hat{Z}_n^{BE} \rightsquigarrow Z_\infty^{BE}$.*

We take the special case of $L = 1$ to illustrate the distribution of Z_∞^{BE} . When the loss function is quadratic, i.e., $\ell(u) = u^2$, Z_∞^{BE} minimizes

$$\int (z - v)^2 f(\tilde{Z}_\infty(k) - v; \xi_0) dv.$$

By the first-order condition and simple calculations, we obtain

$$Z_\infty^{BE} = \tilde{Z}_\infty(k) - \mathbb{E}\tilde{Z}_\infty(k).$$

The new limit Z_∞^{BE} is the demeaned version of the limit (i.e., $\tilde{Z}_\infty(k)$) of the original estimator, exactly because it is designed to minimize the MSE. This illustrates that our approach can automatically correct for the bias of the original estimator. Similarly, when $\ell(u) = |u|$, the quasi-Bayesian estimator is asymptotically median-unbiased, i.e., it minimizes the mean absolute deviation (MAD).

Next, we confirm this property of our estimator for the general case with $L > 1$. Let $\theta_n(\cdot)$ be a generic estimator, i.e., a function of data (z_1, \dots, z_L) and $\hat{\xi}$, and K be a compact subset of \Re . Following [Chernozhukov and Hong \(2003\)](#), we denote the finite average risk of the estimator θ_n in

K as

$$AR_{\ell,K}(\theta_n) = \int_K \int_{\Re^L} \ell(\theta_n(z_1, \dots, z_L; \hat{\xi}) - v) f(z_1 - v, \dots, z_L - v; \hat{\xi}) dz_1 \cdots d_{z_L} dv / \Lambda(K), \quad (4.7)$$

where $\ell(\cdot)$ and $\Lambda(\cdot)$ are the loss function and the Lebesgue measure, respectively. For a generic sequence of estimators $\{\theta_n(\cdot)\}_{n \geq 1}$, the asymptotic average risk is defined as

$$AAR_{\ell}(\{\theta_n\}) = \limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} AR_{\ell,K_t}(\theta_n),$$

in which K_t is defined after (4.6), i.e., $K_t = [-t, t]$. Recall the Bayesian estimator \hat{Z}_n^{BE} is a function of first-stage estimates $(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L))$ and $\hat{\xi}$, i.e., $\hat{Z}_n^{BE} = \hat{Z}_n^{BE}(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L); \hat{\xi})$. The next theorem establishes some optimality property regarding such function (or equivalently, such way of combining first-stage estimates).

Theorem 4.2. *If the assumptions in Theorem 4.1 hold, then*

$$AAR_{\ell}(\{\theta_n^{BE}\}) = \mathbb{E}\ell(Z_{\infty}^{BE}) \text{ a.s.}$$

In addition, let Θ_n be the collection of all estimators based on $(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L))$ and $\hat{\xi}$. Then

$$\inf_{\theta_n \in \Theta_n} AAR_{\ell}(\{\theta_n\}) = AAR_{\ell}(\{\theta_n^{BE}\}) \text{ a.s.}$$

Theorem 4.2 shows that the quasi-Bayesian estimator achieves the infimum of the asymptotic average risk over the class of estimators constructed based on $(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L))$ and $\hat{\xi}$. It is possible to find other better estimators outside this class. Searching for the best estimator for the production frontier is out of the scope of this paper. The main purpose of establishing Theorem 4.2 is the following corollary: the confidence interval constructed using the posterior quantiles controls size asymptotically.

Corollary 4.1. *Let $\hat{q}^{BE}(0.5)$, $\hat{q}^{BE}(\tau')$, and $\hat{q}^{BE}(\tau'')$ be the quasi-Bayesian estimators that solve (4.1) with the loss function $\tilde{\ell}_{\tau}(u) = (\mathbf{1}\{u > 0\} - \tau)u = u - \rho_{\tau}(u)$ and $\tau = 0.5$, τ' and τ'' , respectively. Let $Z_{\infty}^{BE}(0.5)$, $Z_{\infty}^{BE}(\tau')$ and $Z_{\infty}^{BE}(\tau'')$ be the limits of $\hat{\alpha}_n(\hat{q}^{BE}(0.5) - q(1))$, $\hat{\alpha}_n(\hat{q}^{BE}(\tau') - q(1))$ and $\hat{\alpha}_n(\hat{q}^{BE}(\tau'') - q(1))$, respectively. If $0 < \tau' < \tau'' < 1$ and $Z_{\infty}^{BE}(0.5)$, $Z_{\infty}^{BE}(\tau')$ and $Z_{\infty}^{BE}(\tau'')$ are continuously distributed at zero, then*

$$\mathbb{P}(q(1) \leq \hat{q}^{BE}(0.5)) \rightarrow 0.5 \quad \text{and} \quad \mathbb{P}(q(1) \in CI^{BE}(\tau'' - \tau')) \rightarrow \tau'' - \tau',$$

where $CI^{BE}(\tau'' - \tau') = (\hat{q}^{BE}(\tau'), \hat{q}^{BE}(\tau''))$.

The quasi-Bayesian estimator $\hat{q}^{BE}(\tau)$ is just the τ -th posterior quantile. Corollary 4.1 shows we can construct a median-unbiased estimator and a valid confidence interval based on posterior quantiles. To implement the MCMC method (such as the Metropolis-Hastings algorithm) and obtain the posterior distribution, we have to evaluate $f(\cdot; \xi)$, the joint PDF of $(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$ at

$$(\hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - \bar{q}), \dots, \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - \bar{q})).$$

Next, we derive an analytical form for $f(u_1, \dots, u_L; \xi)$.

Assumption 7. $\lceil k_0 \rceil < \lceil mk_0 \rceil < \lceil k_1 \rceil < \dots < \lceil k_L \rceil$.

The order of k 's is needed to derive a simple formula for the joint PDF but is not required for Theorem 4.1. Essentially, Assumption 7 requires that $\lceil k_0 \rceil$ and $\lceil mk_0 \rceil$ are smaller than all the other k 's, which makes it much easier to handle the common denominator $Z_\infty(mk_0) - Z_\infty(k_0)$ in $\tilde{Z}_\infty(k_l)$ for $l = 1, \dots, L$.

Proposition 4.1. *Let f_h be the PDF of a gamma random variable with shape and scale parameters being equal to h and 1, respectively. If Assumption 7 holds, then*

$$\begin{aligned} & f(u_1, \dots, u_L; \xi) \\ &= \int (-1/\xi)^L \tilde{u}(t, s)^{-L/\xi} \left[\prod_{l=1}^L u_l^{-1/\xi-1} f_{h_l-h_{l-1}}(v_l - v_{l-1}) \right] f_{\lceil k_0 \rceil}(s) f_{\lceil mk_0 \rceil - \lceil k_0 \rceil}(t) ds dt, \end{aligned}$$

where $h_l = \lceil k_l \rceil$ for $1 \leq l \leq L$, $h_0 = \lceil mk_0 \rceil$, $v_l = (u_l \tilde{u}(t, s))^{-1/\xi}$ for $1 \leq l \leq L$, $\tilde{u}(t, s) = (t+s)^{-\xi} - s^{-\xi}$, and $v_0 = t + s$.

Given the analytical form of $f(u_1, \dots, u_L; \xi)$, the estimates $(\hat{q}_n(\hat{\tau}_{n1}), \dots, \hat{q}_n(\hat{\tau}_{nL}))$, and the feasible convergence rate $\hat{\alpha}_n$, we can generate MCMC draws from the posterior

$$f(\hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - \bar{q}), \dots, \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - \bar{q}); \xi) \pi(\bar{q}) \phi\left(\frac{\xi - \hat{\xi}}{\hat{\sigma}}\right) 1\{\xi \in \Gamma\}.$$

Then, we can use these MCMC draws to construct point estimator and confidence interval for $q(1)$. The quasi-Bayesian approach requires several tuning parameters, namely L , (k_0, \dots, k_L) , and m . We discuss the choices of these tuning parameters in Section 5.2. We also describe the detail of the MCMC procedure in Section 5.3. The R code for the quasi-Bayesian inference is available upon request.

5 Simulations

In this section, we investigate the finite-sample performance of our estimation and inference procedures.

5.1 Data Generating Processes

The data generating process (DGP) is based on the model

$$Y_i = \psi(X_i)\mathcal{U}_i, X_i \sim \text{Unif}(0, 6), i = 1, 2, \dots, n, \quad (5.1)$$

where $\psi(X)$ is a function representing the frontier, \mathcal{U} is the error term, and $\text{Unif}(0, 6)$ denotes the uniform distribution over $[0, 6]$. We consider three different $\psi(X)$'s:

$$(1) \psi(X) = X^{0.5}, \quad (2) \psi(X) = X, \quad \text{and} \quad (3) \psi(X) = \frac{X^2}{6},$$

which are concave, linear, and convex, respectively.

The first two functional forms have been investigated in simulations in previous papers, e.g., [Aragon et al. \(2005\)](#) and [Daouia et al. \(2010\)](#). Convex frontiers, like (3), were adopted in simulations in [Park et al. \(2000\)](#) and [Martins-Filho and Yao \(2008\)](#) among others.

We combine the above three $\psi(X)$'s with the following five distributions of \mathcal{U} .

- (1) $\mathcal{U} \sim \text{Unif}(0, 1)$ with density evenly distributed over the support $[0, 1]$.
- (2) $\mathcal{U} \sim \exp(-u), u \sim \text{exponential}\left(\frac{1}{3}\right)$ with density skewed to the left over the support $[0, 1]$.
- (3) $\mathcal{U} \sim \text{Beta}\left(\frac{1}{2}, \frac{3}{2}\right)$ with density skewed more to the left compared to the density in (2).
- (4) $\mathcal{U} \sim \text{Beta}\left(\frac{3}{2}, \frac{1}{2}\right)$ with density skewed to the right over the support $[0, 1]$.
- (5) $\mathcal{U} \sim \text{truncated normal}$ with density $\frac{2\phi\left(\frac{u-1/2}{1/2}\right)}{\Phi(1)-\Phi(-1)}$ for $u \in [0, 1]$ (ϕ and Φ denote the PDF and the CDF of standard normal, respectively), which is concentrated in the middle over the support $[0, 1]$.

The five distributions above exhibit different types of tail behaviors. Specifically, in our DGPs, ξ_0 only depends on the density of \mathcal{U} . Some simple calculation further shows that $\xi_0 = -\frac{1}{2}$ for the DGPs with distributions (1), (2), and (5), $\xi_0 = -\frac{2}{5}$ for the DGPs with distribution (3), and $\xi_0 = -\frac{2}{3}$ for the DGPs with distribution (4). Note that $\xi_0 = -\frac{1}{2}$, $\xi_0 > -\frac{1}{2}$, and $\xi_0 < -\frac{1}{2}$ when the density of \mathcal{U} is bounded and bounded away from zero, decays to zero, and diverges to infinity at the boundary 1, respectively. Overall, we consider DGPs using all the combinations of the functional forms of ψ and the distributions of \mathcal{U} , which results in 15 DGPs in total. We denote them as $\text{DGP}(i, j)$, where $i = 1, 2, 3$ represents the functional forms of $\psi(\cdot)$ and $j = 1, \dots, 5$ denotes distributions of \mathcal{U} .

Since the data in our empirical application contains four outliers, we also add four outliers to each DGP to check the impact of outliers on our estimation and inference procedure. Note [Aragon et al. \(2005\)](#) also introduce outliers in their simulation setup. Specifically, the four outliers in our

DGPs are

$$(\text{Unif}(0, 1.5), \psi(2.25)), \quad (1.5 + \text{Unif}(0, 1.5), \psi(3.5)), \quad (1.5 + \text{Unif}(0, 1.5), \psi(4)), \\ \text{and } (3 + \text{Unif}(0, 1.5), \psi(5.25)).$$

We report the results for $x = 1.5, 3.0$, and 4.5 . Thus, our procedure faces 1, 3, and 4 outliers at $1.5, 3.0$, and 4.5 , respectively.

5.2 Tuning parameters

The tuning parameters used in our procedures are the lower quantile index k_0 , the spacing parameter m , and the upper quantile index k_L . How to choose those tuning parameters optimally is an important yet challenging question. Just as argued in Müller and Wang (2017, Section 5), “*under $k_n \rightarrow \infty$, the determination of k_n in a given sample size n is widely recognized as a difficult issue. But the problem is arguably even harder under fixed- k asymptotics, as there cannot exist a procedure based on the largest k order statistics that consistently determines whether, say, $k_1 < k$ or $k_2 < k_1$ is appropriate*”. Here, we provide some rules of thumb for k_0, m , and k_L based on the existing literature and some unique features of our own procedure. We leave the formal analysis on the higher-order impact of the tuning parameters to future research.

Note that the spacing parameter m and the upper quantile index k_L have been well studied in Chernozhukov and Fernández-Val (2011). We choose m and k_L based on their recommendation. The role of k_0 is to guard against outliers. We detail our rule-of-thumb choice of the tuning parameters below.

(1) To be robust against outliers, we set k_0 as

$$k_0 = \text{Number of spotted outliers} + 2.$$

In the simulation, we set $k_0 = 3, 5$, and 6 for $X = 1.5, 3.0$, and 4.5 , respectively.

- (2) As for m , Chernozhukov and Fernández-Val (2011) suggest using $m = 1 + \frac{sp}{\lceil k_0 \rceil}$, where $sp \in [2, 20]$,³ and they set $sp = 5$ for simulations and applications. We follow them and set $sp = 5$. Then, $m = 1 + \frac{5}{\lceil k_0 \rceil}$, which implies $k_1 = k_0 + 5$.
- (3) Chernozhukov and Fernández-Val (2011) point out that the fixed- k asymptotics has a better approximation of the finite sample distribution when k_L is within the range $[40, 80]$. In addition, denote the effective sample size for each n and x as $n\hat{p}$, where $\hat{p} = \frac{1}{n} \sum_{i=1}^n 1\{X_i \leq x\}$. Then, $k_L/(n\hat{p})$ is the quantile index of the k_L -th order statistic in the effective sample. As

³The original formula in Chernozhukov and Fernández-Val (2011) is $m = 1 + \frac{d+sp}{\lceil k_0 \rceil}$, where d is the dimension of the regressors. In our case, there is no regressor so $d = 0$.

our theories rely on the extreme quantile asymptotics, we require such quantile index to be close to zero. In practice, we require $k_L \leq 0.1n\hat{p}$. Therefore, our rule-of-thumb choice of k_L is $k_L = \min(0.1n\hat{p}, 40)$. Note as $n \rightarrow \infty$, $\min(0.1n\hat{p}, 40)$ reduces to 40, which fits the extreme quantile asymptotics. For robustness check, we also consider $k'_L = \min(0.1n\hat{p}, 35)$ and $k''_L = \min(0.1n\hat{p}, 45)$ for all 15 DGPs. All the simulation results are very close.

- (4) The more quantiles are used, the more efficient is our quasi-Bayesian estimator. Thus, we use all the integers between k_1 and k_L for estimation, i.e., we let

$$\{k_l\}_{l=0}^L = \{k_0, k_1, k_1 + 1, k_1 + 2, \dots, k_L - 1, k_L\}.$$

Once k_0 , m , and k_L are determined, the whole sequence $\{k_l\}_{l=0}^L$ is determined.

To show the robustness of our procedure to k_0 and sp (or m), we experiment $k_0 = \text{Number of spotted outliers} + 3$ or $k_0 = \text{Number of spotted outliers} + 4$ and $sp = 6$ or 7 for DGP(1,1) and DGP(2,1). These results can be found in Sections J.3 and J.4 in the supplement.

5.3 Detail about the MCMC procedure

The numerical evaluation of the joint density function established in Proposition 4.1 is detailed in Section H in the supplement. The length of burn-in sequence and MCMC sequence should be set as large as computationally possible. We use 4,000 and 10,000, respectively. Second, we need to determine the initial values of the MCMC. Given x , the initial value \bar{q}^* is computed by

$$\bar{q}^* = \arg \min_q \sum_{i=1}^n \rho_\tau(Y_i - q) \mathbf{1}\{X_i \leq x\},$$

where $\tau = 0.99$. The initial value of ξ_0 is $\hat{\xi}$ computed by the **rho_momt_pick** function in the R package **npbr**. We will provide more detail about the estimation of ξ_0 in Section 5.6 below.

5.4 Estimators for comparison

Based on the characterization in Daouia et al. (2017, Table 2), our paper considers point-wise and robust estimation of the production frontier under the assumption that the frontier is monotone only. Among all the estimators mentioned in Daouia et al. (2017, Table 2), the moment- and Pickands-type estimations proposed by Daouia et al. (2010) and the probability-weighted moment frontier estimation proposed by Daouia et al. (2012) are in the same category as ours and produce not only point estimates but also confidence intervals. Therefore, we will compare our method to them. The four methods are labelled as follows:

- (1) “Quasi-Bayesian”: our quasi-Bayesian method,

- (2) “Mom”: the moment frontier estimator,
- (3) “Momt_pick”: the Pickands frontier estimator,
- (4) “Pwm”: the probability-weighted moment frontier estimator.

The estimation procedures for “Mom”, “Momt_pick”, and “Pwm” are described in Section [G](#) in the supplement.

5.5 Prior of \bar{q}

For all simulations, we simply set $\pi(\cdot) = 1$, which is the uninformative prior for \bar{q} . We experiment $\pi(\cdot)$ as normal with the mean as the initial value \bar{q}^* and the variance as 1 or 1.5 for DGPs(1,1) and (2,1). The simulation results show our inference method is insensitive to the choice of prior distributions. The detail can be found in Section [J.2](#) in the supplement.

5.6 Estimation of ξ_0

All four estimation methods above require the estimation of the EV index ξ_0 . For fair comparison, for each replication, we force all estimators to share the same EV index estimate $\hat{\xi}$, which is the negative reciprocal of the output of the function **rho_momt_pick** in **npbr** with arguments **method** = “**Pickands**” and support intervals (1, 3), (0.5, 2.5) and (1.5, 3.5), when the true values of $-1/\xi_0$ are 2 (error distributions (1), (2), (5)), 2.5 (error distribution (3)), and 1.5 (error distribution (4)), respectively. When the effective sample size is small, occasionally, the function **rho_momt_pick** may return NA value. In this case, we propose to use the following equation to compute $\hat{\xi}$:

$$\hat{\xi} = \text{Med} \left[\left\{ -\frac{1}{r \log(l)} \log \left[\frac{\hat{q}_n \left(\tau \left(\tilde{m} l^r \tilde{k} \right) \right) - \hat{q}_n \left(\tau \left(l^r \tilde{k} \right) \right)}{\hat{q}_n \left(\tau \left(\tilde{m} \tilde{k} \right) \right) - \hat{q}_n \left(\tau \left(\tilde{k} \right) \right)} \right] \right\}_{\tilde{k}=K-c\hat{p}}^{K+c\hat{p}} \right], \quad (5.2)$$

where $\text{Med}(\cdot)$ denotes the median operator, $\tau(k) = 1 - \frac{k}{np}$, $l = 2$, $r = 2$, $\tilde{m} = 1.5$, $c = 20$ and the tuning parameter $K = np/10$. Once the estimated $\hat{\xi}$ is outside Γ , we directly assume it equals to the closest boundary.

For our quasi-Bayesian method, we use the truncated normal prior $\phi(\frac{\xi - \hat{\xi}}{\hat{\sigma}}) 1\{\xi \in \Gamma\}$, where $\hat{\sigma}$ is obtained via bootstrap. Specifically, for the s -th bootstrap sample, we can generate $\{\zeta_i^{(s)}\}_{i=1}^n$, which is a sequence of i.i.d. standard exponentially distributed random variables. For a generic quantile index τ , we can compute

$$\hat{q}_n^{(s)}(\tau) = \arg \min_q \sum_{i=1}^n \zeta_i^{(s)} \rho_\tau(Y_i - q) 1\{X_i \leq x\}.$$

Then, the EV index estimator $\hat{\xi}^{(s)}$ for the s -th bootstrap sample can be computed similarly using (5.2) with $\hat{q}_n(\cdot)$ and K replaced by $\hat{q}_n^{(s)}(\cdot)$ and \hat{K} , respectively, where \hat{K} is the optimal tuning parameter associated with the EV index $\hat{\xi}$ obtained by function **rho_momt_pick**. For some replication, when **rho_momt_pick** returns NA value, we instead set $\hat{K} = n\hat{p}/10$. For each replication, we repeat the above procedure for $s = 1, \dots, S$, where S is a sufficiently large positive integer and obtain $\{\hat{\xi}^{(s)}\}_{s=1}^S$. We let

$$\hat{\sigma} = \frac{c_\sigma(c_{0.75} - c_{0.25})}{\text{normal inverse}(0.75) - \text{normal inverse}(0.25)},$$

where $c_\sigma = 1.5 + 1.5 \cdot 1\{\hat{\xi} > -0.5\}$, $c_{0.75}$ and $c_{0.25}$ are the 75% and 25% quantiles of $\{\hat{\xi}^{(s)}\}_{s=1}^S$, and $\text{normal inverse}(0.75)$ and $\text{normal inverse}(0.25)$ are the 75% and 25% quantiles of the standard normal distribution, respectively.

5.7 Results

We construct 95% confidence intervals for the four estimation methods. We report the results of the coverage probabilities and average lengths of the CIs for $\psi(x)$ at $x = 1.5, 3.0, 4.5$. Due to the length limit, we report results for DGPs(1,1), (2,1), and (3,1) in Tables 1–3. The results for the rest 12 DGPs and various robustness checks are relegated to the supplement. Given the value of x , we report the performance when the sample size $n = 500, 1,000, 2,000$ and $4,000$. All simulations are repeated 1,000 times.

We can make several observations. First, the quasi-Bayesian method controls size well, even when the effective sample size is small. Meanwhile, the CIs for the Pickands, the moment, and the probability-weighed frontier methods over- or under-cover quite a bit in the majority of cases. The simulation results in Section J.5 in the supplement further show that even we use the true EV index, the inferences using these methods still have the same issue. This may be due to the fact that their tuning parameters selected by **npbr** are not optimal for inference purpose. Second, the average length of the quasi-Bayesian method is in general the shortest among all four methods, despite the fact that its coverage rate is also closest to the nominal rate. Third, both the coverage rates and average lengths of our method are stable across different values of k_L . In addition, in Sections J.2–J.4 in the supplement, we show the performance of quasi-Bayesian method is insensitive to the choices of k_0 , sp (or equivalently m) and the prior $\pi(\bar{q})$. Fourth, the average lengths for the quasi-Bayesian method decrease as the sample size increases. This indicates the validity of the fixed-k type asymptotics, which our theory relies on.

Table 1: DGP(1,1)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9660	0.9650	0.9660	1.0000	0.9680	0.9570
$np_0 = 125$	(0.5328)	(0.5332)	(0.5283)	(0.8451)	(1.9062)	(1.1307)
$n = 1000$	0.9800	0.9820	0.9820	0.9940	0.9730	0.9880
$np_0 = 250$	(0.3062)	(0.3069)	(0.3068)	(0.4978)	(1.2312)	(0.8488)
$n = 2000$	0.9730	0.9700	0.9710	0.9930	0.9800	0.9930
$np_0 = 500$	(0.2138)	(0.2117)	(0.2119)	(0.3267)	(0.8659)	(0.5960)
$n = 4000$	0.9650	0.9580	0.9550	0.9880	0.9850	0.9970
$np_0 = 1000$	(0.1477)	(0.1467)	(0.1458)	(0.2303)	(0.6343)	(0.4158)

Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9390	0.9280	0.9360	1.0000	0.9900	0.7020
$np_0 = 250$	(0.5485)	(0.5490)	(0.5495)	(0.8204)	(1.9218)	(0.9080)
$n = 1000$	0.9500	0.9590	0.9450	0.9990	0.9830	0.8150
$np_0 = 500$	(0.3616)	(0.3585)	(0.3559)	(0.5384)	(1.3046)	(0.7082)
$n = 2000$	0.9520	0.9530	0.9520	0.9990	0.9880	0.8930
$np_0 = 1000$	(0.2461)	(0.2424)	(0.2410)	(0.3669)	(0.9549)	(0.5213)
$n = 4000$	0.9390	0.9440	0.9380	0.9990	0.9950	0.9390
$np_0 = 2000$	(0.1715)	(0.1686)	(0.1667)	(0.2567)	(0.7050)	(0.3703)

Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9540	0.9550	0.9550	0.9980	0.9850	0.8430
$np_0 = 375$	(0.5187)	(0.5167)	(0.5171)	(0.6377)	(1.7471)	(0.4787)
$n = 1000$	0.9320	0.9290	0.9200	0.9880	0.9900	0.9250
$np_0 = 750$	(0.3730)	(0.3665)	(0.3632)	(0.4262)	(1.2530)	(0.3641)
$n = 2000$	0.9760	0.9720	0.9720	0.9930	0.9920	0.9400
$np_0 = 1500$	(0.2432)	(0.2396)	(0.2384)	(0.3124)	(0.9603)	(0.2622)
$n = 4000$	0.9830	0.9740	0.9750	0.9960	0.9990	0.9660
$np_0 = 3000$	(0.1719)	(0.1680)	(0.1665)	(0.2285)	(0.7277)	(0.1814)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 2: DGP(2,1)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9700	0.9660	0.9720	0.9980	0.9820	0.9330
$np_0 = 125$	(0.840)	(0.8180)	(0.830)	(1.4814)	(3.9622)	(1.4388)
$n = 1000$	0.9780	0.9800	0.9770	0.9970	0.9940	0.9630
$np_0 = 250$	(0.4799)	(0.4805)	(0.4793)	(1.0809)	(3.1801)	(1.1207)
$n = 2000$	0.9690	0.9620	0.9610	0.9760	0.9870	0.9730
$np_0 = 500$	(0.3457)	(0.3419)	(0.3376)	(0.7511)	(2.3309)	(0.8267)
$n = 4000$	0.9770	0.9680	0.9700	0.9550	0.9740	0.9790
$np_0 = 1000$	(0.2509)	(0.2481)	(0.2459)	(0.4650)	(1.4488)	(0.5810)

Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9040	0.9030	0.9060	1.0000	0.9960	0.3990
$np_0 = 250$	(0.8849)	(0.8871)	(0.8882)	(1.8113)	(4.8315)	(1.1930)
$n = 1000$	0.9080	0.9020	0.8900	0.9990	0.9940	0.5420
$np_0 = 500$	(0.5986)	(0.5873)	(0.5768)	(1.2152)	(3.4421)	(1.0037)
$n = 2000$	0.9610	0.9600	0.9640	0.9830	0.9840	0.7430
$np_0 = 1000$	(0.4145)	(0.4075)	(0.4019)	(0.7482)	(2.1716)	(0.7682)
$n = 4000$	0.9410	0.9410	0.9350	0.9750	0.9780	0.8010
$np_0 = 2000$	(0.2976)	(0.2915)	(0.2883)	(0.4875)	(1.4713)	(0.5490)

Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9550	0.9570	0.9530	0.9980	0.9930	0.5590
$np_0 = 375$	(0.8537)	(0.8396)	(0.8420)	(1.5060)	(4.9829)	(0.7829)
$n = 1000$	0.9270	0.9280	0.9250	0.9810	0.9910	0.6990
$np_0 = 750$	(0.6194)	(0.6046)	(0.5975)	(0.9935)	(3.1977)	(0.6487)
$n = 2000$	0.9610	0.9640	0.9600	0.9650	0.9760	0.8410
$np_0 = 1500$	(0.4328)	(0.4241)	(0.4176)	(0.6307)	(2.0447)	(0.5009)
$n = 4000$	0.9680	0.9650	0.9640	0.9550	0.9710	0.8640
$np_0 = 3000$	(0.3019)	(0.2963)	(0.2925)	(0.4454)	(1.4648)	(0.3756)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 3: DGP(3,1)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9670	0.9660	0.9640	1.0000	0.9950	0.9400
$np_0 = 125$	(0.9531)	(0.9424)	(0.9367)	(2.0546)	(5.0625)	(1.7818)
$n = 1000$	0.9780	0.9810	0.9800	0.9960	0.9930	0.9680
$np_0 = 250$	(0.5472)	(0.5478)	(0.5480)	(1.4625)	(3.9807)	(1.3816)
$n = 2000$	0.9770	0.9740	0.9770	0.9940	0.9990	0.9770
$np_0 = 500$	(0.4063)	(0.3952)	(0.3872)	(1.0924)	(3.0601)	(1.0267)
$n = 4000$	0.9770	0.9760	0.9780	0.9840	0.9920	0.9890
$np_0 = 1000$	(0.3210)	(0.3137)	(0.3078)	(0.7476)	(2.1403)	(0.7322)

Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.8990	0.8970	0.8980	1.0000	0.9990	0.3320
$np_0 = 250$	(0.9964)	(1.0069)	(1.0045)	(2.7008)	(6.4429)	(1.4510)
$n = 1000$	0.9110	0.9140	0.9150	1.0000	0.9990	0.4930
$np_0 = 500$	(0.70)	(0.6728)	(0.6505)	(1.8219)	(4.5428)	(1.2463)
$n = 2000$	0.9560	0.9610	0.9560	0.9990	0.9980	0.6240
$np_0 = 1000$	(0.5249)	(0.5067)	(0.4960)	(1.2479)	(3.3172)	(0.9707)
$n = 4000$	0.9610	0.9670	0.9680	0.9940	0.9970	0.7610
$np_0 = 2000$	(0.4035)	(0.3944)	(0.3842)	(0.8147)	(2.2525)	(0.7152)

Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9670	0.9610	0.9690	1.0000	0.9990	0.4430
$np_0 = 375$	(0.9845)	(0.9624)	(0.9619)	(2.4174)	(7.0347)	(1.0180)
$n = 1000$	0.9590	0.9630	0.9640	1.0000	1.0000	0.6010
$np_0 = 750$	(0.7604)	(0.7392)	(0.7236)	(1.6713)	(4.9101)	(0.8610)
$n = 2000$	0.9570	0.9630	0.9570	0.9870	0.9960	0.7430
$np_0 = 1500$	(0.5651)	(0.5496)	(0.5368)	(1.1008)	(3.2691)	(0.6784)
$n = 4000$	0.9840	0.9800	0.9870	0.9780	0.9990	0.8360
$np_0 = 3000$	(0.4302)	(0.4164)	(0.4069)	(0.7426)	(2.3010)	(0.5187)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

To sum up, the quasi-Bayesian method works well and is not sensitive to reasonable choices of

tuning parameters. However, we also want to emphasize that these results do not mean our method outperforms the existing methods in the literature in all respects. First, the performance of other three existing estimators can still be improved. Second, the three existing methods are based on the intermediate, rather than extreme, quantile estimations. Therefore, they can tolerate more outliers. As put by Daouia et al. (2010), “ *It is difficult to imagine one procedure being preferable in all contexts. Hence, a sensible practice is not to restrict the frontier analysis to one procedure . . .* ” We view our quasi-Bayesian method as an alternative to the existing inference procedures in the literature. The simulation study above shows our method has a better control of size in finite samples with small or moderate sample sizes.

6 An Empirical Application

We apply our inference approach to the frontier analysis of French post offices observed in 1994. The same dataset is also studied in Daouia et al. (2010). In this context, X and Y denote the quantity of labor and volume of the delivered mails, respectively. The total number of observations is 4,000, which is close to what we consider in our simulations. Table 4 contains the summary statistics of the data.

Table 4: Summary Statistics

	MEAN	STD	MIN	LQ	MEDIAN	UQ	MAX
X	1592	790	177	1128	1338	1730	4405
Y	7.709	0.612	3.829	7.349	7.698	8.062	9.576

Notes: STD = standard errors, LQ = 25% quantile, UQ = 75% quantile.

There are four data points deviating from the rest of the sample (shown as circles in Figures 1 and 2). We view them as outliers. We use the same sets of tuning parameters as in the simulations. Specifically, we set $k_0 = \text{number of spotted outliers} + 2$, $m = 1 + \frac{5}{\lceil k_0 \rceil}$ (or, equivalently, $sp = 5$), and $k_L = \min \{0.1n\hat{\rho}, 40\}$. In Figure 1, we report the point estimators and the associated 95% confidence intervals of the production frontier for labor between 800 and 4400. Note the number of observations with labor less than 800 is 187. For comparison, we also report the point estimates of three existing methods considered in Section 5, namely “Mom”, “Momt_pick” and “Pwm.”

Cazals et al. (2002) consider the estimation and inference for the cost function.⁴ Denote the cost function as $C(y) = \inf\{X : Y \geq y\}$, where (X, Y) follow the joint distribution of input and output. Let $\tilde{X} = M_1 - Y$, $\tilde{Y} = M_2 - X$, and $\tilde{x} = M_1 - y$, where M_1 and M_2 are two large positive constants such that \tilde{X} and \tilde{Y} are always positive. Then, we have $C(y) = M_2 - \tilde{\psi}(\tilde{x})$ where $\tilde{\psi}(\tilde{x}) \equiv \sup\{\tilde{Y} : \tilde{X} \leq \tilde{x}\}$. This means we can transform the cost function $C(y)$ to a production function $\tilde{\psi}(\tilde{x})$. We first estimate and infer $\tilde{\psi}(\tilde{x})$. The corresponding point estimator and confidence interval are denoted

⁴We thank a referee for pointing it out.

as $\hat{\psi}_m(\tilde{x})$ and \widetilde{CI}_m , respectively, where $m \in \{\text{“Quasi-Bayesian”}, \text{“Mom”}, \text{“Momt_pick”}, \text{“Pwm”}\}$. Then, we can obtain the point estimate and confidence interval for $C(y)$ as $M_2 - \hat{\psi}_m(\tilde{x})$ and $M_2 - \widetilde{CI}_m$, respectively. We emphasize that our quasi-Bayesian point estimate and confidence interval are numerically invariant to the choices of M_1 and M_2 . We maintain Assumptions in Theorem 4.1 for (\tilde{X}, \tilde{Y}) . In implementation, we set $M_1 = \max_{i=1,\dots,n} Y_i$ and $M_2 = \max_{i=1,\dots,n} X_i$ and use the same sets of tuning parameters as discussed above. In Figure 2, we report the results for the cost function when the volume of delivered mails ranges between 0 and 7,500. Note the effective sample for the cost function estimation at output y is all the observations with output $Y \geq y$. Therefore, the effective sample size for estimating the cost of 7,500 mails is 120. We rescale the input between 8,000 and 14,000 in Figure 2 to better present our results. We also report the “FDH” point estimates in both Figures 1 and 2.

Figure 1: Estimation and Inference for the Production Function

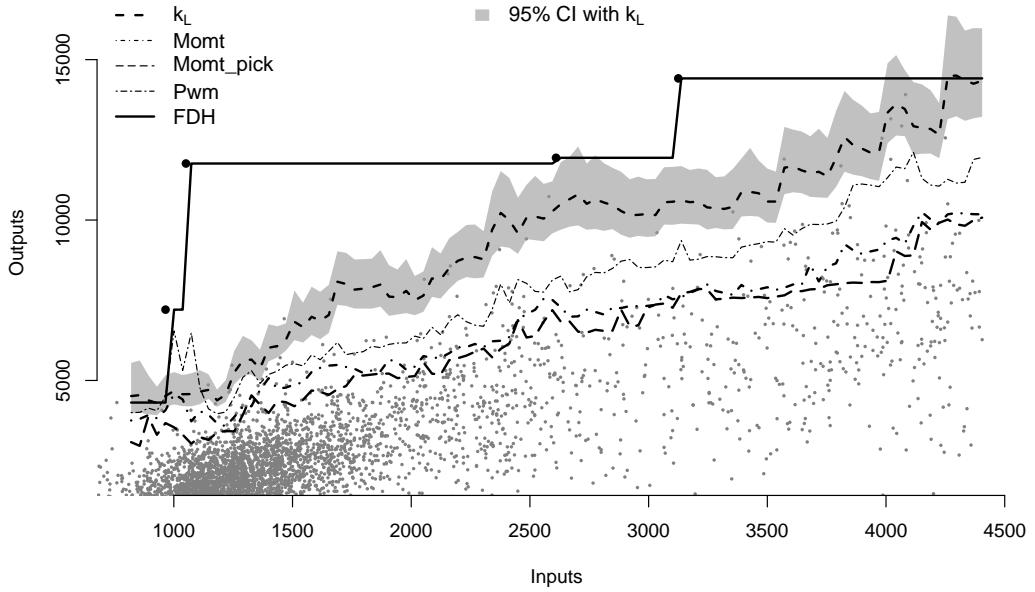
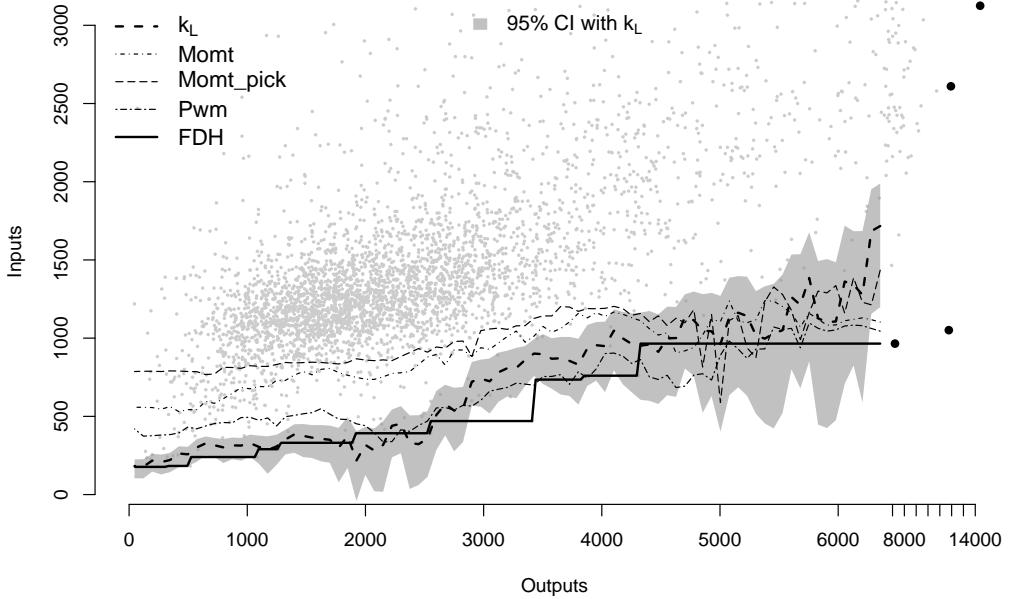


Figure 2: Estimation and Inference for the Cost Function



Several comments regarding Figures 1 and 2 are in order. First, our point estimators and confidence intervals are all clearly away from the outliers. This confirms the robustness of our method to outliers. In contrast, the FDH estimator is greatly influenced by the outliers in Figure 1. Second, the point estimators of our method in general envelope the data in Figures 1 and 2. In Figure 1, it is above all the other estimators except when the input is around 1,000. In Figure 2, it is below all the other estimators when the output ranges from 0 to 3,000. Third, in Figure 2, the confidence intervals become wider as the output grows. This is because the effective sample size for the estimation of the cost function shrinks as output grows. In contrast, in Figure 1, as the input grows, more observations are used for estimation and inference of the production frontier, which results in shorter confidence intervals. Fourth, it appears that the majority of French offices deliver mails well below the frontiers (lower than the lower bound of the confidence intervals). This indicates that most French offices are not efficient. An interesting direction for research is to investigate the relationship between the relative efficiency and other demographic variables and identify the key factors that contribute to the inefficiency. Last, the point estimators for the production and cost functions are not monotonic in both figures. We note that this is a common feature for point-wise estimation methods. For example, point estimators proposed by Cazals et al. (2002), Aragon et al. (2005), Daouia et al. (2010) and Daouia et al. (2012) cannot guarantee monotonicity either. To ensure monotonicity, one can possibly take our estimates as initial estimates and monotonize them following Daouia et al. (2014, Section 2.4.2). The inference

procedure, however, does need to change accordingly. We believe this direction is rather interesting, but is outside the scope of this paper. We leave it for future research.

7 Conclusion

In this article, we propose a quasi-Bayesian method to estimate and infer the production frontier. Our procedure is based on extreme quantile estimators, and thus is robust to a few outliers. The asymptotic validity of our method is theoretically justified. The application to the French post offices dataset shows that our method can be a practical alternative to existing inference methods in the literature.

Supplement to “Quasi-Bayesian Inference for Production Frontiers”

Abstract

This paper gathers the supplementary material to the original paper. Section A introduces a convexity lemma which will be used later. Sections B, C, D, E, and F prove Theorems 3.1, 4.1, 4.2, Corollary 4.1, and Proposition 4.1, respectively. Section G describes the computation of three existing methods considered in Section 5. Section H illustrates how to evaluate the density $f(u_1, \dots, u_L; \xi)$ in our MCMC procedure. Section I provides some calculation of ξ for production and cost frontiers. Section J contains additional simulation results.

Keywords: Approximate Bayesian Computation, Extreme Value Theory, Fixed-k Asymptotics

A The Convexity Lemma due to Geyer (1996) and Knight (1999)

We first state the convexity lemma attributed to Geyer (1996) and Knight (1999).

Lemma A.1. Suppose (i) a sequence of convex lower-semicontinuous functions $Q_n: \mathfrak{R} \mapsto \bar{\mathfrak{R}} = \mathfrak{R} \cup \{\pm\infty\}$ marginally converges to $Q_\infty: \mathfrak{R} \mapsto \bar{\mathfrak{R}}$ over a dense subset of \mathfrak{R} , (ii) Q_∞ is finite over a non-empty open set \mathcal{Z}_0 , and (iii) Q_∞ is uniquely minimized at a random variable Z_∞ , then any minimizer of Q_n , denoted $\hat{Z}_n(1)$, converges in distribution of Z_∞ .

B Proof of Theorem 3.1

Let $\alpha_n = 1/(q(1) - q(1 - 1/(np_0)))$, $\hat{Z}_n(k) = \alpha_n(\hat{q}_n(\hat{\tau}_n) - q(1))$, $\hat{Z}_n^c(k) = \alpha_n(\hat{q}_n(\hat{\tau}_n) - q(\tau_n))$. We divide the proof into two steps. In the first step, we show that

$$\begin{pmatrix} \hat{Z}_n(k_0) \\ \hat{Z}_n(mk_0) \\ \hat{Z}_n(k_1) \\ \vdots \\ \hat{Z}_n(k_L) \end{pmatrix} \rightsquigarrow \begin{pmatrix} Z_\infty(k_0) \\ Z_\infty(mk_0) \\ Z_\infty(k_1) \\ \vdots \\ Z_\infty(k_L) \end{pmatrix}. \quad (\text{B.1})$$

In the second step, we derive the desired results in theorem.

Step 1:

Denote $\mathbb{L}(u, v) = (v - u)\mathbb{1}\{u < v\}$.

$$\begin{aligned}
\widehat{Z}_n(k) &= \arg \min_z \sum_{i=1}^n \frac{1}{\alpha_n} \left[\alpha_n(Y_i - q(1)) - z \right] \left[1 - \frac{k}{n\hat{p}} - \mathbb{1}\{\alpha_n(Y_i - q(1)) \leq z\} \right] \mathbb{1}\{X_i \leq x\} \\
&= \arg \min_z \sum_{i=1}^n \frac{1}{\alpha_n} \left[\alpha_n(Y_i - q(1)) - z \right] \left[\mathbb{1}\{\alpha_n(Y_i - q(1)) > z\} - \frac{k}{n\hat{p}} \right] \mathbb{1}\{X_i \leq x\} \\
&= \arg \min_z kz + \sum_{i=1}^n \mathbb{L}(-\alpha_n(Y_i - q(1)), -z) \mathbb{1}\{X_i \leq x\} \\
&= \arg \min_z kz + \int \mathbb{L}(u, -z) d\hat{N}_n \\
&= \arg \min_{-z} -kz + \int \mathbb{L}(u, z) d\hat{N}_n \\
&= -\arg \min_z -kz + \int \mathbb{L}(u, z) d\hat{N}_n,
\end{aligned} \tag{B.2}$$

where $\hat{N}_n = \sum_{i=1}^n \mathbb{1}\{-\alpha_n(Y_i - q(1)) \in \cdot, X_i \leq x\}$ is a point process and the second last inequality is due to a change of variables. We denote

$$Q_n(z, k) = -kz + \int \mathbb{L}(u, z) d\hat{N}_n \tag{B.3}$$

as the sample objective function. Then

$$\begin{aligned}
&(-\widehat{Z}_n(k_0), -\widehat{Z}_n(mk_0), -\widehat{Z}_n(k_1), \dots, -\widehat{Z}_n(k_L)) \\
&= \arg \min_{z_0, \tilde{z}_0, z_1, \dots, z_L} Q_n(z_0, k_0) + Q_n(\tilde{z}_0, mk_0) + \sum_{l=1}^L Q_n(z_l, k_l)
\end{aligned}$$

We first derive the limit of the sample objective function

$$Q_n(z_0, k_0) + Q_n(\tilde{z}_0, mk_0) + \sum_{l=1}^L Q_n(z_l, k_l) \tag{B.4}$$

point-wise in $(z_0, \tilde{z}_0, z_1, \dots, z_L)$. Since the check function $\ell_\tau(u)$ and thus the sample objective function are convex, the point-wise convergence in $(z_0, \tilde{z}_0, z_1, \dots, z_L)$ is sufficient for the uniform convergence in $(z_0, \tilde{z}_0, z_1, \dots, z_L)$. Given the uniform convergence of the sample objective function, in the second step we show that the limiting objective function has a unique minimizer

$$(-Z_\infty(k_0), -Z_\infty(mk_0), -Z_\infty(k_1), \dots, -Z_\infty(k_L))$$

with probability one. Then, by Lemma A.1, we have

$$(\widehat{Z}_n(k_0), \widehat{Z}_n(mk_0), \widehat{Z}_n(k_1), \dots, \widehat{Z}_n(k_L)) \rightsquigarrow (Z_\infty(k_0), Z_\infty(mk_0), Z_\infty(k_1), \dots, Z_\infty(k_L)).$$

We focus on deriving the limit of $Q_n(z, k)$ in (B.3) with generic (z, k) such that k is not an integer. Then, the limit of (B.4) is just the sum of the limits of each term in it.

For the second term of $Q_n(z, k)$ in (B.3), we can show that the point process $\hat{N}_n(\cdot)$ weakly converges to $N(\cdot)$, a Poisson random measure with mean measure $\mu([a, b]) = \eta^{-1}(b) - \eta^{-1}(a)$. In addition, note that both $\hat{N}_n(\cdot)$ and $N(\cdot)$ are random measures on $\Re^+ = [0, \infty)$ because $Y_i \leq q(1)$ for any $i \geq 1$. Then for any fixed $z \geq 0$ and $u \in \Re^+$, $|\mathbb{L}(u, z)|$ is bounded by z , vanishes for $u \geq z$, and is continuous in u . By the continuous mapping theorem, we have, point-wise in z ,

$$\int \mathbb{L}(u, z) d\hat{N}_n \rightsquigarrow \int \mathbb{L}(u, z) dN.$$

Now we show

$$\hat{N}_n(\cdot) \rightsquigarrow N(\cdot).$$

Let $T_i = \alpha_n(Y_i - q(1))$. By Chernozhukov (2005, Lemma 9.3 and 9.4), it suffices to show that, for any $0 \leq a < b < \infty$,

$$n\mathbb{P}(-T_i \in [a, b], X_i \leq x) \rightarrow \eta^{-1}(b) - \eta^{-1}(a).$$

Note that $F(y/x) = \mathbb{P}(Y \leq y | X \leq x)$ and

$$\frac{1}{np_0} = 1 - F(q(1) - \frac{1}{\alpha_n}/x).$$

Then,

$$\begin{aligned} n\mathbb{P}(-T_i \in [a, b], X_i \leq x) &= np_0 \mathbb{P}(-T_i \in [a, b] | X_i \leq x) \\ &= \mathbb{P}(Y_i \in [q(1) - \frac{b}{\alpha_n}, q(1) - \frac{a}{\alpha_n}] | X_i \leq x) / (1 - F(q(1) - \frac{1}{\alpha_n}/x)) \\ &= \frac{F(q(1) - \frac{a}{\alpha_n}/x) - F(q(1) - \frac{b}{\alpha_n}/x)}{1 - F(q(1) - \frac{1}{\alpha_n}/x)} \\ &\rightarrow \eta^{-1}(b) - \eta^{-1}(a), \end{aligned}$$

where the last convergence follows Assumption 4. By Resnick (1987, Propositions 3.7 and 3.8), $N(\cdot)$ can be written as $\sum_{i=1}^{\infty} \mathbf{1}\{\mathcal{J}_i \in \cdot\}$, where $\mathcal{J}_i = (\sum_{j=1}^i \mathcal{E}_j)^{-\xi_0}$ and $\{\mathcal{E}_i\}_{i \geq 1}$ is a sequence of i.i.d. standard exponential random variables. Therefore, the sample objective function will converge to

$$Q_\infty(z, k) = -kz + \int \mathbb{L}(u, z) dN = -kz + \sum_{i=1}^{\infty} \mathbb{L}(\mathcal{J}_i, z)$$

weakly and uniformly over $z \in \Re^+$.

In addition, from the first-order condition of the limit objective function, we have

$$\begin{aligned} & (-Z_\infty(k_0), -Z_\infty(mk_0), -Z_\infty(k_1), \dots, -Z_\infty(k_L)) \\ &= \arg \min_{z_0, \tilde{z}_0, z_1, \dots, z_L} Q_\infty(z_0, k_0) + Q_\infty(\tilde{z}_0, mk_0) + \sum_{l=1}^L Q_\infty(z_l, k_l). \end{aligned}$$

This establishes (B.1).

Step 2:

By (B.1), we have

$$\begin{pmatrix} \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{n1}) - q(1)) \\ \vdots \\ \hat{\alpha}_n(\hat{q}_n(\hat{\tau}_{nL}) - q(1)) \end{pmatrix} \rightsquigarrow \begin{pmatrix} Z_\infty(k_1)/(Z_\infty(k_0) - Z_\infty(mk_0)) \\ \vdots \\ Z_\infty(k_L)/(Z_\infty(k_0) - Z_\infty(mk_0)) \end{pmatrix} = \begin{pmatrix} \tilde{Z}_\infty(k_1) \\ \vdots \\ \tilde{Z}_\infty(k_L) \end{pmatrix}.$$

The denominator $Z_\infty(k_0) - Z_\infty(mk_0)$ is nonzero because $Z_\infty(k_0) = -\mathcal{J}_{h_0}$ and $Z_\infty(mk_0) = -\mathcal{J}_{h'_0}$ for $h_0 \in (k_0, k_0 + 1)$ and $h'_0 \in (mk_0, mk_0 + 1)$, respectively, and by Assumption 5, $h_0 \neq h'_0$ because $mk_0 > k_0 + 1$.

C Proof of Theorem 4.1

By Theorem 3.1, we have

$$(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L)) \rightsquigarrow (\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L)) = O_p(1).$$

Therefore, for any $\varepsilon > 0$, we can choose a constant M sufficiently large such that

$$\mathbb{P}\left((\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L)) \in [-M, M]^L\right) \geq 1 - \varepsilon.$$

It suffices to show that

$$\theta_n^{BE}(z_{1n}, \dots, z_{Ln}; \hat{\xi}) \xrightarrow{p} \theta_\infty^{BE}(z_1, \dots, z_L),$$

where both (z_{1n}, \dots, z_{Ln}) and (z_1, \dots, z_L) are two deterministic sequences that belong to $[-M, M]^L$ and

$$(z_{1n}, \dots, z_{Ln}) \rightarrow (z_1, \dots, z_L).$$

Note

$$\begin{aligned}
& \theta_n^{BE}(z_{1n}, \dots, z_{Ln}; \hat{\xi}) \\
&= \arg \min_z Q_n(z, z_{1n}, \dots, z_{Ln}; \hat{\xi}) \\
&= \arg \min_z \frac{1}{\hat{\sigma}} \int \int \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \xi) \pi(q(1) + v/\hat{\alpha}_n) \phi(\frac{\xi - \hat{\xi}}{\hat{\sigma}}) 1\{\xi \in \Gamma\} d\xi dv \\
&= \arg \min_z \int \int \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi} + u\hat{\sigma}) \pi(q(1) + v/\hat{\alpha}_n) \phi(u) 1\{u \in \Gamma_n\} du dv,
\end{aligned}$$

where $\Gamma_n = (\Gamma - \hat{\xi})/\hat{\sigma}$. In addition, we have

$$\begin{aligned}
& \int \int \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi} + u\hat{\sigma}) \pi(q(1) + v/\hat{\alpha}_n) \phi(u) 1\{u \in \Gamma_n\} du dv \\
&= C_n \int \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi}) \pi(q(1) + v/\hat{\alpha}_n) dv \\
&\quad + \int \int \ell(z - v) (f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi} + u\hat{\sigma}) - f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi})) \\
&\quad \times \pi(q(1) + v/\hat{\alpha}_n) \phi(u) 1\{u \in \Gamma_n\} du dv,
\end{aligned} \tag{C.1}$$

where $C_n = \int_{\Gamma_n} \phi(u) du \rightarrow 1$ as $n \rightarrow \infty$. By Assumption 6.4, $f(z_1, \dots, z_L; \xi)$ is continuous in all its arguments, $\pi(q(1) + v/\hat{\alpha}_n) \rightarrow \pi(q(1))$. Therefore, point-wise in v ,

$$C_n \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi}) \pi(q(1) + v/\hat{\alpha}_n) \xrightarrow{p} \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \xi_0) \pi(q(1)).$$

In addition, we have $\mathbb{P}(\hat{\xi} \in \Gamma) \geq 1 - \varepsilon$ as n being sufficiently large. Therefore, by Assumption 6.4 and with probability greater than $1 - \varepsilon$,

$$\int |\ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi}) \pi(q(1) + v/\hat{\alpha}_n)| dv \lesssim \int |\ell(z - v)| H_{1M}(v) dv < \infty.$$

By the dominated convergence theorem, we have, point-wise in z ,

$$\begin{aligned}
& C_n \int \ell(z - v) f(z_{1n} - v, \dots, z_{Ln} - v; \hat{\xi}) \pi(q(1) + v/\hat{\alpha}_n) \{v \in \Omega_n\} dv \\
&\xrightarrow{p} \int \ell(z - v) f(z_1 - v, \dots, z_L - v; \xi_0) \pi(q(1)) dv \equiv Q_\infty(z, z_1, \dots, z_L).
\end{aligned}$$

Let $\Gamma'_n = [-\hat{\sigma}^{-1/2}, \hat{\sigma}^{-1/2}]$. For the second term on the RHS of (C.1), we have

$$\begin{aligned}
& \left| \int \int \ell(z-v) (f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi} + u\hat{\sigma}) - f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi})) \right. \\
& \quad \times \pi(q(1) + v/\hat{\alpha}_n) \phi(u) 1\{u \in \Gamma_n\} 1\{v \in \Omega_n\} dv du \Big| \\
& \leq \int \int |\ell(z-v) (f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi} + u\hat{\sigma}) - f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi}))| \\
& \quad \times \pi(q(1) + v/\hat{\alpha}_n) \phi(u) 1\{u \in \Gamma'_n\} dv du \\
& \quad + \int \int |\ell(z-v) (f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi} + u\hat{\sigma}) - f(z_{1n}-v, \dots, z_{Ln}-v; \hat{\xi}))| \\
& \quad \times \pi(q(1) + v/\hat{\alpha}_n) \phi(u) (1 - 1\{u \in \Gamma'_n\}) dv du \\
& \leq \hat{\sigma} \int \int |\ell(z-v)| H_{2M}(v) |u| \phi(u) 1\{u \in \Gamma'_n\} dudv + 2 \int \int |\ell(z-v)| H_1(v) \phi(u) (1 - 1\{u \in \Gamma'_n\}) dudv \\
& \xrightarrow{p} 0,
\end{aligned}$$

where the last inequality is due to Assumption 6.4 and the convergence in the last line holds because $\hat{\sigma} \xrightarrow{p} 0$ and that

$$\int \phi(u) (1 - 1\{u \in \Gamma'_n\}) du \xrightarrow{p} 0.$$

Therefore, point-wise in z ,

$$Q_n(z, z_{1n}, \dots, z_{Ln}; \hat{\xi}) \xrightarrow{p} Q_\infty(z, z_1, \dots, z_L).$$

In addition, since $\ell(\cdot)$ is convex in z , so be $Q_n(\cdot; \hat{\xi})$ and $Q_\infty(\cdot)$. In view of Lemma A.1, we have verified (i) and assumed (ii) and (iii) in Assumption 6.7. Therefore, by Lemma A.1,

$$\theta_n^{BE}(z_{1n}, \dots, z_{Ln}; \hat{\xi}) \xrightarrow{p} \theta_\infty^{BE}(z, z_1, \dots, z_L)$$

where $\theta_n^{BE}(\cdot)$ and $\theta_\infty^{BE}(\cdot)$ are defined in (4.2) and (4.5), respectively. Since the sequence (z_{1n}, \dots, z_{Ln}) is arbitrary, we have

$$\theta_n^{BE}(z_1, \dots, z_L; \hat{\xi}) \xrightarrow{p} \theta_\infty^{BE}(z_1, \dots, z_L)$$

uniformly over (z_1, \dots, z_L) in any compact subset of the joint support of $(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$. In addition, we note that

$$\begin{pmatrix} \tilde{Z}_n(k_1) \\ \vdots \\ \tilde{Z}_n(k_L) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \tilde{Z}_\infty(k_1) \\ \vdots \\ \tilde{Z}_\infty(k_L) \end{pmatrix}.$$

Therefore, by the continuous mapping theorem,

$$\hat{Z}_n^{BE} \equiv \theta_n^{BE}(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L); \hat{\xi}) \rightsquigarrow Z_\infty^{BE} \equiv \theta_\infty^{BE}(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L)).$$

This concludes the proof.

D Proof of Theorem 4.2

First, the proof of Theorem 4.1 implies, uniformly over $(z_1, \dots, z_L) \in [-M, M]^L$,

$$\theta_n^{BE}(z_1, \dots, z_L; \hat{\xi}) \xrightarrow{p} \theta_\infty^{BE}(z_1, \dots, z_L),$$

where $\theta_\infty^{BE}(z_1, \dots, z_L)$ is defined in (4.5). In addition, by Assumptions 6.3 and 6.5, with probability approaching one,

$$\sup_{v \in K_t} \ell(\theta_n^{BE}(z_1, \dots, z_L; \hat{\xi}) - v) f(z_1 - v, \dots, z_L - v; \xi_0)$$

is dominated by

$$C \left(\sum_{l=1}^L |z_l^{d_2}| + t \right)^{d_1} H_{3t}(z_1, \dots, z_L),$$

which is an integrable function w.r.t. (z_1, \dots, z_L) for fixed t . Therefore, by the dominated convergence theorem, as $n \rightarrow \infty$

$$\begin{aligned} & \int_{\Re^L} \ell(\theta_n^{BE}(z_1, \dots, z_L; \hat{\xi}) - v) f(z_1 - v, \dots, z_L - v; \xi_0) dz_1 \cdots dz_L \\ & \xrightarrow{p} \int_{\Re^L} \ell(\theta_\infty^{BE}(z_1, \dots, z_L) - v) f(z_1 - v, \dots, z_L - v; \xi_0) dz_1 \cdots dz_L. \end{aligned} \tag{D.1}$$

By (4.5) and a change of variable argument, we have, for any v ,

$$\theta_\infty^{BE}(z_1, \dots, z_L) - v = \theta_\infty^{BE}(z_1 - v, \dots, z_L - v).$$

Furthermore, by construction, $f(\cdot; \xi_0)$ is the joint PDF of $(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))$. Therefore,

$$\begin{aligned} \text{the RHS of (D.1)} &= \int_{\Re^L} \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L \\ &= \mathbb{E} \ell(\theta_\infty^{BE}(\tilde{Z}_\infty(k_1), \dots, \tilde{Z}_\infty(k_L))) = \mathbb{E} \ell(Z_\infty^{BE}), \end{aligned}$$

where the last equality holds because $Z_\infty^{BE} = \theta_\infty^{BE}(Z_\infty(k_1), \dots, Z_\infty(k_L))$. Then, we have, for every fixed t ,

$$\begin{aligned} & \int_{K_t} \int_{\Re^L} \ell(\theta_n^{BE}(z_1, \dots, z_L; \hat{\xi}) - v) f(z_1 - v, \dots, z_L - v; \xi_0) dz_1 \cdots dz_L dv / \Lambda(K_t) \\ & \xrightarrow{p} \int_{K_t} \mathbb{E} \ell(Z_\infty^{BE}) dv / \Lambda(K_t) = \mathbb{E} \ell(Z_\infty^{BE}) \end{aligned},$$

Taking $\limsup_{t \rightarrow \infty}$ on both sides, we have

$$AAR_\ell(\{\theta_n^{BE}\}) = \mathbb{E} \ell(Z_\infty^{BE}).$$

To prove the second result, for each $t \geq 1$, we denote $\tilde{q}_{n,t}^{BE}$ as the quasi-Bayesian estimator with prior $\pi(\bar{q}) = \mathbb{1}\{\hat{\alpha}_n(\bar{q} - q(1)) \in K_t\}$, i.e.,

$$\hat{\alpha}_n(\tilde{q}_{n,t}^{BE} - q(1)) = \tilde{\theta}_t^{BE}(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_1); \hat{\xi}),$$

where $\tilde{\theta}_t^{BE}(z_1, \dots, z_L)$ is defined in (4.6). Next, we aim to show

$$\limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} AR_{\ell, K_t}(\tilde{\theta}_t^{BE}) = \mathbb{E} \ell(Z_\infty^{BE}). \quad (\text{D.2})$$

Note that,

$$\begin{aligned} & AR_{\ell, K_t}(\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \hat{\xi})) \\ &= \int_{-t}^t \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \hat{\xi}) - v) f(z_1 - v, \dots, z_L - v; \hat{\xi}) dz_1 \cdots dz_L dv / 2t \\ &= \int_{-1}^1 \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \hat{\xi}) - tu) f(z_1 - tu, \dots, z_L - tu; \hat{\xi}) dz_1 \cdots dz_L du / 2 \\ &\xrightarrow{p} \int_{-1}^1 \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du / 2, \end{aligned} \quad (\text{D.3})$$

where the last convergence follows the same argument in (D.1). By the definition of $\tilde{\theta}_t^{BE}$ in (4.6),

$$\begin{aligned} & \tilde{\theta}_t^{BE}(w_1 + tu, \dots, w_L + tu; \xi_0) - tu \\ &= \arg \min_{\gamma} \int_{-t}^t \ell(\gamma + tu - v) f(w_1 + tu - v, \dots, w_L + tu - v; \xi_0) dv \\ &= \arg \min_{\gamma} \int_{\Re} \mathbb{1}\{v \in (t - tu, -t - tu)\} \ell(\gamma - v) f(w_1 - v, \dots, w_L - v; \xi_0) dv. \end{aligned}$$

Since $u \in (-1, 1)$, as $t \rightarrow \infty$, $\mathbb{1}\{v \in (t - tu, -t - tu)\} \uparrow 1$. Therefore, by the monotone convergence

theorem, point-wise in γ ,

$$\begin{aligned} & \int_{\Re} \mathbb{1}\{v \in (t-tu, -t-tu)\} \ell(\gamma-v) f(w_1-v, \dots, w_L-v; \xi_0) dv \\ & \rightarrow \int_{\Re} \ell(\gamma-v) f(w_1-v, \dots, w_L-v; \xi_0) dv. \end{aligned}$$

Then, by Lemma A.1, as $t \rightarrow \infty$

$$\tilde{\theta}_t^{BE}(z_1 + tu, z_L + tu; \xi_0) - tu \rightarrow \theta_\infty^{BE}(z_1, \dots, z_L). \quad (\text{D.4})$$

Following (D.3), in order to show (D.2), it suffices to show, as $t \rightarrow \infty$

$$\begin{aligned} & \int_{-1}^1 \int_{\Re^L} \left| \ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) - \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) \right| \\ & \quad \times f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du / 2 \\ & = \int_{-1}^1 \int_{\Re^L} \left[\ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) - \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) \right]^- \\ & \quad \times f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du / 2 \\ & \quad + \int_{-1}^1 \int_{\Re^L} \left[\ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) - \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) \right]^+ \\ & \quad \times f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du / 2 \\ & = I_t + II_t \rightarrow 0. \end{aligned}$$

For I_t , we have

$$\left[\ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) - \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) \right]^- \leq \ell(\theta_\infty^{BE}(z_1, \dots, z_L))$$

which, by Assumption 6.4, is integrable w.r.t. $f(z_1, \dots, z_L; \xi_0) \mathbb{1}\{|u| < 1\} dz_1 \cdots dz_L du$. Therefore, by (D.4) and the dominated convergence theorem, we have $I_t \rightarrow 0$.

In addition, by (4.6),

$$\begin{aligned} & \int_{-1}^1 \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du \\ & = \int_{-t}^t \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1, \dots, z_L; \xi_0) - v) f(z_1 - v, \dots, z_L - v; \xi_0) dz_1 \cdots dz_L dv / 2t \\ & \leq \int_{-t}^t \int_{\Re^L} \ell(\theta_\infty^{BE}(z_1, \dots, z_L) - v) f(z_1 - v, \dots, z_L - v; \xi_0) dz_1 \cdots dz_L dv / 2t \\ & = \int_{-1}^1 \int_{\Re^L} \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du / 2. \end{aligned}$$

Therefore,

$$\begin{aligned}
& 2(II_t - I_t) \\
&= \int_{-1}^1 \int_{\Re^L} \ell(\theta_\infty^{BE}(z_1, \dots, z_L)) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du \\
&\quad - \int_{-1}^1 \int_{\Re^L} \ell(\tilde{\theta}_t^{BE}(z_1 + tu, \dots, z_L + tu; \xi_0) - tu) f(z_1, \dots, z_L; \xi_0) dz_1 \cdots dz_L du \\
&\leq 0,
\end{aligned}$$

or equivalently,

$$0 \leq II_t \leq I_t \rightarrow 0.$$

This concludes (D.2). If there exists a sequence of estimators, denoted as $\{\check{\theta}_n\}$, such that $\check{\theta}_n \in \Theta_n$ and it achieves strictly smaller asymptotic average risk than the quasi-Bayesian estimator θ_n^{BE} , then for infinitely many t and n ,

$$AR_{\ell, K_t}(\check{\theta}_n) < AR_{\ell, K_t}(\tilde{\theta}_t^{BE}).$$

This is a contradiction because, by construction,

$$\tilde{\theta}_t^{BE}(\cdot) \in \arg \min_{\theta \in \Theta_n} AR_{\ell, K_t}(\theta).$$

This concludes the proof.

E Proof of Corollary 4.1

Denote $\hat{Z}_n^{BE}(\tau') = \hat{\alpha}_n(\hat{q}^{BE}(\tau') - q(1)) = \theta_n^{BE}(\tilde{Z}_n(k_1), \dots, \tilde{Z}_n(k_L); \hat{\xi})$. Then we have

$$\mathbb{P}(\hat{q}^{BE}(\tau') > q(1)) = \mathbb{P}(\hat{Z}_n^{BE}(\tau') > 0) \rightarrow \mathbb{P}(Z_\infty^{BE}(\tau') > 0).$$

Next, we show

$$\mathbb{P}(Z_\infty^{BE}(\tau') > 0) = \tau'.$$

Suppose not, then there exists a nonzero constant c such that $\mathbb{P}(Z_\infty^{BE}(\tau') > c) = \tau'$ or equivalently, by the first order condition,

$$\mathbb{E}\tilde{\ell}_{\tau'}(Z_\infty^{BE}(\tau') - c) < \mathbb{E}\tilde{\ell}_{\tau'}(Z_\infty^{BE}(\tau')),$$

where the loss function $\tilde{\ell}_{\tau'}(\cdot)$ is defined in Corollary 4.1. Similar to the proof of the first result in Theorem 4.2, we can show $\mathbb{E}\tilde{\ell}_{\tau'}(Z_\infty^{BE}(\tau') - c)$ is the asymptotic average risk for the estimator

$\theta_n^{BE}(\cdot; \hat{\xi}) - c$, i.e.,

$$AAR_{\tilde{\ell}_{\tau'}}(\{\theta_n^{BE}(\cdot; \hat{\xi}) - c\}) = \mathbb{E}\tilde{\ell}_{\tau'}(Z_\infty^{BE}(\tau') - c) < \mathbb{E}\tilde{\ell}_{\tau'}(Z_\infty^{BE}(\tau')) = AAR_{\tilde{\ell}_{\tau'}}(\{\theta_n^{BE}(\cdot; \hat{\xi})\}).$$

On the other hand, $\theta_n^{BE}(\cdot; \hat{\xi}) - c \in \Theta_n$. Therefore, we reach a contradiction to the second result in Theorem 4.2. This implies

$$\mathbb{P}(Z_\infty^{BE}(\tau') > 0) = \tau'.$$

Then, for $\tau' < \tau''$

$$\begin{aligned} & \mathbb{P}(\hat{q}^{BE}(\tau') \leq q(1) \leq \hat{q}^{BE}(\tau'')) \\ &= 1 - \mathbb{P}(q(1) > \hat{q}^{BE}(\tau'') \text{ or } q(1) < \hat{q}^{BE}(\tau')) \\ &= 1 - \mathbb{P}(q(1) > \hat{q}^{BE}(\tau'')) - \mathbb{P}(q(1) < \hat{q}^{BE}(\tau')) + \mathbb{P}(\hat{q}^{BE}(\tau') > q(1) > \hat{q}^{BE}(\tau'')) \\ &= \mathbb{P}(\hat{Z}_n^{BE}(\tau'') > 0) - \mathbb{P}(\hat{Z}_n^{BE}(\tau') > 0) \\ &\rightarrow \mathbb{P}(Z_\infty^{BE}(\tau'') > 0) - \mathbb{P}(Z_\infty^{BE}(\tau') > 0) = \tau'' - \tau', \end{aligned}$$

where the third equality holds due to the fact that, by construction, $\hat{q}^{BE}(\tau')$, the τ' -th posterior quantile, is less than or equal to $\hat{q}^{BE}(\tau'')$, the τ'' -th posterior quantile, as $\tau'' > \tau'$.

F Proof of Proposition 4.1

We consider the CDF evaluated at (u_1, \dots, u_L) such that $u_1 < u_2, \dots, < u_L$. Note that

$$Z_\infty(k) = -\mathcal{J}_{\lceil k \rceil} = -(\gamma_1^{\lceil k \rceil}/p)^{-\xi},$$

where $\gamma_i^j = \sum_{l=i}^j \mathcal{E}_l$. Therefore,

$$\begin{aligned} & \mathbb{P}(\tilde{Z}_\infty(k_1) \leq u_1, \dots, \tilde{Z}_\infty(k_L) \leq u_L) \\ &= \mathbb{E}\mathbb{P}(\tilde{Z}_\infty(k_1) \leq u_1, \dots, \tilde{Z}_\infty(k_L) \leq u_L | \gamma_1^{\lceil k_0 \rceil}, \gamma_1^{\lceil mk_0 \rceil}) \\ &= \mathbb{E}\mathbb{P}\left(\frac{(\gamma_1^{\lceil k_1 \rceil})^{-\xi}}{(\gamma_1^{\lceil mk_0 \rceil})^{-\xi} - (\gamma_1^{\lceil k_0 \rceil})^{-\xi}} \leq u_1, \dots, \frac{(\gamma_1^{\lceil k_L \rceil})^{-\xi}}{(\gamma_1^{\lceil mk_0 \rceil})^{-\xi} - (\gamma_1^{\lceil k_0 \rceil})^{-\xi}} \leq u_L \middle| \gamma_1^{\lceil k_0 \rceil}, \gamma_1^{\lceil mk_0 \rceil}\right) \\ &= \mathbb{E}\mathbb{P}\left(\gamma_{\lceil mk_0 \rceil+1}^{\lceil k_1 \rceil} \leq [u_1((\gamma_1^{\lceil mk_0 \rceil})^{-\xi} - (\gamma_1^{\lceil k_0 \rceil})^{-\xi})]^{-1/\xi} - \gamma_1^{\lceil mk_0 \rceil}, \dots, \right. \\ & \quad \left. \gamma_{\lceil mk_0 \rceil+1}^{\lceil k_L \rceil} \leq [u_L((\gamma_1^{\lceil mk_0 \rceil})^{-\xi} - (\gamma_1^{\lceil k_0 \rceil})^{-\xi})]^{-1/\xi} - \gamma_1^{\lceil mk_0 \rceil} | \gamma_1^{\lceil k_0 \rceil}, \gamma_1^{\lceil mk_0 \rceil}\right) \end{aligned} \tag{F.1}$$

Notice that

$$(\gamma_{\lceil mk_0 \rceil+1}^{\lceil k_1 \rceil}, \dots, \gamma_{\lceil mk_0 \rceil+1}^{\lceil k_L \rceil}) \perp\!\!\!\perp (\gamma_1^{\lceil k_0 \rceil}, \gamma_1^{\lceil mk_0 \rceil}).$$

Let $s = \gamma_1^{[k_0]}$, $t = \gamma_{\lceil k_0 \rceil + 1}^{[mk_0]}$, $\tilde{u} = (t + s)^{-\xi} - s^{-\xi}$, respectively. Then,

$$\begin{aligned} & \text{The RHS of (F.1)} \\ &= \int \mathbb{P}\left(\gamma_{\lceil mk_0 \rceil + 1}^{[k_1]} \leq (u_1 \tilde{u}(t, s))^{-1/\xi} - t, \dots, \gamma_{\lceil mk_0 \rceil + 1}^{[k_L]} \leq (u_L \tilde{u}(t, s))^{-1/\xi} - t\right) \\ & \quad \times f_{\lceil k_0 \rceil}(s) f_{\lceil mk_0 \rceil - \lceil k_0 \rceil}(t) ds dt. \end{aligned}$$

Take derivatives w.r.t. (u_1, \dots, u_L) , we obtain that

$$\begin{aligned} & f(u_1, \dots, u_L; \xi) \\ &= \int (-1/\xi)^L \tilde{u}(t, s)^{-L/\xi} \left[\prod_{l=1}^L u_l^{-1/\xi-1} f_{h_l-h_{l-1}}(v_l - v_{l-1}) \right] f_{\lceil k_0 \rceil}(s) f_{\lceil mk_0 \rceil - \lceil k_0 \rceil}(t) ds dt, \end{aligned}$$

where $h_l = \lceil k_l \rceil$ for $L \geq l \geq 1$, $h_0 = \lceil mk_0 \rceil$, $v_l = (u_l \tilde{u}(t, s))^{-1/\xi}$ for $L \geq l \geq 1$, and $v_0 = t$.

G The Computation of the Three Existing Methods

We compute the three estimators in the literature based on the instructions in [Daouia et al. \(2017\)](#). The details are listed below.

- Moment frontier estimator (“Mom”)
 - The built-in EV index estimator is computed using the function **rho_momt_pick** with argument **method = “moment”**.
 - The tuning parameter k_n involved in the estimation of the EV index is computed by the function **kopt_momt_pick** with **method = “moment”** and estimated EV index.
 - Based on the above estimation, **dfs_momt** is used to compute the estimator and the corresponding 95% confidence interval for the production frontier.
- Pickands frontier estimator (“Momt_pick”)
 - The built-in EV index is estimated using function **rho_momt_pick** with argument **method = “pickands”**.
 - The tuning parameter k_n involved in the estimation of the EV index is computed by the function **kopt_momt_pick** with **method = “pickands”** and the estimated EV index.
 - Based on the above estimation, **dfs_pick** is used to compute the estimator and the corresponding 95% confidence interval for the production frontier.¹

¹Based on [Daouia et al. \(2010\)](#), the expressions for the asymptotic variance of the Pickands frontier estimator are different depending on whether the EV index is estimated or not. Since we estimate the EV index, we use the expression of $V_2(\rho_x)$ in [Daouia et al. \(2010, Theorem 2.5\)](#).

- Probability-weighted moment frontier estimator ("Pwm")
 - The built-in EV index estimator is computed using **rho_pwm** with the default arguments.
 - The tuning parameter k_n involved in the estimation of the EV index is computed by **mopt_pwm** with default arguments.
 - Based on the above estimation, **dfs_pwm** is used to compute the estimator and the corresponding 95% confidence interval for the production frontier is constructed via bootstrap following the procedure described in [Daouia et al. \(2012\)](#).²

H The Numerical Evaluation of the Density $f(u_1, \dots, u_L; \xi)$

In this section, we introduce the procedure to evaluate the value of $f(u_1, \dots, u_L; \xi)$ established in [Proposition 4.1](#). We use the simple Trapezoid rule to evaluate the integrals with fine grids. The detailed procedure is as follows.

- Let $p^{right} = 0.99999$ and $p^{left} = 0.00001$. Obtain the (p^{left}, p^{right}) quantiles of random variables with densities $f_{\lceil k_0 \rceil}(s)$ and $f_{\lceil m k_0 \rceil - \lceil k_0 \rceil}(t)$, and denote them as $(Q_1^{left}, Q_1^{right})$ and $(Q_2^{left}, Q_2^{right})$, respectively.
- Construct a $I_1 \times I_1$ grid G for the rectangle area $[Q_1^{left}, Q_1^{right}] \times [Q_2^{left}, Q_2^{right}]$. Further denote $g_i^1 = Q_1^{left} + i \times \frac{Q_1^{right} - Q_1^{left}}{I_1 - 1}$, $g_j^2 = Q_2^{left} + j \times \frac{Q_2^{right} - Q_2^{left}}{I_1 - 1}$ for $i, j = 0, \dots, I_1 - 1$, and

$$\begin{aligned}\tilde{f}_{i,j} &= \left(-\frac{1}{\xi}\right)^L \tilde{u}(g_i^1, g_j^2)^{-\frac{L}{\xi}} \left[\prod_{l=1}^L u_l^{-\frac{1}{\xi}-1} f_{h_l-h_{l-1}}(v_l - v_{l-1}) \right] \\ &\quad \times f_{\lceil k_0 \rceil}(g_i^1) f_{\lceil m k_0 \rceil - \lceil k_0 \rceil}(g_j^2).\end{aligned}$$

- Evaluate the density $f(u_1, \dots, u_L; \xi)$ numerically, i.e.,

$$\begin{aligned}\hat{f}(u_1, \dots, u_L; \xi) &= \frac{(Q_1^{right} - Q_1^{left})(Q_2^{right} - Q_2^{left})}{(I_1 - 1)(I_1 - 1)} \\ &\quad \times \sum_{i=0}^{I_1-1} \sum_{j=0}^{I_1-1} \frac{1}{4} [\tilde{f}_{i,j} + \tilde{f}_{i,j+1} + \tilde{f}_{i+1,j} + \tilde{f}_{i+1,j+1}].\end{aligned}$$

For implementation, we let $I_1 = 100$. Based on our simulation experience, such numerical integration is much faster and more accurate than the usual Monte Carlo method with 100,000 random

²Although the R package **npbr** produces the analytical confidence interval for the probability-weighted estimator as established in [Daouia et al. \(2012\)](#), we follow the practice in [Daouia et al. \(2012\)](#) and conduct bootstrap inference. In our simulation study, we find that the bootstrap inference has better performance in terms of coverage rates.

draws.

I Some calculation of ξ for production and cost frontiers

In this section, we show the calculation of ξ for DGP 1 using the first f and the first \mathcal{U} . We also obtain ξ for DGP 1 when we try to estimate the cost function, based on what we propose in the application. Throughout this section, we assume all functions are smooth enough, and limits exist so that we could apply L'Hospital's rule. The calculations here can be easily extended to other DGPs. It turns our ξ are the same for production frontier and cost frontier for all 15 DGPs. We omit the details, due to similarity.

Before the calculations, we show a general result.

Suppose $Y = f(X)\mathcal{U}$. The density of \mathcal{U} is $g(u)$. Let $f^{-1}(y)$ denote the inverse of $f(x)$. Furture $X \sim \text{Unif}(0, 1)$. Then by the definition,

$$\begin{aligned} F(y|x) &= P(f(X)\mathcal{U} \leq y | X \leq x) \\ &= P\left(X \leq f^{-1}\left(\frac{y}{\mathcal{U}}\right) | X \leq x\right) \\ &= \int_0^1 \int_0^{f^{-1}\left(\frac{y}{u}\right) \wedge x} \frac{1}{x} dt g(u) du = \frac{1}{x} \int_0^1 \left(f^{-1}\left(\frac{y}{u}\right) \wedge x\right) g(u) du \\ &= \frac{1}{x} \int_0^{\frac{y}{f(x)}} x g(u) du + \frac{1}{x} \int_{\frac{y}{f(x)}}^1 f^{-1}\left(\frac{y}{u}\right) g(u) du \\ &= \int_0^{\frac{y}{f(x)}} g(u) du + \frac{1}{x} \int_{\frac{y}{f(x)}}^1 f^{-1}\left(\frac{y}{u}\right) g(u) du \end{aligned}$$

Further,

$$\begin{aligned} &\lim_{z \rightarrow 0} \frac{1 - F(f(x) - vz)}{1 - F(f(x) - z)} \\ &= \lim_{z \rightarrow 0} \frac{1 - \int_0^{1 - \frac{vz}{f(x)}} g(u) du - \frac{1}{x} \int_{1 - \frac{vz}{f(x)}}^1 f^{-1}\left(\frac{f(x)-vz}{u}\right) g(u) du}{1 - \int_0^{1 - \frac{z}{f(x)}} g(u) du - \frac{1}{x} \int_{1 - \frac{z}{f(x)}}^1 f^{-1}\left(\frac{f(x)-z}{u}\right) g(u) du} \\ &\stackrel{\text{L'Hospital's rule}}{=} \lim_{z \rightarrow 0} \frac{\frac{v}{f(x)} g\left(1 - \frac{vz}{f(x)}\right) - \frac{1}{x} \frac{v}{f(x)} f^{-1}(f(x)) g\left(1 - \frac{vz}{f(x)}\right) - \frac{1}{x} \int_{1 - \frac{vz}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x)-vz}{u}\right) g(u) du}{\frac{1}{f(x)} g\left(1 - \frac{z}{f(x)}\right) - \frac{1}{x} \frac{1}{f(x)} f^{-1}(f(x)) g\left(1 - \frac{z}{f(x)}\right) - \frac{1}{x} \int_{1 - \frac{z}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x)-z}{u}\right) g(u) du} \\ &= \lim_{z \rightarrow 0} \frac{\int_{1 - \frac{vz}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x)-vz}{u}\right) g(u) du}{\int_{1 - \frac{z}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x)-z}{u}\right) g(u) du} \end{aligned}$$

Therefore,

$$\lim_{z \rightarrow 0} \frac{1 - F(f(x) - vz)}{1 - F(f(x) - z)} = \lim_{z \rightarrow 0} \frac{\int_{1 - \frac{vz}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x) - vz}{u}\right) g(u) du}{\int_{1 - \frac{z}{f(x)}}^1 \frac{\partial}{\partial z} f^{-1}\left(\frac{f(x) - z}{u}\right) g(u) du}$$

For DGP 1, $Y = XU$, $U \sim \text{Unif}(0, 1)$, $g(u) = 1$, $f(x) = x$ and $f^{-1}(y) = y$. Using the above result,

$$\lim_{z \rightarrow 0} \frac{\int_{1 - \frac{vz}{x}}^1 \frac{\partial}{\partial z} (x - vz) u^{-1} du}{\int_{1 - \frac{z}{x}}^1 \frac{\partial}{\partial z} (x - z) u^{-1} du} = \lim_{z \rightarrow 0} \frac{-v \int_{1 - \frac{vz}{x}}^1 u^{-1} du}{-\int_{1 - \frac{z}{x}}^1 u^{-1} du} = \lim_{z \rightarrow 0} \frac{-\frac{v^2}{x} (1 - \frac{vz}{x})^{-1}}{-\frac{1}{x} (1 - \frac{z}{x})^{-1}} = v^2$$

Thus, by the definition of ξ ,

$$\xi = -\frac{1}{2}.$$

Now, we turn to the cost function. Again, we first present a general result, then we apply it to DGP 1.

Let $\check{Y} = -X$, $\check{X} = -Y$. Let $C(\check{X}) = -f^{-1}(-\check{X})$. Then, by $Y = f(X)U$,

$$\check{Y} = C(\check{X}U^{-1}), \quad U \in [0, 1].$$

We can alternatively let $\tilde{Y} = -X + M$, $\tilde{X} = -Y + M$, where M is a large positive constant. Note this transformation gives the same value of ξ as that from \check{X} and \check{Y} . We obtain ξ based on \check{X} and \check{Y} .

By definition,

$$\begin{aligned} F(\check{y}|\check{x}) &= P(\check{Y} \leq \check{y} | \check{X} \leq \check{x}) = P(-X \leq \check{y} | -Y \leq \check{x}) \\ &= P(X \geq -\check{y} | Y \geq -\check{x}) \\ &= P(X \geq -\check{y} | f(X)U \geq -\check{x}) \\ &= \frac{P(X \geq -\check{y}, f(X)U \geq -\check{x})}{P(f(X)U \geq -\check{x})} \\ &= \frac{1}{P(f(X)U \geq -\check{x})} P\left(X \geq -\check{y}, X \geq f^{-1}\left(\frac{-\check{x}}{U}\right)\right) \\ &= \frac{1}{P(f(X)U \geq -\check{x})} \int_0^1 \int_{-\check{y} \vee f^{-1}(\frac{-\check{x}}{U})}^1 dt g(u) du \\ &= \frac{1}{P(f(X)U \geq -\check{x})} \int_0^1 \left(1 - \left(-\check{y} \vee f^{-1}\left(\frac{-\check{x}}{u}\right)\right)\right) g(u) du \\ &= \frac{1}{P(f(X)U \geq -\check{x})} \left[\int_0^{-\frac{\check{x}}{f(-\check{y})}} \left(1 - f^{-1}\left(\frac{-\check{x}}{u}\right)\right) g(u) du + (1 + \check{y}) \int_{-\frac{\check{x}}{f(-\check{y})}}^1 g(u) du \right] \end{aligned}$$

By the definition of ξ ,

$$\frac{1 - F(-f^{-1}(-\check{x}) - vz)}{1 - F(-f^{-1}(-\check{x}) - z)} \rightarrow v^{-1/\xi},$$

and (note $P(f(X)U \geq -\check{x})$ is cancelled out)

$$\begin{aligned} & \lim_{z \rightarrow 0} \frac{1 - F(-f^{-1}(-\check{x}) - vz)}{1 - F(-f^{-1}(-\check{x}) - z)} \\ &= \lim_{z \rightarrow 0} \frac{1 - \int_0^{-\frac{\check{x}}{f(f^{-1}(-\check{x})+vz)}} (1 - f^{-1}(\frac{-\check{x}}{u})) g(u) du - (1 - f^{-1}(-\check{x}) - vz) \int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+vz)}}^1 g(u) du}{1 - \int_0^{-\frac{\check{x}}{f(f^{-1}(-\check{x})+z)}} (1 - f^{-1}(\frac{-\check{x}}{u})) g(u) du - (1 - f^{-1}(-\check{x}) - z) \int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+z)}}^1 g(u) du} \\ &\stackrel{\text{L'Hospital's rule}}{=} \lim_{z \rightarrow 0} \frac{v \int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+vz)}}^1 g(u) du}{\int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+z)}}^1 g(u) du} \quad (\text{first two derivatives cancelled out, similar to before}). \end{aligned}$$

Therefore,

$$\lim_{z \rightarrow 0} \frac{1 - F(-f^{-1}(-\check{x}) - vz)}{1 - F(-f^{-1}(-\check{x}) - z)} = \lim_{z \rightarrow 0} \frac{v \int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+vz)}}^1 g(u) du}{\int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+z)}}^1 g(u) du}.$$

We apply the above result for DGP 1 where $Y = X\mathcal{U}$, $\mathcal{U} \sim \text{Unif}(0, 1)$, $g(u) = 1$, $f(x) = x$ and $f^{-1}(y) = y$. Then

$$\lim_{z \rightarrow 0} \frac{v \int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+vz)}}^1 g(u) du}{\int_{-\frac{\check{x}}{f(f^{-1}(-\check{x})+z)}}^1 g(u) du} = \lim_{z \rightarrow 0} \frac{v \int_{-\frac{-\check{x}}{vz}}^1 du}{\int_{-\frac{-\check{x}}{vz}}^1 du} = \lim_{z \rightarrow 0} \frac{v^2 \frac{-\check{x}}{(-\check{x}+vz)^2}}{\frac{-\check{x}}{(-\check{x}+z)^2}} = v^2$$

So

$$\xi = -\frac{1}{2}.$$

J Additional Simulation Results

J.1 Addition Simulation Results

In this section, we report simulation results for DGPs(1, 2), (2, 2), (3, 2), (1, 3), \dots , (3, 5) in Tables 5–16.

Table 5: DGP(1,2)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9680	0.9760	0.9710	0.8670	0.8880	0.9740
$np_0 = 125$	(0.2223)	(0.2222)	(0.2236)	(0.2835)	(0.5236)	(0.7557)
$n = 1000$	0.9500	0.9590	0.9540	0.9070	0.8920	1.0000
$np_0 = 250$	(0.1236)	(0.1239)	(0.1230)	(0.1406)	(0.2827)	(0.5458)
$n = 2000$	0.9350	0.9430	0.9340	0.9110	0.8990	1.0000
$np_0 = 500$	(0.0728)	(0.0734)	(0.0747)	(0.0864)	(0.1833)	(0.3875)
$n = 4000$	0.9480	0.9480	0.9390	0.8600	0.8870	1.0000
$np_0 = 1000$	(0.0396)	(0.0402)	(0.0406)	(0.0545)	(0.1240)	(0.2686)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9870	0.9870	0.9890	0.3810	0.9510	0.8090
$np_0 = 250$	(0.2058)	(0.2048)	(0.2055)	(0.2240)	(0.4296)	(0.5777)
$n = 1000$	0.9560	0.9580	0.9490	0.4210	0.9470	0.9530
$np_0 = 500$	(0.1184)	(0.1201)	(0.1206)	(0.1337)	(0.2652)	(0.4385)
$n = 2000$	0.9450	0.9420	0.9480	0.5230	0.9240	0.9930
$np_0 = 1000$	(0.0698)	(0.0705)	(0.0716)	(0.0836)	(0.1725)	(0.3198)
$n = 4000$	0.9460	0.9340	0.9420	0.5660	0.9090	0.9980
$np_0 = 2000$	(0.0360)	(0.0358)	(0.0362)	(0.0535)	(0.1170)	(0.2223)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9740	0.9760	0.9760	0.9930	0.9390	0.9860
$np_0 = 375$	(0.2059)	(0.2066)	(0.2057)	(0.1389)	(0.3951)	(0.2230)
$n = 1000$	0.9560	0.9640	0.9550	0.9730	0.9260	0.9970
$np_0 = 750$	(0.1119)	(0.1123)	(0.1131)	(0.0870)	(0.2483)	(0.1580)
$n = 2000$	0.9240	0.9220	0.9300	0.9680	0.9260	0.9980
$np_0 = 1500$	(0.0595)	(0.0597)	(0.060)	(0.0577)	(0.1696)	(0.1084)
$n = 4000$	0.9650	0.9680	0.9640	0.9360	0.8970	1.0000
$np_0 = 3000$	(0.0366)	(0.0360)	(0.0359)	(0.0379)	(0.1129)	(0.0719)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 6: DGP(2,2)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9740	0.9720	0.9710	1.0000	0.9810	0.9380
$np_0 = 125$	(0.6393)	(0.6427)	(0.6415)	(1.1016)	(2.6322)	(1.2612)
$n = 1000$	0.9820	0.9810	0.9790	0.9870	0.9560	0.9860
$np_0 = 250$	(0.3708)	(0.3692)	(0.3704)	(0.5646)	(1.4879)	(0.9243)
$n = 2000$	0.9740	0.9720	0.9770	0.9560	0.9510	0.9940
$np_0 = 500$	(0.2549)	(0.2540)	(0.2514)	(0.3520)	(1.1942)	(0.6325)
$n = 4000$	0.9750	0.9780	0.9710	0.9740	0.9720	0.9930
$np_0 = 1000$	(0.1809)	(0.1791)	(0.1787)	(0.2513)	(0.8610)	(0.4343)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9270	0.9310	0.9310	1.0000	0.9820	0.7450
$np_0 = 250$	(0.6665)	(0.6679)	(0.6669)	(0.9576)	(2.3058)	(1.0034)
$n = 1000$	0.9400	0.9260	0.9130	0.9980	0.9740	0.8380
$np_0 = 500$	(0.4457)	(0.4402)	(0.4325)	(0.6068)	(1.6316)	(0.7577)
$n = 2000$	0.9650	0.9640	0.9610	0.9970	0.9880	0.9400
$np_0 = 1000$	(0.2994)	(0.2945)	(0.2925)	(0.3964)	(1.1999)	(0.5570)
$n = 4000$	0.9570	0.9530	0.9490	0.9980	0.9870	0.9650
$np_0 = 2000$	(0.2121)	(0.2079)	(0.2052)	(0.2871)	(0.8722)	(0.3917)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9720	0.9660	0.9690	0.9910	0.9870	0.8630
$np_0 = 375$	(0.6293)	(0.6237)	(0.6229)	(0.7477)	(2.1069)	(0.5684)
$n = 1000$	0.9550	0.9470	0.9430	0.9800	0.9810	0.9140
$np_0 = 750$	(0.4501)	(0.4437)	(0.4393)	(0.5214)	(1.5795)	(0.4232)
$n = 2000$	0.9820	0.9790	0.9840	0.9780	0.9900	0.9540
$np_0 = 1500$	(0.2995)	(0.2937)	(0.2928)	(0.3638)	(1.1538)	(0.3062)
$n = 4000$	0.9730	0.9790	0.9820	0.9880	0.9940	0.9830
$np_0 = 3000$	(0.2073)	(0.2042)	(0.2017)	(0.2556)	(0.8206)	(0.2102)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 7: DGP(3,2)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9740	0.9730	0.9730	1.0000	0.9830	0.9530
$np_0 = 125$	(0.8416)	(0.8272)	(0.8289)	(1.8886)	(3.8828)	(3.0487)
$n = 1000$	0.9800	0.9820	0.9780	0.9990	0.9820	0.9960
$np_0 = 250$	(0.4854)	(0.4864)	(0.4847)	(1.1938)	(2.8319)	(2.3919)
$n = 2000$	0.9680	0.9710	0.9630	0.9950	0.9800	1.0000
$np_0 = 500$	(0.3450)	(0.3419)	(0.3384)	(0.7545)	(1.9118)	(1.7313)
$n = 4000$	0.9550	0.9480	0.9510	0.9910	0.9720	1.0000
$np_0 = 1000$	(0.2482)	(0.2459)	(0.2449)	(0.5150)	(1.3444)	(1.2218)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9780	0.9770	0.9750	1.0000	0.9890	0.4830
$np_0 = 250$	(1.2304)	(1.2279)	(1.2310)	(2.8498)	(6.3030)	(3.4320)
$n = 1000$	0.9770	0.9760	0.9800	0.9990	0.9900	0.6890
$np_0 = 500$	(0.8119)	(0.7995)	(0.7891)	(1.7404)	(4.0388)	(2.7388)
$n = 2000$	0.9510	0.9560	0.9530	1.0000	0.9870	0.7980
$np_0 = 1000$	(0.5749)	(0.5640)	(0.5573)	(1.1643)	(2.8646)	(2.0439)
$n = 4000$	0.9430	0.9400	0.9380	0.9990	0.9870	0.8440
$np_0 = 2000$	(0.4249)	(0.4173)	(0.4122)	(0.8487)	(2.2158)	(1.4797)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9520	0.9550	0.9520	0.9970	0.9820	0.6780
$np_0 = 375$	(1.4704)	(1.4419)	(1.4353)	(2.5178)	(7.1443)	(1.8635)
$n = 1000$	0.9560	0.9580	0.9540	0.9920	0.9860	0.8180
$np_0 = 750$	(1.0701)	(1.0441)	(1.0320)	(1.6198)	(4.6928)	(1.4353)
$n = 2000$	0.9690	0.9720	0.9690	0.9830	0.9880	0.8380
$np_0 = 1500$	(0.7097)	(0.6981)	(0.6914)	(1.1881)	(3.6423)	(1.0679)
$n = 4000$	0.9450	0.9490	0.9470	0.9930	0.9930	0.8060
$np_0 = 3000$	(0.5166)	(0.5060)	(0.5003)	(0.9086)	(2.8784)	(0.7983)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 8: DGP(1,3)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9510	0.9470	0.9490	0.9950	0.9610	0.9590
$np_0 = 125$	(1.1584)	(1.1606)	(1.1519)	(2.2188)	(5.4216)	(3.4788)
$n = 1000$	0.9580	0.9520	0.9590	0.9780	0.9690	0.9850
$np_0 = 250$	(0.6645)	(0.6664)	(0.6673)	(1.5651)	(4.2438)	(2.7218)
$n = 2000$	0.9710	0.9670	0.9620	0.9880	0.9910	1.0000
$np_0 = 500$	(0.4965)	(0.4870)	(0.4741)	(1.2254)	(3.4312)	(2.0395)
$n = 4000$	0.9810	0.9740	0.9760	0.9800	0.9830	0.9990
$np_0 = 1000$	(0.3846)	(0.3779)	(0.3710)	(0.9431)	(2.7106)	(1.4748)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9500	0.9450	0.9450	1.0000	0.9900	0.4470
$np_0 = 250$	(1.7438)	(1.7454)	(1.7518)	(3.8273)	(9.5945)	(3.9034)
$n = 1000$	0.9510	0.9440	0.9320	0.9990	0.9840	0.5420
$np_0 = 500$	(1.2449)	(1.2019)	(1.1618)	(2.8144)	(7.2778)	(3.2288)
$n = 2000$	0.9700	0.9700	0.9680	0.9960	0.9880	0.7000
$np_0 = 1000$	(0.9012)	(0.8758)	(0.8521)	(2.0669)	(5.6281)	(2.5067)
$n = 4000$	0.9490	0.9470	0.9410	0.9880	0.9880	0.7500
$np_0 = 2000$	(0.6788)	(0.6605)	(0.6508)	(1.4570)	(4.1683)	(1.8238)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9600	0.9630	0.9640	0.9950	0.9910	0.5880
$np_0 = 375$	(2.0724)	(2.0203)	(2.0257)	(4.1978)	(11.7651)	(2.4990)
$n = 1000$	0.9430	0.9420	0.9420	0.9900	0.9920	0.6560
$np_0 = 750$	(1.6309)	(1.5576)	(1.5303)	(3.1555)	(9.2127)	(2.0170)
$n = 2000$	0.9870	0.9880	0.9840	0.9900	0.9870	0.6710
$np_0 = 1500$	(1.1557)	(1.1258)	(1.1066)	(2.3103)	(7.1616)	(1.5719)
$n = 4000$	0.9780	0.9830	0.9820	0.9690	0.9790	0.7540
$np_0 = 3000$	(0.8429)	(0.8294)	(0.8148)	(1.5585)	(5.0660)	(1.2132)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 9: DGP(2,3)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9590	0.9580	0.9600	1.0000	0.9850	0.9670
$np_0 = 125$	(1.2570)	(1.2608)	(1.2656)	(3.0654)	(6.9954)	(4.1390)
$n = 1000$	0.9680	0.9720	0.9680	0.9990	0.9910	0.9960
$np_0 = 250$	(0.7282)	(0.7302)	(0.7276)	(2.0424)	(5.1007)	(3.2040)
$n = 2000$	0.9690	0.9710	0.9600	0.9880	0.9950	1.0000
$np_0 = 500$	(0.5621)	(0.5410)	(0.5223)	(1.6244)	(4.2416)	(2.4056)
$n = 4000$	0.9790	0.9800	0.9820	0.9880	0.9960	1.0000
$np_0 = 1000$	(0.4553)	(0.4443)	(0.4350)	(1.2363)	(3.3682)	(1.7375)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9340	0.9310	0.9340	1.0000	0.9990	0.4010
$np_0 = 250$	(1.8895)	(1.8931)	(1.8920)	(5.2153)	(12.0894)	(4.5569)
$n = 1000$	0.9390	0.9390	0.9380	0.9990	0.9950	0.5330
$np_0 = 500$	(1.3633)	(1.2968)	(1.2508)	(3.7687)	(8.7902)	(3.8196)
$n = 2000$	0.9730	0.9800	0.9830	0.9980	0.9970	0.6800
$np_0 = 1000$	(1.0554)	(1.0226)	(0.9857)	(2.8166)	(6.9658)	(2.9655)
$n = 4000$	0.9750	0.9780	0.9800	0.9990	0.9980	0.7570
$np_0 = 2000$	(0.8499)	(0.8223)	(0.8015)	(2.1130)	(5.5997)	(2.1860)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9810	0.9780	0.9810	0.9990	0.9970	0.5640
$np_0 = 375$	(2.2554)	(2.1930)	(2.1879)	(5.5002)	(15.0285)	(3.1276)
$n = 1000$	0.9670	0.9740	0.9750	0.9940	0.9980	0.6420
$np_0 = 750$	(1.8409)	(1.770)	(1.7079)	(4.2624)	(11.6848)	(2.5595)
$n = 2000$	0.9880	0.9900	0.9880	0.9910	1.0000	0.7150
$np_0 = 1500$	(1.3676)	(1.3241)	(1.2927)	(3.2516)	(8.9144)	(1.9690)
$n = 4000$	0.9830	0.9800	0.9870	0.9770	0.9960	0.7580
$np_0 = 3000$	(1.0811)	(1.0623)	(1.0419)	(2.3628)	(6.8181)	(1.5001)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 10: DGP(3,3)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9630	0.9690	0.9660	0.8640	0.9380	0.9560
$np_0 = 125$	(0.4289)	(0.4309)	(0.4328)	(0.8002)	(1.3381)	(2.2473)
$n = 1000$	0.9680	0.9720	0.9640	0.9200	0.9560	0.9970
$np_0 = 250$	(0.2483)	(0.2485)	(0.2496)	(0.4577)	(0.9028)	(1.7155)
$n = 2000$	0.9530	0.9350	0.9410	0.8880	0.9700	0.9980
$np_0 = 500$	(0.1552)	(0.1564)	(0.1566)	(0.3054)	(0.6532)	(1.2793)
$n = 4000$	0.9290	0.9310	0.9230	0.7850	0.9940	1.0000
$np_0 = 1000$	(0.0994)	(0.10)	(0.1002)	(0.2102)	(0.4783)	(0.9187)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9600	0.9660	0.9630	0.5250	0.9710	0.5720
$np_0 = 250$	(0.5851)	(0.5864)	(0.5868)	(1.0592)	(1.9112)	(2.4441)
$n = 1000$	0.9510	0.9420	0.9430	0.4820	0.9890	0.7630
$np_0 = 500$	(0.3654)	(0.3658)	(0.3668)	(0.6822)	(1.3430)	(1.9589)
$n = 2000$	0.9290	0.9230	0.9160	0.3370	0.9910	0.8930
$np_0 = 1000$	(0.2432)	(0.2432)	(0.2426)	(0.4655)	(0.9709)	(1.4748)
$n = 4000$	0.9160	0.9130	0.9110	0.1100	0.9990	0.9500
$np_0 = 2000$	(0.1556)	(0.1544)	(0.1543)	(0.3241)	(0.7116)	(1.0567)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9730	0.9720	0.9740	0.9950	0.9860	0.8570
$np_0 = 375$	(0.7113)	(0.7099)	(0.7080)	(0.8229)	(2.3778)	(1.0761)
$n = 1000$	0.9640	0.9600	0.9550	0.9970	0.9900	0.9200
$np_0 = 750$	(0.4496)	(0.4503)	(0.4469)	(0.5399)	(1.6277)	(0.7950)
$n = 2000$	0.9640	0.9610	0.9660	0.9900	1.0000	0.9250
$np_0 = 1500$	(0.2933)	(0.2892)	(0.2889)	(0.3890)	(1.2176)	(0.5679)
$n = 4000$	0.9550	0.9520	0.9540	0.9880	1.0000	0.9220
$np_0 = 3000$	(0.1798)	(0.1793)	(0.1783)	(0.2697)	(0.8681)	(0.3897)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 11: DGP(1,4)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9740	0.9740	0.9690	0.9990	0.9790	0.9560
$np_0 = 125$	(0.9623)	(0.9610)	(0.9614)	(2.1086)	(4.6819)	(3.2110)
$n = 1000$	0.9710	0.9700	0.9710	0.9960	0.9690	0.9980
$np_0 = 250$	(0.5612)	(0.5618)	(0.5620)	(1.3366)	(3.3795)	(2.5349)
$n = 2000$	0.9720	0.9760	0.9720	0.9850	0.9780	0.9990
$np_0 = 500$	(0.4025)	(0.3946)	(0.3884)	(0.9069)	(2.6871)	(1.8556)
$n = 4000$	0.9750	0.9690	0.9700	0.9780	0.9700	1.0000
$np_0 = 1000$	(0.3005)	(0.2954)	(0.2936)	(0.6073)	(2.3723)	(1.2899)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9500	0.9520	0.9530	1.0000	0.9970	0.4620
$np_0 = 250$	(1.4362)	(1.4411)	(1.4422)	(3.3134)	(7.8654)	(3.6376)
$n = 1000$	0.9740	0.9750	0.9590	1.0000	0.9900	0.6130
$np_0 = 500$	(0.9871)	(0.9643)	(0.9449)	(2.2222)	(6.5572)	(2.9438)
$n = 2000$	0.9660	0.9620	0.9610	0.9990	0.9860	0.7980
$np_0 = 1000$	(0.6947)	(0.6820)	(0.6715)	(1.3886)	(3.9875)	(2.1911)
$n = 4000$	0.9330	0.9280	0.9230	0.9940	0.9830	0.8370
$np_0 = 2000$	(0.5146)	(0.5036)	(0.4934)	(0.9951)	(2.9033)	(1.5660)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9550	0.9540	0.9560	0.9970	0.9860	0.6780
$np_0 = 375$	(1.6847)	(1.6648)	(1.6566)	(3.1221)	(9.0108)	(2.1454)
$n = 1000$	0.9540	0.9470	0.9460	0.9840	0.9900	0.7940
$np_0 = 750$	(1.2894)	(1.2615)	(1.2315)	(2.0033)	(6.2287)	(1.6640)
$n = 2000$	0.9700	0.9830	0.9840	0.9720	0.9830	0.8560
$np_0 = 1500$	(0.8711)	(0.8509)	(0.8372)	(1.4040)	(4.5268)	(1.2413)
$n = 4000$	0.9620	0.9650	0.9650	0.9750	0.9880	0.8300
$np_0 = 3000$	(0.6386)	(0.6248)	(0.6145)	(1.0613)	(3.4934)	(0.9235)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 12: DGP(2,4)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9660	0.9640	0.9680	0.9990	0.9810	0.9620
$np_0 = 125$	(0.2670)	(0.2674)	(0.2670)	(0.6926)	(1.2594)	(2.0065)
$n = 1000$	0.9710	0.9770	0.9700	1.0000	0.9830	1.0000
$np_0 = 250$	(0.1553)	(0.1556)	(0.1555)	(0.4422)	(0.9558)	(1.6196)
$n = 2000$	0.9670	0.9640	0.9660	1.0000	0.9880	1.0000
$np_0 = 500$	(0.1128)	(0.1106)	(0.1081)	(0.3270)	(0.7721)	(1.2250)
$n = 4000$	0.9590	0.9620	0.9610	0.9950	0.9770	1.0000
$np_0 = 1000$	(0.0837)	(0.0819)	(0.0808)	(0.2207)	(0.5373)	(0.8642)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9920	0.9940	0.9900	0.9990	0.9940	0.3980
$np_0 = 250$	(0.7504)	(0.7509)	(0.7524)	(2.2102)	(4.4224)	(4.1140)
$n = 1000$	0.9770	0.9760	0.9770	0.9990	0.9940	0.5780
$np_0 = 500$	(0.5166)	(0.5057)	(0.4938)	(1.5119)	(3.2712)	(3.3877)
$n = 2000$	0.9530	0.9550	0.9500	1.0000	0.9860	0.8510
$np_0 = 1000$	(0.3894)	(0.3807)	(0.3741)	(0.9752)	(2.2767)	(2.5438)
$n = 4000$	0.9410	0.9460	0.9370	0.9990	0.9880	0.9280
$np_0 = 2000$	(0.2882)	(0.2809)	(0.2779)	(0.6420)	(1.5477)	(1.7799)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9780	0.9780	0.9770	1.0000	0.9980	0.6240
$np_0 = 375$	(1.4110)	(1.3782)	(1.380)	(3.0772)	(8.2952)	(2.5734)
$n = 1000$	0.9830	0.9830	0.9830	0.9910	0.9890	0.7580
$np_0 = 750$	(1.0569)	(1.0272)	(1.0062)	(2.0931)	(6.0219)	(2.0002)
$n = 2000$	0.9800	0.9770	0.9840	0.9890	0.9850	0.8060
$np_0 = 1500$	(0.7331)	(0.7154)	(0.7044)	(1.4315)	(4.2559)	(1.4679)
$n = 4000$	0.9730	0.9740	0.9790	0.9720	0.9690	0.8680
$np_0 = 3000$	(0.5257)	(0.5143)	(0.5102)	(0.8776)	(2.7159)	(1.0339)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 13: DGP(3,4)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9360	0.9380	0.9330	0.9900	0.9250	0.9890
$np_0 = 125$	(0.3203)	(0.3228)	(0.3186)	(0.6860)	(1.5014)	(2.1898)
$n = 1000$	0.9220	0.9250	0.9260	0.9640	0.9490	1.0000
$np_0 = 250$	(0.1851)	(0.1848)	(0.1848)	(0.4224)	(1.0371)	(1.6840)
$n = 2000$	0.9470	0.9450	0.9430	0.9570	0.9690	1.0000
$np_0 = 500$	(0.1463)	(0.140)	(0.1348)	(0.3337)	(0.8660)	(1.2607)
$n = 4000$	0.9700	0.9690	0.9620	0.9770	0.9860	1.0000
$np_0 = 1000$	(0.1198)	(0.1158)	(0.1122)	(0.2716)	(0.7183)	(0.9206)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9850	0.9860	0.9840	0.9990	0.9720	0.5230
$np_0 = 250$	(0.9739)	(0.9740)	(0.9714)	(2.2293)	(5.1238)	(4.4394)
$n = 1000$	0.9880	0.9860	0.9850	0.9980	0.9830	0.6620
$np_0 = 500$	(0.7067)	(0.6712)	(0.6408)	(1.5532)	(3.7708)	(3.5509)
$n = 2000$	0.9730	0.9680	0.9710	0.9930	0.9920	0.8460
$np_0 = 1000$	(0.5509)	(0.5291)	(0.5129)	(1.2162)	(3.0338)	(2.7241)
$n = 4000$	0.9640	0.9590	0.9580	0.9970	0.9960	0.8960
$np_0 = 2000$	(0.4365)	(0.4221)	(0.4097)	(1.0004)	(2.6463)	(1.9730)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9630	0.9600	0.9570	0.9680	0.9670	0.7290
$np_0 = 375$	(1.7439)	(1.6775)	(1.6798)	(3.4327)	(9.2865)	(3.0586)
$n = 1000$	0.9830	0.9750	0.9730	0.9670	0.9780	0.7750
$np_0 = 750$	(1.4784)	(1.4198)	(1.3557)	(2.6812)	(7.5204)	(2.4019)
$n = 2000$	0.9850	0.9850	0.9850	0.9860	0.9930	0.7960
$np_0 = 1500$	(1.0767)	(1.0403)	(1.0157)	(2.0941)	(6.2039)	(1.7837)
$n = 4000$	0.9880	0.9890	0.9870	0.9940	0.9980	0.7660
$np_0 = 3000$	(0.8422)	(0.8159)	(0.7978)	(1.6545)	(5.0648)	(1.2786)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 14: DGP(1,5)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9290	0.9250	0.9280	1.0000	0.9640	0.9970
$np_0 = 125$	(0.3370)	(0.3358)	(0.3368)	(0.9546)	(1.8920)	(2.5371)
$n = 1000$	0.9040	0.8980	0.9000	0.9950	0.9630	1.0000
$np_0 = 250$	(0.1934)	(0.1932)	(0.1939)	(0.5695)	(1.3096)	(1.9455)
$n = 2000$	0.9420	0.9390	0.9370	0.9780	0.9830	1.0000
$np_0 = 500$	(0.1537)	(0.1469)	(0.1409)	(0.4194)	(1.0223)	(1.4481)
$n = 4000$	0.9650	0.9570	0.9640	0.9800	0.9940	1.0000
$np_0 = 1000$	(0.1320)	(0.1274)	(0.1232)	(0.3480)	(0.8797)	(1.0493)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9540	0.9540	0.9520	1.0000	0.9910	0.4550
$np_0 = 250$	(1.0248)	(1.0261)	(1.0203)	(3.1054)	(6.6101)	(5.0743)
$n = 1000$	0.9800	0.9790	0.9770	1.0000	0.9890	0.6800
$np_0 = 500$	(0.7549)	(0.7142)	(0.6767)	(1.9824)	(4.3994)	(4.0805)
$n = 2000$	0.9760	0.9810	0.9730	1.0000	0.9980	0.8290
$np_0 = 1000$	(0.6110)	(0.5864)	(0.5676)	(1.6036)	(3.7310)	(3.1246)
$n = 4000$	0.9810	0.9790	0.9850	0.9960	0.9970	0.9400
$np_0 = 2000$	(0.5068)	(0.4887)	(0.4719)	(1.2493)	(3.0515)	(2.2757)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9450	0.9430	0.9420	0.9950	0.9860	0.6800
$np_0 = 375$	(1.7857)	(1.7186)	(1.7239)	(4.5003)	(10.8424)	(3.6719)
$n = 1000$	0.9760	0.9680	0.9670	0.9880	0.9920	0.7720
$np_0 = 750$	(1.5884)	(1.4967)	(1.4216)	(3.4665)	(8.9477)	(2.9001)
$n = 2000$	0.9710	0.9690	0.9640	0.9860	0.9980	0.8320
$np_0 = 1500$	(1.1929)	(1.1490)	(1.1178)	(2.7701)	(7.5159)	(2.1623)
$n = 4000$	0.9810	0.9850	0.9720	0.9870	0.9990	0.8130
$np_0 = 3000$	(0.9974)	(0.9647)	(0.9339)	(2.1998)	(6.1795)	(1.5627)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 15: DGP(2,5)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9750	0.9680	0.9680	0.8950	0.9590	0.9620
$np_0 = 125$	(0.1659)	(0.1654)	(0.1658)	(0.4583)	(0.7104)	(1.6608)
$n = 1000$	0.9460	0.9560	0.9450	0.9000	0.9480	1.0000
$np_0 = 250$	(0.0937)	(0.0937)	(0.0935)	(0.2429)	(0.4537)	(1.3028)
$n = 2000$	0.9490	0.9520	0.9430	0.8540	0.9470	1.0000
$np_0 = 500$	(0.0606)	(0.0608)	(0.0609)	(0.1422)	(0.2909)	(0.9406)
$n = 4000$	0.9250	0.9210	0.9220	0.7360	0.9510	1.0000
$np_0 = 1000$	(0.0396)	(0.0398)	(0.0398)	(0.0944)	(0.2058)	(0.6686)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9690	0.9600	0.9570	0.6140	0.9580	0.5220
$np_0 = 250$	(0.4512)	(0.4497)	(0.4494)	(1.1313)	(1.8356)	(3.3187)
$n = 1000$	0.9390	0.9260	0.9160	0.4640	0.9560	0.7470
$np_0 = 500$	(0.2850)	(0.2851)	(0.2858)	(0.6401)	(1.1551)	(2.5805)
$n = 2000$	0.9170	0.9200	0.9010	0.2780	0.9760	0.8600
$np_0 = 1000$	(0.1930)	(0.1920)	(0.1913)	(0.4356)	(0.8223)	(1.9184)
$n = 4000$	0.9150	0.9110	0.9000	0.0570	0.9840	0.9420
$np_0 = 2000$	(0.1239)	(0.1227)	(0.1234)	(0.3066)	(0.6338)	(1.3889)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9640	0.9600	0.9570	0.9960	0.9710	0.7790
$np_0 = 375$	(0.8155)	(0.8197)	(0.8194)	(1.1820)	(3.4385)	(1.7884)
$n = 1000$	0.9400	0.9420	0.9240	0.9890	0.9780	0.8820
$np_0 = 750$	(0.5310)	(0.530)	(0.5319)	(0.7408)	(2.2031)	(1.2994)
$n = 2000$	0.9510	0.9530	0.9490	0.9880	0.9860	0.9010
$np_0 = 1500$	(0.3468)	(0.3441)	(0.3440)	(0.5227)	(1.5975)	(0.9328)
$n = 4000$	0.9490	0.9500	0.9540	0.9790	0.9900	0.8540
$np_0 = 3000$	(0.2226)	(0.2207)	(0.2203)	(0.3922)	(1.2192)	(0.6570)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 16: DGP(3,5)

Panel A: $x = 1.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9660	0.9680	0.9700	0.9990	0.9610	0.9740
$np_0 = 125$	(0.2897)	(0.2908)	(0.2917)	(0.6908)	(1.3902)	(2.0587)
$n = 1000$	0.9650	0.9610	0.9640	0.9980	0.9810	0.9990
$np_0 = 250$	(0.1729)	(0.1728)	(0.1729)	(0.450)	(1.0407)	(1.6376)
$n = 2000$	0.9680	0.9650	0.9540	0.9960	0.9870	1.0000
$np_0 = 500$	(0.1258)	(0.1222)	(0.1195)	(0.3451)	(1.1745)	(1.2522)
$n = 4000$	0.9610	0.9630	0.9670	0.9890	0.9830	1.0000
$np_0 = 1000$	(0.0978)	(0.0953)	(0.0941)	(0.2520)	(0.7611)	(0.8988)
Panel B: $x = 3.0$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9880	0.9860	0.9890	1.0000	0.9850	0.4500
$np_0 = 250$	(0.8559)	(0.8508)	(0.8492)	(2.2227)	(4.8326)	(4.2059)
$n = 1000$	0.9880	0.9880	0.9820	1.0000	0.9970	0.6160
$np_0 = 500$	(0.6021)	(0.5808)	(0.5653)	(1.6202)	(5.0796)	(3.4570)
$n = 2000$	0.9760	0.9750	0.9700	0.9990	0.9900	0.8300
$np_0 = 1000$	(0.4561)	(0.4425)	(0.4307)	(1.1778)	(3.4365)	(2.6573)
$n = 4000$	0.9440	0.9510	0.9520	0.9980	0.9840	0.9240
$np_0 = 2000$	(0.3482)	(0.3384)	(0.3317)	(0.8205)	(2.3834)	(1.8895)
Panel C: $x = 4.5$						
	Quasi-Bayesian			Pickands		
	k'_L	k_L	k''_L	Mom	Momt-pick	Pwm
$n = 500$	0.9750	0.9720	0.9790	0.9960	0.9890	0.6370
$np_0 = 375$	(1.5712)	(1.5248)	(1.5297)	(3.3904)	(9.6408)	(2.7729)
$n = 1000$	0.9910	0.9870	0.9860	0.9890	0.9910	0.7330
$np_0 = 750$	(1.2442)	(1.1967)	(1.1629)	(2.4786)	(7.7734)	(2.1640)
$n = 2000$	0.9820	0.9860	0.9810	0.9810	0.9860	0.7970
$np_0 = 1500$	(0.8781)	(0.8533)	(0.8341)	(1.8032)	(5.9840)	(1.6152)
$n = 4000$	0.9770	0.9780	0.9800	0.9790	0.9830	0.8380
$np_0 = 3000$	(0.6431)	(0.6276)	(0.6143)	(1.2068)	(4.1326)	(1.1778)

Notes: $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$. The coverage rates and average lengths of the CIs (in parentheses) are reported.

J.2 Sensitivity of $\pi(\bar{q})$

In this section, we examine the sensitive of the choice of $\pi(\bar{q})$ in DGPs(1,1) and (2,1). The first and second main columns in Tables 17 and 18 report the results for the quasi-Bayesian method when we use $\pi_1(\bar{q})$ and $\pi_2(\bar{q})$ as the prior respectively, where $\pi_1(\bar{q})$ and $\pi_2(\bar{q})$ are normal with mean \bar{q}^* and variance 1 and 1.5, respectively. Note we set $k_0 = \text{number of outliers} + 2$, $sp = 5$, and

$k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$.

Table 17: Robustness check of $\pi(\bar{q})$ for DGP(1,1)

Panel A: $x = 1.5$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9650	0.9670	0.9650	0.9650	0.9670	0.9650
$np_0 = 125$	(0.5322)	(0.5302)	(0.5325)	(0.5322)	(0.5302)	(0.5325)
$n = 1000$	0.9800	0.9810	0.9800	0.9800	0.9810	0.9800
$np_0 = 250$	(0.3073)	(0.3072)	(0.3061)	(0.3073)	(0.3072)	(0.3061)
$n = 2000$	0.9730	0.9710	0.9710	0.9730	0.9710	0.9710
$np_0 = 500$	(0.2138)	(0.2117)	(0.2120)	(0.2138)	(0.2117)	(0.2120)
$n = 4000$	0.9650	0.9580	0.9550	0.9650	0.9580	0.9550
$np_0 = 1000$	(0.1477)	(0.1467)	(0.1458)	(0.1477)	(0.1467)	(0.1458)
Panel B: $x = 3.0$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9540	0.9550	0.9540	0.9540	0.9550	0.9540
$np_0 = 250$	(0.5348)	(0.5343)	(0.5360)	(0.5348)	(0.5343)	(0.5360)
$n = 1000$	0.9800	0.9750	0.9720	0.9800	0.9750	0.9720
$np_0 = 500$	(0.3479)	(0.3457)	(0.3423)	(0.3479)	(0.3457)	(0.3423)
$n = 2000$	0.9520	0.9530	0.9510	0.9520	0.9530	0.9510
$np_0 = 1000$	(0.2461)	(0.2425)	(0.2409)	(0.2461)	(0.2425)	(0.2409)
$n = 4000$	0.9390	0.9440	0.9380	0.9390	0.9440	0.9380
$np_0 = 2000$	(0.1715)	(0.1686)	(0.1667)	(0.1715)	(0.1686)	(0.1667)
Panel C: $x = 4.5$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9680	0.9660	0.9630	0.9680	0.9660	0.9630
$np_0 = 375$	(0.5136)	(0.5121)	(0.5095)	(0.5136)	(0.5121)	(0.5095)
$n = 1000$	0.9750	0.9720	0.9730	0.9750	0.9720	0.9730
$np_0 = 1000$	(0.3528)	(0.3486)	(0.3465)	(0.3528)	(0.3486)	(0.3465)
$n = 2000$	0.9760	0.9720	0.9720	0.9760	0.9720	0.9720
$np_0 = 1500$	(0.2431)	(0.2395)	(0.2384)	(0.2431)	(0.2395)	(0.2384)
$n = 4000$	0.9830	0.9740	0.9750	0.9830	0.9740	0.9750
$np_0 = 3000$	(0.1719)	(0.1680)	(0.1665)	(0.1719)	(0.1680)	(0.1665)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 18: Robustness check of $\pi(\bar{q})$ for DGP(2,1)

Panel A: $x = 1.5$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9690	0.9710	0.9720	0.9720	0.9730	0.9770
$np_0 = 125$	(0.8276)	(0.8277)	(0.8376)	(0.8293)	(0.8359)	(0.8269)
$n = 1000$	0.9810	0.9820	0.9770	0.9810	0.9810	0.9810
$np_0 = 250$	(0.4848)	(0.4826)	(0.4803)	(0.4836)	(0.4868)	(0.4861)
$n = 2000$	0.9620	0.9610	0.9590	0.9680	0.9710	0.9620
$np_0 = 500$	(0.3443)	(0.3405)	(0.3371)	(0.3450)	(0.3419)	(0.3384)
$n = 4000$	0.9500	0.9550	0.9500	0.9550	0.9480	0.9510
$np_0 = 1000$	(0.2466)	(0.2447)	(0.2415)	(0.2482)	(0.2459)	(0.2449)
Panel B: $x = 3.0$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9790	0.9790	0.9770	0.9840	0.9820	0.9830
$np_0 = 250$	(1.1597)	(1.1670)	(1.1541)	(1.1641)	(1.1572)	(1.1646)
$n = 1000$	0.9720	0.9690	0.9650	0.9700	0.9730	0.9680
$np_0 = 500$	(0.7865)	(0.7687)	(0.7593)	(0.7842)	(0.7705)	(0.7622)
$n = 2000$	0.9470	0.9450	0.9470	0.9510	0.9560	0.9530
$np_0 = 1000$	(0.5727)	(0.5644)	(0.5564)	(0.5749)	(0.5640)	(0.5573)
$n = 4000$	0.9360	0.9320	0.9360	0.9430	0.9400	0.9380
$np_0 = 2000$	(0.4238)	(0.4151)	(0.4128)	(0.4249)	(0.4173)	(0.4122)
Panel C: $x = 4.5$						
	$\pi_1(\bar{q})$			$\pi_2(\bar{q})$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9810	0.9820	0.9770	0.9800	0.9760	0.9850
$np_0 = 375$	(1.3457)	(1.3233)	(1.3391)	(1.3657)	(1.3399)	(1.3423)
$n = 1000$	0.9790	0.9760	0.9760	0.9830	0.9790	0.9800
$np_0 = 1000$	(0.9763)	(0.9581)	(0.9473)	(0.9830)	(0.9680)	(0.9507)
$n = 2000$	0.9680	0.9710	0.9640	0.9690	0.9720	0.9690
$np_0 = 1500$	(0.7086)	(0.6982)	(0.6920)	(0.7097)	(0.6981)	(0.6915)
$n = 4000$	0.9450	0.9440	0.9380	0.9450	0.9490	0.9470
$np_0 = 3000$	(0.5142)	(0.5032)	(0.4980)	(0.5166)	(0.5060)	(0.5003)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

J.3 Sensitivity to k_0

In this section, we examine the sensitivity of the choice of k_0 in DGPs(1,1) and (2,1). The first and second main columns in Tables 19 and 20 report the results for the quasi-Bayesian method with $k_0 = \text{Number of outliers} + 3$ and $k_0 = \text{Number of outliers} + 4$, respectively. Note we set $sp = 5$ and $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$.

Table 19: Robustness check of k_0 for DGP(1,1)

Panel A: $x = 1.5$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k_L'	k_L	k_L''	k_L'	k_L	k_L''
$n = 500$	0.9870	0.9850	0.9870	0.9830	0.9860	0.9830
$np_0 = 125$	(0.4944)	(0.4972)	(0.4943)	(0.5786)	(0.580)	(0.5802)
$n = 1000$	0.9880	0.9880	0.9900	0.9910	0.9870	0.9900
$np_0 = 250$	(0.3174)	(0.3168)	(0.3173)	(0.3551)	(0.3526)	(0.3524)
$n = 2000$	0.9830	0.9850	0.9820	0.9910	0.9890	0.9890
$np_0 = 500$	(0.2273)	(0.2267)	(0.2263)	(0.2537)	(0.2504)	(0.2487)
$n = 4000$	0.9730	0.9710	0.9700	0.9850	0.9820	0.9840
$np_0 = 1000$	(0.1578)	(0.1563)	(0.1550)	(0.1756)	(0.1733)	(0.1717)

Panel B: $x = 3.0$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k_L'	k_L	k_L''	k_L'	k_L	k_L''
$n = 500$	0.9560	0.9550	0.9540	0.9910	0.9930	0.9940
$np_0 = 250$	(0.5405)	(0.5339)	(0.5354)	(0.5722)	(0.5683)	(0.5675)
$n = 1000$	0.9630	0.9670	0.9620	0.9800	0.9810	0.9820
$np_0 = 500$	(0.3776)	(0.3730)	(0.3697)	(0.4062)	(0.4003)	(0.3950)
$n = 2000$	0.9620	0.9700	0.9650	0.9640	0.9710	0.9690
$np_0 = 1000$	(0.2671)	(0.2613)	(0.2582)	(0.2884)	(0.2817)	(0.2769)
$n = 4000$	0.9590	0.9610	0.9620	0.9670	0.9660	0.9700
$np_0 = 2000$	(0.1802)	(0.1779)	(0.1755)	(0.1994)	(0.1941)	(0.1916)

Panel C: $x = 4.5$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k_L'	k_L	k_L''	k_L'	k_L	k_L''
$n = 500$	0.9720	0.9710	0.9690	0.9860	0.9870	0.9860
$np_0 = 375$	(0.5363)	(0.5282)	(0.5271)	(0.5721)	(0.5599)	(0.5577)
$n = 1000$	0.9610	0.9600	0.9600	0.9750	0.9780	0.9790
$np_0 = 1000$	(0.3875)	(0.3797)	(0.3756)	(0.4119)	(0.4011)	(0.3965)
$n = 2000$	0.9500	0.9470	0.9480	0.9650	0.9680	0.9640
$np_0 = 1500$	(0.2744)	(0.2687)	(0.2655)	(0.2910)	(0.2844)	(0.2816)
$n = 4000$	0.9420	0.9380	0.9410	0.9440	0.9460	0.9440
$np_0 = 3000$	(0.1897)	(0.1858)	(0.1830)	(0.2045)	(0.1986)	(0.1947)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 20: Robustness check of k_0 for DGP(2,1)

Panel A: $x = 1.5$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9770	0.9750	0.9730	0.9670	0.9740	0.9770
$np_0 = 125$	(0.7416)	(0.7415)	(0.7450)	(0.8529)	(0.8477)	(0.8501)
$n = 1000$	0.9840	0.9810	0.9820	0.9880	0.9890	0.9890
$np_0 = 250$	(0.4906)	(0.4899)	(0.490)	(0.5414)	(0.5373)	(0.5355)
$n = 2000$	0.9830	0.9840	0.9850	0.9940	0.9940	0.9910
$np_0 = 500$	(0.3668)	(0.3615)	(0.3594)	(0.4041)	(0.3977)	(0.3932)
$n = 4000$	0.9710	0.9690	0.9690	0.9830	0.9860	0.9870
$np_0 = 1000$	(0.2644)	(0.2615)	(0.2586)	(0.2916)	(0.2873)	(0.2843)

Panel B: $x = 3.0$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9610	0.9710	0.9690	0.9920	0.9870	0.9870
$np_0 = 250$	(1.1746)	(1.1580)	(1.1509)	(1.2520)	(1.2390)	(1.2286)
$n = 1000$	0.9910	0.9870	0.9860	0.9850	0.9910	0.9880
$np_0 = 500$	(0.8485)	(0.8354)	(0.8230)	(0.920)	(0.8957)	(0.8831)
$n = 2000$	0.9760	0.9740	0.9730	0.9860	0.9850	0.9880
$np_0 = 1000$	(0.6072)	(0.5980)	(0.5888)	(0.6650)	(0.6517)	(0.6399)
$n = 4000$	0.9360	0.9380	0.9370	0.9660	0.9630	0.9700
$np_0 = 2000$	(0.4363)	(0.4293)	(0.4249)	(0.4692)	(0.4594)	(0.4521)

Panel C: $x = 4.5$						
	$k_0 = 3 + \#\text{(outliers)}$			$k_0 = 4 + \#\text{(outliers)}$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9690	0.9710	0.9630	0.9840	0.9830	0.9800
$np_0 = 375$	(1.5038)	(1.4763)	(1.4495)	(1.6091)	(1.5643)	(1.530)
$n = 1000$	0.9580	0.9590	0.9590	0.9700	0.9740	0.9750
$np_0 = 1000$	(1.1084)	(1.0815)	(1.0649)	(1.1840)	(1.1445)	(1.1299)
$n = 2000$	0.9540	0.9460	0.9500	0.9600	0.9550	0.9570
$np_0 = 1500$	(0.7966)	(0.7761)	(0.7646)	(0.8544)	(0.8266)	(0.8133)
$n = 4000$	0.9150	0.9150	0.9150	0.9270	0.9220	0.9330
$np_0 = 3000$	(0.5627)	(0.5470)	(0.5381)	(0.6103)	(0.5880)	(0.5784)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

J.4 Sensitivity to sp

In this section, we examine the sensitivity of the choice of sp in DGPs(1,1) and (2,1). The first and second main columns in Tables 21 and 22 report the results for $sp = 6$ and $sp = 7$, respectively. Note we set $k_0 = \text{number of outliers} + 2$ and $k'_L = \min\{\lceil 0.10n\hat{p} \rceil, 35\}$, $k_L = \min\{\lceil 0.10n\hat{p} \rceil, 40\}$, and $k''_L = \min\{\lceil 0.10n\hat{p} \rceil, 45\}$.

Table 21: Robustness check of sp for DGP(1,1)

Panel A: $x = 1.5$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9760	0.9800	0.9780	0.9760	0.9800	0.9780
$np_0 = 125$	(0.4537)	(0.4552)	(0.4559)	(0.4537)	(0.4552)	(0.4559)
$n = 1000$	0.9810	0.9750	0.9780	0.9810	0.9750	0.9780
$np_0 = 250$	(0.2964)	(0.2968)	(0.2964)	(0.2964)	(0.2968)	(0.2964)
$n = 2000$	0.9650	0.9650	0.9620	0.9650	0.9650	0.9620
$np_0 = 500$	(0.2124)	(0.2115)	(0.2106)	(0.2124)	(0.2115)	(0.2106)
$n = 4000$	0.9590	0.9620	0.9630	0.9590	0.9620	0.9630
$np_0 = 1000$	(0.1475)	(0.1453)	(0.1452)	(0.1475)	(0.1453)	(0.1452)

Panel B: $x = 3.0$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9140	0.9140	0.9160	0.9140	0.9140	0.9160
$np_0 = 250$	(0.5182)	(0.5130)	(0.5138)	(0.5182)	(0.5130)	(0.5138)
$n = 1000$	0.9600	0.9520	0.9500	0.9600	0.9520	0.9500
$np_0 = 500$	(0.3608)	(0.3581)	(0.3551)	(0.3608)	(0.3581)	(0.3551)
$n = 2000$	0.9550	0.9650	0.9610	0.9550	0.9650	0.9610
$np_0 = 1000$	(0.2522)	(0.2479)	(0.2458)	(0.2522)	(0.2479)	(0.2458)
$n = 4000$	0.9480	0.9480	0.9450	0.9480	0.9480	0.9450
$np_0 = 2000$	(0.1708)	(0.1684)	(0.1669)	(0.1708)	(0.1684)	(0.1669)

Panel C: $x = 4.5$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9550	0.9500	0.9440	0.9550	0.9500	0.9440
$np_0 = 375$	(0.5176)	(0.5125)	(0.5114)	(0.5176)	(0.5125)	(0.5114)
$n = 1000$	0.9400	0.9430	0.9330	0.9400	0.9430	0.9330
$np_0 = 1000$	(0.3735)	(0.3656)	(0.3618)	(0.3735)	(0.3656)	(0.3618)
$n = 2000$	0.9370	0.9310	0.9270	0.9370	0.9310	0.9270
$np_0 = 1500$	(0.2625)	(0.2575)	(0.2551)	(0.2625)	(0.2575)	(0.2551)
$n = 4000$	0.9440	0.9450	0.9380	0.9440	0.9450	0.9380
$np_0 = 3000$	(0.1808)	(0.1769)	(0.1747)	(0.1808)	(0.1769)	(0.1747)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 22: Robustness check of sp for DGP(2,1)

Panel A: $x = 1.5$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9750	0.9720	0.9660	0.9750	0.9720	0.9660
$np_0 = 125$	(0.6902)	(0.6913)	(0.6889)	(0.6902)	(0.6913)	(0.6889)
$n = 1000$	0.9770	0.9790	0.9740	0.9770	0.9790	0.9740
$np_0 = 250$	(0.460)	(0.4586)	(0.4563)	(0.460)	(0.4586)	(0.4563)
$n = 2000$	0.9670	0.9710	0.9630	0.9670	0.9710	0.9630
$np_0 = 500$	(0.3442)	(0.3383)	(0.3368)	(0.3442)	(0.3383)	(0.3368)
$n = 4000$	0.9590	0.9610	0.9580	0.9590	0.9610	0.9580
$np_0 = 1000$	(0.2476)	(0.2448)	(0.2425)	(0.2476)	(0.2448)	(0.2425)
Panel B: $x = 3.0$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9680	0.9650	0.9640	0.9680	0.9650	0.9640
$np_0 = 250$	(1.1332)	(1.1175)	(1.1199)	(1.1332)	(1.1175)	(1.1199)
$n = 1000$	0.9770	0.9700	0.9740	0.9770	0.9700	0.9740
$np_0 = 500$	(0.8101)	(0.7948)	(0.7862)	(0.8101)	(0.7948)	(0.7862)
$n = 2000$	0.9610	0.9530	0.9570	0.9610	0.9530	0.9570
$np_0 = 1000$	(0.5774)	(0.5678)	(0.5616)	(0.5774)	(0.5678)	(0.5616)
$n = 4000$	0.9000	0.9030	0.9060	0.9000	0.9030	0.9060
$np_0 = 2000$	(0.4197)	(0.4134)	(0.4072)	(0.4197)	(0.4134)	(0.4072)
Panel C: $x = 4.5$						
	Quasi-Bayesian with $sp = 6$			Quasi-Bayesian with $sp = 7$		
	k'_L	k_L	k''_L	k'_L	k_L	k''_L
$n = 500$	0.9470	0.9480	0.9400	0.9470	0.9480	0.9400
$np_0 = 375$	(1.4608)	(1.4285)	(1.3940)	(1.4608)	(1.4285)	(1.3940)
$n = 1000$	0.9550	0.9530	0.9520	0.9550	0.9530	0.9520
$np_0 = 1000$	(1.0613)	(1.0457)	(1.0291)	(1.0613)	(1.0457)	(1.0291)
$n = 2000$	0.9470	0.9510	0.9570	0.9470	0.9510	0.9570
$np_0 = 1500$	(0.7644)	(0.7449)	(0.7316)	(0.7644)	(0.7449)	(0.7316)
$n = 4000$	0.9470	0.9550	0.9610	0.9470	0.9550	0.9610
$np_0 = 3000$	(0.5162)	(0.5041)	(0.5006)	(0.5162)	(0.5041)	(0.5006)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

J.5 Additional Simulation Results for Methods “Mom”, “Momt_pick”, and “Pwm”

In this section, we report additional results for “Mom”, “Momt_pick”, and “Pwm” with different EV index estimators in Tables 23–37. Specifically, in those tables, “Pickands” and “Built-in” mean the EV index is computed by `rho_momt_pick` with the argument `method = “pickands”` and each estimation method’s built-in function as illustrated in Section G, respectively. The “True

Value” method indicates that we simply use the infeasible true value of the EV index for inference. The results in Tables 23–37 show that the three existing estimators mostly either under-cover or over-cover quite a bit even when the true index is used.

Table 23: DGP(1,1)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9680	0.9570	1.0000	0.9680	1.0000	1.0000	1.0000	1.0000
$np_0 = 125$	(0.8451)	(1.9062)	(1.1307)	(1.0030)	(1.9062)	(0.8803)	(1.0228)	(2.3633)	(1.1420)
$n = 1000$	0.9940	0.9730	0.9880	0.9980	0.9730	1.0000	1.0000	0.9990	1.0000
$np_0 = 250$	(0.4979)	(1.2312)	(0.8488)	(0.6093)	(1.2312)	(0.6345)	(0.6363)	(1.6115)	(0.8707)
$n = 2000$	0.9930	0.9800	0.9930	0.9970	0.9800	1.0000	1.0000	0.9990	1.0000
$np_0 = 500$	(0.3267)	(0.8659)	(0.5960)	(0.3736)	(0.8659)	(0.4350)	(0.4150)	(1.1212)	(0.6267)
$n = 4000$	0.9880	0.9850	0.9970	0.9920	0.9850	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.2303)	(0.6343)	(0.4158)	(0.2388)	(0.6343)	(0.2910)	(0.2758)	(0.7698)	(0.4353)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9900	0.7020	1.0000	0.9900	0.9860	1.0000	1.0000	0.0970
$np_0 = 250$	(0.8204)	(1.9218)	(0.9080)	(1.0725)	(1.9218)	(0.6704)	(1.0535)	(2.5257)	(0.8646)
$n = 1000$	0.9990	0.9830	0.8150	1.0000	0.9830	1.0000	1.0000	1.0000	0.5860
$np_0 = 500$	(0.5384)	(1.3046)	(0.7082)	(0.6369)	(1.3046)	(0.4924)	(0.6614)	(1.6522)	(0.6993)
$n = 2000$	0.9990	0.9880	0.8930	0.9990	0.9880	0.9160	1.0000	1.0000	0.8650
$np_0 = 1000$	(0.3669)	(0.9549)	(0.5213)	(0.4037)	(0.9549)	(0.3561)	(0.4309)	(1.1412)	(0.5213)
$n = 4000$	0.9990	0.9950	0.9390	0.9970	0.9950	1.0000	1.0000	1.0000	0.9790
$np_0 = 2000$	(0.2567)	(0.7050)	(0.3703)	(0.2645)	(0.7050)	(0.2438)	(0.2856)	(0.7894)	(0.3695)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9980	0.9850	0.8430	0.9990	0.9850	0.8520	1.0000	1.0000	0.4520
$np_0 = 375$	(0.6377)	(1.7471)	(0.4787)	(0.7209)	(1.7471)	(0.3887)	(0.8213)	(2.2790)	(0.4152)
$n = 1000$	0.9880	0.9900	0.9250	0.9950	0.9900	0.9940	1.0000	1.0000	0.6680
$np_0 = 750$	(0.4262)	(1.2530)	(0.3641)	(0.4764)	(1.2530)	(0.2909)	(0.5365)	(1.6323)	(0.3170)
$n = 2000$	0.9930	0.9920	0.9400	0.9910	0.9920	1.0000	1.0000	1.0000	0.7970
$np_0 = 1500$	(0.3124)	(0.9603)	(0.2622)	(0.3213)	(0.9603)	(0.2344)	(0.3402)	(1.1245)	(0.2233)
$n = 4000$	0.9960	0.9990	0.9660	0.9870	0.9990	1.0000	1.0000	1.0000	0.9690
$np_0 = 3000$	(0.2285)	(0.7277)	(0.1814)	(0.2299)	(0.7277)	(0.1994)	(0.2317)	(0.7839)	(0.1424)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 24: DGP(2,1)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9830	0.9530	1.0000	0.9830	1.0000	1.0000	0.9990	1.0000
$np_0 = 125$	(1.8886)	(3.8828)	(3.0487)	(1.7846)	(3.8828)	(2.2567)	(1.7452)	(3.5711)	(2.9555)
$n = 1000$	0.9990	0.9820	0.9960	0.9990	0.9820	1.0000	1.0000	0.9990	1.0000
$np_0 = 250$	(1.1938)	(2.8319)	(2.3919)	(1.0357)	(2.8319)	(1.6693)	(1.0599)	(2.5173)	(2.2898)
$n = 2000$	0.9950	0.9800	1.0000	0.9950	0.9800	1.0000	1.0000	0.9960	1.0000
$np_0 = 500$	(0.7545)	(1.9118)	(1.7313)	(0.7211)	(1.9118)	(1.1631)	(0.7092)	(1.7663)	(1.6687)
$n = 4000$	0.9910	0.9720	1.0000	0.9770	0.9720	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.5150)	(1.3444)	(1.2218)	(0.5906)	(1.3444)	(0.7936)	(0.4710)	(1.2116)	(1.1666)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9890	0.4830	1.0000	0.9890	0.9820	1.0000	1.0000	0.1160
$np_0 = 250$	(2.8498)	(6.3030)	(3.4320)	(2.2688)	(6.3030)	(2.3812)	(2.5030)	(5.4479)	(3.1066)
$n = 1000$	0.9990	0.9900	0.6890	0.9980	0.9900	1.0000	1.0000	1.0000	0.7900
$np_0 = 500$	(1.7404)	(4.0388)	(2.7388)	(1.3848)	(4.0388)	(1.8194)	(1.5631)	(3.6593)	(2.5066)
$n = 2000$	1.0000	0.9870	0.7980	0.9910	0.9870	0.9640	1.0000	1.0000	0.9860
$np_0 = 1000$	(1.1643)	(2.8646)	(2.0439)	(1.1706)	(2.8646)	(1.3048)	(1.0473)	(2.5212)	(1.8721)
$n = 4000$	0.9990	0.9870	0.8440	0.9970	0.9870	1.0000	1.0000	1.0000	0.9980
$np_0 = 2000$	(0.8487)	(2.2158)	(1.4797)	(1.0736)	(2.2158)	(0.8720)	(0.6982)	(1.7766)	(1.3173)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9970	0.9820	0.6780	1.0000	0.9820	0.9920	1.0000	1.0000	0.6210
$np_0 = 375$	(2.5178)	(7.1443)	(1.8635)	(2.9579)	(7.1443)	(1.2756)	(2.3074)	(6.4701)	(1.5360)
$n = 1000$	0.9920	0.9860	0.8180	0.9980	0.9860	1.0000	1.0000	1.0000	0.8370
$np_0 = 750$	(1.6198)	(4.6929)	(1.4353)	(2.2721)	(4.6929)	(0.9238)	(1.4185)	(4.4482)	(1.1625)
$n = 2000$	0.9830	0.9880	0.8380	1.0000	0.9880	1.0000	0.9990	1.0000	0.9410
$np_0 = 1500$	(1.1881)	(3.6423)	(1.0679)	(1.6526)	(3.6423)	(0.7000)	(0.9534)	(3.0812)	(0.8041)
$n = 4000$	0.9930	0.9930	0.8060	1.0000	0.9930	1.0000	0.9990	1.0000	0.9980
$np_0 = 3000$	(0.9086)	(2.8784)	(0.7983)	(1.1660)	(2.8784)	(0.5469)	(0.6680)	(2.1533)	(0.5229)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 25: DGP(3,1)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990	0.9810	0.9620	1.0000	0.9810	1.0000	1.0000	0.9980	1.0000
$np_0 = 125$	(0.6926)	(1.2594)	(2.0065)	(1.0749)	(1.2594)	(1.4580)	(0.6077)	(1.1008)	(1.9234)
$n = 1000$	1.0000	0.9830	1.0000	1.0000	0.9830	1.0000	1.0000	0.9950	1.0000
$np_0 = 250$	(0.4422)	(0.9558)	(1.6196)	(0.6258)	(0.9558)	(1.1014)	(0.3577)	(0.7668)	(1.5110)
$n = 2000$	1.0000	0.9880	1.0000	0.9970	0.9880	1.0000	1.0000	1.0000	1.0000
$np_0 = 500$	(0.3270)	(0.7721)	(1.2250)	(0.3837)	(0.7721)	(0.7833)	(0.2392)	(0.5600)	(1.1124)
$n = 4000$	0.9950	0.9770	1.0000	0.9810	0.9770	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.2207)	(0.5373)	(0.8642)	(0.2111)	(0.5373)	(0.5417)	(0.1618)	(0.3880)	(0.7821)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990	0.9940	0.3980	0.9990	0.9940	0.9610	1.0000	1.0000	0.1760
$np_0 = 250$	(2.2102)	(4.4224)	(4.1140)	(3.0513)	(4.4224)	(2.8234)	(1.7029)	(3.2635)	(3.7198)
$n = 1000$	0.9990	0.9940	0.5780	0.9990	0.9940	1.0000	1.0000	0.9980	0.9240
$np_0 = 500$	(1.5119)	(3.2712)	(3.3877)	(1.7281)	(3.2712)	(2.1849)	(1.0715)	(2.2519)	(2.9893)
$n = 2000$	1.0000	0.9860	0.8510	1.0000	0.9860	0.9860	1.0000	1.0000	0.9980
$np_0 = 1000$	(0.9752)	(2.2767)	(2.5439)	(0.9829)	(2.2767)	(1.5748)	(0.7246)	(1.6244)	(2.2292)
$n = 4000$	0.9990	0.9880	0.9280	0.9910	0.9880	1.0000	1.0000	1.0000	1.0000
$np_0 = 2000$	(0.6420)	(1.5477)	(1.7799)	(0.4926)	(1.5477)	(1.0667)	(0.4946)	(1.1679)	(1.5651)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9980	0.6240	0.9970	0.9980	1.0000	1.0000	1.0000	0.8670
$np_0 = 375$	(3.0773)	(8.2952)	(2.5734)	(3.2919)	(8.2952)	(1.7223)	(2.0093)	(5.8439)	(2.1646)
$n = 1000$	0.9910	0.9890	0.7580	0.9800	0.9890	1.0000	1.0000	1.0000	0.9640
$np_0 = 750$	(2.0931)	(6.0219)	(2.0002)	(2.0914)	(6.0219)	(1.1757)	(1.3825)	(4.0914)	(1.6229)
$n = 2000$	0.9890	0.9850	0.8060	0.9100	0.9850	1.0000	1.0000	1.0000	0.9970
$np_0 = 1500$	(1.4315)	(4.2559)	(1.4679)	(1.1794)	(4.2559)	(0.8275)	(0.9673)	(2.9548)	(1.1087)
$n = 4000$	0.9720	0.9690	0.8680	0.8370	0.9690	1.0000	1.0000	1.0000	1.0000
$np_0 = 3000$	(0.8776)	(2.7159)	(1.0339)	(0.7713)	(2.7159)	(0.6016)	(0.6763)	(2.1080)	(0.7326)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 26: DGP(1,2)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9980	0.9820	0.9330	1.0000	0.9820	1.0000	1.0000	0.9980	0.9940
$np_0 = 125$	(1.4814)	(3.9622)	(1.4388)	(1.9523)	(3.9622)	(1.0401)	(1.3366)	(3.6022)	(1.3374)
$n = 1000$	0.9970	0.9940	0.9630	0.9950	0.9940	1.0000	1.0000	1.0000	1.0000
$np_0 = 250$	(1.0809)	(3.1801)	(1.1207)	(1.3875)	(3.1801)	(0.7404)	(0.8605)	(2.5052)	(1.0086)
$n = 2000$	0.9760	0.9870	0.9730	0.9920	0.9870	1.0000	1.0000	0.9990	1.0000
$np_0 = 500$	(0.7511)	(2.3309)	(0.8267)	(0.9485)	(2.3309)	(0.5027)	(0.5825)	(1.7822)	(0.7147)
$n = 4000$	0.9550	0.9740	0.9790	0.9870	0.9740	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.4650)	(1.4488)	(0.5810)	(0.6185)	(1.4488)	(0.3278)	(0.3995)	(1.2341)	(0.4913)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9960	0.3990	1.0000	0.9960	0.9980	1.0000	1.0000	0.1190
$np_0 = 250$	(1.8113)	(4.8315)	(1.1930)	(2.3759)	(4.8315)	(0.7832)	(1.4633)	(3.8752)	(0.9840)
$n = 1000$	0.9990	0.9940	0.5420	0.9980	0.9940	1.0000	1.0000	1.0000	0.5110
$np_0 = 500$	(1.2152)	(3.4420)	(1.0037)	(1.5705)	(3.4420)	(0.5604)	(0.9337)	(2.6234)	(0.8026)
$n = 2000$	0.9830	0.9840	0.7430	0.9970	0.9840	0.9820	1.0000	1.0000	0.8290
$np_0 = 1000$	(0.7482)	(2.1716)	(0.7682)	(1.0374)	(2.1716)	(0.4157)	(0.6212)	(1.8183)	(0.6022)
$n = 4000$	0.9750	0.9780	0.8010	0.9940	0.9780	1.0000	1.0000	1.0000	0.9440
$np_0 = 2000$	(0.4875)	(1.4713)	(0.5490)	(0.6673)	(1.4713)	(0.2914)	(0.4204)	(1.2839)	(0.4248)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9980	0.9930	0.5590	1.0000	0.9930	0.9410	1.0000	1.0000	0.4280
$np_0 = 375$	(1.5060)	(4.9829)	(0.7829)	(1.6908)	(4.9829)	(0.5165)	(1.1924)	(3.8543)	(0.5577)
$n = 1000$	0.9810	0.9910	0.6990	0.9990	0.9910	0.9270	1.0000	1.0000	0.6670
$np_0 = 750$	(0.9935)	(3.1977)	(0.6487)	(1.1236)	(3.1977)	(0.4427)	(0.8142)	(2.5769)	(0.4337)
$n = 2000$	0.9650	0.9760	0.8410	0.9940	0.9760	0.9500	1.0000	1.0000	0.7990
$np_0 = 1500$	(0.6307)	(2.0447)	(0.5009)	(0.8049)	(2.0447)	(0.3884)	(0.5532)	(1.7958)	(0.3103)
$n = 4000$	0.9550	0.9710	0.8640	0.9960	0.9710	0.9940	1.0000	1.0000	0.9500
$np_0 = 3000$	(0.4454)	(1.4648)	(0.3756)	(0.5908)	(1.4648)	(0.3267)	(0.3695)	(1.2600)	(0.2003)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 27: DGP(2,2)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9950	0.9610	0.9590	1.0000	0.9610	1.0000	1.0000	0.9940	1.0000
$np_0 = 125$	(2.2188)	(5.4216)	(3.4788)	(3.4650)	(5.4216)	(2.4951)	(1.9792)	(4.8058)	(3.2518)
$n = 1000$	0.9780	0.9690	0.9850	0.9930	0.9690	1.0000	1.0000	0.9940	1.0000
$np_0 = 250$	(1.5651)	(4.2438)	(2.7218)	(2.3905)	(4.2438)	(1.8323)	(1.2702)	(3.4136)	(2.5034)
$n = 2000$	0.9880	0.9910	1.0000	0.9930	0.9910	1.0000	1.0000	0.9980	1.0000
$np_0 = 500$	(1.2254)	(3.4312)	(2.0395)	(1.7470)	(3.4312)	(1.2629)	(0.8927)	(2.4580)	(1.8055)
$n = 4000$	0.9800	0.9830	0.9990	0.9830	0.9830	1.0000	0.9980	0.9990	1.0000
$np_0 = 1000$	(0.9431)	(2.7106)	(1.4748)	(1.1897)	(2.7106)	(0.8491)	(0.6316)	(1.7796)	(1.2531)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9900	0.4470	1.0000	0.9900	0.9960	1.0000	1.0000	0.1810
$np_0 = 250$	(3.8273)	(9.5945)	(3.9034)	(4.3673)	(9.5945)	(2.6152)	(3.0333)	(7.7149)	(3.3697)
$n = 1000$	0.9990	0.9840	0.5420	0.9930	0.9840	1.0000	1.0000	0.9990	0.7700
$np_0 = 500$	(2.8143)	(7.2778)	(3.2288)	(3.3399)	(7.2778)	(1.9684)	(1.9722)	(5.2322)	(2.7333)
$n = 2000$	0.9960	0.9880	0.7000	0.9930	0.9880	0.9920	1.0000	1.0000	0.9700
$np_0 = 1000$	(2.0669)	(5.6281)	(2.5067)	(2.4638)	(5.6281)	(1.4298)	(1.3731)	(3.7262)	(2.0465)
$n = 4000$	0.9880	0.9880	0.7500	0.9710	0.9880	1.0000	1.0000	1.0000	0.9930
$np_0 = 2000$	(1.4570)	(4.1683)	(1.8239)	(1.7020)	(4.1683)	(0.9563)	(0.9561)	(2.6663)	(1.4439)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9950	0.9910	0.5880	0.9940	0.9910	0.9570	1.0000	1.0000	0.6310
$np_0 = 375$	(4.1978)	(11.7651)	(2.4990)	(5.3171)	(11.7651)	(1.7404)	(3.1879)	(8.9517)	(1.9331)
$n = 1000$	0.9900	0.9920	0.6560	0.9830	0.9920	0.9870	1.0000	1.0000	0.8280
$np_0 = 750$	(3.1554)	(9.2127)	(2.0170)	(3.7883)	(9.2127)	(1.2939)	(2.0597)	(6.3268)	(1.4758)
$n = 2000$	0.9900	0.9870	0.6710	0.9820	0.9870	1.0000	0.9990	1.0000	0.9160
$np_0 = 1500$	(2.3103)	(7.1616)	(1.5719)	(2.6580)	(7.1616)	(1.0355)	(1.4018)	(4.5931)	(1.0447)
$n = 4000$	0.9690	0.9790	0.7540	0.9500	0.9790	0.9960	0.9970	1.0000	0.9960
$np_0 = 3000$	(1.5585)	(5.0660)	(1.2132)	(1.7273)	(5.0660)	(0.8612)	(0.9959)	(3.3014)	(0.6724)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 28: DGP(3,2)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9900 (0.6860)	0.9250 (1.5014)	0.9890 (2.1898)	1.0000 (1.2009)	0.9250 (1.5014)	1.0000 (1.5300)	1.0000 (0.5864)	0.9760 (1.2579)	1.0000 (2.0148)
$np_0 = 125$									
$n = 1000$	0.9640 (0.4224)	0.9490 (1.0371)	1.0000 (1.6840)	0.9980 (0.7605)	0.9490 (1.0371)	1.0000 (1.1529)	1.0000 (0.3691)	0.9870 (0.9179)	1.0000 (1.5778)
$np_0 = 250$									
$n = 2000$	0.9570 (0.3337)	0.9690 (0.8660)	1.0000 (1.2607)	0.9960 (0.5470)	0.9690 (0.8660)	1.0000 (0.8144)	0.9990 (0.2637)	0.9980 (0.6844)	1.0000 (1.1555)
$np_0 = 500$									
$n = 4000$	0.9770 (0.2716)	0.9860 (0.7182)	1.0000 (0.9206)	0.9970 (0.3992)	0.9860 (0.7182)	1.0000 (0.5594)	1.0000 (0.1938)	0.9960 (0.5105)	1.0000 (0.8090)
$np_0 = 1000$									

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990 (2.2293)	0.9720 (5.1238)	0.5230 (4.4393)	1.0000 (3.7285)	0.9720 (5.1238)	0.9930 (2.9733)	1.0000 (1.7626)	0.9960 (4.0675)	0.2510 (3.8912)
$np_0 = 250$									
$n = 1000$	0.9980 (1.5533)	0.9830 (3.7708)	0.6620 (3.5509)	1.0000 (2.5337)	0.9830 (3.7708)	1.0000 (2.2984)	1.0000 (1.1752)	0.9990 (2.8513)	0.9390 (3.1341)
$np_0 = 500$									
$n = 2000$	0.9930 (1.2162)	0.9920 (3.0338)	0.8460 (2.7241)	0.9990 (1.7831)	0.9920 (3.0338)	0.9930 (1.6587)	1.0000 (0.8407)	1.0000 (2.0998)	0.9980 (2.3442)
$np_0 = 1000$									
$n = 4000$	0.9970 (1.0004)	0.9960 (2.6463)	0.8960 (1.9730)	0.9920 (1.2654)	0.9960 (2.6463)	1.0000 (1.1113)	1.0000 (0.6124)	1.0000 (1.5985)	1.0000 (1.6449)
$np_0 = 2000$									

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9680 (3.4327)	0.9670 (9.2865)	0.7290 (3.0586)	0.9710 (4.0245)	0.9670 (9.2865)	1.0000 (2.0784)	0.9840 (2.4200)	0.9940 (7.4570)	0.8590 (2.5292)
$np_0 = 375$									
$n = 1000$	0.9670 (2.6812)	0.9780 (7.5204)	0.7750 (2.4019)	0.9660 (3.0880)	0.9780 (7.5204)	1.0000 (1.4926)	0.9890 (1.7613)	0.9980 (5.3341)	0.9640 (1.9084)
$np_0 = 750$									
$n = 2000$	0.9860 (2.0941)	0.9930 (6.2039)	0.7960 (1.7837)	0.9850 (2.4227)	0.9930 (6.2039)	1.0000 (1.1206)	0.9850 (1.2854)	1.0000 (3.9366)	0.9960 (1.3160)
$np_0 = 1500$									
$n = 4000$	0.9940 (1.6545)	0.9980 (5.0648)	0.7660 (1.2786)	0.9820 (1.8811)	0.9980 (5.0648)	0.9990 (0.8256)	0.9790 (0.9431)	1.0000 (2.9245)	1.0000 (0.8696)
$np_0 = 3000$									

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 29: DGP(1,3)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9950	0.9400	1.0000	0.9950	1.0000	1.0000	1.0000	0.9940
$np_0 = 125$	(2.0546)	(5.0625)	(1.7819)	(2.4149)	(5.0625)	(1.3876)	(1.9314)	(4.8767)	(1.6855)
$n = 1000$	0.9960	0.9930	0.9680	0.9960	0.9930	1.0000	1.0000	0.9980	0.9980
$np_0 = 250$	(1.4625)	(3.9807)	(1.3816)	(1.7172)	(3.9807)	(1.0127)	(1.2847)	(3.4725)	(1.2765)
$n = 2000$	0.9940	0.9990	0.9770	0.9890	0.9990	1.0000	1.0000	1.0000	0.9990
$np_0 = 500$	(1.0924)	(3.0601)	(1.0266)	(1.1551)	(3.0601)	(0.6930)	(0.8954)	(2.4858)	(0.9140)
$n = 4000$	0.9840	0.9920	0.9890	0.9830	0.9920	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.7476)	(2.1403)	(0.7322)	(0.7658)	(2.1403)	(0.4520)	(0.6348)	(1.7979)	(0.6310)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9990	0.3320	1.0000	0.9990	0.9300	1.0000	1.0000	0.0430
$np_0 = 250$	(2.7008)	(6.4429)	(1.4509)	(2.8440)	(6.4429)	(1.0746)	(2.3035)	(5.4941)	(1.2373)
$n = 1000$	1.0000	0.9990	0.4930	1.0000	0.9990	0.9980	1.0000	1.0000	0.3860
$np_0 = 500$	(1.8219)	(4.5428)	(1.2463)	(1.8593)	(4.5428)	(0.8412)	(1.5027)	(3.7367)	(1.0383)
$n = 2000$	0.9990	0.9980	0.6240	0.9990	0.9980	1.0000	1.0000	1.0000	0.7000
$np_0 = 1000$	(1.2479)	(3.3172)	(0.9707)	(1.1767)	(3.3172)	(0.5804)	(1.0209)	(2.6924)	(0.7838)
$n = 4000$	0.9940	0.9970	0.7610	0.9860	0.9970	0.9790	1.0000	1.0000	0.8710
$np_0 = 2000$	(0.8147)	(2.2526)	(0.7152)	(0.7111)	(2.2526)	(0.3753)	(0.6970)	(1.9506)	(0.5572)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9990	0.4430	1.0000	0.9990	0.9990	1.0000	1.0000	0.2730
$np_0 = 375$	(2.4174)	(7.0347)	(1.0180)	(2.6033)	(7.0347)	(0.6829)	(1.9899)	(5.6971)	(0.7814)
$n = 1000$	1.0000	1.0000	0.6010	0.9990	1.0000	0.9780	1.0000	1.0000	0.5840
$np_0 = 750$	(1.6713)	(4.9101)	(0.8610)	(1.6568)	(4.9101)	(0.5498)	(1.3528)	(3.9035)	(0.6323)
$n = 2000$	0.9870	0.9960	0.7430	0.9930	0.9960	0.9780	1.0000	1.0000	0.7610
$np_0 = 1500$	(1.1008)	(3.2691)	(0.6784)	(1.0483)	(3.2691)	(0.4360)	(0.9327)	(2.7540)	(0.4631)
$n = 4000$	0.9780	0.9990	0.8360	0.9840	0.9990	0.9810	1.0000	1.0000	0.8630
$np_0 = 3000$	(0.7426)	(2.3010)	(0.5187)	(0.8092)	(2.3010)	(0.3712)	(0.6483)	(2.0185)	(0.3211)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 30: DGP(2,3)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9850	0.9670	1.0000	0.9850	1.0000	1.0000	0.9980	1.0000
$np_0 = 125$	(3.0654)	(6.9954)	(4.1390)	(4.4747)	(6.9954)	(3.2049)	(2.7782)	(6.1927)	(3.8966)
$n = 1000$	0.9990	0.9910	0.9960	1.0000	0.9910	1.0000	1.0000	0.9980	1.0000
$np_0 = 250$	(2.0424)	(5.1007)	(3.2040)	(3.0133)	(5.1007)	(2.3863)	(1.8146)	(4.4999)	(3.0169)
$n = 2000$	0.9880	0.9950	1.0000	0.9960	0.9950	1.0000	1.0000	1.0000	1.0000
$np_0 = 500$	(1.6244)	(4.2416)	(2.4056)	(2.0937)	(4.2416)	(1.6657)	(1.2929)	(3.3576)	(2.1935)
$n = 4000$	0.9880	0.9960	1.0000	0.9800	0.9960	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(1.2363)	(3.3682)	(1.7375)	(1.3762)	(3.3682)	(1.1055)	(0.9269)	(2.4891)	(1.5290)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9990	0.4010	1.0000	0.9990	0.9390	1.0000	1.0000	0.0570
$np_0 = 250$	(5.2154)	(12.0894)	(4.5569)	(6.0994)	(12.0894)	(3.3878)	(4.5405)	(10.4261)	(4.0248)
$n = 1000$	0.9990	0.9950	0.5330	0.9990	0.9950	1.0000	1.0000	1.0000	0.5960
$np_0 = 500$	(3.7688)	(8.7902)	(3.8196)	(3.9635)	(8.7902)	(2.6501)	(2.9125)	(7.0511)	(3.3376)
$n = 2000$	0.9980	0.9970	0.6800	0.9970	0.9970	1.0000	1.0000	1.0000	0.9110
$np_0 = 1000$	(2.8166)	(6.9658)	(2.9655)	(2.6856)	(6.9658)	(1.8935)	(2.0635)	(5.2298)	(2.5124)
$n = 4000$	0.9990	0.9980	0.7570	0.9830	0.9980	1.0000	1.0000	1.0000	0.9870
$np_0 = 2000$	(2.1130)	(5.5997)	(2.1860)	(1.7428)	(5.5997)	(1.2075)	(1.4529)	(3.8637)	(1.7892)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990	0.9970	0.5640	0.9990	0.9970	0.9990	1.0000	1.0000	0.4850
$np_0 = 375$	(5.5002)	(15.0285)	(3.1276)	(6.1859)	(15.0285)	(2.1895)	(4.6872)	(12.6130)	(2.5517)
$n = 1000$	0.9940	0.9980	0.6420	0.9910	0.9980	0.9940	1.0000	1.0000	0.7730
$np_0 = 750$	(4.2624)	(11.6848)	(2.5595)	(4.3261)	(11.6848)	(1.7446)	(3.2505)	(8.8763)	(2.0104)
$n = 2000$	0.9910	1.0000	0.7150	0.9690	1.0000	0.9950	1.0000	1.0000	0.8960
$np_0 = 1500$	(3.2516)	(8.9144)	(1.9690)	(2.9199)	(8.9144)	(1.2309)	(2.2048)	(6.3937)	(1.4482)
$n = 4000$	0.9770	0.9960	0.7580	0.9080	0.9960	1.0000	0.9990	1.0000	0.9740
$np_0 = 3000$	(2.3628)	(6.8180)	(1.5001)	(1.8548)	(6.8180)	(0.9287)	(1.5639)	(4.8275)	(0.9479)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 31: DGP(3,3)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9640	0.9970	1.0000	0.9640	1.0000	1.0000	0.9960	1.0000
$np_0 = 125$	(0.9546)	(1.8920)	(2.5371)	(1.4896)	(1.8920)	(1.9158)	(0.8056)	(1.5757)	(2.3371)
$n = 1000$	0.9950	0.9630	1.0000	1.0000	0.9630	1.0000	1.0000	0.9960	1.0000
$np_0 = 250$	(0.5695)	(1.3096)	(1.9455)	(0.9498)	(1.3096)	(1.4521)	(0.5112)	(1.1739)	(1.8432)
$n = 2000$	0.9780	0.9830	1.0000	0.9970	0.9830	1.0000	1.0000	1.0000	1.0000
$np_0 = 500$	(0.4194)	(1.0223)	(1.4481)	(0.6732)	(1.0223)	(1.0347)	(0.3630)	(0.8919)	(1.3589)
$n = 4000$	0.9800	0.9940	1.0000	0.9960	0.9940	1.0000	1.0000	1.0000	1.0000
$np_0 = 1000$	(0.3480)	(0.8797)	(1.0493)	(0.4947)	(0.8797)	(0.7025)	(0.2710)	(0.6843)	(0.9555)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9910	0.4550	1.0000	0.9910	0.9420	1.0000	0.9970	0.0770
$np_0 = 250$	(3.1054)	(6.6101)	(5.0743)	(4.5170)	(6.6101)	(3.7070)	(2.4616)	(5.2894)	(4.5133)
$n = 1000$	1.0000	0.9890	0.6800	1.0000	0.9890	1.0000	1.0000	0.9990	0.8790
$np_0 = 500$	(1.9824)	(4.3994)	(4.0805)	(3.1320)	(4.3994)	(2.8962)	(1.6308)	(3.7073)	(3.6922)
$n = 2000$	1.0000	0.9980	0.8290	1.0000	0.9980	1.0000	1.0000	1.0000	0.9950
$np_0 = 1000$	(1.6036)	(3.7310)	(3.1246)	(2.2719)	(3.7310)	(2.0972)	(1.1933)	(2.8406)	(2.7718)
$n = 4000$	0.9960	0.9970	0.9400	0.9930	0.9970	1.0000	1.0000	1.0000	1.0000
$np_0 = 2000$	(1.2494)	(3.0515)	(2.2757)	(1.5956)	(3.0515)	(1.3706)	(0.8743)	(2.1370)	(1.9744)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9950	0.9860	0.6800	0.9970	0.9860	1.0000	0.9970	1.0000	0.6910
$np_0 = 375$	(4.5003)	(10.8424)	(3.6719)	(4.9684)	(10.8424)	(2.5718)	(3.5454)	(9.5272)	(3.1332)
$n = 1000$	0.9880	0.9920	0.7720	0.9780	0.9920	1.0000	0.9980	1.0000	0.9130
$np_0 = 750$	(3.4665)	(8.9477)	(2.9001)	(3.6559)	(8.9477)	(1.9634)	(2.4415)	(7.0436)	(2.4262)
$n = 2000$	0.9860	0.9980	0.8320	0.9570	0.9980	1.0000	0.9980	0.9990	0.9800
$np_0 = 1500$	(2.7701)	(7.5158)	(2.1623)	(2.7839)	(7.5158)	(1.3407)	(1.8346)	(5.3614)	(1.7302)
$n = 4000$	0.9870	0.9990	0.8130	0.9520	0.9990	1.0000	0.9950	1.0000	1.0000
$np_0 = 3000$	(2.1998)	(6.1795)	(1.5627)	(2.2526)	(6.1795)	(0.9351)	(1.3796)	(4.0632)	(1.1081)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 32: DGP(1,4)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.8670	0.8880	0.9740	0.8720	0.8880	1.0000	0.7260	0.7120	1.0000
$np_0 = 125$	(0.2835)	(0.5236)	(0.7557)	(0.2892)	(0.5236)	(0.5891)	(0.1402)	(0.2131)	(0.6310)
$n = 1000$	0.9070	0.8920	1.0000	0.9140	0.8920	1.0000	0.8730	0.6000	1.0000
$np_0 = 250$	(0.1406)	(0.2827)	(0.5458)	(0.1396)	(0.2827)	(0.4400)	(0.0792)	(0.1321)	(0.4730)
$n = 2000$	0.9110	0.8990	1.0000	0.9110	0.8990	1.0000	0.9210	0.4620	1.0000
$np_0 = 500$	(0.0864)	(0.1833)	(0.3875)	(0.0818)	(0.1833)	(0.3299)	(0.0476)	(0.0847)	(0.3369)
$n = 4000$	0.8600	0.8870	1.0000	0.8660	0.8870	1.0000	0.8830	0.2820	1.0000
$np_0 = 1000$	(0.0545)	(0.1240)	(0.2686)	(0.0521)	(0.1240)	(0.2273)	(0.0298)	(0.0559)	(0.2315)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.3810	0.9510	0.8090	0.3980	0.9510	0.9120	0.3910	0.6610	0.9900
$np_0 = 250$	(0.2240)	(0.4296)	(0.5777)	(0.2290)	(0.4296)	(0.4727)	(0.1284)	(0.1954)	(0.5029)
$n = 1000$	0.4210	0.9470	0.9530	0.4440	0.9470	1.0000	0.6510	0.5250	1.0000
$np_0 = 500$	(0.1337)	(0.2652)	(0.4385)	(0.1289)	(0.2652)	(0.3706)	(0.0762)	(0.1260)	(0.3842)
$n = 2000$	0.5230	0.9240	0.9930	0.5500	0.9240	1.0000	0.7810	0.3340	1.0000
$np_0 = 1000$	(0.0836)	(0.1725)	(0.3198)	(0.0797)	(0.1725)	(0.2819)	(0.0471)	(0.0809)	(0.2770)
$n = 4000$	0.5660	0.9090	0.9980	0.5900	0.9090	1.0000	0.8160	0.1760	1.0000
$np_0 = 2000$	(0.0535)	(0.1170)	(0.2223)	(0.0512)	(0.1170)	(0.1739)	(0.0298)	(0.0537)	(0.1906)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9930	0.9390	0.9860	0.9790	0.9390	1.0000	0.6380	0.5360	1.0000
$np_0 = 375$	(0.1389)	(0.3951)	(0.2230)	(0.1330)	(0.3951)	(0.1951)	(0.0743)	(0.1790)	(0.1992)
$n = 1000$	0.9730	0.9260	0.9970	0.9540	0.9260	1.0000	0.4790	0.3740	1.0000
$np_0 = 750$	(0.0870)	(0.2483)	(0.1580)	(0.0834)	(0.2483)	(0.1415)	(0.0476)	(0.1169)	(0.1367)
$n = 2000$	0.9680	0.9260	0.9980	0.8940	0.9260	1.0000	0.2750	0.1900	1.0000
$np_0 = 1500$	(0.0577)	(0.1696)	(0.1084)	(0.0541)	(0.1696)	(0.0875)	(0.0307)	(0.0771)	(0.0902)
$n = 4000$	0.9360	0.8970	1.0000	0.8590	0.8970	1.0000	0.1430	0.0900	0.9980
$np_0 = 3000$	(0.0379)	(0.1129)	(0.0719)	(0.0362)	(0.1129)	(0.0995)	(0.0200)	(0.0512)	(0.0586)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 33: DGP(2,4)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.8640	0.9380	0.9560	0.9200	0.9380	1.0000	0.5350	0.6920	1.0000
$np_0 = 125$	(0.8002)	(1.3381)	(2.2473)	(1.0034)	(1.3381)	(1.6313)	(0.2876)	(0.3861)	(1.7405)
$n = 1000$	0.9200	0.9560	0.9970	0.9480	0.9560	1.0000	0.7850	0.5760	1.0000
$np_0 = 250$	(0.4577)	(0.9028)	(1.7155)	(0.5700)	(0.9028)	(1.2318)	(0.1585)	(0.2440)	(1.3171)
$n = 2000$	0.8880	0.9700	0.9980	0.9260	0.9700	1.0000	0.8830	0.4300	1.0000
$np_0 = 500$	(0.3054)	(0.6532)	(1.2793)	(0.3401)	(0.6532)	(0.9270)	(0.0952)	(0.1576)	(0.9441)
$n = 4000$	0.7850	0.9940	1.0000	0.7770	0.9940	1.0000	0.8830	0.2600	1.0000
$np_0 = 1000$	(0.2102)	(0.4783)	(0.9187)	(0.2127)	(0.4783)	(0.6417)	(0.0596)	(0.1048)	(0.6514)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.5250	0.9710	0.5720	0.6790	0.9710	0.8710	0.2100	0.6360	0.9800
$np_0 = 250$	(1.0592)	(1.9112)	(2.4441)	(1.3426)	(1.9112)	(1.7838)	(0.3722)	(0.5102)	(1.8773)
$n = 1000$	0.4820	0.9890	0.7630	0.5560	0.9890	1.0000	0.3930	0.4940	1.0000
$np_0 = 500$	(0.6822)	(1.3430)	(1.9589)	(0.7648)	(1.3430)	(1.4100)	(0.2176)	(0.3259)	(1.4399)
$n = 2000$	0.3370	0.9910	0.8930	0.3270	0.9910	1.0000	0.6110	0.2990	1.0000
$np_0 = 1000$	(0.4655)	(0.9709)	(1.4748)	(0.4657)	(0.9709)	(1.0649)	(0.1356)	(0.2155)	(1.0419)
$n = 4000$	0.1100	0.9990	0.9500	0.1010	0.9990	1.0000	0.7070	0.1480	1.0000
$np_0 = 2000$	(0.3241)	(0.7116)	(1.0567)	(0.3065)	(0.7116)	(0.6705)	(0.0857)	(0.1429)	(0.7192)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9950	0.9860	0.8570	0.9980	0.9860	1.0000	0.6440	0.5050	1.0000
$np_0 = 375$	(0.8229)	(2.3778)	(1.0761)	(0.8226)	(2.3778)	(0.7663)	(0.2461)	(0.5713)	(0.8134)
$n = 1000$	0.9970	0.9900	0.9200	0.9960	0.9900	1.0000	0.4650	0.3260	1.0000
$np_0 = 750$	(0.5399)	(1.6277)	(0.7951)	(0.5334)	(1.6277)	(0.5697)	(0.1578)	(0.3721)	(0.5578)
$n = 2000$	0.9900	1.0000	0.9250	0.9950	1.0000	1.0000	0.3130	0.1820	1.0000
$np_0 = 1500$	(0.3890)	(1.2176)	(0.5679)	(0.3566)	(1.2176)	(0.3584)	(0.1042)	(0.2502)	(0.3699)
$n = 4000$	0.9880	1.0000	0.9220	0.9840	1.0000	1.0000	0.1660	0.0800	1.0000
$np_0 = 3000$	(0.2697)	(0.8681)	(0.3897)	(0.2441)	(0.8681)	(0.3590)	(0.0676)	(0.1661)	(0.2412)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 34: DGP(3,4)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.8950	0.9590	0.9620	0.8310	0.9590	1.0000	0.3980	0.6220	1.0000
$np_0 = 125$	(0.4583)	(0.7104)	(1.6608)	(0.5132)	(0.7104)	(1.1148)	(0.1174)	(0.1361)	(1.1847)
$n = 1000$	0.9000	0.9480	1.0000	0.8590	0.9480	1.0000	0.6430	0.5330	1.0000
$np_0 = 250$	(0.2429)	(0.4537)	(1.3028)	(0.2474)	(0.4537)	(0.8494)	(0.0642)	(0.0888)	(0.9040)
$n = 2000$	0.8540	0.9470	1.0000	0.8570	0.9470	1.0000	0.7960	0.3850	1.0000
$np_0 = 500$	(0.1422)	(0.2909)	(0.9406)	(0.1255)	(0.2909)	(0.6419)	(0.0384)	(0.0587)	(0.6519)
$n = 4000$	0.7360	0.9510	1.0000	0.8570	0.9510	1.0000	0.8550	0.2410	1.0000
$np_0 = 1000$	(0.0944)	(0.2058)	(0.6686)	(0.0779)	(0.2058)	(0.4465)	(0.0241)	(0.0397)	(0.4517)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.6140	0.9580	0.5220	0.5630	0.9580	0.7650	0.1060	0.5800	0.9480
$np_0 = 250$	(1.1313)	(1.8356)	(3.3187)	(1.2188)	(1.8356)	(2.1994)	(0.3083)	(0.3731)	(2.3073)
$n = 1000$	0.4640	0.9560	0.7470	0.4510	0.9560	1.0000	0.2240	0.4600	1.0000
$np_0 = 500$	(0.6401)	(1.1551)	(2.5805)	(0.5514)	(1.1551)	(1.7385)	(0.1783)	(0.2441)	(1.7749)
$n = 2000$	0.2780	0.9760	0.8600	0.4650	0.9760	1.0000	0.3910	0.2780	1.0000
$np_0 = 1000$	(0.4356)	(0.8223)	(1.9184)	(0.3224)	(0.8223)	(1.2967)	(0.1111)	(0.1613)	(1.2870)
$n = 4000$	0.0570	0.9840	0.9420	0.2320	0.9840	1.0000	0.5180	0.1700	1.0000
$np_0 = 2000$	(0.3066)	(0.6338)	(1.3889)	(0.3316)	(0.6338)	(0.8560)	(0.0699)	(0.1096)	(0.8904)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9960	0.9710	0.7790	0.9950	0.9710	1.0000	0.6310	0.4630	1.0000
$np_0 = 375$	(1.1819)	(3.4385)	(1.7884)	(1.4261)	(3.4385)	(1.1358)	(0.2833)	(0.6273)	(1.2313)
$n = 1000$	0.9890	0.9780	0.8820	0.9910	0.9780	1.0000	0.4370	0.3100	1.0000
$np_0 = 750$	(0.7408)	(2.2031)	(1.2994)	(1.0190)	(2.2031)	(0.8529)	(0.1836)	(0.4200)	(0.8456)
$n = 2000$	0.9880	0.9860	0.9010	0.9970	0.9860	1.0000	0.3060	0.1590	1.0000
$np_0 = 1500$	(0.5227)	(1.5975)	(0.9328)	(0.7002)	(1.5975)	(0.5470)	(0.1216)	(0.2821)	(0.5641)
$n = 4000$	0.9790	0.9900	0.8540	0.9910	0.9900	0.9760	0.1680	0.0760	1.0000
$np_0 = 3000$	(0.3922)	(1.2192)	(0.6570)	(0.4707)	(1.2192)	(0.4903)	(0.0800)	(0.1914)	(0.3696)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 35: DGP(1,5)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9810	0.9380	1.0000	0.9810	1.0000	1.0000	0.9980	0.9990
$np_0 = 125$	(1.1016)	(2.6322)	(1.2612)	(1.0221)	(2.6322)	(0.9387)	(1.1406)	(2.7074)	(1.2140)
$n = 1000$	0.9870	0.9560	0.9860	0.9930	0.9560	1.0000	1.0000	0.9990	1.0000
$np_0 = 250$	(0.5646)	(1.4879)	(0.9243)	(0.7100)	(1.4879)	(0.6732)	(0.7201)	(1.9619)	(0.9214)
$n = 2000$	0.9560	0.9510	0.9940	0.9780	0.9510	1.0000	1.0000	0.9990	1.0000
$np_0 = 500$	(0.3520)	(1.1942)	(0.6325)	(0.4646)	(1.1942)	(0.4623)	(0.4815)	(1.5341)	(0.6620)
$n = 4000$	0.9740	0.9720	0.9930	0.9710	0.9720	1.0000	1.0000	0.9960	1.0000
$np_0 = 1000$	(0.2513)	(0.8610)	(0.4343)	(0.3047)	(0.8610)	(0.3041)	(0.3266)	(1.1461)	(0.4556)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9820	0.7450	1.0000	0.9820	0.9950	1.0000	0.9990	0.1030
$np_0 = 250$	(0.9576)	(2.3058)	(1.0034)	(1.2517)	(2.3058)	(0.7125)	(1.2162)	(3.0011)	(0.9105)
$n = 1000$	0.9980	0.9740	0.8380	1.0000	0.9740	1.0000	1.0000	1.0000	0.5290
$np_0 = 500$	(0.6068)	(1.6316)	(0.7577)	(0.8199)	(1.6316)	(0.5137)	(0.7856)	(2.1780)	(0.7357)
$n = 2000$	0.9970	0.9880	0.9400	0.9980	0.9880	0.9700	1.0000	1.0000	0.8560
$np_0 = 1000$	(0.3964)	(1.1999)	(0.5570)	(0.5199)	(1.1999)	(0.3769)	(0.5059)	(1.5693)	(0.5509)
$n = 4000$	0.9980	0.9870	0.9650	0.9900	0.9870	1.0000	1.0000	0.9960	0.9690
$np_0 = 2000$	(0.2871)	(0.8722)	(0.3917)	(0.3305)	(0.8722)	(0.2601)	(0.3377)	(1.0504)	(0.3911)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9910	0.9870	0.8630	0.9950	0.9870	0.9060	1.0000	1.0000	0.4680
$np_0 = 375$	(0.7477)	(2.1069)	(0.5684)	(0.9541)	(2.1069)	(0.4505)	(0.9746)	(2.8608)	(0.4776)
$n = 1000$	0.9800	0.9810	0.9140	0.9870	0.9810	0.9330	1.0000	1.0000	0.6600
$np_0 = 750$	(0.5215)	(1.5795)	(0.4232)	(0.6222)	(1.5795)	(0.3584)	(0.6531)	(2.0285)	(0.3585)
$n = 2000$	0.9780	0.9900	0.9540	0.9810	0.9900	1.0000	1.0000	0.9990	0.8070
$np_0 = 1500$	(0.3637)	(1.1537)	(0.3062)	(0.4184)	(1.1537)	(0.2909)	(0.4248)	(1.4273)	(0.2556)
$n = 4000$	0.9880	0.9940	0.9830	0.9580	0.9940	0.9980	1.0000	0.9930	0.9720
$np_0 = 3000$	(0.2555)	(0.8206)	(0.2102)	(0.2838)	(0.8206)	(0.2495)	(0.2864)	(0.9924)	(0.1647)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 36: DGP(2,5)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990	0.9790	0.9560	1.0000	0.9790	1.0000	1.0000	0.9970	1.0000
$np_0 = 125$	(2.1086)	(4.6819)	(3.2110)	(1.9884)	(4.6819)	(2.3459)	(1.8575)	(4.0159)	(3.0659)
$n = 1000$	0.9960	0.9690	0.9980	0.9940	0.9690	1.0000	1.0000	0.9980	1.0000
$np_0 = 250$	(1.3366)	(3.3795)	(2.5349)	(1.0409)	(3.3795)	(1.7297)	(1.1622)	(2.9040)	(2.3685)
$n = 2000$	0.9850	0.9780	0.9990	0.9530	0.9780	1.0000	1.0000	1.0000	1.0000
$np_0 = 500$	(0.9069)	(2.6871)	(1.8555)	(0.6912)	(2.6871)	(1.2026)	(0.7796)	(2.7740)	(1.7246)
$n = 4000$	0.9780	0.9700	1.0000	0.8950	0.9700	1.0000	1.0000	0.9980	1.0000
$np_0 = 1000$	(0.6073)	(2.3723)	(1.2899)	(0.5314)	(2.3723)	(0.8115)	(0.5359)	(1.9240)	(1.1973)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9970	0.4620	1.0000	0.9970	0.9940	1.0000	1.0000	0.1460
$np_0 = 250$	(3.3134)	(7.8654)	(3.6376)	(2.6047)	(7.8654)	(2.4751)	(2.7303)	(6.3455)	(3.2120)
$n = 1000$	1.0000	0.9900	0.6130	1.0000	0.9900	1.0000	1.0000	1.0000	0.7620
$np_0 = 500$	(2.2222)	(6.5572)	(2.9438)	(1.4570)	(6.5572)	(1.8728)	(1.7596)	(4.9369)	(2.5856)
$n = 2000$	0.9990	0.9860	0.7980	0.9770	0.9860	0.9910	1.0000	0.9990	0.9770
$np_0 = 1000$	(1.3886)	(3.9875)	(2.1911)	(0.9750)	(3.9875)	(1.3514)	(1.1773)	(3.3872)	(1.9366)
$n = 4000$	0.9940	0.9830	0.8370	0.9140	0.9830	1.0000	1.0000	0.9970	0.9990
$np_0 = 2000$	(0.9951)	(2.9033)	(1.5660)	(0.8678)	(2.9033)	(0.9023)	(0.8032)	(2.2559)	(1.3647)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9970	0.9860	0.6780	0.9950	0.9860	0.9750	1.0000	1.0000	0.6280
$np_0 = 375$	(3.1221)	(9.0108)	(2.1454)	(3.0972)	(9.0108)	(1.4644)	(2.6918)	(7.5257)	(1.7017)
$n = 1000$	0.9840	0.9900	0.7940	0.9850	0.9900	1.0000	1.0000	1.0000	0.8290
$np_0 = 750$	(2.0033)	(6.2287)	(1.6640)	(2.4022)	(6.2287)	(1.0470)	(1.6746)	(5.5606)	(1.2715)
$n = 2000$	0.9720	0.9830	0.8560	0.9860	0.9830	1.0000	1.0000	0.9960	0.9300
$np_0 = 1500$	(1.4040)	(4.5268)	(1.2413)	(1.8810)	(4.5268)	(0.8300)	(1.1313)	(3.8913)	(0.8929)
$n = 4000$	0.9750	0.9880	0.8300	0.9990	0.9880	1.0000	0.9990	0.9940	0.9960
$np_0 = 3000$	(1.0613)	(3.4934)	(0.9235)	(1.4003)	(3.4934)	(0.6670)	(0.7947)	(2.7608)	(0.5799)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

Table 37: DGP(3,5)

Panel A: $x = 1.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9990	0.9610	0.9740	0.9990	0.9610	1.0000	1.0000	0.9960	1.0000
$np_0 = 125$	(0.6908)	(1.3902)	(2.0587)	(1.0690)	(1.3902)	(1.4859)	(0.6110)	(1.1944)	(1.9589)
$n = 1000$	0.9980	0.9810	0.9990	0.9960	0.9810	1.0000	1.0000	0.9990	1.0000
$np_0 = 250$	(0.4500)	(1.0407)	(1.6376)	(0.6424)	(1.0407)	(1.1205)	(0.3693)	(0.8390)	(1.5352)
$n = 2000$	0.9960	0.9870	1.0000	0.9870	0.9870	1.0000	1.0000	0.9990	1.0000
$np_0 = 500$	(0.3451)	(1.1745)	(1.2522)	(0.4181)	(1.1745)	(0.7955)	(0.2510)	(0.8403)	(1.1305)
$n = 4000$	0.9890	0.9830	1.0000	0.9550	0.9830	1.0000	1.0000	0.9960	1.0000
$np_0 = 1000$	(0.2520)	(0.7611)	(0.8988)	(0.2496)	(0.7611)	(0.5470)	(0.1759)	(0.5907)	(0.7911)

Panel B: $x = 3.0$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	1.0000	0.9850	0.4500	1.0000	0.9850	0.9850	1.0000	1.0000	0.2040
$np_0 = 250$	(2.2227)	(4.8326)	(4.2059)	(3.1484)	(4.8326)	(2.8868)	(1.7483)	(3.7378)	(3.7916)
$n = 1000$	1.0000	0.9970	0.6160	1.0000	0.9970	1.0000	1.0000	1.0000	0.9250
$np_0 = 500$	(1.6202)	(5.0796)	(3.4570)	(1.9491)	(5.0796)	(2.2240)	(1.1327)	(3.0182)	(3.0385)
$n = 2000$	0.9990	0.9900	0.8300	0.9920	0.9900	0.9950	1.0000	1.0000	0.9990
$np_0 = 1000$	(1.1777)	(3.4365)	(2.6573)	(1.1478)	(3.4365)	(1.6065)	(0.7803)	(2.1789)	(2.2719)
$n = 4000$	0.9980	0.9840	0.9240	0.9430	0.9840	1.0000	1.0000	0.9970	1.0000
$np_0 = 2000$	(0.8205)	(2.3834)	(1.8895)	(0.5865)	(2.3834)	(1.0822)	(0.5422)	(1.6780)	(1.5943)

Panel C: $x = 4.5$									
	Pickands			Built-in			True		
	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm	Mom	Momt-pick	Pwm
$n = 500$	0.9960	0.9890	0.6370	0.9910	0.9890	1.0000	0.9990	1.0000	0.8590
$np_0 = 375$	(3.3904)	(9.6407)	(2.7729)	(3.5803)	(9.6407)	(1.8653)	(2.1894)	(6.8933)	(2.3140)
$n = 1000$	0.9890	0.9910	0.7330	0.9630	0.9910	1.0000	0.9980	0.9960	0.9620
$np_0 = 750$	(2.4786)	(7.7734)	(2.1640)	(2.4114)	(7.7734)	(1.2897)	(1.5404)	(5.0917)	(1.7231)
$n = 2000$	0.9810	0.9860	0.7970	0.8760	0.9860	1.0000	1.0000	0.9940	0.9930
$np_0 = 1500$	(1.8032)	(5.9840)	(1.6152)	(1.4059)	(5.9840)	(0.9270)	(1.0979)	(3.6194)	(1.1857)
$n = 4000$	0.9790	0.9830	0.8380	0.7120	0.9830	1.0000	0.9970	0.9940	1.0000
$np_0 = 3000$	(1.2068)	(4.1326)	(1.1778)	(0.7840)	(4.1326)	(0.6749)	(0.7785)	(2.5798)	(0.7839)

Notes: The coverage rates and average lengths of the CIs (in parentheses) are reported.

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