

Double Robust Semi-Supervised Inference for the Mean: Selection Bias under MAR Labeling with Decaying Overlap*

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Abstract

Semi-supervised (SS) inference has received much attention in recent years. Apart from a moderate-sized labeled data, \mathcal{L} , the SS setting is characterized by an additional, *much larger sized*, unlabeled data, \mathcal{U} . The setting of $|\mathcal{U}| \gg |\mathcal{L}|$, makes SS inference unique and different from the standard missing data problems, owing to natural violation of the so-called “positivity” or “overlap” assumption. However, most of the SS literature implicitly assumes \mathcal{L} and \mathcal{U} to be equally distributed, i.e., no selection bias in the labeling. Inferential challenges in missing at random (MAR) type labeling allowing for selection bias, are inevitably exacerbated by the decaying nature of the propensity score (PS). We address this gap for a prototype problem, the estimation of the response’s mean. We propose a double robust SS (DRSS) mean estimator and give a complete characterization of its asymptotic properties. The proposed estimator is consistent as long as either the outcome or the PS model is correctly specified. When both models are correctly specified, we provide inference results with a non-standard consistency rate that depends on the smaller size $|\mathcal{L}|$. The results are also extended to causal inference with imbalanced treatment groups. Further, we provide several novel choices of models and estimators of the decaying PS, including a novel offset logistic model and a stratified labeling model. We present their properties under both high and low dimensional settings. These may be of independent interest. Lastly, we present extensive simulations and also a real data application.

Keywords: Selection bias, Missing data, Causal inference, Decaying overlap, Imbalanced classification, Average treatment effects.

1. Introduction

Inference in semi-supervised (SS) settings has received substantial attention in recent times. Unlike traditional statistical learning settings that are usually either supervised or unsupervised, an SS setting represents a confluence of these two settings. A typical SS setting has two types of available data: apart from a small or moderate-sized *labeled* (or supervised) data $\mathcal{L} = (Y_i, \mathbf{X}_i)_{i=1}^n$, one has access to a *much larger sized unlabeled* (or unsupervised) data $\mathcal{U} = (\mathbf{X}_i)_{i=n+1}^N$ with $N \gg n$. Here, $Y_i \in \mathbb{R}$ and

*Accepted by *Information and Inference: A Journal of the IMA*.

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$\mathbf{X}_i \in \mathbb{R}^p$ denote the outcome of interest and a covariate vector (possibly high dimensional), respectively. To integrate the notation, we use $R_i \in \{0, 1\}$ to denote the missingness/labeling indicator and use $\mathbb{S} = \mathcal{L} \cup \mathcal{U} = (R_i, R_i Y_i, \mathbf{X}_i)_{i=1}^N$ to denote the full data, a collection of N i.i.d. (independent and identically distributed) observations of (R, RY, \mathbf{X}) , where throughout this paper, we let (R, RY, \mathbf{X}) denote an independent copy of $(R_i, R_i Y_i, \mathbf{X}_i)$.

SS settings arise naturally whenever the covariates are easily available for a large cohort (so that \mathcal{U} is plentiful), but the corresponding response is expensive and/or difficult to obtain due to various practical constraints (thus limiting the size of \mathcal{L}), a frequent scenario in modern studies involving large databases in the ‘big data’ era. Examples of such settings are ubiquitous across various scientific disciplines, including machine learning problems like speech recognition, text mining etc. [Chapelle et al., 2006, Zhu, 2005], as well as more recent (and relevant to our work) biomedical applications, like electronic health records (EHR) and integrative genomic studies [Cai and Guo, 2020, Chakrabortty and Cai, 2018]. It is important to note that while SS settings can be viewed as a missing data problem of sorts, the fact that $|\mathcal{U}| \gg |\mathcal{L}|$ is a key distinguishing feature of SS settings (for instance, $|\mathcal{L}|$ could be of the order of hundreds, while $|\mathcal{U}|$ could be in the order of tens of thousands!). This condition, a natural consequence of the underlying practical situations leading to these data, implies that the proportion of labeled observations in SS settings converges to 0 as the sample sizes $|\mathcal{L}|, |\mathcal{U}| \rightarrow \infty$. This makes SS settings unique and fundamentally different from any standard missing data problem where this proportion is always assumed to be bounded away from 0, a condition also known as the positivity (or overlap) assumption in the missing data literature [Imbens, 2004, Tsiatis, 2007], which is naturally violated here.

Most of the SS literature, however, implicitly assumes that \mathbf{X} is equally distributed in \mathcal{L} and \mathcal{U} samples, that is, a missing completely at random (MCAR) setting, where $R \perp\!\!\!\perp (Y, \mathbf{X})$, and the goal is to improve efficiency over an (already valid) supervised estimator based on \mathcal{L} . A biased, covariate-dependent, missing at random (MAR) type labeling mechanism has not been studied much, although they are much more realistic in practice, especially in biomedical applications (including the examples discussed earlier) where selection bias is common. For instance, in EHR data, relatively ‘sicker’ patients may often be more likely to be labeled, especially if the labeling is for a disease response. We work in this type of a ‘decaying’ MAR domain, which we name MAR-SS for short, under the typical “ignorability” assumption:

$$R \perp\!\!\!\perp Y | \mathbf{X},$$

thereby allowing for a selection bias in the process. It is important to note that the traditional MAR setting amongst the missing data literature is typically studied together with an overlap (positivity) condition that bounds away the propensity score (PS) $\mathbb{E}(R|\mathbf{X})$ uniformly from zero [Bang and Robins, 2005]. Compared to such MAR settings, our MAR-SS setting is significantly more challenging due to the inevitably decaying nature of the PS. We also interchangeably refer to this setting as *decaying overlap*. As $N \gg n$ here, positivity is automatically excluded, thus leading to a *non-standard* asymptotic regime.

SUBTLETIES To work with such unbalanced labeling, we denote the PS as $\pi_N(\mathbf{X}) := \mathbb{E}(R|\mathbf{X}) \equiv \mathbb{P}(R = 1|\mathbf{X})$ and let $\bar{\pi}_N := \mathbb{E}(R) \equiv \mathbb{P}(R = 1)$. It is important to note that in order to allow a non-degenerate PS with $\mathbb{E}(R) \rightarrow 0$ as $N \rightarrow \infty$, we must allow R , $\pi_N(\mathbf{X})$ and $\bar{\pi}_N$ to depend on N (otherwise forcing $n/N \rightarrow 0$ would lead to a degenerate situation with $\mathbb{E}(R) = 0$ and $\mathbb{E}(R|\mathbf{X}) = 0$ almost surely (a.s.)). Hence, both $\{R_{N,i}\}_{N,i}$ and $\{\pi_N(\mathbf{X}_i)\}_{N,i}$ form triangular arrays. We suppress the dependence of R_N on N throughout for notational simplicity.

Under such a *decaying MAR-SS setting*, we study the fundamental problem of estimation and inference towards the mean response, defined as:

$$\theta_0 := \mathbb{E}(Y).$$

The mean estimation problem above is a canonical problem in classical missing data as well as causal inference literature, and we consider it here mainly as a prototype problem. The bigger purpose of this paper is to provide a deeper understanding of this *MAR-SS* setting and all its subtleties, where the main challenge is to allow for the uniform decay of the PS with the sample size and handle the non-standard asymptotics that arises inevitably. Moreover, unlike “traditional” SS settings (with MCAR), the goal here is *not* to “improve” over a supervised estimator from \mathcal{L} (which is no longer valid under selection bias) but rather *develop from scratch a consistent and rate-optimal estimator along with inferential tools for it*. The contributions of this work therefore constitutes advances both in the literature of classical missing data and causal inference as well as that of traditional SS inference. We first provide an overview of the existing literature(s), followed by a summary of our contributions.

1.1 Related Literature

SS literature on prediction problems is vast, typically under the name of semi-supervised learning; see [Zhu \[2005\]](#) and [Chapelle et al. \[2006\]](#) for a review. SS inference has attracted a lot of recent attention. [Zhang et al. \[2019\]](#) and [Zhang and Bradic \[2022\]](#) proposed SS mean estimators. The estimators in [Zhang and Bradic \[2022\]](#) can be roughly seen as a special (MCAR) case of the MAR-SS setting here. [Azriel et al. \[2022\]](#) and [Chakrabortty and Cai \[2018\]](#) tackled the SS linear regression problem, while [Kawakita and Kanamori \[2013\]](#) considered likelihood based SS inference. [Cai and Guo \[2020\]](#) studied SS inference of the explained variance in high dimensional linear regression. However, they all require a MCAR assumption, i.e., $R \perp\!\!\!\perp (Y, \mathbf{X})$. MCAR is practically too strong, and these estimators lead to doubtful results once the dependency of R on \mathbf{X} occurs causing a selection bias in the labeling.

Works that remove some of the MCAR restrictions have been proposed recently. A special stratified labeling in a SS framework was studied in [Gronsbell et al. \[2020\]](#) with a focus on prediction performance measures. Stratified labeling was also studied in [Hong et al. \[2020\]](#), though their setting is very specific in that their only source of randomness arises from the treatment assignment. [Ryan and Culp \[2015\]](#) considered regression problems under a SS framework with covariate shift; however, certain manifold conditions were imposed therein that we avoid altogether. More recently, [Liu et al. \[2020\]](#) used an approach based on density ratios for the same setup, albeit working with semi-nonparametric models and a non-decaying PS. A problem setup closest to us was considered only in [Kallus and Mao \[2020\]](#). Their main focus, however, was on treatment effects estimation and efficiency theory when surrogate variables occur in the MAR setting with positivity. They do provide a semiparametric efficiency bound under a decaying PS setting. However, we provide a *complete characterization* (see Sections 3.1–3.2) of the estimators’ asymptotic properties and the inferential tools (see Section 3.3), and further, such results are provided under much weaker conditions. For instance, we only require $N\bar{\pi}_N \rightarrow \infty$ (while they require $N\bar{\pi}_N^2 \rightarrow \infty$) and we allow an unbounded support for \mathbf{X} , which is essentially violated under the uniformly bounded density ratio condition $\bar{\pi}_N/\pi_N(\mathbf{X}) < C$ assumed in [Kallus and Mao \[2020\]](#). Moreover, the authors therein did not provide any results and/or methodology on the decaying PS’s estimation which is an essential component of the problem here.

Our work is also naturally connected to the rich missing data (and causal inference) literature on semi-parametric methods, and especially to so-called doubly robust (DR) inference; see [Robins et al. \[1994\]](#), [Robins and Rotnitzky \[1995\]](#), [Bang and Robins \[2005\]](#), [Tsiatis \[2007\]](#), [Kang and Schafer \[2007\]](#),

and [Graham \[2011\]](#) for a review. High-dimensional DR equivalents have been presented recently as well; see for example [Belloni et al. \[2014b\]](#), [Badic et al. \[2019\]](#), [Chernozhukov et al. \[2018\]](#), [Farrell \[2015\]](#), [Smucler et al. \[2019\]](#). They work on a low-dimensional parameter estimation problem that involves high-dimensional nuisance parameters. On the other hand, [Semenova and Chernozhukov \[2017\]](#) and [Chakrabortty et al. \[2019\]](#) work on problems where the parameters of interest themselves are high-dimensional. However, the positivity assumption is always assumed. Our work is a direct extension of the above literature where we now include a decaying PS, and therefore a setting of imbalanced treatment mechanisms.

Another related setting to our decaying PS setting is the so-called “limited overlap” setting. A few notable prior works on limited overlap include [Crump et al. \[2009\]](#), [Khan and Tamer \[2010\]](#), [Rothe \[2017\]](#), [Visconti and Zubizarreta \[2018\]](#), [Yang and Ding \[2017\]](#) among others, where a truncation of the PS is introduced and a restricted analysis to the portions of the treatment groups such that overlap holds is performed. The “limited overlap” condition is also weaker than the usual overlap condition, but very *different* from our decaying PS situation. The limited overlap allows the PS to approach zero on some specific regions in the support of \mathbf{X} , while we allow $\mathbb{E}(R|\mathbf{X})$ to shrink to zero (with N) uniformly in \mathbf{X} . Moreover, they assume that $\mathbb{E}(R|\mathbf{X})$ is independent of N . By allowing R to depend on N , we allow \mathbb{P}_R and $\mathbb{P}_{R|\mathbf{X}}$ to depend on N so that $\bar{\pi}_N = \mathbb{E}(R) \rightarrow 0$ is permissible (a necessity under our settings of interest), much unlike the existing limited overlap literature.

1.2 Our Contributions

Contributions of our work are three fold: on (i) *double robust estimation with decaying PS*, (ii) *estimation of decaying PS*, and (iii) *average treatment effect (ATE) estimation with imbalanced groups*.

DOUBLE ROBUST ESTIMATION WITH DECAYING PROPENSITY We believe this work fills in an important gap in both the SS literature and the missing data literature. A selection bias in the labeling mechanism is allowed, therefore parting with the SS literature. A PS is allowed to decay to zero uniformly, consequently enriching the MAR literature. We propose a *double robust semi-supervised* (DRSS) mean estimator (see Sections 3.1-3.2), which can be viewed as an adaptation of the standard DR estimator [[Robins et al., 1994](#)] to our MAR-SS setting. Theorem 3.2, our main result for this part, provides a full characterization of the DRSS estimator and its asymptotic expansion when at least one of the nuisance functions is correctly specified. Throughout, our results bring in a new set of rate-adjusted high-level estimation error conditions on the nuisance estimators that are agnostic to their mode of construction. When both nuisance models are correctly specified, we derive the asymptotic normality of our estimator if a product rate condition for the estimation errors is further assumed, with an asymptotic variance reaching the semi-parametric efficiency bound derived in [Kallus and Mao \[2020\]](#). We also construct a corresponding confidence interval (see Section 3.3) that adapts to the rate of decay of the PS. Adaptivity here implies that the confidence sets are wider for the cases of faster decay without changing the estimators themselves. The analyses and the methods are considerably more involved here compared to the standard problems, due to the decaying nature of the PS. For example, we establish that the rate of convergence is no longer governed by N solely; rather the effective rate is identified to be in terms of Na_N , where $a_N^{-1} = \mathbb{E}\{\pi_N^{-1}(\mathbf{X})\}$. In high dimensions, and using standard parametric nuisance models, the product rate condition required for the asymptotic normality is $s_m s_\pi \{\log(p)\}^2 = o(Na_N)$, where s_m and s_π are the sparsity levels of the (linear/logistic) nuisance functions $m(\mathbf{X}) = \mathbb{E}(Y|\mathbf{X})$ and $\pi_N(\mathbf{X}) = \mathbb{E}(R|\mathbf{X})$, respectively. When $a_N \asymp 1$, such a condition coincides with the usual product condition [[Chernozhukov et al., 2018](#)] where the positivity condition is assumed. However, whenever $a_N \rightarrow 0$,

the condition is stricter in order to compensate for the decay of the PS.

ESTIMATION OF THE DECAYING PROPENSITY A key challenge for any methodological development in our MAR-SS setting is the modeling of the decaying PS. We propose several choices and associated results in this regard, including (i) *stratified labeling* (see Section 4.2) as well as (ii) a novel *offset based imbalanced logistic regression model* (see Section 4.1), under both low and high dimensional settings. The first approach, (i), is often practically relevant in the presence of *a priori* information available on a stratifying variable. The second approach, (ii), on the other hand, is applicable quite generally and constitutes a natural extension of logistic models to our case of a decaying PS. Related to the latter model, imbalanced classification in low-dimensions was recently studied by [Owen \[2007\]](#) and [Wang \[2020\]](#). Our offset based model is closely related to their diverging intercept model, and yet has distinct methodological advantages (see Appendix A and Remark A.1 therein). We provide theoretical results about estimation rates and other properties of these models under both high and low dimensional settings. These results may be more generally useful and are of independent interest; for example, our results on estimation of decaying PS under a logistic model in high dimensions are the *first* such results to our knowledge. We demonstrate that, for a sub-Gaussian \mathbf{X} , the estimation error of $\pi_N(\cdot)$ is $O_p(\sqrt{s_\pi \log(p)/(N\bar{\pi}_N)})$, where s_π is the sparsity level of the logistic model parameter. Such a result is non-trivial as, per Theorem 4.2, an appropriate choice of the regularization parameter is non-standard with $\lambda_N \asymp \sqrt{\bar{\pi}_N \log(p)/N}$. We also obtain a regular and asymptotically linear (RAL) expansion for the estimator of the logistic regression parameter in the low-dimensional case; see Theorem 4.1. Moreover, we showcase that the estimator reaches the asymptotic variance as established in [Wang \[2020\]](#) for low-dimensional problems. For the cases where the outcome model is misspecified, we further construct an adjusted RAL expansion of our DRSS estimator. Lastly, in Appendix B, we also consider the special case of the MCAR model and the corresponding results in that setting.

AVERAGE TREATMENT EFFECT (ATE) ESTIMATION WITH IMBALANCED GROUPS Drawing on a natural connection between the causal inference and missing data settings (see the discussions in Section 1.1 of [Chakrabortty et al. \[2019\]](#) for instance) we extend our results to a corresponding ATE estimation problem. Our results allow for an extremely imbalanced treatment or control groups, in that $\bar{\pi}_N = \mathbb{P}(R = 1) \rightarrow 0$ (or alternatively, $\bar{\pi}_N \rightarrow 1$) as $N \rightarrow \infty$.

We establish a RAL expansion for the proposed ATE estimator with a non-standard consistency rate, $O_p(1/\sqrt{N\bar{\pi}_N})$, where without loss of generality we assume $\bar{\pi}_N \rightarrow 0$. A sufficient condition for the expansion's validity is correctness of the model for the treatment group's outcome as well as that of the PS model. Notably, the control group's outcome and PS models can be (even both) misspecified if $\bar{\pi}_N \rightarrow 0$ fast enough. Such a condition is different from most of the recent results, such as [Farrell \[2015\]](#) and [Chernozhukov et al. \[2018\]](#), where the nuisance functions in both of the groups need to be correctly specified for valid inference results. It is also different from the recent work of [Smucler et al. \[2019\]](#) and [Tan \[2020\]](#), where they used specific parametric working models and they required at least one of the nuisance functions to be correctly specified for both of the groups.

The PS setting, the parameter of interest, and the methodology are also different from the limited overlap literature, e.g., [[Crump et al., 2009](#)]. As shown in [Khan and Tamer \[2010\]](#), the information bound for the ATE estimation is 0 if only under the ignorability assumption and a.s., $\pi_N(\cdot) \in (0, 1)$. As a result, a common approach in the limited overlap literature is to re-target the parameter of interest by considering a “shifted” ATE induced by the truncation of the PS [[Crump et al., 2009](#)]. In our paper, we show that it is in fact possible to estimate the ATE directly when we have additional information that the

inverse PS has well-behaved tails, e.g., $\pi_N(\cdot)$ follows an offset logistic model and \mathbf{X} is sub-Gaussian; see Theorems 3.2, 4.1, and 4.2.

1.3 Organization

The rest of this paper is organized as follows. In Section 2, we formulate the decaying PS setting and the mean estimation problem. In Section 3, we construct the mean estimators and provide a complete characterization of the asymptotic properties of our proposed estimators. In Section 3.1, we first consider a special case that the PS function is known, and in Section 3.2, we consider a general unknown $\pi_N(\cdot)$ case and provide results on our final proposed DRSS estimator. In Section 4, we analyze three specific decaying PS models, including an offset logistic model under both low and high dimensions (Section 4.1), a stratified labeling model (Section 4.2), and a MCAR model (Section B). For each of the PS models, we propose a corresponding PS estimator and establish their asymptotic properties as well as the asymptotic results of the DRSS estimators based on them. In Section 5, we extend our results to an ATE estimation problem where the treatment groups are extremely imbalanced. Simulation results and an application to a NHEFS data are presented in Section 6, followed by a concluding discussion in Section 7. Further discussions, additional theoretical and numerical results, as well as the proofs of the main results are provided in the [Supplement](#) (Appendices A-D).

1.4 Notation

We use the following notation throughout. Let $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ denote the probability measure and expectation characterizing the joint distribution of the underlying (possibly unobserved) random vector $\mathbb{Z} := (R, Y, \mathbf{X})$, respectively, where $R \in \{0, 1\}$, $Y \in \mathbb{R}$, and $\mathbf{X} \in \mathbb{R}^p$. Let $\mathbb{P}_{\mathbf{X}}$ denote the marginal distribution of \mathbf{X} . For any $r > 0$, let $\|f(\cdot)\|_{r,\mathbb{P}} := \{\mathbb{E}|f(\mathbb{Z})|^r\}^{1/r}$ and $\|f(\cdot)\|_{r,\mathbb{P}_{\mathbf{X}}} := \{\mathbb{E}_{\mathbf{X}}|f(\mathbf{X})|^r\}^{1/r}$. For any vector $\mathbf{z} \in \mathbb{R}^p$, we denote $\mathbf{z}(j)$ as the j -th coordinate of \mathbf{z} . For $r \geq 1$, define the l_r -norm of a vector \mathbf{z} with $\|\mathbf{z}\|_r := (\sum_{j=1}^p |\mathbf{z}(j)|^r)^{1/r}$, $\|\mathbf{z}\|_0 := |\{j : \mathbf{z}(j) \neq 0\}|$, and $\|\mathbf{z}\|_\infty := \max_j |\mathbf{z}(j)|$. For a matrix $A \in \mathbb{R}^{p \times p}$, $\|A\|_r := \sup_{\mathbf{z} \neq \mathbf{0}} \|A\mathbf{z}\|_r / \|\mathbf{z}\|_r$ and $\lambda_{\min}(A)$ denotes the smallest eigenvalue of A . For sequences a_N and b_N , we denote $a_N \asymp b_N$ if there exists constants $c, C, N_0 > 0$ such that $cb_N < a_N < Cb_N$ for all $N > N_0$. Lastly, we define the logit function as $\text{logit}(u) := \log\{u/(1-u)\}$ for any $u \in (0, 1)$.

2. Problem setup

Let the entire dataset be denoted as: $\mathbb{S} := \{\mathbb{Z}_i = (R_i, R_i Y_i, \mathbf{X}_i), i = 1, \dots, N\}$. The dimension of the covariates p can be either fixed or growing with N in that $p = p_N \rightarrow \infty$ as $N \rightarrow \infty$. We assume the following ignorability condition throughout.

ASSUMPTION 2.1 (Ignorability or MAR condition) R, Y and \mathbf{X} are such that $R \perp\!\!\!\perp Y | \mathbf{X}$.

The ignorability condition is standard in the missing data literature [Bang and Robins, 2005, Tsiatis, 2007]. Let $m(\mathbf{x}) := \mathbb{E}(Y | \mathbf{X} = \mathbf{x})$ and $\pi_N(\mathbf{x}) := \mathbb{E}(R | \mathbf{X} = \mathbf{x})$ denote the conditional mean of Y and the conditional PS, respectively. Assume $\mathbb{E}\{\pi_N^{-1}(\mathbf{X})\} < \infty$ for each N . We then define a positive sequence of real numbers, a_N as:

$$a_N^{-1} := \mathbb{E}\{\pi_N^{-1}(\mathbf{X})\}, \quad (2.1)$$

which plays a key role in determining the rates of any inverse-probability weighting type estimator. We are primarily interested in the case of $a_N \rightarrow 0$, although our results hold more broadly. We assume $\mathbb{E}\{\pi_N^{-1}(\mathbf{X})\} < \infty$ for each N , i.e. $a_N > 0$ for each N , but we still allow $a_N \rightarrow 0$ as $N \rightarrow \infty$. The value

a_N shrinks when the distribution of $\pi_N(\cdot)$ has too much mass concentrated around 0; see Remark 3.5 for more details. Notice that the usual positivity (overlap) condition, $\pi_N(\mathbf{X}) > c > 0$, is NOT assumed throughout the paper, and we allow a uniformly decaying PS in that $\pi_N(\mathbf{x}) \rightarrow 0$ as $N \rightarrow \infty$, for every \mathbf{x} in the support \mathcal{X} of \mathbf{X} .

EXAMPLE 2.1 (Offset based PS model) As an illustration of a decaying PS model with a dependence of $\pi_N(\mathbf{X})$ onto N , we consider a general *offset based PS model* as follows

$$\pi_N(\mathbf{X}) = g(f(\mathbf{X}) + \log(\bar{\pi}_N)), \quad \text{with some } f : \mathbb{R}^p \rightarrow \mathbb{R} \text{ and } g(u) := \frac{\exp(u)}{1 + \exp(u)},$$

where $\log(\bar{\pi}_N)$ is considered as an “offset”. The model above constitutes a fairly general way of incorporating the uniformly decaying PS model. Further details on the rationale behind and the analysis of such an offset model are discussed in Section 4.1, where we allow a linear f with a sub-Gaussian \mathbf{X} , thus clearly violating the positivity condition. Uniform decay of $\pi_N(\mathbf{X})$ can also be seen whenever \mathbf{X} has a compact support \mathcal{X} , leading to $c_1 \bar{\pi}_N \leq \pi_N(\mathbf{x}) \leq c_2 \bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$ with constants $0 < c_1 < c_2$.

PRELIMINARIES: IDENTIFICATION AND ALTERNATIVE REPRESENTATIONS We have the following three alternative representations or identifications of $\theta_0 = \mathbb{E}(Y)$ based on the observable variables and some unknown (but estimable) nuisance functions, i.e., $m(\mathbf{X})$ and $\pi_N(\mathbf{X})$.

(Reg) Regression based representation: $\theta_0 = \mathbb{E}\{m(\mathbf{X})\}$.

(IPW) Inverse probability weighting representation: $\theta_0 = \mathbb{E}\{\pi_N^{-1}(\mathbf{X})RY\}$.

(DR) Doubly robust representation: $\theta_0 = \mathbb{E}[m(\mathbf{X}) + \pi_N^{-1}(\mathbf{X})\{RY - Rm(\mathbf{X})\}]$.

A natural estimator of θ_0 would be the empirical mean of the observed responses, $\bar{Y}_{\text{labeled}} := \sum_{i=1}^N R_i Y_i / \sum_{i=1}^N R_i$. Under a MCAR setting, \bar{Y}_{labeled} is a consistent estimator. However, under the MAR setting, \bar{Y}_{labeled} is no longer a consistent estimator; $\bar{Y}_{\text{labeled}} \xrightarrow{P} \mathbb{E}(Y|R=1) \neq \mathbb{E}(Y)$ in general. According to the above representations, with $\hat{m}(\cdot)$ and $\hat{\pi}_N(\cdot)$ estimating $m(\cdot)$ and $\pi_N(\cdot)$, respectively, we could consider $\hat{\theta}_{\text{Reg}} := N^{-1} \sum_{i=1}^N \hat{m}(\mathbf{X}_i)$ and $\hat{\theta}_{\text{IPW}} := N^{-1} \sum_{i=1}^N R_i Y_i \hat{\pi}_N^{-1}(\mathbf{X}_i)$. For the sake of simplicity, here we consider an ideal case that $\hat{m}(\cdot)$ and $\hat{\pi}_N(\cdot)$ are trained on another additional set so that $(\hat{m}(\cdot), \hat{\pi}_N(\cdot)) \perp\!\!\!\perp (\mathbf{X}_i)_{i=1}^N$.

It is then not hard to show that

$$\begin{aligned} \hat{\theta}_{\text{Reg}} - \theta_0 &= O_p \left(\|\hat{m}(\mathbf{X}) - m(\mathbf{X})\|_{1, \mathbb{P}_{\mathbf{X}}} + N^{-1/2} \right), \\ \hat{\theta}_{\text{IPW}} - \theta_0 &= O_p \left(\|1 - \pi_N(\mathbf{X})/\hat{\pi}_N(\mathbf{X})\|_{2, \mathbb{P}_{\mathbf{X}}} + N^{-1/2} \right). \end{aligned}$$

Hence, the Reg and IPW estimators are not even consistent when the corresponding nuisance model is misspecified. Even when the corresponding nuisances are correctly specified, estimators directly depend on the estimation error of $\hat{m}(\cdot)$ and $\hat{\pi}_N(\cdot)$, respectively, which are not \sqrt{N} -consistent (nor $\sqrt{N}\bar{\pi}_N$ -consistent) in the high-dimensional or non-parametric settings.

The DR representation of θ_0 , viewed as a combination of the Reg and IPW representations [Accomando, 1974], leads to double robustness. DR estimators are consistent as long as at least one of the models are correctly specified (this property is called “double robustness”, see, e.g., Theorem 2 of Farrell [2015]). When both models are correctly specified, the estimation errors of the DR estimators depend on the product of estimation errors of the nuisance functions; this property is called “rate double robustness,”

as defined in Definition 2 of [Smucler et al. \[2019\]](#). Moreover, DR estimators are known to be semi-parametrically optimal when both models are correct [[Bang and Robins, 2005](#)], as well as first order insensitive to the estimation errors of the nuisance functions [[Chernozhukov et al., 2018](#)]; see the discussions in [Chakrabortty et al. \[2019\]](#). In Section 3, we propose estimators based on the above DR representation.

3. Semi-supervised inference under a MAR-SS setting

3.1 Known PS $\pi_N(\cdot)$

We first consider an oracle case where the PS, $\pi_N(\cdot)$, is known. In other words, the missing mechanism is designed and controlled by the researcher. This is also closely related to the randomized controlled trials in causal inference literature. Based on the DR representation, we consider the following SS estimator:

$$\tilde{\theta} := N^{-1} \sum_{i=1}^N \widehat{m}(\mathbf{X}_i) + N^{-1} \sum_{i=1}^N \frac{R_i}{\pi_N(\mathbf{X}_i)} \{Y_i - \widehat{m}(\mathbf{X}_i)\}, \quad (3.1)$$

where $\widehat{m}(\mathbf{X}_i)$ is a *cross-fitted* estimator established as follows: 1) for any fixed $\mathbb{K} \geq 2$, let $\{\mathcal{I}_k\}_{k=1}^{\mathbb{K}}$ be a random partition of $\mathcal{I} := \{1, \dots, N\}$; 2) for each $k \leq \mathbb{K}$, obtain the estimator $\widehat{m}(\cdot; \mathbb{S}_{-k})$ using the training set $\mathbb{S}_{-k} := \{\mathbf{Z}_i : i \in \mathcal{I} \setminus \mathcal{I}_k\}$, where for typical supervised methods, $\widehat{m}(\cdot; \mathbb{S}_{-k})$ only depends on the labeled observations, $\{\mathbf{Z}_i : i \in \mathcal{I} \setminus \mathcal{I}_k, R_i = 1\}$; 3) for each $i = 1, \dots, N$, let $\widehat{m}(\mathbf{X}_i) := \widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k(i)})$, where $k(i)$ denotes the unique k such that $i \in \mathcal{I}_k$. The proposed $\tilde{\theta}$ can be seen as a debiased $\widehat{\theta}_{\text{Reg}}$ estimator, where the misspecification or estimation bias of $\widehat{m}(\cdot)$ is removed by the knowledge of $\pi_N(\cdot)$. On the other hand, $\widehat{\theta}_{\text{IPW}}$ is a special case of $\tilde{\theta}$ with $\widehat{\pi}_N(\cdot) = \pi_N(\cdot)$ and $\widehat{m}(\cdot) \equiv 0$. However, $\tilde{\theta}$ with a “good” estimator for the outcome model improves the efficiency of the IPW estimator; see e.g., Remark 3.3. The cross-fitting is vital for the bias correction; see discussions in [Chernozhukov et al. \[2018\]](#) and [Chakrabortty et al. \[2019\]](#). By the cross-fitting construction, $\widehat{m}(\cdot; \mathbb{S}_{-k(i)}) \perp\!\!\!\perp \mathbf{Z}_i$ for each $i \leq N$. As a result,

$$\mathbb{E}_{\mathbf{X}} \left[\widehat{m}(\mathbf{X}) + \frac{R}{\pi_N(\mathbf{X})} \{Y - \widehat{m}(\mathbf{X})\} \right] = \mathbb{E}_{\mathbf{X}} \left[\widehat{m}(\mathbf{X}) + \frac{\pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{Y - \widehat{m}(\mathbf{X})\} \right] = \theta_0,$$

and hence the proposed estimator $\tilde{\theta}$ is unbiased for θ_0 , even if $m(\cdot)$ is misspecified. We denote $\mu(\cdot)$ as a “limit” (potentially *misspecified*) of $\widehat{m}(\cdot)$, i.e., in general, $\mu(\cdot) \neq m(\cdot)$ is allowed.

ASSUMPTION 3.1 (Basic assumption) (a) \mathbf{Z} has finite 2nd moments and $\Sigma \equiv \text{Var}(\mathbf{X})$ is positive definite. (b) Let $\mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X} = \mathbf{x}] \geq \sigma_{\zeta,1}^2 > 0$ and $\mathbb{E}[\{Y - \mu(\mathbf{X})\}^2 | \mathbf{X} = \mathbf{x}] \leq \sigma_{\zeta,2}^2 < \infty$ for all \mathbf{x} in the support \mathcal{X} of $\mathbb{P}_{\mathbf{X}}$. Moreover, $\text{Var}(Y) \leq \sigma_{\zeta,2}^2$.

ASSUMPTION 3.2 (Tail condition) Let $a_N^{-1} \mathbb{E} \left[\psi_{\mu,\pi}^2(\mathbf{Z}) \mathbf{1} \left\{ |\psi_{\mu,\pi}(\mathbf{Z})| > c \sqrt{N/a_N} \right\} \right] \rightarrow 0$, for any $c > 0$ as $N \rightarrow \infty$, where recall that a_N is defined in (2.1), and with $\psi_{\mu,\pi}(\mathbf{Z})$ as:

$$\psi_{\mu,\pi}(\mathbf{Z}) := \mu(\mathbf{X}) + \frac{R}{\pi_N(\mathbf{X})} \{Y - \mu(\mathbf{X})\} - \theta_0 = Y - \theta_0 + \left\{ \frac{R}{\pi_N(\mathbf{X})} - 1 \right\} \{Y - \mu(\mathbf{X})\}. \quad (3.2)$$

REMARK 3.1 (Discussion on Assumptions 3.1 and 3.2) Assumption 3.1 imposes some mild moment conditions; similar versions can be found in [Zhang et al. \[2019\]](#), [Zhang and Bradic \[2022\]](#). Assumption 3.2 is needed only for the asymptotic normality and is satisfied if 1) $\pi_N(\cdot)$ follows an offset propensity model as in Example 2.1 with sub-Gaussian $f(\mathbf{X})$ (see Section 4.1 where we analyzed a special case

of the offset model); 2) $\mathbb{E}\{|Y - \mu(\mathbf{X})|^{2+\delta}|X\} < C$, $\mathbb{E}(|Y - \theta_0|^{2+\delta}) < C$ with constants $\delta, C > 0$; and 3) $N\bar{\pi}_N \rightarrow \infty$ as $N \rightarrow \infty$. A sufficient condition for Assumption 3.2 is given in Assumption 3.3.

In the result below, we analyze the theoretical properties of $\tilde{\theta}$ including its consistency, convergence rate, asymptotic normality and robustness properties.

THEOREM 3.1 Let Assumptions 2.1 and 3.1 hold. Let $Na_N \rightarrow \infty$ as $N \rightarrow \infty$. Let $\mu(\cdot)$ be a well-defined limit of the cross-fitted $\hat{m}(\cdot)$, that satisfy:

$$\mathbb{E}_{\mathbf{X}} \left[\frac{a_N}{\pi_N(\mathbf{X})} \{ \hat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X}) \}^2 \right] = O_p(c_{\mu,N}^2), \text{ with sequence } c_{\mu,N} = o(1), \quad (3.3)$$

for $k \leq K$. Then,

$$\tilde{\theta} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) + O_p \left(\frac{c_{\mu,N}}{\sqrt{Na_N}} \right) \text{ and } V_N(\mu) := \text{Var}\{\psi_{\mu,\pi}(\mathbf{Z})\} \asymp a_N^{-1},$$

where $\psi_{\mu,\pi}(\mathbf{Z}_i)$ is defined in (3.2). Alternatively, we also have the following asymptotically linear representation:

$$\tilde{\theta} - \theta_0 = N^{-1} \sum_{i=1}^N \tilde{\psi}_{\mu}(\mathbf{Z}_i) + O_p \left(\frac{c_{\mu,N}}{\sqrt{Na_N}} + \frac{1}{\sqrt{N}} \right) \text{ and } \tilde{V}_N(\mu) := \text{Var}\{\tilde{\psi}_{\mu}(\mathbf{Z})\} \asymp a_N^{-1},$$

where $\tilde{\psi}_{\mu}(\mathbf{Z}) := R/\pi_N(\mathbf{X})\{Y - \mu(\mathbf{X})\} - \mathbb{E}[R/\pi_N(\mathbf{X})\{Y - \mu(\mathbf{X})\}]$ and $\mathbb{E}\{\tilde{\psi}_{\mu}(\mathbf{Z})\} = 0$. Additionally, as long as Assumption 3.2 holds, we have:

$$(Na_N)^{1/2}(\tilde{\theta} - \theta_0) = O_p(1), \text{ and } N^{1/2}V_N^{-1/2}(\mu)(\tilde{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1).$$

Moreover, if $a_N \rightarrow 0$ as $N \rightarrow \infty$, then,

$$N^{1/2}\tilde{V}_N^{-1/2}(\mu)(\tilde{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1), \text{ and } \frac{V_N(\mu)}{\tilde{V}_N(\mu)} = 1 + O(a_N).$$

REMARK 3.2 (Discussion on condition (3.3)) As per Theorem 3.1, consistency and asymptotic normality of $\tilde{\theta}$ depend on (3.3), a condition that involves 1) the convergence rate of $\hat{m}(\cdot)$ towards some $\mu(\cdot)$, depending on the (expected) labeled sample size $(N\bar{\pi}_N)$, and 2) the tail of $\pi_N^{-1}(\mathbf{X})$, that is, how much of the mass of the distribution of $\pi_N(\mathbf{X})$ concentrates around zero. For a special case of $\pi_N(\mathbf{X}) \equiv \bar{\pi}_N$, MCAR, (3.3) is equivalent to $\|\hat{m}(\cdot; \mathbb{S}_{-k}) - \mu(\cdot)\|_{2, \mathbb{P}_{\mathbf{X}}} = o_p(1)$ coinciding with Zhang and Bradic [2022]. On the other hand, when $\pi_N(\cdot)$ follows the offset model (Example 2.1) with sub-Gaussian $f(\mathbf{X})$, we have $a_N \asymp \bar{\pi}_N$, and (3.3) holds once $\mathbb{E}_{\mathbf{X}}\{|\hat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})|^{2+\delta}\} = o_p(1)$ with $\delta > 0$.

REMARK 3.3 (Efficiency of $\tilde{\theta}$ and the choice of $\mu(\cdot)$) Although the choice of $\hat{m}(\cdot)$ is arbitrary as long as it converges to some $\mu(\cdot)$ as in (3.3), the efficiency of $\tilde{\theta}$ does depend on the limit $\mu(\cdot)$, and hence also on the choice of $\hat{m}(\cdot)$. For a simple case of $\hat{m}(\mathbf{x}) = \mu(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X}$, $\tilde{\theta}$ can be written as $\tilde{\theta} = N^{-1} \sum_{i=1}^N R_i Y_i / \pi_N(\mathbf{X}_i)$, which coincides with the IPW estimator, an estimator independent of $m(\cdot)$. However, an appropriate estimator $\hat{m}(\cdot)$ will provide a better efficiency for $\tilde{\theta}$. The optimal choice of $\mu(\cdot)$ that minimizes the asymptotic variance $V_N(\mu)$ is $\mu(\cdot) = m(\cdot)$ indicating that the outcome model is correctly specified.

REMARK 3.4 (Intuition behind the influence functions (IFs)) Two separate IFs $\psi_{\mu,\pi}(\mathbf{Z})$ and $\tilde{\psi}_{\mu}(\mathbf{Z})$ appear in the expansions of $\tilde{\theta}$ in Theorem 3.1. The first IF $\psi_{\mu,\pi}(\mathbf{Z})$ is an “accurate influence function” in that $\tilde{\theta} - \theta_0 = T_1 + o_p((Na_N)^{-1/2})$ with $T_1 = N^{-1} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) \asymp (Na_N)^{-1/2}$. Whenever $a_N \rightarrow 0$ and $N \rightarrow \infty$, the second IF $\tilde{\psi}_{\mu}(\mathbf{Z})$ captures the main contribution of $\psi_{\mu,\pi}(\mathbf{Z})$. It only involves the labeled samples and hence one can clearly see that the rate of $\tilde{\theta}$ is effectively determined by the smaller sized, labeled data only. When the outcome model is correctly specified, the second IF $\tilde{\psi}_{\mu}(\mathbf{Z})$ coincides with the efficient IF of Kallus and Mao [2020]; see Theorem 4.1 therein.

REMARK 3.5 (Convergence rate and “effective sample size”) Suppose the conditions in Theorem 3.1 hold, then $\tilde{\theta}$ is a $(Na_N)^{1/2}$ -consistent estimator for θ_0 . The value Na_N can be seen as an “effective sample size” having a similar role as the sample size in supervised learning. Below is a discussion on the value Na_N . By Jensen’s inequality, $Na_N \leq N\bar{\pi}_N$, where the difference between the two rates is related to the tail of $\pi_N^{-1}(\mathbf{X})$. Here, $N\bar{\pi}_N$ is the expected sample size as $N\bar{\pi}_N = \mathbb{E}(n)$, where $n := \sum_{i=1}^N R_i$. Therefore, the effective sample size, Na_N , depends on 1) how much of the mass of the distribution of $\pi_N(\mathbf{X})$ concentrates around 0 and 2) the (expected) size of the labeled sample. MCAR is a special case with $\pi_N(\cdot)$ being a constant and therefore $Na_N = N\bar{\pi}_N$. In another example, the offset based model in Example 2.1 and Section 4.1, we have $a_N \asymp \bar{\pi}_N$ for sub-Gaussian \mathbf{X} ; see Theorems 4.1 and 4.2.

3.2 Unknown PS $\pi_N(\cdot)$ and the general version of the DRSS estimator

With $\pi_N(\cdot)$ being unknown in general observational studies, we propose our final estimator, a *doubly robust semi-supervised* (DRSS) estimator of the mean θ_0 , given by:

$$\hat{\theta}_{\text{DRSS}} := N^{-1} \sum_{i=1}^N \hat{m}(\mathbf{X}_i) + N^{-1} \sum_{i=1}^N \frac{R_i}{\hat{\pi}_N(\mathbf{X}_i)} \{Y_i - \hat{m}(\mathbf{X}_i)\}, \quad (3.4)$$

where $\hat{\pi}_N(\mathbf{X}_i)$ is a cross-fitted estimator of $\pi_N(\mathbf{X}_i)$ constructed similarly as $\hat{m}(\mathbf{X}_i)$, as discussed below (3.1) in Section 3.1. The proposed estimator (3.4) is a plug-in version of (3.1). We denote with $e_N(\cdot)$ a “limit” of $\hat{\pi}_N(\cdot)$, which is possibly *misspecified*, i.e., $e_N(\cdot)$ is not necessarily the same as $\pi_N(\cdot)$. Define the following generalization of (3.2), i.e., a DR score (influence) function:

$$\psi_{\mu,e}(\mathbf{Z}) := \mu(\mathbf{X}) + \frac{R}{e_N(\mathbf{X})} \{Y - \mu(\mathbf{X})\} - \theta_0 = Y - \theta_0 + \left\{ \frac{R}{e_N(\mathbf{X})} - 1 \right\} \{Y - \mu(\mathbf{X})\}. \quad (3.5)$$

We have the following asymptotic results under the two cases: (a) both $\pi_N(\cdot)$ and $m(\cdot)$ are correctly specified; (b) one of $\pi_N(\cdot)$ and $m(\cdot)$ is correctly specified.

THEOREM 3.2 Let Assumptions 2.1 and 3.1 hold and let $Na_N \rightarrow \infty$, as $N \rightarrow \infty$. Suppose the cross-fitted versions of $\hat{m}(\cdot)$ and $\hat{\pi}_N(\cdot)$ have well-defined (possibly misspecified) limits $\mu(\cdot)$ and $e_N(\cdot)$, respectively, such that (3.3) holds for $k \leq \mathbb{K}$ as well as

$$\mathbb{E}_{\mathbf{X}} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{e_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\}^2 \right] = O_p(c_{e,N}^2) \text{ with sequence } c_{e,N} = o(1), \quad (3.6)$$

$$\mathbb{E}_{\mathbf{X}} \{ \hat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X}) \}^2 = O_p(r_{\mu,N}^2) \text{ with sequence } r_{\mu,N} = o(1), \text{ and} \quad (3.7)$$

$$\mathbb{E}_{\mathbf{X}} \left\{ 1 - \frac{e_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\}^2 = O_p(r_{e,N}^2) \text{ with sequence } r_{e,N} = o(1). \quad (3.8)$$

The properties of $\hat{\theta}_{\text{DRSS}}$ under different cases are as follows:

(a) Suppose both $\mu(\cdot) = m(\cdot)$ and $e_N(\cdot) = \pi_N(\cdot)$ hold. Then, as $N \rightarrow \infty$, $\hat{\theta}_{\text{DRSS}}$ satisfies the following asymptotic linear expansion:

$$\hat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + O_p \left(\frac{c_{\mu,N}}{\sqrt{Na_N}} + \frac{c_{e,N}}{\sqrt{Na_N}} + r_{\mu,N} r_{e,N} \right),$$

and $V_N(\mu, e) \asymp a_N^{-1}$, where $V_N(\mu, e) := \text{Var}\{\psi_{\mu,e}(\mathbf{Z})\}$. Hence, as long as the product rate $r_{\mu,N} r_{e,N}$ from (3.7) and (3.8) further satisfies $r_{\mu,N} r_{e,N} = o(1/\sqrt{Na_N})$, and Assumption 3.2 holds, we have:

$$(Na_N)^{1/2} (\hat{\theta}_{\text{DRSS}} - \theta_0) = O_p(1), \text{ and } N^{1/2} V_N^{-1/2}(\mu, e) (\hat{\theta}_{\text{DRSS}} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1). \quad (3.9)$$

(b) Suppose now that either $\mu(\cdot) = m(\cdot)$ or $e_N(\cdot) = \pi_N(\cdot)$ holds. Moreover, if $e_N(\cdot) \neq \pi_N(\cdot)$, we assume $c \leq \pi_N(\mathbf{X})/e_N(\mathbf{X}) \leq C$ a.s. for some constants $c, C > 0$. Then, as $N \rightarrow \infty$, $\hat{\theta}_{\text{DRSS}}$ satisfies the following asymptotic linear expansion:

$$\hat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + O_p \left(\frac{c_{\mu,N}}{\sqrt{Na_N}} + \frac{c_{e,N}}{\sqrt{Na_N}} + r_{\mu,N} r_{e,N} \right) + \hat{\Delta}_N,$$

with $\hat{\Delta}_N$ satisfying (3.10) or (3.11):

$$\hat{\Delta}_N := N^{-1} \sum_{i=1}^N \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - \frac{R_i}{\hat{\pi}_N(\mathbf{X}_i)} \right\} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} \quad \text{if } e_N(\cdot) = \pi_N(\cdot), \quad (3.10)$$

$$\hat{\Delta}_N := N^{-1} \sum_{i=1}^N \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{ \hat{m}(\mathbf{X}_i) - m(\mathbf{X}_i) \} \quad \text{if } \mu(\cdot) = m(\cdot). \quad (3.11)$$

Suppose for case (3.10), $\|m(\cdot) - \mu(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} < C$, while for case (3.11), $\|1 - \pi_N(\cdot)/e_N(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} < C$, with a constant $C < \infty$. Then, $\hat{\theta}_{\text{DRSS}}$ satisfies:

$$\hat{\theta}_{\text{DRSS}} - \theta_0 = O_p \left(\frac{1 + c_{\mu,N} + c_{e,N}}{\sqrt{Na_N}} + r_{\mu,N} r_{e,N} + r_{e,N} \mathbb{1}\{\mu(\cdot) \neq m(\cdot)\} + r_{\mu,N} \mathbb{1}\{e_N(\cdot) \neq \pi_N(\cdot)\} \right).$$

A few remarks pertaining to the estimation rates are presented next.

REMARK 3.6 (Conditions in Theorem 3.2) Here we discuss the rate conditions (3.3), (3.6), (3.7), and (3.8) required in Theorem 3.2. The rate (3.7) is a standard estimation error of the outcome model; see for example, Zhang et al. [2019]. The other rates, (3.3), (3.6), and (3.8), are rescaled or self-normalized versions of conditions in Chernozhukov et al. [2018]. They are needed as the price of violating the positivity condition. The rate (3.8), a rescaled version of the usually considered $\mathbb{E}_{\mathbf{X}}\{\hat{\pi}_N(\mathbf{X}) - e_N(\mathbf{X})\}^2$, is a change needed to properly address a decaying PS estimator. Then, (3.3) and (3.6) can be seen as self-normalized versions with the normalization factor being $\omega(\mathbf{X}) := a_N/\pi_N(\mathbf{X})$. Notice that $\mathbb{E}\{\omega(\mathbf{X})\} = 1$, so these weights $\omega(\cdot)$ can be viewed as reweighing or redistribution factor. Then, the estimation errors of $\hat{\pi}_N(\mathbf{X})$ and $\hat{m}(\mathbf{X})$ at \mathbf{X} , with a smaller PS, contribute more to rates (3.7) and (3.8). The rates of the reweighted versions, $c_{\mu,N}$ and $c_{e,N}$ in (3.3) and (3.6), only need to be $o(1)$; whereas $r_{\mu,N}$ and $r_{e,N}$ in (3.7) and (3.8) appear in the final rate for $\hat{\theta}_{\text{DRSS}}$. In high dimensions, assume $\pi_N(\cdot)$ follows an offset based model as in Example 2.1. Suppose $m(\cdot)$ and $f(\cdot)$ in Example 2.1 are linear with sparsity levels s_m

and s_π , respectively. Then, for sub-Gaussian \mathbf{X} , we demonstrate in Theorem 4.2 that $a_N \asymp \bar{\pi}_N$ as long as $r_{e,N} = \sqrt{s_\pi \log(p)/(N\bar{\pi}_N)}$ and $r_{\mu,N} = \sqrt{s_m \log(p)/(N\bar{\pi}_N)}$, therefore coming close to the simplest missingness pattern, that of MCAR.

REMARK 3.7 (Double robustness, rates and efficiency) Here, we discuss the double robustness and the efficiency of the proposed estimator. Whenever $\pi_N(\cdot)$ and $m(\cdot)$ are correctly specified, the asymptotic normality with a rate of consistency $(Na_N)^{-1/2}$ is guaranteed if a product rate condition $r_{\mu,N}r_{e,N} = o(1/\sqrt{Na_N})$ is satisfied. We can see that our product rate condition is an analog of the usual product rate condition in the literature [Chernozhukov et al., 2018], if the sample size is replaced with Na_N , the “effective sample size” in our case; see Remark 3.5. In addition, when the asymptotic normality occurs, our estimator reaches the semi-parametric efficiency bound proposed in Kallus and Mao [2020] when $\bar{\pi}_N \rightarrow 0$ as $N \rightarrow \infty$. When one of $\pi_N(\cdot)$ and $m(\cdot)$ is misspecified, we obtain a consistency rate of $O_p(r_{e,N})$ if $\pi_N(\cdot)$ is correctly specified, whereas the rate is $O_p(r_{\mu,N})$ if $m(\cdot)$ is correctly specified. Therefore, the consistency rate of $\hat{\theta}_{DRSS}$ directly depends on the estimation error rate of the correct model. As a special case, $\hat{\theta}_{DRSS}$ is consistent as long as the correct model is consistently estimated. Additionally, we can see that $\hat{\theta}_{DRSS}$ can still be $(Na_N)^{1/2}$ -consistent as long as the correct model is estimated with an error rate $O_p(Na_N)^{-1/2}$, which is reachable in low dimensions. For instance, for a (correctly specified) low-dimensional offset logistic PS model as introduced in Section 4.1.1, as shown in Theorem 4.1, not only do we reach the error rate $O_p(Na_N)^{-1/2}$ but are able to construct a RAL expansion for $\hat{\theta}_{DRSS}$.

REMARK 3.8 (Unbounded support for \mathbf{X}) We do not enforce a bounded support for \mathbf{X} , which is typically an assumption assumed (implicitly) in missing data and causal inference literature. For instance, suppose $\pi_N(\cdot)$ follows an (offset based) logistic model as in Example 2.1. Both the usual positivity condition $\mathbb{P}(\pi_N(\mathbf{X}) > c > 0) = 1$ in the standard missing data literature [Imbens, 2004, Imbens and Rubin, 2015, Tsiatis, 2007] and the uniform bounded density ratio condition, $\bar{\pi}_N/\pi_N(\mathbf{X}) < C$, in Kallus and Mao [2020], which tackles a MAR-SS setting, essentially require a compact support for \mathbf{X} . However, our results only require a sub-Gaussian \mathbf{X} as in Theorems 4.1 and 4.2.

REMARK 3.9 (Asymptotic linearity and $(Na_N)^{1/2}$ -consistency under misspecification) Moreover, in Section 4, we demonstrate that $\hat{\theta}_{DRSS}$ can still be asymptotically normal even if $m(\cdot)$ is misspecified. Such an asymptotic normality is constructed based on a careful analysis to obtain the regular and asymptotically linear (RAL) expansion and the IF for the additional error term $\hat{\Delta}_N$ in (3.10), in that

$$\hat{\Delta}_N = N^{-1} \sum_{j=1}^N \text{IF}_\pi(\mathbf{Z}_j) + o_p\left((Na_N)^{-1/2}\right),$$

for some $\text{IF}_\pi(\cdot)$ with $\mathbb{E}\{\text{IF}_\pi(\mathbf{Z})\} = 0$ and $\mathbb{E}\{\text{IF}_\pi^2(\mathbf{Z})\} \asymp a_N^{-1}$. The final IF of $\hat{\theta}_{DRSS}$ involves the extra IF contributed from the estimation error of $\hat{\pi}_N(\cdot)$. Consequently, the RAL expansion and the asymptotic normality of $\hat{\theta}_{DRSS}$ are also affected accordingly. Using the above expansion for $\hat{\Delta}_N$ and the general expansion of $\hat{\theta}_{DRSS}$ from Theorem 3.2, we have a RAL expansion of $\hat{\theta}_{DRSS}$ as:

$$\begin{aligned} \hat{\theta}_{DRSS} - \theta_0 &= N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + O_p\left(\frac{c_{\mu,N}}{\sqrt{Na_N}} + \frac{c_{e,N}}{\sqrt{Na_N}} + r_{\mu,N}r_{e,N}\right) + \hat{\Delta}_N \\ &= N^{-1} \sum_{i=1}^N \{\psi_{\mu,e}(\mathbf{Z}_i) + \text{IF}_\pi(\mathbf{Z}_i)\} + o_p\left((Na_N)^{-1/2}\right). \end{aligned}$$

The function $\Psi(\mathbf{Z}) := \psi_{\mu,e}(\mathbf{Z}) + \text{IF}_\pi(\mathbf{Z})$ is the final *adjusted* IF of $\hat{\theta}_{DRSS}$ with $\mathbb{E}\{\Psi(\mathbf{Z})\} = 0$ and

$\text{Var}\{\Psi(\mathbf{Z})\} \asymp a_N^{-1}$. Consequently, we also have:

$$N^{1/2}[\text{Var}\{\Psi(\mathbf{Z})\}]^{-1/2}(\hat{\theta}_{\text{DRSS}} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1).$$

REMARK 3.10 (Estimation when both models are misspecified) In this remark, we briefly discuss a few aspects of the proposed estimator's behavior when both nuisance models are misspecified.

Firstly, as one of the anonymous reviewers pointed out, we do not need at least one nuisance model to be correctly specified; in fact, the identification of the mean parameter only requires

$$\mathbb{E} \left[\left\{ 1 - \frac{\pi_N(\mathbf{X})}{e_N(\mathbf{X})} \right\} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] = 0. \quad (3.12)$$

For instance, (3.12) holds when $\mu(\mathbf{x}) = m(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}_1$ and $e_N(\mathbf{x}) = \pi_N(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}_2$, for some \mathcal{X}_1 and \mathcal{X}_2 with $\mathcal{X}_1 \cup \mathcal{X}_2 = \mathcal{X}$. This is indeed weaker than requiring one of $\pi_N(\cdot)$ and $m(\cdot)$ to be correctly specified, and, of course, there are also other ways possible to achieve the weaker condition (3.12). However, for the sake of interpretation, we consider the simpler and sufficient (but not necessary) condition that either $\mu(\cdot) = m(\cdot)$ or $e_N(\cdot) = \pi_N(\cdot)$ throughout the paper; this is not unlike the missing data or causal inference literature, e.g., Bang and Robins [2005], Chernozhukov et al. [2018], Farrell [2015], Robins and Rotnitzky [1995], Rubin [1974], Tan [2020], Tsiatis [2007].

Secondly, if (3.12) does not hold and both models have non-ignorable misspecification errors with $\mu(\cdot) - m(\cdot) \asymp 1$ and $1 - \pi_N(\cdot)/e_N(\cdot) \asymp 1$, we do not expect any consistent estimate for θ_0 to exist in general. On the other hand, if at least one model is misspecified but with a *decaying* misspecification error, we can still obtain consistent estimates. For instance, consider a “weakly misspecified” linear outcome model with

$$m(\mathbf{X}) = \mathbf{X}^T \boldsymbol{\beta}_0 + \xi(\mathbf{X}),$$

where $\xi(\mathbf{X}) = \xi_N(\mathbf{X})$ is the misspecification error whose distribution is dependent on N . The case of misspecification of the PS model follows analogously. If, for the sake of simplicity, we assume that $\mathbb{E}\{\xi(\mathbf{X})\} = 0$ and $\mathbb{E}\{\mathbf{X}\xi(\mathbf{X})\} = \mathbf{0}$, then whenever the linear model is approximately correct with $\mathbb{E}\{\xi^2(\mathbf{X})\} = o(1)$ as $N \rightarrow \infty$, the proposed DRSS mean estimator is still consistent; see, e.g., Belloni et al. [2012; 2014a;b], where the authors also only require the true nuisance functions to be close enough to the linear approximations. Moreover, another indirect situation with “weakly misspecified” models is when the nuisance function is “weakly sparse”; see, e.g., Section 4.3 of Negahban et al. [2012] and Example 10 of Smucler et al. [2019]. In general, let $\mathbb{E}\{1 - \pi_N(\mathbf{X})/e_N(\mathbf{X})\}^2 \asymp \bar{r}_{e,N}^2 = O(1)$ and $\mathbb{E}\{\mu(\mathbf{X}) - m(\mathbf{X})\}^2 \asymp \bar{r}_{\mu,N}^2 = O(1)$. The additional bias of our DRSS mean estimator originating from model misspecification is

$$\text{BiasDR} := \mathbb{E}\{\psi_{\mu,e}(\mathbf{Z})\} - \theta_0 = \mathbb{E}\{[1 - \pi_N(\mathbf{X})/e_N(\mathbf{X})]\{\mu(\mathbf{X}) - m(\mathbf{X})\}\} = O(\bar{r}_{e,N}\bar{r}_{\mu,N}) = o(1),$$

as long as either $\bar{r}_{e,N} = o(1)$ or $\bar{r}_{\mu,N} = o(1)$. For comparison, if we consider the corresponding regression-based estimator $\hat{\theta}_{\text{Reg}}$ or the IPW estimator $\hat{\theta}_{\text{IPW}}$ (see definitions in Section 2), the additional biases of these estimators from model misspecification are, respectively,

$$\text{Bias}_{\text{Reg}} := \mathbb{E}\{\mu(\mathbf{X})\} - \theta_0 = O(\bar{r}_{\mu,N}), \quad \text{Bias}_{\text{IPW}} := \mathbb{E}\{\pi_N(\mathbf{X})m(\mathbf{X})/e_N(\mathbf{X})\} - \theta_0 = O(\bar{r}_{e,N}).$$

That is, $\hat{\theta}_{\text{Reg}}$ is consistent if $\bar{r}_{\mu,N} = o(1)$; $\hat{\theta}_{\text{IPW}}$ is consistent if $\bar{r}_{e,N} = o(1)$. We can see $\hat{\theta}_{\text{DRSS}}$ is more robust than $\hat{\theta}_{\text{Reg}}$ and $\hat{\theta}_{\text{IPW}}$ as we allow both $\bar{r}_{e,N} = o(1)$ and $\bar{r}_{\mu,N} = o(1)$. In addition, $\hat{\theta}_{\text{DRSS}}$ also has a faster consistency rate when both misspecification errors shrink; see empirical comparisons of the mean estimators in Section 6.1.4.

3.3 Asymptotic variance estimation

In this section, we consider the estimation of the asymptotic variances $V_N(\mu)$ in Theorem 3.1 (with $\pi_N(\cdot)$ known) and $V_N(\mu, e)$ in Theorem 3.2 (with $\pi_N(\cdot)$ unknown and both $m(\cdot)$ and $\pi_N(\cdot)$ are correctly specified). These facilitate inference on θ_0 (via confidence intervals, hypothesis tests etc.) using $\tilde{\theta}$ and $\hat{\theta}_{\text{DRSS}}$. We assume the following tail condition.

ASSUMPTION 3.3 (Tail condition) With $N \rightarrow \infty$, for a constant $\delta > 0$, let

$$N^{-\delta/2} a_N^{1+\delta/2} \mathbb{E}\{|\psi_{\mu, \pi}(\mathbf{Z})|^{2+\delta}\} \rightarrow 0.$$

The Assumption 3.3 is a sufficient condition for Assumption 3.2. Under the setting in Theorem 3.1 and part (a) of Theorem 3.2, we have:

$$N^{1/2} V_N^{-1/2}(\mu)(\tilde{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1), \quad N^{1/2} V_N^{-1/2}(\mu, e)(\hat{\theta}_{\text{DRSS}} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1).$$

We propose the following plug-in estimates, $\hat{V}_N(\mu) = \hat{V}_N(\hat{m}, \bar{\pi}_N, \tilde{\theta})$ and $\hat{V}_N(\mu, e) = \hat{V}_N(\hat{m}, \hat{\pi}_N, \hat{\theta}_{\text{DRSS}})$, where

$$\hat{V}_N(a, b, c) := N^{-1} \sum_{i=1}^N \left[a(\mathbf{X}_i) - c + \frac{R_i}{b(\mathbf{X}_i)} \{Y_i - a(\mathbf{X}_i)\} \right]^2.$$

THEOREM 3.3 (a) Let Assumptions in Theorem 3.1 hold. Then, as $N \rightarrow \infty$, $\hat{V}_N(\mu) = V_N(\mu)\{1 + o_p(1)\}$.
(b) Let Assumptions (a) of Theorem 3.2 hold. Further let Assumption 3.3 hold and

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\}^2 \{\hat{m}(\mathbf{X}; \mathbb{S}_{-k}) - m(\mathbf{X})\}^2 \right] = o_p(1). \quad (3.13)$$

Then, as $N \rightarrow \infty$, $\hat{V}_N(\mu, e) = V_N(\mu, e)\{1 + o_p(1)\}$.

Notice that we only require a $o_p(1)$ condition in (3.13). Such a condition can be satisfied as long as we have upper bounds for the $(2+c)$ -th moment of the estimation errors and the tail of $\pi_N^{-1}(\mathbf{X})$ is well-behaved. Under a standard positivity condition, when $\mu(\cdot) = m(\cdot)$, (3.13) only requires $r_{\mu, N} = o(1)$, which would have been already assumed for consistency.

Under the conditions in Theorem 3.3, asymptotically valid $100(1 - \alpha)\%$ confidence intervals (CIs) for $\tilde{\theta}$ and $\hat{\theta}_{\text{DRSS}}$ at any significance level α can now be obtained as:

$$\begin{aligned} \text{CI}(\tilde{\theta}) &:= \left(\tilde{\theta} - N^{-1/2} \hat{V}_N^{1/2}(\mu) z_{1-\alpha/2}, \tilde{\theta} + N^{-1/2} \hat{V}_N^{1/2}(\mu) z_{1-\alpha/2} \right), \\ \text{CI}(\hat{\theta}_{\text{DRSS}}) &:= \left(\hat{\theta}_{\text{DRSS}} - N^{-1/2} \hat{V}_N^{1/2}(\mu, e) z_{1-\alpha/2}, \hat{\theta}_{\text{DRSS}} + N^{-1/2} \hat{V}_N^{1/2}(\mu, e) z_{1-\alpha/2} \right), \end{aligned} \quad (3.14)$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ -quantile of a standard normal distribution. As shown in Theorems 3.1 and 3.2, $V_N(\mu) \asymp a_N^{-1}$ and $V_N(\mu, e) \asymp a_N^{-1}$. Hence, the length of the proposed confidence intervals are of the order $(Na_N)^{-1/2}$.

It is important to note, that these confidence intervals are valid when both the outcome and propensity score models are correctly specified. Whenever the outcome model is misspecified, they need further adjustment based on an adjusted RAL expansion as discussed in Remark 3.9. Based on the adjusted IF, $\Psi(\mathbf{Z})$, therein, one can estimate the asymptotic variance $\text{Var}\{\Psi(\mathbf{Z})\}$ using a plug-in estimate $N^{-1} \sum_{i=1}^N \hat{\Psi}^2(\mathbf{Z}_i; \hat{\theta}_{\text{DRSS}})$, where $\hat{\Psi}(\cdot; \hat{\theta}_{\text{DRSS}})$ is a consistent estimator of $\Psi(\cdot)$, and obtain the corresponding *adjusted* confidence intervals. We also illustrate the numerical performance of these adjusted confidence intervals in Appendix C of the Supplement.

4. Decaying PS models

In Section 3, we proposed a DR estimator $\hat{\theta}_{\text{DRSS}}$ of $\theta_0 = E(Y)$. Such an estimator is based on an outcome estimator $\hat{m}(\cdot)$ and a PS estimator $\hat{\pi}_N(\cdot)$. Due to the decaying nature of the PS, the estimation of $\pi_N(\cdot)$ itself is also an interesting and challenging problem. In this section, we illustrate three decaying PS models: (i) an *offset logistic model* (Section 4.1), (ii) a *stratified labeling model* (Section 4.2), and (iii) a *MCAR labeling model* (presented in Appendix B of the [Supplement](#) in the interest of space). These are just some natural examples of modeling a decaying PS – our main results are completely general.

We propose PS estimators under each of the three models and establish detailed asymptotic results, especially for the offset logistic model (in both low and high dimensions). Moreover, as discussed in Remark 3.9, for a misspecified $m(\cdot)$, based on a case by case study of $\pi_N(\cdot)$, we further construct an adjusted RAL expansion of $\hat{\theta}_{\text{DRSS}}$ and hence provide an asymptotic normality with an adjusted asymptotic variance.

We begin with a brief remark here that addresses this particular point for the MCAR model, though all other results and detailed discussions for the MCAR setting are deferred to Appendix B of the [Supplement](#) for brevity.

REMARK 4.1 (Adjusted IF and general RAL expansion under MCAR) For a simple case of a MCAR labeling mechanism, i.e., $\pi_N(\mathbf{X}) \equiv \bar{\pi}_N$, we establish a similar result as in [Zhang and Bradic \[2022\]](#) under fairly general conditions. In Theorem A.1 of the [Supplement](#), we provide an adjusted (and general) RAL expansion of $\hat{\theta}_{\text{DRSS}}$ under the MCAR setting, allowing for misspecification of $\hat{m}(\cdot)$. The adjusted IF takes the form of $\Psi(\mathbf{Z}) := \psi_{\mu,\pi}(\mathbf{Z}) + \text{IF}_\pi(\mathbf{Z})$, where $\psi_{\mu,\pi}(\mathbf{Z})$ is as defined in (3.2) and

$$\text{IF}_\pi(\mathbf{Z}) := \left(\frac{R - \bar{\pi}_N}{\bar{\pi}_N} \right) \Delta_\mu, \quad \text{with } \Delta_\mu := \mathbb{E}\{\mu(\mathbf{X}) - m(\mathbf{X})\},$$

and a fast consistency rate of $(Na_N)^{-1/2}$ for $\hat{\theta}_{\text{DRSS}}$ is achievable. We further show that

$$\text{Var}\{\psi_{\mu,\pi}(\mathbf{Z})\} = \text{Var}\{\Psi(\mathbf{Z})\} + (\bar{\pi}_N^{-1} - 1)\Delta_\mu^2 \geq \text{Var}\{\Psi(\mathbf{Z})\};$$

see Appendix B in the [Supplement](#) for more details and formal statements. Recall that, as shown in Theorem 3.1, $\psi_{\mu,\pi}(\mathbf{Z})$ is the IF of $\tilde{\theta}$, the mean estimator constructed based on the *true* PS $\bar{\pi}_N$. Meanwhile, under the MCAR setting, $\Psi(\mathbf{Z})$ is the IF of the DRSS mean estimator $\hat{\theta}_{\text{DRSS}}$, which is based on an *estimated* constant PS; see Theorem A.1 of the [Supplement](#). Hence, $\text{Var}\{\psi_{\mu,\pi}(\mathbf{Z})\}$ and $\text{Var}\{\Psi(\mathbf{Z})\}$ are the asymptotic variances of $\tilde{\theta}$ and $\hat{\theta}_{\text{DRSS}}$, respectively. From the above inequality, we can thus conclude that $\hat{\theta}_{\text{DRSS}}$ is asymptotically more efficient than $\tilde{\theta}$. This therefore suggests that, under the MCAR setting, *even if* $\bar{\pi}_N$ is known, it is *still* worth estimating $\bar{\pi}_N$ instead of directly plugging in the true value $\bar{\pi}_N$, as long as $\Delta_\mu \neq 0$. Formal results of the proposed DRSS mean estimator under the MCAR setting, along with additional connections with the semi-supervised estimators of [Zhang et al. \[2019\]](#) and [Zhang and Bradic \[2022\]](#) can be found in Appendix B of the [Supplement](#).

4.1 Offset logistic regression

In this section, we propose a parametric logistic model for extremely unbalanced outcomes, i.e., $\bar{\pi}_N = \mathbb{P}(R=1) \rightarrow 0$ as $N \rightarrow \infty$, where we consider a PS model (assumed correctly specified) with the form:

$$\pi_N(\mathbf{X}) = \bar{\pi}_N \frac{\exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)}{1 + \bar{\pi}_N \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)} = \frac{\exp\{\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)\}}{1 + \exp\{\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)\}}, \quad (4.1)$$

where $\vec{\mathbf{X}} := (1, \mathbf{X}^T)^T$ and the parameter $\boldsymbol{\gamma}_0 \in \mathbb{R}^{p+1}$ possibly depends on N with $\|\boldsymbol{\gamma}_0\|_2 < C$ for some constant $C > 0$. This model is fairly natural and allows for a general way to incorporate the decaying nature of the labeling fraction. At the same time, it ensures that the dependence of $\pi_N(\mathbf{X})$ on \mathbf{X} is not distorted by the decaying nature of $\bar{\pi}_N$. Model (4.1) could also be viewed as a logistic model with $\log(\bar{\pi}_N)$ (a diverging negative intercept) as an *offset*. If a standard logistic model is used [Owen, 2007, Wang, 2020] instead, i.e., we let

$$\pi_N(\mathbf{X}) = g(\vec{\mathbf{X}}^T \boldsymbol{\beta}), \quad \text{where } g(u) := \frac{\exp(u)}{1 + \exp(u)}, \quad (4.2)$$

then under some standard conditions whenever an extreme imbalance exists, $\exp(-\boldsymbol{\beta}(1)) \asymp \bar{\pi}_N^{-1} \rightarrow \infty$ whenever $N \rightarrow \infty$; see Appendix A and Remark A.1 in the [Supplement](#) for further details. This provides a clear justification for our offset model (4.1) where we precisely extract out $\log(\bar{\pi}_N)$ as an offset to be estimated separately and plugged in apriori to the likelihood equation. In this way, we are able to treat the auxiliary intercept and the slope as well-behaved, i.e., finite and independent or bounded in N .

REMARK 4.2 (Connections with density ratio estimation) There is an intricate connection between the offset model (4.1) and a model for density ratios usually used in the covariate shift literature where R_i s are treated as fixed (or conditioned on) and $\mathbb{P}_{\mathbf{X}} \neq \mathbb{P}_{\mathbf{X}|R=1}$ is allowed [Kawakita and Kanamori, 2013, Liu et al., 2020]. Observe that

$$\text{logit}\{\pi_N(\mathbf{X})\} = \log(\bar{\pi}_N) - \log(1 - \bar{\pi}_N) - \log\{\Lambda_N(\mathbf{X})\},$$

where $\Lambda_N(\mathbf{X}) := f(\mathbf{X}|R=0)/f(\mathbf{X}|R=1)$ and $f(\cdot|R=\cdot)$ is the conditional density of \mathbf{X} given R . However, direct estimation of density ratios is often arduous. The above representation, however, suggests that the *same* model can be fitted by a simple logistic regression of $R|\mathbf{X}$, and further using $\log\{\bar{\pi}_N/(1 - \bar{\pi}_N)\}$ as an *offset*. Therefore, missing data literature related to density ratios can now be enriched with an effective estimation of the decaying PS; see Section 4 of Kallus and Mao [2020] where semi-parametric efficiency is established but no estimator is discussed. In some sense, such a density ratio estimator is also optimal [Qin, 1998]. For more discussions on these connections, see Remark A.2 in the [Supplement](#).

4.1.1 Low-dimensional offset logistic regression. We first consider a low-dimensional setting where p is fixed. We propose a PS estimator $\hat{\pi}_N(\cdot)$ for the offset model (4.1), based on the full sample \mathbb{S} and use its cross-fitted version (based on a subsample \mathbb{S}_{-k}) to construct the DR mean estimator $\hat{\theta}_{\text{DRSS}}$.

We construct $\hat{\pi}_N(\cdot)$ based on an apriori chosen estimate $\hat{\pi}_N := N^{-1} \sum_{i=1}^N R_i$. Let $\hat{\boldsymbol{\gamma}}$ be the minimizer of $\ell_N(\boldsymbol{\gamma}; \hat{\pi}_N)$, where

$$\ell_N(\boldsymbol{\gamma}; a) := -N^{-1} \sum_{i=1}^N \left[R_i \vec{\mathbf{X}}_i^T \boldsymbol{\gamma} - \log\{1 + a \exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma})\} \right], \quad (4.3)$$

where recall that $\vec{\mathbf{X}} = (1, \mathbf{X})^T$. Then, the PS estimate, $\hat{\pi}_N(\cdot)$, can be obtained by plugging $\hat{\pi}_N$ into (4.1), as follows:

$$\hat{\pi}_N(\mathbf{X}) := \frac{\hat{\pi}_N \exp(\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}})}{1 + \hat{\pi}_N \exp(\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}})}. \quad (4.4)$$

Here, for any $a \in (0, 1]$, $\ell_N(\boldsymbol{\gamma}; a)$ is the negative log-likelihood under the offset based model, up to a term $-N^{-1} \sum_{i=1}^N R_i \log(a)$ that is independent of $\boldsymbol{\gamma}$. Existence and uniqueness of $\hat{\boldsymbol{\gamma}}$ is discussed in detail in

Remark A.4 in the [Supplement](#). But, it is worth mentioning that the results of [Owen \[2007\]](#) showcasing the existence of the MLE for the model (4.2) can be extended to guarantee the existence of $\hat{\gamma}$ as well.

The following theorem provides asymptotic results for $\hat{\gamma}$ and $\hat{\pi}_N(\cdot)$, as well as an adjusted RAL expansion of the DRSS estimator $\hat{\theta}_{\text{DRSS}}$ in low-dimensional setting with p being fixed and when $m(\cdot)$ is possibly misspecified. For this result alone we consider the following conditions on the design: $\mathbb{E}\{\exp(t\|\mathbf{X}\|_2)\} < \infty$ for any $t > 0$, $\lambda_{\min}[\mathbb{E}\{\vec{\mathbf{X}}\vec{\mathbf{X}}^T \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\}] > 0$, where $g(\cdot)$ was defined in (4.2) and $\dot{g}(\cdot) = g(\cdot)\{1 - g(\cdot)\}$ is the derivative of $g(\cdot)$.

THEOREM 4.1 Let $N\bar{\pi}_N \rightarrow \infty$ as $N \rightarrow \infty$, and $\|\boldsymbol{\gamma}_0\|_2 < C < \infty$ where $\boldsymbol{\gamma}_0$ was defined in (4.1). Suppose that $\|\mathbb{E}\{\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\vec{\mathbf{X}}\vec{\mathbf{X}}^T\}^{-1}\|_2 < C$ with some constant $C > 0$. Then, as $N \rightarrow \infty$,

$$\begin{aligned}\hat{\gamma} - \boldsymbol{\gamma}_0 &= N^{-1} \sum_{i=1}^N \text{IF}_{\boldsymbol{\gamma}}(\mathbf{Z}_i) + \mathbf{R}_N, \quad \text{with } \|\mathbf{R}_N\|_2 = o_p((N\bar{\pi}_N)^{-1/2}), \\ \text{IF}_{\boldsymbol{\gamma}}(\mathbf{Z}) &:= \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\{R_i - g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\}\vec{\mathbf{X}} - (\bar{\pi}_N^{-1}R - 1)\mathbf{e}_1,\end{aligned}$$

where $\mathbf{e}_1 := (1, 0, \dots, 0)^T \in \mathbb{R}^{p+1}$, $\mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) := \mathbb{E}\{\vec{\mathbf{X}}\vec{\mathbf{X}}^T \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\}$, and $\|\hat{\gamma} - \boldsymbol{\gamma}_0\|_2 = O_p((N\bar{\pi}_N)^{-1/2})$. Further, we also have the following error rates:

$$\begin{aligned}\|\pi_N^{-1}(\mathbf{X})\|_{r, \mathbb{P}_{\mathbf{X}}} &\asymp \bar{\pi}_N^{-1} \quad \forall r > 0, \quad \text{and hence } a_N \asymp \bar{\pi}_N, \\ \left\|1 - \frac{\pi_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X})}\right\|_{2, \mathbb{P}_{\mathbf{X}}} &= O_p((N\bar{\pi}_N)^{-1/2}),\end{aligned}\tag{4.5}$$

$$\mathbb{E}_{\mathbf{X}}\left[\frac{a_N}{\pi_N(\mathbf{X})}\left\{1 - \frac{\pi_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X})}\right\}^2\right] = O_p((N\bar{\pi}_N)^{-1}) = o_p(1).\tag{4.6}$$

If we further assume that $\|m(\cdot) - \mu(\cdot)\|_{2+c, \mathbb{P}_{\mathbf{X}}} < \infty$, then we have a RAL expansion of the term $\hat{\Delta}_N$ defined in (3.10) as follows:

$$\hat{\Delta}_N := N^{-1} \sum_{i=1}^N \text{IF}_{\pi}(\mathbf{Z}_i) + o_p((N\bar{\pi}_N)^{-1/2}), \quad \text{where}\tag{4.7}$$

$$\text{IF}_{\pi}(\mathbf{Z}) := \mathbb{E}\left[\{1 - \pi_N(\mathbf{X})\}\{\mu(\mathbf{X}) - m(\mathbf{X})\}\vec{\mathbf{X}}^T\right]\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\vec{\mathbf{X}}\{R - \pi_N(\mathbf{X})\}.\tag{4.8}$$

Moreover, if we assume $\|\hat{m}(\cdot) - \mu(\cdot)\|_{2+c, \mathbb{P}_{\mathbf{X}}} = o_p(1)$ (we suppressed the dependency of $\hat{m}(\cdot)$ on k as in Theorem 3.2), then we have the following rate:

$$\mathbb{E}_{\mathbf{X}}\left[\frac{a_N}{\pi_N(\mathbf{X})}\{\hat{m}(\mathbf{X}) - \mu(\mathbf{X})\}^2\right] = o_p(1),\tag{4.9}$$

and with it a RAL expansion of $\hat{\theta}_{\text{DRSS}}$ as:

$$\hat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + o_p\left(\frac{1}{\sqrt{N\bar{\pi}_N}}\right), \quad \text{where } \Psi(\mathbf{Z}) := \psi_{\mu, \pi}(\mathbf{Z}) + \text{IF}_{\pi}(\mathbf{Z}),\tag{4.10}$$

and $\psi_{\mu, \pi}(\mathbf{Z})$ is defined in (3.2). Lastly, $\mathbb{E}\{\Psi(\mathbf{Z})\} = 0$, $\mathbb{E}\{\Psi^2(\mathbf{Z})\} = O(\bar{\pi}_N^{-1})$.

The displays (4.5), (4.6) and (4.9) are the conditions we need to guarantee the assumptions of Theorem 3.2, while the result (4.7) on $\widehat{\Delta}_N$ helps characterize the full RAL expansion of $\widehat{\theta}_{\text{DRSS}}$ under misspecification of $\widehat{m}(\cdot)$. Lastly, notice that we do not assume $\pi_N(\mathbf{X})/\bar{\pi}_N$ to be bounded below a.s., which is a condition required in Kallus and Mao [2020].

REMARK 4.3 (Necessity of the RAL expansion's modification) When $\bar{\pi}_N \rightarrow 0$, we observe that part of the additional IF, $\text{IF}_\pi(\mathbf{Z})$, in (4.8) has the following property:

$$\mathbb{E} \left[\{1 - \pi_N(\mathbf{X})\} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \vec{\mathbf{X}} \right] = \mathbb{E} \left[\{\mu(\mathbf{X}) - m(\mathbf{X})\} \vec{\mathbf{X}} \right] + O_p(\bar{\pi}_N).$$

If the outcome model is fitted by a linear model whose limit has a linear form $\mu(\mathbf{X}) = \vec{\mathbf{X}}^T \boldsymbol{\beta}^*$, with $\boldsymbol{\beta}^* := \{\mathbb{E}(\vec{\mathbf{X}} \vec{\mathbf{X}}^T)\}^{-1} \mathbb{E}(\vec{\mathbf{X}} Y)$, then,

$$\mathbb{E} \left[\{\mu(\mathbf{X}) - m(\mathbf{X})\} \vec{\mathbf{X}} \right] = \boldsymbol{\beta}^{*T} \mathbb{E}(\vec{\mathbf{X}} \vec{\mathbf{X}}^T) - \mathbb{E}(\vec{\mathbf{X}} Y) = 0,$$

indicating that the RAL expansion's modification is unnecessary when $\bar{\pi}_N \rightarrow 0$ and $\widehat{m}(\cdot)$ converges to the linear projection $\mu(\cdot)$. Here, $\mu(\cdot) \neq m(\cdot)$. The same argument holds if one performs a linear transformation on some basis function $\{\phi_j(\mathbf{X})\}_{j=1}^d$ with a fixed $d < \infty$. However, when d grows with N in that $d/N \rightarrow c \in (0, 1)$, a least squares estimator leads to a latent misspecification i.e., the limit $\mu(\cdot) \neq m(\cdot)$ even if $m(\cdot)$ is indeed linear on $\{\phi_j(\mathbf{X})\}_{j=1}^d$. Hence, an adjusted RAL would be more appropriate if the outcome model is linear with a growing degree of freedom; see Appendix C of the [Supplement](#) for corresponding simulation results.

REMARK 4.4 (Comparison with alternative PS estimators based on under-sampling of the unlabeled group) Under the decaying PS model, a possible alternative to our offset logistic regression estimator could be the so-called “under-sampled” estimators, as studied by Wang [2020] in low-dimensional settings, where the observations from the large unlabeled data are under-sampled in some way to create a more “balanced” setting. Since the under-sampled data is biased (as the under-sampling is done only for one group, i.e., the unlabeled group), additional bias correction or weight adjustment is needed; see the discussion before Section 3.1 of Wang [2020]. Furthermore, as they have also discussed in their Remarks 3 and 4, even after such techniques are applied, the resulting PS estimator may still be less efficient than the full-data-based estimator, such as our proposed offset-based formulation. The high-dimensional setting for such methods has yet to be studied in depth; however, we expect to see additional bias accumulation due to regularization effects. Our results in high dimensions for the offset logistic model are non-trivial where non-standard rates are proposed and these serve as important extensions of the existing high dimensional literature; see Theorem 4.2 and Remark 4.5 below.

Additionally, the loss of efficiency due to the under-sampling may also affect the finite sample performance or even the asymptotic efficiency of the final mean estimator. When the nuisance models are all correctly specified, the efficiency loss will not affect the final estimator's asymptotic efficiency but will likely impact the estimator's finite sample properties. More importantly, when the outcome model is misspecified, the adjusted RAL expansion of the mean estimator is directly dependent on the estimation error and the IF of the PS estimator; see our results in Theorem 4.1 and Remark 3.9. Hence, a less efficient PS estimator will result in a less efficient mean estimator.

4.1.2 High-dimensional offset logistic regression. Next, we consider a high-dimensional setting with p possibly depending on N and $p \rightarrow \infty$ as N grows. The problem here is challenging as together with

$p \rightarrow \infty$, the labels are extremely imbalanced in that $\bar{\pi}_N = \mathbb{P}(R = 1) \rightarrow 0$. Unlike before, an adjusted RAL expansion for the case when $m(\cdot)$ is misspecified is now not available, as we are no longer able to obtain a parametric rate for the PS estimation. In this section, we provide the consistency rate $r_{e,N}$ in (3.8) for an offset, sparse, logistic PS model and establish asymptotic results for $\hat{\theta}_{\text{DRSS}}$ when both $m(\cdot)$ and $\pi_N(\cdot)$ are correctly specified.

Consider the same parametric offset model (4.1), except here we allow $p \rightarrow \infty$ as $N \rightarrow \infty$. In this subsection, we assume the parameter γ_0 to be sparse with $s := \|\gamma_0\|_0$ denoting its sparsity level. Let $\widehat{\pi}_N := N^{-1} \sum_{i=1}^N R_i$ and for every $\gamma \in \mathbb{R}^{p+1}$ and $a \in (0, 1]$, recall $\ell_N(\gamma; a)$ defined in (4.3). Let $\widehat{\gamma}$ be a minimizer of the convex program:

$$\arg \min_{\gamma \in \mathbb{R}^{p+1}} \{\ell_N(\gamma; \widehat{\pi}_N) + \lambda_N \|\gamma\|_1\}, \quad (4.11)$$

with a sequence $\lambda_N > 0$. Then, $\pi_N(\mathbf{X})$ can be estimated similarly as in (4.4) by $\widehat{\pi}_N(\mathbf{X}) := g(\vec{\mathbf{X}}^T \widehat{\gamma} + \log(\widehat{\pi}_N))$. We establish the theoretical properties of our estimators $\widehat{\gamma}$ and $\widehat{\pi}_N(\cdot)$ in 3 parts: 1) establish a restricted strong convexity (RSC) property; 2) control the l_∞ norm of the gradient of the loss at the true parameter, i.e., $\|\nabla \ell_N(\gamma_0; \widehat{\pi}_N)\|_\infty$; and 3) obtain the final probabilistic bounds on the error rates of our estimator.

RSC property for the offset logistic model We first analyze the RSC property of our high dimensional offset logistic model. Under our imbalanced treatment setting, we show that the RSC condition holds with a parameter of the order of $\bar{\pi}_N \rightarrow 0$ (rather than a constant bounded away from 0), once the RSC condition holds for a balanced logistic model with some constant $\kappa > 0$. For any $\Delta, \gamma \in \mathbb{R}^{p+1}$, define the following:

$$\delta\ell(\Delta; a; \gamma) := \ell_N(\gamma + \Delta; a) - \ell_N(\gamma; a) - \Delta^T \nabla \ell_N(\gamma; a).$$

We say the restricted strong convexity (RSC) property holds for $\delta\ell(\Delta; a; \gamma_0)$ with parameter κ on a given set A if

$$\delta\ell(\Delta; a; \gamma_0) \geq \kappa \|\Delta\|_2^2, \quad \text{for all } \Delta \in A.$$

We have the following *deterministic* result.

LEMMA 4.1 For any $a \in (0, 1]$,

$$\delta\ell(\Delta; a; \gamma_0) \geq a \delta\ell(\Delta; 1; \gamma_0).$$

Hence, for a given set A and for any given realization of the data, if the RSC property holds for $\delta\ell(\Delta; 1; \gamma_0)$ with parameter κ on a set A , then the RSC property also holds for $\delta\ell(\Delta; a; \gamma_0)$ with parameter $a\kappa$ on A .

Notice that

$$\begin{aligned} \delta\ell(\Delta; 1; \gamma_0) &= \ell_N(\gamma_0 + \Delta; 1) - \ell_N(\gamma_0; 1) - \Delta^T \nabla \ell_N(\gamma_0; 1) \\ &= \ell_N^{\text{bal}}(\gamma_0 + \Delta) - \ell_N^{\text{bal}}(\gamma_0) - \Delta^T \nabla \ell_N^{\text{bal}}(\gamma_0), \quad \text{where} \\ \ell_N^{\text{bal}}(\gamma) &:= -N^{-1} \sum_{i=1}^N [R_i^* \vec{\mathbf{X}}^T \gamma - \log\{1 + \exp(\vec{\mathbf{X}}^T \gamma)\}], \quad \forall \gamma \in \mathbb{R}^{p+1}, \end{aligned}$$

with $(R_i^*)_{i=1}^N$ being i.i.d. random variables generated from Bernoulli($g(\vec{\mathbf{X}}^T \gamma_0)$). Here, $\ell_N^{\text{bal}}(\gamma)$ is the negative log-likelihood function under a *balanced* logistic model with the true parameter γ_0 . By Lemma

4.1., we relate the RSC property of our imbalanced model to a standard balanced logistic model. The RSC property for a balanced logistic model has been studied in Negahban et al. [2010], among others. We also present a more general version in this paper; see Lemma 4.3.

Gradient control Now, we control the l_∞ norm of the gradient, $\|\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \widehat{\boldsymbol{\pi}}_N)\|_\infty$, and the following lemma demonstrates that the rate of $\|\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \widehat{\boldsymbol{\pi}}_N)\|_\infty = O_p(\{N^{-1} \bar{\pi}_N \log(p)\}^{1/2})$.

LEMMA 4.2 Let $\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0$ be a sub-Gaussian random variable and $\vec{\mathbf{X}}$ a marginal sub-Gaussian random vector, in that $\|\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0\|_{\psi_2} \leq \sigma_{\boldsymbol{\gamma}_0} < \infty$ and $\max_{1 \leq j \leq p+1} \|\vec{\mathbf{X}}(j)\|_{\psi_2} \leq \sigma < \infty$, respectively. Then, for any $t_1, t_2 \geq 0$ and $t_2 < N\bar{\pi}_N/9$,

$$\begin{aligned} \|\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \widehat{\boldsymbol{\pi}}_N)\|_\infty &\leq C_1(\bar{\pi}_N + \bar{\pi}_N^{1/2}) \sqrt{\frac{\{t_1 + \log(p+1)\}}{N}} + C_4 \left\{ \sqrt{\frac{t_2 \bar{\pi}_N}{N}} + \frac{t_2}{N} \right\} \\ &\quad + (C_2 + C_3 \bar{\pi}_N) \frac{\sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N}, \end{aligned}$$

with probability at least $1 - 6\exp(-t_1) - 2\exp(-t_2)$. The constants $C_1, C_2, C_3, C_4 > 0$ independent of N are defined through equations (D.57)-(D.58) in the [Supplement](#).

Define $S := \{j \leq p+1 : \boldsymbol{\gamma}_0(j) \neq 0\}$, $s = |S|$ and the *cone set*:

$$\mathbb{C}_\delta(S; 3) := \{\Delta \in \mathbb{R}^{p+1} : \|\Delta_{S^c}\|_1 \leq 3\|\Delta_S\|_1, \|\Delta\|_2 = \delta\},$$

where $\Delta_S = \{\Delta(j)\}_{j \in S}$ and $\Delta_{S^c} = \{\Delta(j)\}_{j \notin S}$. Define the *critical tolerance*:

$$\delta_N := \inf \left\{ \delta > 0 : \delta \geq 2\lambda_N s^{1/2} \tilde{\kappa}^{-1}, \text{ RSC holds for } \ell_N(\cdot; \widehat{\boldsymbol{\pi}}_N) \text{ with parameter } \tilde{\kappa} \text{ over } \mathbb{C}_\delta(S; 3) \right\}.$$

Then, any optimal solution $\widehat{\boldsymbol{\gamma}} = \widehat{\boldsymbol{\gamma}}_{\lambda_N}$ to the convex program (4.11) satisfies $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 \leq \delta_N$.

Probabilistic bounds Finally, we now obtain the probabilistic bounds and convergence rates for $\widehat{\boldsymbol{\gamma}}$, and subsequently $\widehat{\boldsymbol{\pi}}_N(\cdot)$, in the following result.

THEOREM 4.2 Assume $\log(p)\log(N) = O(N\bar{\pi}_N)$ and $s\log(p) = o(N\bar{\pi}_N)$ as $N, p \rightarrow \infty$, where $s := \|\boldsymbol{\gamma}_0\|_0$. Assume conditions in Lemma 4.2. Suppose the RSC property holds for $\delta\ell(\Delta; 1; \boldsymbol{\gamma}_0)$ with parameter $\kappa > 0$ on the set $\overline{\mathbb{C}}(S; 3) := \{\Delta \in \mathbb{R}^{p+1} : \|\Delta_{S^c}\|_1 \leq 3\|\Delta_S\|_1, \|\Delta\|_2 \leq 1\}$, with probability at least $1 - \alpha_N$, where $\alpha_N = o(1)$. Let

$$M_N := C_5 \sqrt{\frac{\bar{\pi}_N \log(p+1)}{N}} + C_6 \frac{\sqrt{\log(2N)} \log(p+1)}{N},$$

with some constants $C_5, C_6 > 0$. For any λ_N satisfying $2(1+c)M_N \leq \lambda_N \leq 9\kappa\bar{\pi}_Ns^{-1/2}$ with $c > 0$, whenever $N\bar{\pi}_N > 9c\log(p+1)$,

$$\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 \leq \frac{1}{9} \lambda_N s^{1/2} \bar{\pi}_N^{-1} \kappa^{-1}, \quad \text{with probability at least } 1 - 8(p+1)^{-c} - \alpha_N.$$

Further assume that $\|\vec{\mathbf{X}}^T \mathbf{v}\|_{\psi_2} \leq \sigma \|\mathbf{v}\|_2$ for any $\mathbf{v} \in \mathbb{R}^{p+1}$. Then, for any $r > 0$, with some $\lambda_N \asymp \sqrt{\bar{\pi}_N \log(p)/N}$,

$$\begin{aligned} \|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}_{\mathbf{X}}} &\asymp \bar{\pi}_N^{-1} \quad \forall r > 0, \quad \text{and hence } a_N \asymp \bar{\pi}_N, \\ \left\| 1 - \frac{\pi_N(\cdot)}{\hat{\pi}_N(\cdot)} \right\|_{r,\mathbb{P}_{\mathbf{X}}} &= O_p \left(\sqrt{\frac{s \log(p)}{N \bar{\pi}_N}} \right) \quad \forall r > 0, \end{aligned} \quad (4.12)$$

$$\mathbb{E}_{\mathbf{X}} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\hat{\pi}_N(\mathbf{X})} \right\}^2 \right] = O_p \left(\frac{s \log(p)}{N \bar{\pi}_N} \right) = o_p(1). \quad (4.13)$$

Moreover, if $\|\widehat{m}(\cdot) - m(\cdot)\|_{2+c,\mathbb{P}_{\mathbf{X}}} = o_p(1)$ with constant $c > 0$, then,

$$\mathbb{E}_{\mathbf{X}} \left[\frac{a_N}{\pi_N(\mathbf{X})} \{\widehat{m}(\mathbf{X}) - m(\mathbf{X})\}^2 \right] = o_p(1).$$

REMARK 4.5 (Non-standard rates) The implication of Theorem 4.2 is that for some $\lambda_N \asymp \{N^{-1} \bar{\pi}_N \log(p)\}^{1/2}$,

$$\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 = O_p \left(\sqrt{\frac{s \log(p)}{N \bar{\pi}_N}} \right). \quad (4.14)$$

As long as $\bar{\pi}_N \rightarrow 0$, the rate $\lambda_N \asymp \{N^{-1} \bar{\pi}_N \log(p)\}^{1/2}$, is *faster* than the usual rate of $\{N^{-1} \log(p)\}^{1/2}$ used for tuning parameter choice in a standard (i.e., balanced) ℓ_1 -penalized logistic regression. This in turn implies *slower* than usual rate of convergence in (4.14), as $N a_N$ is much smaller than N . This is also reflected in the error rates of the conditional propensity score, in (4.12). The “effective sample size” here is $N a_N$ rather than N , thus leading to *non-standard* rates. The results above may therefore be seen as a *generalization* of standard high-dimensional logistic regression models (i.e., where positivity holds) to the case of a decaying PS. To our knowledge, these rates are novel for high-dimensional settings.

REMARK 4.6 (Marginal versus “Joint” sub-Gaussianity) In Theorem 4.2, we obtained a non-asymptotic upper bound for $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2$ that only requires a marginal sub-Gaussianity of $\vec{\mathbf{X}}$, that is $\max_{1 \leq j \leq p+1} \|\vec{\mathbf{X}}(j)\|_{\psi_2} \leq \sigma < \infty$. Unfortunately, to show (4.12) and (4.13), we do require a “joint” sub-Gaussianity of $\vec{\mathbf{X}}$ in that $\|\vec{\mathbf{X}}^T \mathbf{v}\|_{\psi_2} \leq \sigma \|\mathbf{v}\|_2$ for any $\mathbf{v} \in \mathbb{R}^{p+1}$. In high-dimensions, the joint sub-Gaussianity is stronger than the marginal sub-Gaussianity in that the latter enforces a weaker dependency among the covariates; see Section 4 of [Kuchibhotla and Chakrabortty \[2022\]](#) for more details.

Note that in Theorem 4.2, we *only* assume the RSC property for a classical balanced logistic regression model, which is standard in the high-dimensional regression (and classification) literature. As shown in Proposition 2 of [Negahban et al. \[2010\]](#), with probability at least $1 - 2\exp(-c_1 N)$,

$$\begin{aligned} \delta\ell(\Delta; 1; \boldsymbol{\gamma}_0) &= \ell_N^{\text{bal}}(\boldsymbol{\gamma}_0 + \Delta) - \ell_N^{\text{bal}}(\boldsymbol{\gamma}_0) - \Delta^T \nabla_{\boldsymbol{\gamma}} \ell_N^{\text{bal}}(\boldsymbol{\gamma}_0) \\ &\geq c_2 \|\Delta\|_2 \left\{ \|\Delta\|_2 - c_3 \sqrt{\frac{\log(p+1)}{N}} \|\Delta\|_1 \right\}, \quad \forall \|\Delta\|_2 \leq 1, \end{aligned} \quad (4.15)$$

with some constants $c_1, c_2, c_3 > 0$, and hence, the RSC property holds for $\delta\ell(\Delta; 1; \boldsymbol{\gamma}_0)$ with some $\kappa > 0$ on the set $\overline{\mathbb{C}}(S; 3)$. The conditions required in [Negahban et al. \[2010\]](#) essentially amount to: $s \log(p) =$

$o(N)$, the intercept term $\gamma_0(1) = 0$, \mathbf{X} is a jointly sub-Gaussian with mean zero, and $\lambda_{\min}\{\text{Cov}(\mathbf{X})\} \geq c > 0$. Similar conditions are also required in Example 9.17 and Theorem 9.36 of Wainwright [2019]. In the following Lemma 4.3, we propose a user-friendly version of RSC condition results for a balanced logistic regression problem that only require a marginal sub-Gaussianity of \mathbf{X} and an additional $(2+c)$ -th moment condition $\sup_{\|\mathbf{v}\|_2 \leq 1} \|\vec{\mathbf{X}}^T \mathbf{v}\|_{2+c, \mathbb{P}_{\mathbf{X}}} \leq M < \infty$. In addition, we do not enforce a mean zero \mathbf{X} , and we do not require a zero intercept term in the logistic model either.

LEMMA 4.3 Assume the smallest eigenvalue $\lambda_{\min}\{\mathbb{E}(\vec{\mathbf{X}} \vec{\mathbf{X}}^T)\} \geq \kappa_l > 0$, a $(2+c)$ -th moment condition $\sup_{\|\mathbf{v}\|_2 \leq 1} \|\vec{\mathbf{X}}^T \mathbf{v}\|_{2+c, \mathbb{P}_{\mathbf{X}}} \leq M < \infty$, a c -th moment condition $\|\vec{\mathbf{X}}^T \gamma_0\|_{c, \mathbb{P}_{\mathbf{X}}} \leq \mu_c < \infty$ and the marginal sub-Gaussianity $\sup_{1 \leq j \leq p+1} \|\vec{\mathbf{X}}(j)\|_{\psi_2} \leq \sigma < \infty$. Then, with probability at least $1 - 2\exp(-c_1 N)$, (4.15) holds, with constants $c_1, c_2, c_3 > 0$. If we further assume that $s \log(p) = o(N)$, then, for large enough N , there exists a constant $\kappa > 0$ such that, with probability at least $1 - 2\exp(-c_1 N)$,

$$\delta\ell(\Delta; 1; \gamma_0) \geq \kappa \|\Delta\|_2^2, \quad \forall \Delta \in \overline{\mathbb{C}}(S; 3). \quad (4.16)$$

Although Lemma 4.3 is based on the logistic loss function, it in fact applies to any loss function $\ell_N^{\text{bal}}(\cdot)$ based on the maximum likelihood of a balanced generalized linear model.

4.2 Stratified labeling

We consider here a stratified labeling mechanism. Here, the labeling indicator R depends on \mathbf{X} , but does so only through an intermediate stratification in \mathbf{X} . Such mechanisms are often of practical relevance in biomedical studies when *prior* information is available on stratification through another observed variable. Specifically, let $\delta \in \{0, 1\}$ denote an observed random stratum indicator and assume that $R \perp\!\!\!\perp \mathbf{X} | \delta$. Note that nothing changes if we were to move from binary to finitely many strata, and while we stick to a binary δ here for simplicity, our work can be easily extended to a multiple-stratum situation. Let $\pi_{j,N} := \mathbb{P}(R = 1 | \delta = j, \mathbf{X}) \equiv \mathbb{P}(R = 1 | \delta = j)$ for each $j = 0, 1$. We assume δ is a “well behaved” indicator whose distribution is independent of N and itself satisfies the overlap condition

$$c < p_\delta(\mathbf{x}) := \mathbb{P}(\delta = 1 | \mathbf{x}) < 1 - c, \quad \text{for all } \mathbf{x} \in \mathcal{X},$$

with a constant $0 < c < 1/2$ independent of N . Then, we have:

$$\pi_N(\mathbf{X}) = \pi_{1,N} p_\delta(\mathbf{X}) + \pi_{0,N} \{1 - p_\delta(\mathbf{X})\}.$$

As long as δ is observed, then $\pi_{j,N}$ for each j can be estimated very easily and at a rate $O_p((N\bar{\pi}_N)^{-1/2})$. Moreover, when $\bar{\pi}_N \rightarrow 0$ as $N \rightarrow \infty$, $p_\delta(\mathbf{X})$ can be estimated at a parametric $N^{-1/2}$ rate if the model is parametric, or at a rate slower than $N^{-1/2}$ but still as a function of N (rather than $N\bar{\pi}_N$) if a non-parametric estimator is performed. Therefore, we will continue to have a fast enough rate for $\hat{\pi}_N(\cdot)$ under this setting, so that the error term $\hat{\Delta}_N$ in (3.10) can potentially have a rate:

$$\hat{\Delta}_N = O_p(r_{e,N}) = O_p\left((N\bar{\pi}_N)^{-1/2}\right).$$

In this section, we propose a PS estimator based on the stratified labeling model above, and provide a full characterization of its properties as well as a RAL expansion for the error $\hat{\Delta}_N$.

With a slight abuse of notation, we define $\mathbf{Z}_i = (R_i, R_i Y_i, \delta_i, \mathbf{X}_i)$ in this section, and let $\mathbb{S} = (\mathbf{Z}_i)_{i=1}^N$, $\mathbb{S}_{-k} = \{\mathbf{Z}_i : i \in \mathcal{I} \setminus \mathcal{I}_k\}$. Suppose $\hat{p}_\delta(\cdot)$ is an estimator of $p_\delta(\cdot)$, and let $\hat{p}_\delta(\mathbf{X}_i) := \hat{p}_\delta(\mathbf{X}_i; \mathbb{S}_{-k(i)})$ be a corresponding cross-fitted version of this estimator.

Define $\widehat{\pi}_1^{-k} := \sum_{i \notin I_k} \delta_i R_i / \sum_{i \notin I_k} \delta_i$ and $\widehat{\pi}_0^{-k} := \sum_{i \notin I_k} (1 - \delta_i) R_i / \sum_{i \notin I_k} (1 - \delta_i)$ to be the cross-fitted estimators of $\pi_{1,N}$ and $\pi_{0,N}$, respectively. The PS $\pi_N(\cdot)$ is then estimated by:

$$\widehat{\pi}_N(\mathbf{X}_i) := \widehat{\pi}_1^{-k(i)} \widehat{p}_\delta(\mathbf{X}_i) + \widehat{\pi}_0^{-k(i)} \{1 - \widehat{p}_\delta(\mathbf{X}_i)\}.$$

THEOREM 4.3 Assume $\bar{\pi}_N \rightarrow 0$ and $N\bar{\pi}_N \rightarrow \infty$ as $N \rightarrow \infty$. Suppose

$$\|\widehat{p}_\delta(\cdot) - p_\delta(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} = O_p(r_{\delta,N}), \quad \text{for some sequence } r_{\delta,N} = o(1). \quad (4.17)$$

Then, for each $k \leq \mathbb{K}$,

$$\mathbb{E}_{\mathbf{X}} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X})} \right\}^2 = O_p \left(r_{\delta,N}^2 + N\bar{\pi}_N \right).$$

Besides, assume $\|\mu(\cdot) - m(\cdot)\|_{\infty,\mathbb{P}_{\mathbf{X}}} < \infty$ and (3.7). Then, the overall RAL expansion of our DRSS estimator $\widehat{\theta}_{\text{DRSS}}$ under a stratified labeling model as above is:

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + O_p \left((N\bar{\pi}_N)^{-1} + N^{-1/2} + r_{\mu,N}(N\bar{\pi}_N)^{-1/2} + r_{\delta,N} \right),$$

where $\Psi(\mathbf{Z}) := \psi_{\mu,\pi}(\mathbf{Z}) + \text{IF}_{\pi}(\mathbf{Z})$ and $\mathbb{E}\{\Psi(\mathbf{Z})\} = 0$ with $\psi_{\mu,\pi}(\mathbf{Z})$ as defined in (3.2) and

$$\begin{aligned} \text{IF}_{\pi}(\mathbf{Z}) &:= \left\{ \frac{\delta R}{p_\delta} - \pi_{1,N} \right\} \mathbb{E}_{\mathbf{X}} \left[\frac{p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &\quad + \left\{ \frac{(1-\delta)R}{1-p_\delta} - \pi_{0,N} \right\} \mathbb{E}_{\mathbf{X}} \left[\frac{1-p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right], \end{aligned}$$

where $p_\delta = \mathbb{E}\{p_\delta(\mathbf{X})\} = \mathbb{E}(\delta)$. If we further assume $r_{\delta,N} = o((N\bar{\pi}_N)^{-1/2})$, then

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + o_p \left((N\bar{\pi}_N)^{-1/2} \right). \quad (4.18)$$

Note that Theorem 4.3 still holds if π_1 and π_0 are estimated without cross-fitting in that $\widehat{\pi}_1 := \sum_{i=1}^N \delta_i R_i / \sum_{i=1}^N \delta_i$ and $\widehat{\pi}_0 := \sum_{i=1}^N (1 - \delta_i) R_i / \sum_{i=1}^N (1 - \delta_i)$.

EXAMPLE 4.4 Here we illustrate a simple logistic model for $p_\delta(\cdot)$ and investigate the conditions we need for $r_{\delta,N}$ to be $o_p((N\bar{\pi}_N)^{-1/2})$, so that the RAL expansion (4.18) holds. For a fixed dimensional \mathbf{X} , let $\widehat{p}_\delta(\cdot)$ be the MLE of the logistic model. Then $r_{\delta,N} = O(N^{-1/2}) = o((N\bar{\pi}_N)^{-1/2})$ as long as $\bar{\pi}_N \rightarrow 0$. As for a high-dimensional \mathbf{X} , consider a sparse logistic model for $p_\delta(\cdot)$, and let $\widehat{p}_\delta(\cdot)$ be the logistic estimator based on a Lasso penalty. Then, $r_{\delta,N} = O((s_\delta \log(p)/N)^{1/2})$, where s_δ is the sparsity level of the logistic models parameter. Hence, $r_{\delta,N} = o_p((N\bar{\pi}_N)^{-1/2})$ if $s_\delta \bar{\pi}_N \log(p) = o(1)$ as $N \rightarrow \infty$.

REMARK 4.7 (Comparisons with other works) We note that similar, yet different, problems are studied in Gronsbell et al. [2020] and Hong et al. [2020]. They both work on decaying stratified labeling propensity score models, but with different types of stratified labeling mechanisms and parameters of interest compared to our setting. Hong et al. [2020] assume a deterministic δ given \mathbf{X} . Essentially, they require $Y \perp\!\!\!\perp R|\delta$, so that δ can be seen as a univariate confounder with a finite support. On the other hand, both Gronsbell et al. [2020] and our work allow additional randomness in δ . Besides, Hong et al. [2020] work on a “finite-population” ATE estimation problem, where the treatment assignment is the

only source of randomness. [Gronsbell et al. \[2020\]](#) focus on the estimation of the regression parameters and prediction performance measures, for low-dimensional covariates and a binary outcome problem; and we are mainly working on the estimation of the mean response while allowing for high-dimensional covariates and real valued outcomes.

5. Average treatment effect estimation with imbalanced treatment groups

One important application of our proposed method of Section 3 is the popular causal inference problem of ATE estimation and hypothesis testing. Our method is particularly suited when extremely *imbalanced* treatment groups occur. The causal inference literature typically accesses ATE inference by imposing an overlap condition by which $\mathbb{P}(c < \mathbb{E}(R|\mathbf{X}) < 1 - c) = 1$ for some constant $c > 0$. Here, $R \in \{0, 1\}$ is a binary treatment indicator. In contrast, we show that our method identifies and performs inference about the ATE without requiring an overlap condition. We extend our results for the MAR-SS setting of Section 3 to a causal inference setting while allowing a decaying PS in that $\bar{\pi}_N := \mathbb{E}(R) \rightarrow 0$ (or alternatively, $\bar{\pi}_N \rightarrow 1$) as $N \rightarrow \infty$. To the best of our knowledge, no previous work has addressed such an extremely imbalanced treatment groups setting in the context of ATE estimation.

We formulate the problem setup first. Suppose we have i.i.d. samples $\mathbb{S} := (R_i, Y_i, \mathbf{X}_i)_{i=1}^N$ with (R, Y, \mathbf{X}) being an independent copy of (R_i, Y_i, \mathbf{X}_i) . Here, $R = R_N \in \{0, 1\}$ is a treatment indicator that, similarly as in Section 3, is allowed to depend on N , i.e., $R = R_N$. As before, $\mathbf{X} \in \mathbb{R}^p$ denotes the covariate vector while $Y = Y(R)$ now denotes the observed potential outcome. Here, $Y(1)$ denotes the potential outcome if the individual have been treated and $Y(0)$ denotes the potential outcome if the individual haven't been treated. For each individual, only one of the potential outcomes $Y(R)$ is observable. Consistency of the potential outcomes is assumed throughout: $Y = Y(R) = RY(1) + (1 - R)Y(0)$; see [Rubin \[1974\]](#) and [Imbens and Rubin \[2015\]](#).

Now we define the parameter of interest, $\theta_{ATE} := \theta^1 - \theta^0$ to be the ATE of R on Y , where with a slight abuse of notation we denote with $\theta^1 := \mathbb{E}\{Y(1)\}$ and $\theta^0 := \mathbb{E}\{Y(0)\}$. Moving forward we assume the usual *unconfoundedness condition* [[Imbens, 2004](#), [Tsiatis, 2007](#)]:

$$\{Y(0), Y(1)\} \perp\!\!\!\perp R \mid \mathbf{X}.$$

Then, $\theta^1 = \mathbb{E}\{m_1(\mathbf{X})\}$ and $\theta^0 = \mathbb{E}\{m_0(\mathbf{X})\}$, where $m_r(\mathbf{X}) := \mathbb{E}(Y|R=r, \mathbf{X}) \equiv \mathbb{E}\{Y(r)|\mathbf{X}\}$ denotes the conditional outcome model, and $r \in \{0, 1\}$. With extremely imbalanced groups, without loss of generality, we assume $\bar{\pi}_N = \mathbb{P}(R=1) \rightarrow 0$, i.e., most of the individuals are likely to be in the control group.

The estimation of θ^1 is the same as the mean estimation problem in the MAR-SS setting, if we set \mathbf{Z}_i s to be $(R_i, R_i Y_i, \mathbf{X}_i)$. Similarly, θ^0 can be identified as a mean with \mathbf{Z}_i s being $(1 - R_i, (1 - R_i) Y_i, \mathbf{X}_i)$. Now, as in (3.4), θ^1 and θ^0 can be estimated by:

$$\begin{aligned} \hat{\theta}^1 &:= N^{-1} \sum_{i=1}^N \left[\hat{m}_1(\mathbf{X}_i) + \frac{R_i}{\hat{\pi}_N(\mathbf{X}_i)} \{Y_i - \hat{m}_1(\mathbf{X}_i)\} \right], \\ \hat{\theta}^0 &:= N^{-1} \sum_{i=1}^N \left[\hat{m}_0(\mathbf{X}_i) + \frac{1 - R_i}{1 - \hat{\pi}_N(\mathbf{X}_i)} \{Y_i - \hat{m}_0(\mathbf{X}_i)\} \right], \end{aligned} \quad (5.1)$$

where, for each $k \leq K$ and $i \in \mathbb{S}_k$, $\hat{m}_1(\mathbf{X}_i) = \hat{m}_1(\mathbf{X}_i; \mathbb{S}_{-k})$, $\hat{m}_0(\mathbf{X}_i) = \hat{m}_0(\mathbf{X}_i; \mathbb{S}_{-k})$, and $\hat{\pi}_N(\mathbf{X}_i) = \hat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})$ are cross-fitted estimators of $m_1(\mathbf{X}_i)$, $m_0(\mathbf{X}_i)$, and $\pi_N(\mathbf{X}_i)$, respectively. Here, $\mathbb{S}_{-k} = \{\mathbf{Z}_i : i \in \mathcal{I} \setminus \mathcal{I}_k\}$ is defined analogously as discussed below (3.1) in Section 3.1, and for $r \in \{0, 1\}$, $\hat{m}_r(\cdot)$ is constructed based on $\{\mathbf{Z}_i : i \in \mathbb{S}_{-k}, R_i = r\}$. Hence, θ_{ATE} can be estimated by the DRSS ATE estimator:

$$\hat{\theta}_{ATE} := \hat{\theta}^1 - \hat{\theta}^0. \quad (5.2)$$

The asymptotic properties of $\widehat{\theta}^1$ follow directly from Theorem 3.2. The following theorem provides the asymptotic results for $\widehat{\theta}^0$. For the sake of a better interpretability, in the following theorem, we suppose $c\bar{\pi}_N < \pi_N(\mathbf{X})$ with some constant $c > 0$ and hence we have $a_N \asymp \bar{\pi}_N$.

THEOREM 5.1 Let Assumption 2.1 hold and Assumption 3.1 hold for $m_0(\cdot)$ and $\mu_0(\cdot)$, let $N \rightarrow \infty$, $\bar{\pi}_N \rightarrow 0$ and $N\bar{\pi}_N \rightarrow \infty$. Suppose for all $\mathbf{x} \in \mathcal{X}$, $c\bar{\pi}_N < \pi_N(\mathbf{x}), e_N(\mathbf{x}) < C\bar{\pi}_N$, for some $c, C > 0$. Let $\varepsilon := Y - Rm_1(\mathbf{X}) - (1-R)m_0(\mathbf{X})$, assume $\|\varepsilon\|_{2,\mathbb{P}} < \infty$, $\|m_0(\cdot) - \mu_0(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} < C < \infty$, $\text{Var}\{m_0(\mathbf{X})\} < \infty$ as well as

$$\|\widehat{m}_0(\cdot) - \mu_0(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} = O_p(r_{\mu,0,N}), \quad \sup_{\mathbf{x} \in \mathcal{X}} \left| \frac{\widehat{\pi}_N(\mathbf{x}) - e_N(\mathbf{x})}{\bar{\pi}_N} \right| = O_p(r_{e,N}), \quad (5.3)$$

for a sequence of positive numbers $r_{\mu,0,N} = o(1)$ and $r_{e,N} = o(1)$. Then,

$$\begin{aligned} \widehat{\theta}^0 - \theta^0 &= N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i) + O_p(N^{-1/2} \bar{\pi}_N^{1/2} r_{\mu,0,N} + N^{-1/2} \bar{\pi}_N r_{e,N} + \bar{\pi}_N r_{e,N} r_{\mu,0,N}) \\ &\quad + 1\{m_0(\cdot) \neq \mu_0(\cdot)\} O_p(\bar{\pi}_N r_{e,N}) + 1\{e_N(\cdot) \neq \pi_N(\cdot)\} O_p(\bar{\pi}_N r_{\mu,0,N}), \end{aligned}$$

where

$$\begin{aligned} \psi_0(\mathbf{Z}) &:= \mu_0(\mathbf{X}) - \theta^0 + \frac{1-R}{1-e_N(\mathbf{X})} \{Y - \mu_0(\mathbf{X})\} \\ &= \frac{e_N(\mathbf{X}) - R}{1-e_N(\mathbf{X})} \{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\} + m_0(\mathbf{X}) - \theta^0 + \frac{\varepsilon(1-R)}{1-e_N(\mathbf{X})}, \end{aligned}$$

with $\mathbb{E}\{\psi_0(\mathbf{Z})\} = 1\{e_N(\cdot) \neq \pi_N(\cdot), \mu_0(\cdot) \neq m_0(\cdot)\} O_p(\bar{\pi}_N)$ and $\text{Var}\{\psi_0(\mathbf{Z})\} = O_p(N^{-1})$. Hence,

$$\widehat{\theta}^0 - \theta^0 = N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i) + o_p(N^{-1/2}), \quad \text{with } \mathbb{E}\{\psi_0(\mathbf{Z})\} = 0,$$

once $\bar{\pi}_N r_{e,N} r_{\mu,0,N} = o(N^{-1/2})$, $\bar{\pi}_N r_{e,N} = o(N^{-1/2})$ if $m_0(\cdot)$ is misspecified, $\bar{\pi}_N r_{\mu,0,N} = o(N^{-1/2})$ if $\pi_N(\cdot)$ is misspecified and at least one of $m_0(\cdot)$ and $\pi_N(\cdot)$ is correctly specified. If both $m_0(\cdot)$ and $\pi_N(\cdot)$ are misspecified, then we have $\widehat{\theta}^0 - \theta^0 = O_p(\bar{\pi}_N + N^{-1/2})$.

REMARK 5.1 (Comparison with the naive estimator) Now we consider the comparison of the doubly robust estimator $\widehat{\theta}^0$ with the empirical average of the response over the control group $\bar{Y}_0 := \sum_{i=1}^N (1-R_i)Y_i / \sum_{i=1}^N (1-R_i)$. The empirical average \bar{Y}_0 can be seen as a special case of the estimator (5.1) (without cross-fitting) in that $\widehat{\pi}_N(\mathbf{X}) = N^{-1} \sum_{i=1}^N (1-R_i)$ and $\widehat{m}_0(\mathbf{X}) = 0$. Notice that when $\bar{\pi}_N \rightarrow 0$,

$$\begin{aligned} \bar{Y}_0 &= \mathbb{E}\{m_0(\mathbf{X}) | R=0\} + O_p(N^{-1/2}), \quad \text{with} \\ \theta^0 - \mathbb{E}\{m_0(\mathbf{X}) | R=0\} &= [\mathbb{E}\{m_0(\mathbf{X}) | R=1\} - \mathbb{E}\{m_0(\mathbf{X}) | R=0\}] \bar{\pi}_N = O(\bar{\pi}_N), \end{aligned}$$

and hence $\bar{Y}_0 - \theta^0 = O_p(\bar{\pi}_N + N^{-1/2})$, which coincides with the case that both $m(\cdot)$ and $\pi_N(\cdot)$ are misspecified in Theorem 5.1.

COROLLARY 5.1 Let the assumptions of Theorem 5.1 hold and the assumptions of Theorems 3.2 (b) hold for $m_1(\cdot)$, $\mu_1(\cdot)$, and $\widehat{m}_1(\cdot)$. Assume that at least one of $e_N(\cdot) = \pi_N(\cdot)$ and $\mu_1(\cdot) = m_1(\cdot)$ holds, and let

$$\|\widehat{m}_1(\mathbf{X}; \mathbb{S}_{-k}) - \mu_1(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} = O_p(r_{\mu,1,N}), \quad \text{with } r_{\mu,1,N} = o(1).$$

Then,

$$\begin{aligned}\widehat{\theta}_{\text{ATE}} - \theta_{\text{ATE}} &= N^{-1} \sum_{i=1}^N \psi_1(\mathbf{Z}_i) + \widehat{\Delta}_N + O_p(r_{e,N} r_{\mu,1,N} + \bar{\pi}_N r_{e,N} r_{\mu,0,N}) \\ &\quad + 1\{m_0(\cdot) \neq \mu_0(\cdot)\} O_p(\bar{\pi}_N r_{e,N}) + 1\{e_N(\cdot) \neq \pi_N(\cdot)\} O_p(\bar{\pi}_N r_{\mu,0,N}) \\ &\quad + 1\{e_N(\cdot) \neq \pi_N(\cdot), \mu_0(\cdot) \neq m_0(\cdot)\} O_p(\bar{\pi}_N) + o_p((N\bar{\pi}_N)^{-1/2}),\end{aligned}$$

where

$$\psi_1(\mathbf{Z}) := \mu_1(\mathbf{X}) - \theta^1 + \frac{R}{e_N(\mathbf{X})} \{Y - \mu_1(\mathbf{X})\},$$

with $\mathbb{E}\{\psi_1(\mathbf{Z})\} = 0$, $\mathbb{E}\{\psi_1^2(\mathbf{Z})\} \asymp \bar{\pi}_N^{-1}$, and

$$\begin{aligned}\widehat{\Delta}_N &:= \frac{1}{N} \sum_{i=1}^N \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - \frac{R_i}{\widehat{\pi}_N(\mathbf{X}_i)} \right\} \{\mu_1(\mathbf{X}_i) - m_1(\mathbf{X}_i)\} = O_p(r_{e,N}) \text{ if } e_N(\cdot) = \pi_N(\cdot), \\ \widehat{\Delta}_N &:= \frac{1}{N} \sum_{i=1}^N \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{\widehat{m}_1(\mathbf{X}_i) - m_1(\mathbf{X}_i)\} = O_p(r_{\mu,1,N}) \text{ if } \mu_1(\cdot) = m_1(\cdot).\end{aligned}$$

Moreover, if $r_{e,N} r_{\mu,1,N} = o_p((N\bar{\pi}_N)^{-1/2})$,

$$\widehat{\theta}_{\text{ATE}} - \theta_{\text{ATE}} = N^{-1} \sum_{i=1}^N \psi_1(\mathbf{Z}_i) + o_p((N\bar{\pi}_N)^{-1/2}) + \widehat{\Delta}_N,$$

as long as one of the following holds:

- (a) both $m_0(\cdot)$ and $\pi_N(\cdot)$ are correctly specified, and $r_{e,N} r_{\mu,0,N} = o(N^{-1/2} \bar{\pi}_N^{-3/2})$; or
- (b) $\pi_N(\cdot)$ is correctly specified, $m_0(\cdot)$ is misspecified, and $r_{e,N} = o(N^{-1/2} \bar{\pi}_N^{-3/2})$; or
- (c) $m_0(\cdot)$ is correctly specified, $\pi_N(\cdot)$ is misspecified, and $r_{\mu,0,N} = o(N^{-1/2} \bar{\pi}_N^{-3/2})$; or
- (d) both $m_0(\cdot)$ and $\pi_N(\cdot)$ are misspecified, and $N\bar{\pi}_N^3 = o(1)$.

When $\bar{\pi}_N \rightarrow 0$, the control group has a much larger sample size compared with the treatment group. As a result, the estimation of θ^0 is relatively “simpler” than θ^1 . As shown in Theorem 3.2, the consistency of $\widehat{\theta}^1$ requires at least one nuisance model to be correctly specified. On the other hand, $\widehat{\theta}^0$ is consistent even when both nuisance models are misspecified; see Theorem 5.1. In fact, since most of the samples are in the control group and we can observe most of the potential outcomes $Y_i(0)$, the missing data problem is trivial here, and a naive sample mean estimator over the control group is also consistent as discussed in Remark 5.1. Although obtaining a consistent estimator for θ^0 is a trivial problem, the consistency rate still depends on the correctness of the nuisance models. As a result, the RAL expansion of the whole ATE parameter still requires a number of conditions corresponding to the control arm’s estimation; see Corollary 5.1. The smaller the probability $\bar{\pi}_N$ is, the weaker conditions we need for the model correctness and estimation errors of the control arm. When $N\bar{\pi}_N^3 = o(1)$, we can get the RAL expansion for the ATE parameter even when $m_0(\cdot)$ and $\pi_N(\cdot)$ are both misspecified; however, when $\bar{\pi}_N$ is not small enough, we still need either $m_0(\cdot)$ or $\pi_N(\cdot)$ to be correctly specified to get the RAL expansion of the whole ATE parameter.

6. Numerical studies

6.1 Simulation studies

We illustrate the performance of our DRSS estimators through extensive simulations under various data generating processes (DGPs). We first provide our main simulation results in Section 6.1.1, where the double robustness (in the sense of consistency or inference) shows up in different misspecification settings. Then, in Section 6.1.2, we show the simulation results under a special stratified labeling PS model that was discussed in Section 4.2. We further focus on sparse linear models in high dimensions, and provide results under different sparsity levels in Section 6.1.3.

6.1.1 *Main simulation results.* We consider the following choices of parameters p , N and $\bar{\pi}_N$:

$$p \in \{10, 500\}, \quad (N, \bar{\pi}_N) \in \{(10000, 0.01), (50000, 0.01), (10000, 0.1)\}.$$

We generate i.i.d. Gaussian covariates $\mathbf{X}_i \sim \text{iid } \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$ and residuals $\varepsilon_i \sim \text{iid } \mathcal{N}(0, 1)$. Given \mathbf{X}_i , we generate $R_i | \mathbf{X}_i \sim \text{Bernoulli}(\pi_N(\mathbf{X}_i))$, with the following PS models:

P1. (Constant PS) $\pi_N(\cdot) \equiv \bar{\pi}_N$.

P2. (Offset logistic PS) $\pi_N(\mathbf{x}) = g(\vec{\mathbf{x}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))$, where $g(\cdot)$ is defined in (4.2).

We consider the following outcome models for Y_i given \mathbf{X}_i :

O1. (Linear outcome) $Y_i = \vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0 + \varepsilon_i$.

O2. (Quadratic outcome) $Y_i = \vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0 + \sum_{j=1}^{p+1} \boldsymbol{\alpha}_0(j) \vec{\mathbf{X}}_i(j)^2 + \varepsilon_i$.

The parameter values are chosen as:

$$\boldsymbol{\beta}_0 = (-0.5, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T, \quad \boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_0(1), 1, \mathbf{0}_{1 \times (p-1)})^T, \quad \boldsymbol{\alpha}_0 = (0, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T,$$

where $\boldsymbol{\gamma}_0(1)$ is chosen so that $\mathbb{E}(R) = \bar{\pi}_N$ for each $\bar{\pi}_N$. The following DGPs are considered: Setting a: P1+O1, Setting b: P1+O2, Setting c: P2+O1, and Setting d: P2+O2. For each DGP, we compare the performance of the following estimators: (1) A naive mean estimator over the labeled samples $\bar{Y}_{\text{labeled}} := \sum_{i=1}^N R_i Y_i / \sum_{i=1}^N R_i$; (2) An oracle case of the mean estimator $\hat{\theta}_{\text{DRSS}}$ in (3.4) with $\pi_N(\cdot)$ and $m(\cdot)$ treated as known; (3) The proposed DRSS mean estimator $\hat{\theta}_{\text{DRSS}}$ in (3.4), with $K = 5$.

We consider several different choices on the outcome and propensity estimators. In low dimensions ($p = 10$), we consider two parametric outcome model estimators: least squares (LS) linear regression and a polynomial (poly) regression with degree 2 (without interaction terms), and two non-parametric outcome estimators: random forest (RF) and reproducing kernel Hilbert space (RKHS) regression using a Gaussian kernel. In high dimensions ($p = 500$), we consider two ℓ_1 -regularized parametric outcome model estimators: Lasso and a degree-2 polynomial regression with a Lasso-type penalty (poly-Lasso). As for the PS estimators, we consider a constant estimator that essentially corresponds to a MCAR estimator, and an offset based logistic estimator (or its ℓ_1 -regularized version, log-Lasso, when $p = 500$). Note that the DRSS estimators based on a constant PS estimate are essentially the same as the SS mean estimators proposed by [Zhang and Bradic \[2022\]](#), except here, we use a cross-fitted version of $\hat{\pi}_N(\cdot)$. In addition, the SS mean estimators proposed by [Zhang et al. \[2019\]](#) based on a linear outcome model estimated via least squares can be further seen as a special case of our proposed DRSS estimator

Table 6.1. Simulation setting a with $p = 10$. Bias: empirical bias; RMSE: root mean square error; Length: average length of the 95% confidence intervals; Coverage: average coverage of the 95% confidence intervals; ESD: empirical standard deviation; ASD: average of estimated standard deviations. The blue color in the tables denotes the “smallest” and correctly specified parametric model for each of the settings.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
\bar{Y}_{labeled}		0.013	0.204	0.789	0.932	0.203	0.201
oracle		0.003	0.106	0.397	0.942	0.106	0.101
constant	LS	0.003	0.115	0.436	0.938	0.115	0.111
	poly	0.002	0.127	0.482	0.934	0.127	0.123
	RF	0.008	0.155	0.604	0.926	0.155	0.154
	RKHS	0.009	0.145	0.547	0.936	0.145	0.139
logistic	LS	0.005	0.127	0.534	0.972	0.127	0.136
	poly	0.003	0.144	0.600	0.972	0.144	0.153
	RF	0.003	0.152	0.762	0.990	0.152	0.194
	RKHS	0.007	0.161	0.698	0.976	0.161	0.178
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
\bar{Y}_{labeled}		0.008	0.092	0.352	0.950	0.092	0.090
oracle		0.002	0.045	0.179	0.948	0.045	0.046
constant	LS	0.003	0.045	0.182	0.956	0.045	0.046
	poly	0.003	0.046	0.185	0.952	0.046	0.047
	RF	0.004	0.056	0.218	0.942	0.056	0.056
	RKHS	0.003	0.056	0.213	0.942	0.056	0.054
logistic	LS	0.003	0.046	0.189	0.960	0.046	0.048
	poly	0.003	0.047	0.192	0.962	0.046	0.049
	RF	0.002	0.052	0.227	0.974	0.052	0.058
	RKHS	0.001	0.054	0.223	0.960	0.054	0.057
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
\bar{Y}_{labeled}		0.002	0.066	0.260	0.958	0.066	0.066
oracle		-0.001	0.037	0.146	0.958	0.037	0.037
constant	LS	-0.001	0.038	0.147	0.956	0.038	0.038
	poly	-0.001	0.038	0.148	0.954	0.038	0.038
	RF	0.000	0.041	0.164	0.958	0.041	0.042
	RKHS	-0.001	0.042	0.163	0.938	0.042	0.042
logistic	LS	0.000	0.038	0.150	0.960	0.038	0.038
	poly	0.000	0.038	0.150	0.956	0.038	0.038
	RF	-0.001	0.039	0.167	0.964	0.039	0.043
	RKHS	-0.001	0.041	0.166	0.950	0.041	0.042

essentially, where a (non-cross-fitted) constant PS and a (non-cross-fitted) least squares estimate for a linear outcome model are used; see further discussions in Appendix B of the [Supplement](#). The tuning parameters in the regularized estimators are chosen via 5-fold cross-validation. The hyperparameters in the RF are chosen by minimizing the out-of-bag (OOB) error. The bandwidth parameter for the Gaussian kernel in RKHS regression is set to be p .

The simulations are repeated 500 times and the results are presented in Tables 6.1–6.8. We report the bias, the root mean square error (RMSE), the average length and average coverage of the 95% confidence

Table 6.2. Simulation setting b with $p = 10$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD	
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$								
\bar{Y} labeled oracle	LS	0.008	0.334	1.251	0.936	0.334	0.319	
	poly	0.001	0.103	0.410	0.946	0.103	0.105	
	RF	0.009	0.311	1.167	0.938	0.311	0.298	
	RKHS	0.002	0.253	0.939	0.934	0.253	0.239	
	LS	0.004	0.245	0.930	0.928	0.245	0.237	
	constant	0.154	0.420	1.540	0.950	0.391	0.393	
logistic	poly	0.001	0.143	0.615	0.968	0.144	0.157	
	RF	0.075	0.319	1.222	0.960	0.310	0.312	
	RKHS	0.107	0.327	1.216	0.950	0.310	0.310	
	$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
	\bar{Y} labeled oracle	LS	0.005	0.138	0.558	0.958	0.138	0.142
		poly	0.001	0.046	0.184	0.952	0.046	0.047
		RF	0.008	0.119	0.478	0.952	0.119	0.122
		RKHS	0.001	0.076	0.304	0.942	0.076	0.077
		LS	0.001	0.076	0.306	0.952	0.076	0.078
		constant	0.032	0.128	0.505	0.954	0.124	0.129
logistic	poly	0.000	0.048	0.196	0.956	0.048	0.050	
	RF	0.009	0.079	0.320	0.958	0.079	0.082	
	RKHS	0.014	0.080	0.324	0.960	0.079	0.083	
	$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
	\bar{Y} labeled oracle	LS	0.001	0.107	0.411	0.944	0.107	0.105
		poly	0.000	0.045	0.176	0.946	0.045	0.045
		RF	0.000	0.091	0.353	0.958	0.091	0.090
		RKHS	-0.001	0.057	0.228	0.952	0.057	0.058
		LS	-0.001	0.058	0.232	0.952	0.058	0.059
		constant	0.012	0.093	0.363	0.958	0.092	0.093
logistic	poly	0.000	0.046	0.179	0.944	0.046	0.046	
	RF	0.003	0.058	0.233	0.956	0.058	0.059	
	RKHS	-0.005	0.058	0.237	0.956	0.058	0.061	

intervals, the empirical standard error and the averaged estimated standard error for all settings. The **blue** color in the tables denotes the “smallest” (i.e., most parsimonious) and correctly specified parametric model for each of the settings.

We first check the proposed estimator’s double robustness in terms of inference. As per Theorem 3.2, the asymptotic normality results hold when the product rate condition is satisfied and when both of $\pi_N(\cdot)$ and $m(\cdot)$ are correct, in which case, the proposed estimator is $(N\bar{\pi}_N)^{1/2}$ -consistent with the asymptotic efficiency matching that of the oracle estimator. In low dimensions ($p = 10$), the product rate condition always holds since $\hat{\pi}_N(\cdot)$ has an estimation error of rate $(N\bar{\pi}_N)^{-1/2}$ and $\hat{m}(\cdot)$ (parametric or non-parametric) is consistent. As in Tables 6.1-6.4, the coverage of $\hat{\theta}_{DRSS}$ based on correct $\pi_N(\cdot)$

Table 6.3. Simulation setting c with $p = 10$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
constant	\bar{Y}_{labeled}	0.980	0.999	0.783	0.002	0.194	0.200
	oracle	0.003	0.106	0.397	0.942	0.106	0.101
	LS	0.005	0.151	0.434	0.850	0.151	0.111
	poly	-0.004	0.194	0.480	0.792	0.194	0.122
	RF	0.570	0.610	0.596	0.108	0.218	0.152
	RKHS	0.403	0.439	0.543	0.210	0.175	0.139
logistic	LS	0.015	0.234	0.884	0.952	0.234	0.226
	poly	-0.002	0.365	1.083	0.922	0.365	0.276
	RF	-0.171	0.705	1.708	0.950	0.685	0.436
	RKHS	-0.140	0.675	1.525	0.938	0.661	0.389
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
constant	\bar{Y}_{labeled}	0.968	0.972	0.350	0.000	0.090	0.089
	oracle	0.000	0.070	0.281	0.968	0.070	0.072
	LS	0.001	0.063	0.181	0.868	0.063	0.046
	poly	0.001	0.070	0.183	0.812	0.070	0.047
	RF	0.341	0.351	0.215	0.004	0.085	0.055
	RKHS	0.240	0.251	0.211	0.028	0.072	0.054
logistic	LS	0.000	0.072	0.297	0.964	0.073	0.076
	poly	0.000	0.076	0.307	0.956	0.076	0.078
	RF	-0.016	0.121	0.491	0.956	0.120	0.125
	RKHS	-0.015	0.123	0.464	0.938	0.122	0.118
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
constant	\bar{Y}_{labeled}	0.820	0.822	0.244	0.000	0.063	0.062
	oracle	0.001	0.052	0.200	0.946	0.052	0.051
	LS	0.001	0.046	0.142	0.868	0.046	0.036
	poly	0.001	0.051	0.143	0.864	0.051	0.036
	RF	0.241	0.248	0.156	0.006	0.058	0.040
	RKHS	0.149	0.158	0.154	0.102	0.053	0.039
logistic	LS	0.001	0.052	0.203	0.936	0.052	0.052
	poly	0.001	0.053	0.206	0.930	0.053	0.053
	RF	-0.009	0.070	0.294	0.970	0.069	0.075
	RKHS	-0.005	0.069	0.277	0.942	0.069	0.071

and $m(\cdot)$ is close to 95% even with a small $N\bar{\pi}_N = 100$. In high dimensions ($p = 500$), the product rate condition depends on the true PS model. When the true PS model is P1 (MCAR), the corresponding $\hat{\pi}_N(\cdot)$ still has an estimation error of rate $(N\bar{\pi}_N)^{-1/2}$ and hence the product rate condition holds; see Tables 6.5 and 6.6. When the true PS model is P2 (offset logistic), the product rate condition requires $s_m s_\pi = o(N\bar{\pi}_N \{\log(p)\}^{-2})$, where $s_m := \|\boldsymbol{\beta}_0\|_0$ and $s_\pi := \|\boldsymbol{\gamma}_0\|_0$. We can see the coverages are slowly growing towards 95% as $N\bar{\pi}_N$ increases in Tables 6.7 and 6.8. More results with different sparsity levels in the high dimensions can be found in Section 6.1.3. Here, in Tables 6.2, 6.4, and 6.6, we can see fairly good coverages *even if* the outcome model is misspecified and the confidence interval is constructed without a modification. This coincides with the Remarks B.1 and 4.3; see more details in Appendix C

Table 6.4. Simulation setting d with $p = 10$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
\bar{Y}_{labeled}		1.885	1.934	1.617	0.002	0.432	0.412
oracle		0.008	0.168	0.625	0.952	0.168	0.160
constant	LS	-0.929	1.040	1.148	0.226	0.469	0.293
	poly	0.002	0.189	0.492	0.808	0.189	0.125
	RF	0.317	0.443	1.033	0.752	0.309	0.264
	RKHS	0.392	0.472	1.022	0.664	0.263	0.261
	LS	0.378	1.386	3.996	0.910	1.335	1.019
	poly	0.012	0.255	1.011	0.932	0.255	0.258
logistic	RF	0.026	0.573	1.828	0.954	0.573	0.466
	RKHS	-0.014	0.570	1.727	0.950	0.570	0.441
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
\bar{Y}_{labeled}		1.901	1.910	0.728	0.000	0.192	0.186
oracle		-0.002	0.074	0.287	0.944	0.074	0.073
constant	LS	-0.951	0.972	0.473	0.000	0.202	0.121
	poly	-0.002	0.076	0.189	0.798	0.076	0.048
	RF	0.074	0.123	0.323	0.816	0.099	0.082
	RKHS	0.163	0.191	0.328	0.500	0.100	0.084
	LS	0.078	0.485	1.661	0.924	0.479	0.424
	poly	-0.002	0.082	0.318	0.940	0.082	0.081
logistic	RF	0.002	0.181	0.594	0.924	0.181	0.152
	RKHS	0.003	0.178	0.557	0.936	0.178	0.142
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
\bar{Y}_{labeled}		1.371	1.377	0.471	0.000	0.120	0.120
oracle		-0.001	0.053	0.225	0.980	0.053	0.057
constant	LS	-0.746	0.756	0.342	0.000	0.122	0.087
	poly	-0.001	0.052	0.172	0.914	0.052	0.044
	RF	0.010	0.065	0.227	0.924	0.064	0.058
	RKHS	0.053	0.084	0.231	0.838	0.065	0.059
	LS	0.019	0.234	0.951	0.944	0.233	0.243
	poly	-0.001	0.054	0.226	0.976	0.054	0.058
logistic	RF	-0.005	0.100	0.379	0.940	0.099	0.097
	RKHS	0.000	0.094	0.358	0.940	0.094	0.091

of the [Supplement](#). On the other hand, as shown in Tables 6.3, 6.4, 6.7, and 6.8, all the semi-supervised mean estimators based on a constant PS model provide coverages below the desired 95% when the true PS model is logistic. Recall that estimators based on constant PS estimators are essentially the estimators proposed by [Zhang and Bradic \[2022\]](#) (the ones with least squares linear outcome model estimators are further essentially the estimators proposed by [Zhang et al. \[2019\]](#)), where an MCAR setting is considered – this is why we can see poor coverages as selection bias is ignored.

Regarding efficiency, as in Tables 6.1-6.8, we observe that the proposed estimators based on correct parametric models provide “optimal” RMSEs that are close to the oracle estimator. In Tables 6.1-6.4, the RMSEs based on non-parametric (RF and RKHS) $\hat{m}(\cdot)$ are worse than those based on (correctly

Table 6.5. Simulation setting a with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
\bar{Y}_{labeled}	-0.003	0.195	0.788	0.960	0.195	0.201	
	-0.001	0.103	0.400	0.954	0.103	0.102	
constant	Lasso	0.000	0.119	0.478	0.950	0.119	0.122
	poly-Lasso	-0.002	0.120	0.493	0.950	0.120	0.126
log-Lasso	Lasso	0.000	0.120	0.487	0.960	0.120	0.124
	poly-Lasso	-0.002	0.121	0.502	0.952	0.121	0.128
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
\bar{Y}_{labeled}	0.003	0.093	0.352	0.944	0.093	0.090	
	oracle	0.001	0.044	0.178	0.948	0.044	0.046
constant	Lasso	0.001	0.046	0.184	0.952	0.046	0.047
	poly-Lasso	0.001	0.046	0.185	0.952	0.046	0.047
log-Lasso	Lasso	0.001	0.046	0.185	0.948	0.046	0.047
	poly-Lasso	0.001	0.046	0.186	0.952	0.046	0.047
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
\bar{Y}_{labeled}	-0.003	0.063	0.260	0.958	0.063	0.066	
	oracle	-0.002	0.037	0.147	0.956	0.037	0.037
constant	Lasso	-0.002	0.038	0.149	0.960	0.038	0.038
	poly-Lasso	-0.002	0.038	0.149	0.960	0.038	0.038
log-Lasso	Lasso	-0.002	0.038	0.149	0.956	0.038	0.038
	poly-Lasso	-0.002	0.038	0.149	0.960	0.038	0.038

specified) parametric models with the difference arising from the product rate of the estimation errors of $\hat{\pi}_N(\cdot)$ and $\hat{m}(\cdot)$. For a (correctly specified) $\pi_N(\cdot)$ with an estimation error $O_p((N\bar{\pi}_N)^{-1/2})$, such a difference is not significant and the RMSE is first order insensitive to estimation error of $\hat{m}(\cdot)$.

We also check the double robustness in terms of consistency of the proposed estimators, when only one of $\pi_N(\cdot)$ and $m(\cdot)$ is correct. As seen in Tables 6.3-6.6, the naive mean estimator, \bar{Y}_{labeled} , is *not* consistent when the selection bias occurs, i.e., the PS is not a constant. Nevertheless, as suggested by Theorem 3.2, the proposed DRSS estimator is still consistent, and its consistency rate depends on the estimation error rate of the correct one among $\hat{\pi}_N(\cdot)$ and $\hat{m}(\cdot)$. The proposed $\hat{\theta}_{\text{DRSS}}$ can still be $(N\bar{\pi}_N)^{1/2}$ -consistent when the correct estimator has an estimation error of rate $(N\bar{\pi}_N)^{-1/2}$, which is typically true when the correct model is a low dimensional parametric model: see Tables 6.2, 6.4, and 6.6 for correct $\pi_N(\cdot)$; see Tables 6.3 and 6.4 for correct $m(\cdot)$. If the correct estimator is linear (offset logistic) in high dimensions, $\hat{\theta}_{\text{DRSS}} - \theta_0$ is of the order $O_p(\{(N\bar{\pi}_N)^{-1}s \log(p)\}^{1/2})$, where s is the sparsity of the correct model; see results in Tables 6.7 and 6.8. If the correct estimator is non-parametric, we would expect a non-parametric rate on the proposed estimator, matching results in Tables 6.3 and 6.4.

6.1.2 Results under the stratified labeling model. Now we work on a special PS model, that of stratified labeling, as discussed in Section 4.2:

P3. (Stratified PS) Suppose we further observe the stratum indicators $\delta_i \in \{0, 1\}$, with the following model: $\delta_i | \mathbf{X}_i \sim \text{Bernoulli}(p_\delta(\mathbf{X}_i))$ and $R_i | \delta_i \sim \text{Bernoulli}(0.5\bar{\pi}_N\delta_i + 1.5\bar{\pi}_N(1 - \delta_i))$, where $p_\delta(\mathbf{x}) = g(\mathbf{x}(1))$ and $g(u) = \exp(u)/\{1 + \exp(u)\}$.

We consider the same choices of p , N , and $\bar{\pi}_N$ as in Section 6.1.1, and we focus on the following

Table 6.6. Simulation setting b with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
constant	\bar{Y}_{labeled}	0.003	0.306	1.244	0.948	0.306	0.317
	oracle	-0.003	0.107	0.411	0.934	0.107	0.105
	Lasso	0.004	0.296	1.220	0.966	0.296	0.311
	poly-Lasso	0.000	0.174	0.668	0.954	0.175	0.171
log-Lasso	Lasso	0.008	0.297	1.241	0.966	0.298	0.317
	poly-Lasso	0.002	0.175	0.680	0.954	0.175	0.173
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
constant	\bar{Y}_{labeled}	-0.004	0.147	0.554	0.926	0.147	0.141
	oracle	-0.002	0.054	0.215	0.954	0.054	0.055
	Lasso	-0.005	0.129	0.483	0.928	0.129	0.123
	poly-Lasso	-0.002	0.051	0.195	0.934	0.051	0.050
log-Lasso	Lasso	-0.004	0.129	0.484	0.926	0.129	0.124
	poly-Lasso	-0.002	0.051	0.196	0.936	0.051	0.050
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
constant	\bar{Y}_{labeled}	-0.001	0.104	0.411	0.948	0.104	0.105
	oracle	0.001	0.042	0.175	0.964	0.043	0.045
	Lasso	-0.001	0.091	0.357	0.944	0.092	0.091
	poly-Lasso	0.001	0.043	0.178	0.960	0.043	0.046
log-Lasso	Lasso	-0.001	0.091	0.357	0.948	0.091	0.091
	poly-Lasso	0.001	0.043	0.179	0.962	0.043	0.046

DGP of Setting e, i.e., P3+O2. Furthermore, we consider an additional PS estimator based on the stratified labeling of Section 4.2, where $p_\delta(\cdot)$ is estimated by a logistic regression (with a Lasso-type regularization when $p = 500$). The results are shown in Tables 6.9 and 6.10 for the case $p = 10$ and $p = 500$, respectively.

By Theorem 3.2, the estimators based on a correctly specified $\pi_N(\cdot)$ (stratified) and $m(\cdot)$ (poly/RF/RKHS) provide $(N\bar{\pi}_N)^{1/2}$ -consistent estimations and valid asymptotic confidence intervals. Additionally, when $\pi_N(\cdot)$ is correctly specified (stratified) and $m(\cdot)$ is misspecified (linear), the proposed DRSS mean estimator has a consistency rate of $O_p((N\bar{\pi}_N)^{-1/2} + r_{\delta,N})$, with $r_{\delta,N}$, defined in (4.17), satisfying $r_{\delta,N} = O(N^{-1/2})$ in low dimensions and $r_{\delta,N} = O(\{s_\delta \log(p)\}^{1/2} N^{-1/2})$, in high dimensions, as discussed in Example 4.4. The simulation results in Tables 6.9 and 6.10 support our theoretical arguments: we can see the stratified+poly/RF/RKHS estimators provide coverages close to 95%, and all the estimators based on a stratified $\hat{\pi}_N(\cdot)$ provide RMSEs of a similar magnitude.

6.1.3 Results for high dimensional sparse models: Investigating performance under varying sparsity levels. Here we focus on the high dimensional case ($p = 500$) with different sparsity levels. We consider the following PS and outcome models:

P2'. (Offset logistic PS with different sparsity levels) Let $\pi_N(\mathbf{x}) = g(\vec{\mathbf{x}}^T \boldsymbol{\gamma}_{s_\pi} + \log(\bar{\pi}_N))$ and $R_i | \mathbf{X}_i \sim \text{Bernoulli}(\pi_N(\mathbf{X}_i))$, where $g(u) = \exp(u)/\{1 + \exp(u)\}$.

O1'. (Linear outcome with different sparsity levels) Let $Y_i = \vec{\mathbf{X}}_i^T \boldsymbol{\beta}_{s_m} + \varepsilon_i$.

Table 6.7. Simulation setting c with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
constant	$\tilde{Y}_{\text{labeled}}$	0.977	0.997	0.782	0.006	0.198	0.199
	oracle	-0.003	0.160	0.612	0.970	0.160	0.156
	Lasso	0.275	0.320	0.480	0.412	0.164	0.123
	poly-Lasso	0.324	0.368	0.496	0.320	0.173	0.127
log-Lasso	Lasso	0.099	0.210	0.563	0.782	0.186	0.144
	poly-Lasso	0.117	0.229	0.602	0.778	0.197	0.154
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
constant	$\tilde{Y}_{\text{labeled}}$	0.975	0.979	0.350	0.000	0.087	0.089
	oracle	-0.003	0.071	0.282	0.956	0.071	0.072
	Lasso	0.115	0.132	0.184	0.370	0.065	0.047
	poly-Lasso	0.130	0.146	0.185	0.306	0.067	0.047
log-Lasso	Lasso	0.022	0.072	0.241	0.884	0.069	0.061
	poly-Lasso	0.026	0.075	0.243	0.886	0.070	0.062
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
constant	$\tilde{Y}_{\text{labeled}}$	0.822	0.824	0.245	0.000	0.065	0.062
	oracle	0.005	0.051	0.198	0.958	0.050	0.050
	Lasso	0.077	0.090	0.143	0.438	0.047	0.036
	poly-Lasso	0.086	0.098	0.143	0.386	0.047	0.036
log-Lasso	Lasso	0.019	0.053	0.173	0.894	0.049	0.044
	poly-Lasso	0.021	0.054	0.174	0.896	0.050	0.044

The parameter values are:

$$\boldsymbol{\beta}_{s_m} = (-0.5, \sqrt{3/s_m} \mathbf{1}_{1 \times s_m}, \mathbf{0}_{1 \times (p-s_m)})^T \text{ and } \boldsymbol{\gamma}_{s_\pi} = (\boldsymbol{\gamma}_0(1), \sqrt{1/s_\pi} \mathbf{1}_{1 \times s_\pi}, \mathbf{0}_{1 \times (p-s_\pi)})^T.$$

We consider a DGP, Setting c': P2'+O1', with the following choices of $p, N, \bar{\pi}_N, s_m$ and s_π :

$$p = 500, N \in \{50000, 200000\}, \bar{\pi}_N = 0.01, (s_m, s_\pi) \in \{(3, 15), (15, 3)\}.$$

We illustrate the performance of the same estimators that we considered in Section 6.1.1 (for $p = 500$); the results are presented in Table 6.11.

In Table 6.11, we observe that the RMSEs of $\hat{\theta}_{\text{DRSS}}$ based on log-Lasso PS estimators are smaller than those based on constant PS estimators. This coincides with our Remark 3.6, as well as Theorems 3.2 and 4.2 - if both of the nuisance functions are correctly specified, we have $\hat{\theta}_{\text{DRSS}} - \theta_0 = O_p((N\bar{\pi}_N)^{-1/2} + \sqrt{s_m s_\pi} \log(p)/(N\bar{\pi}_N))$; if only the outcome model is correctly specified, we have a slower upper bound $\hat{\theta}_{\text{DRSS}} - \theta_0 = O_p(\sqrt{s_m \log(p)/(N\bar{\pi}_N)})$. For the DRSS estimators based on log-Lasso PS estimators, we can see that the biases of the estimators, originating from the product rate $\sqrt{s_m s_\pi} \log(p)/(N\bar{\pi}_N)$, are non-ignorable compared to the RMSEs, especially for smaller N . As N grows, however, the coverages of the confidence intervals start getting closer to the desired 95% level. This also coincides with our Remark 3.6 that we expect a valid inference result when $s_m s_\pi \log(p) = o(N\bar{\pi}_N)$ for correctly specified models.

6.1.4 Results when both nuisance models are misspecified. In this section, we further investigate the case that both nuisance models (i.e., the outcome and PS models) are misspecified and compare the performance of various SS estimators in such cases. We consider the following nuisance models:

Table 6.8. Simulation setting d with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
constant	$\tilde{Y}_{\text{labeled}}$	1.875	1.917	1.620	0.000	0.401	0.413
	oracle	-0.002	0.157	0.613	0.970	0.157	0.156
log-Lasso	Lasso	-0.183	0.535	1.225	0.756	0.503	0.313
	poly-Lasso	0.273	0.360	0.625	0.566	0.235	0.160
log-Lasso	Lasso	-0.229	0.621	1.511	0.736	0.578	0.386
	poly-Lasso	0.090	0.263	0.708	0.824	0.247	0.181
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
constant	$\tilde{Y}_{\text{labeled}}$	1.890	1.900	0.731	0.000	0.191	0.187
	oracle	-0.006	0.074	0.286	0.954	0.074	0.073
log-Lasso	Lasso	-0.657	0.686	0.481	0.020	0.200	0.123
	poly-Lasso	0.086	0.113	0.194	0.562	0.074	0.049
log-Lasso	Lasso	-0.254	0.377	0.968	0.684	0.279	0.247
	poly-Lasso	0.010	0.076	0.247	0.888	0.075	0.063
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
constant	$\tilde{Y}_{\text{labeled}}$	1.355	1.360	0.470	0.000	0.118	0.120
	oracle	0.000	0.058	0.224	0.964	0.058	0.057
log-Lasso	Lasso	-0.551	0.566	0.343	0.002	0.128	0.088
	poly-Lasso	0.055	0.080	0.175	0.706	0.057	0.045
log-Lasso	Lasso	-0.179	0.250	0.649	0.706	0.175	0.165
	poly-Lasso	0.008	0.058	0.198	0.910	0.058	0.051

P3. (Offset logistic PS with quadratic signals) $\pi_N(\mathbf{x}) = g(\vec{\mathbf{x}}^T \boldsymbol{\gamma}_0 + \sum_{j=1}^{p+1} \boldsymbol{\delta}_0(j) \vec{\mathbf{X}}_i(j)^2 + \log(\bar{\pi}_N))$, where $g(\cdot)$ is defined in (4.2).

O2. (Quadratic outcome) $Y_i = \vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0 + \sum_{j=1}^{p+1} \boldsymbol{\alpha}_0(j) \vec{\mathbf{X}}_i(j)^2 + \varepsilon_i$.

Under P3+O2, three new DGPs, g1, g2, and g3, are considered. The DGPs g1-g3 only varies from the parameter values, and we choose the parameters as follows:

g1. $\boldsymbol{\gamma}_0 = (\gamma_1, 0.3, 0.3, 0.3, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\delta}_0 = (0, 0.3, 0.3, 0.3, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\beta}_0 = (2, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T + r_1 \boldsymbol{\gamma}_0$, $\boldsymbol{\alpha}_0 = r_1 \boldsymbol{\delta}_0$.

g2. $\boldsymbol{\gamma}_0 = (\gamma_2, 0.5, 0.5, 0.5, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\delta}_0 = r_2 (0, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\beta}_0 = (2, 0.3, 0.3, 0.3, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\alpha}_0 = (0, 0.3, 0.3, 0.3, \mathbf{0}_{1 \times (p-3)})^T$.

g3. $\boldsymbol{\gamma}_0 = (\gamma_3, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\delta}_0 = \boldsymbol{\alpha}_0 = r_3 (0, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T$, $\boldsymbol{\beta}_0 = (2, 1, 1, 1, \mathbf{0}_{1 \times (p-3)})^T$.

In the above, γ_1 , γ_2 , and γ_3 are chosen so that $\mathbb{E}(R) = \bar{\pi}_N = 0.05$ for each DGP; r_1 , r_2 , and r_3 denote the misspecification levels. For Setting g1, we choose $r_1 \in \{1, 0.5, 0.3, 0.1\}$; the PS model has a non-ignorable misspecification error, and the outcome model's misspecification error decays as r_1 decreases. For Setting g2, we choose $r_2 \in \{0.3, 0.2, 0.1, 0.05\}$; the outcome model has a non-ignorable misspecification error, and the PS model's misspecification error decays as r_2 decreases. For Setting g3, we choose $r_3 \in \{0.3, 0.2, 0.1, 0.05\}$; both model's misspecification error decays as r_3 decreases.

As in Section 6.1.1, we also consider the naive labeled mean estimator $\tilde{Y}_{\text{labeled}}$, the proposed estimator $\hat{\theta}_{\text{DRSS}}$ (based on a Lasso estimate for the outcome model and a log-Lasso/ constant estimate for

Table 6.9. Simulation setting e with $p = 10$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
constant	\bar{Y}_{labeled}	-0.205	0.360	1.203	0.876	0.297	0.307
	oracle	-0.002	0.101	0.415	0.964	0.101	0.106
	LS	-0.082	0.318	1.149	0.916	0.308	0.293
	poly	-0.009	0.125	0.493	0.954	0.125	0.126
	RF	-0.100	0.259	0.919	0.908	0.239	0.234
	RKHS	-0.087	0.248	0.909	0.918	0.232	0.232
logistic	LS	0.129	0.419	1.639	0.960	0.399	0.418
	poly	-0.011	0.146	0.633	0.976	0.146	0.162
	RF	0.075	0.328	1.346	0.960	0.320	0.343
	RKHS	0.109	0.341	1.337	0.974	0.323	0.341
	LS	0.009	0.324	1.256	0.948	0.324	0.320
	poly	-0.010	0.126	0.510	0.958	0.126	0.130
stratified	RF	0.021	0.268	1.062	0.956	0.268	0.271
	RKHS	0.016	0.262	1.042	0.960	0.262	0.266
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
constant	\bar{Y}_{labeled}	-0.210	0.251	0.532	0.652	0.138	0.136
	oracle	-0.001	0.048	0.187	0.970	0.048	0.048
	LS	-0.078	0.150	0.469	0.882	0.128	0.120
	poly	-0.001	0.050	0.188	0.942	0.050	0.048
	RF	-0.053	0.101	0.296	0.832	0.086	0.075
	RKHS	-0.053	0.098	0.299	0.866	0.083	0.076
logistic	LS	-0.014	0.135	0.531	0.952	0.135	0.135
	poly	-0.001	0.051	0.199	0.946	0.051	0.051
	RF	-0.011	0.091	0.349	0.944	0.090	0.089
	RKHS	-0.002	0.090	0.353	0.942	0.090	0.090
	LS	-0.001	0.132	0.508	0.950	0.132	0.130
	poly	-0.001	0.050	0.193	0.944	0.050	0.049
stratified	RF	-0.006	0.089	0.333	0.956	0.089	0.085
	RKHS	-0.005	0.088	0.336	0.940	0.088	0.086
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
constant	\bar{Y}_{labeled}	-0.202	0.224	0.377	0.428	0.097	0.096
	oracle	0.001	0.041	0.173	0.970	0.041	0.044
	LS	-0.076	0.120	0.334	0.836	0.093	0.085
	poly	0.001	0.042	0.172	0.968	0.042	0.044
	RF	-0.033	0.066	0.215	0.868	0.057	0.055
	RKHS	-0.034	0.067	0.219	0.906	0.057	0.056
logistic	LS	-0.018	0.098	0.370	0.928	0.096	0.094
	poly	0.001	0.042	0.176	0.968	0.042	0.045
	RF	-0.004	0.060	0.240	0.954	0.060	0.061
	RKHS	0.001	0.061	0.245	0.952	0.061	0.063
	LS	0.002	0.095	0.363	0.938	0.095	0.093
	poly	0.001	0.042	0.175	0.968	0.042	0.045
stratified	RF	0.002	0.059	0.236	0.952	0.059	0.060
	RKHS	0.002	0.060	0.241	0.948	0.060	0.061

the PS model) and its oracle version. Here we use ‘‘DRSS’’ to refer in particular to the estimator $\hat{\theta}_{\text{DRSS}}$ based on Lasso and log-Lasso estimates of the respective nuisance models; ‘‘SS’’ denotes the one based

Table 6.10. Simulation setting e with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$N = 10000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 100)$							
\bar{Y}_{labeled} oracle	-0.189	0.363	1.193	0.880	0.311	0.304	
	0.005	0.098	0.417	0.960	0.098	0.106	
constant	Lasso	-0.167	0.352	1.186	0.888	0.310	0.303
	poly-Lasso	-0.103	0.220	0.696	0.894	0.195	0.178
log-Lasso	Lasso	-0.160	0.351	1.215	0.904	0.312	0.310
	poly-Lasso	-0.096	0.218	0.717	0.894	0.196	0.183
stratified	Lasso	0.005	0.339	1.318	0.926	0.339	0.336
	poly-Lasso	0.010	0.200	0.780	0.948	0.200	0.199
$N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
\bar{Y}_{labeled} oracle	-0.196	0.239	0.535	0.702	0.136	0.136	
	0.000	0.048	0.187	0.950	0.048	0.048	
constant	Lasso	-0.134	0.187	0.478	0.776	0.130	0.122
	poly-Lasso	-0.032	0.061	0.195	0.896	0.052	0.050
log-Lasso	Lasso	-0.101	0.163	0.491	0.844	0.128	0.125
	poly-Lasso	-0.022	0.056	0.196	0.914	0.052	0.050
stratified	Lasso	0.001	0.137	0.530	0.946	0.137	0.135
	poly-Lasso	0.000	0.052	0.202	0.944	0.052	0.051
$N = 10000, \bar{\pi}_N = 0.1 (N\bar{\pi}_N = 1000)$							
\bar{Y}_{labeled} oracle	-0.205	0.226	0.377	0.428	0.095	0.096	
	-0.001	0.044	0.173	0.944	0.044	0.044	
constant	Lasso	-0.124	0.151	0.335	0.688	0.087	0.086
	poly-Lasso	-0.024	0.051	0.173	0.900	0.045	0.044
log-Lasso	Lasso	-0.082	0.119	0.347	0.854	0.087	0.088
	poly-Lasso	-0.013	0.047	0.174	0.928	0.045	0.044
stratified	Lasso	-0.013	0.090	0.367	0.956	0.089	0.094
	poly-Lasso	-0.002	0.045	0.176	0.948	0.045	0.045

on a Lasso outcome and constant PS estimates, as it is the same as the semi-supervised (SS) estimator of [Zhang and Bradic \[2022\]](#), except here we use cross-fitted PS estimates. In addition, we further implement the following estimators that have been considered in [Kang and Schafer \[2007\]](#): 1) two variants of the IPW estimator, IPW-POP (re-weighting the respondents to resemble the full population) and IPW-NR (re-weighting the respondents to resemble the nonrespondents' population), where the PS models are estimated by log-Lasso, and 2) the regression-based (Reg) estimator $\hat{\theta}_{\text{Reg}}$ using a Lasso based outcome model estimate, which can be seen as a regularized version of the OLS regression estimate used therein. We choose $N = 2000$, $p = 100$, and repeat the simulations 500 times. In Table 6.12, we report the bias and the root mean square error (RMSE) based on the considered estimators.

Similarly to Section 6.1, the naive estimator \bar{Y}_{labeled} has a poor performance since it converges to $\mathbb{E}(Y | R = 1) \neq \mathbb{E}(Y)$. When both models have non-ignorable misspecification errors, none of the estimators has a good performance, and as also noticed by [Kang and Schafer \[2007\]](#), a doubly-robust-type estimator may perform even worse than both the IPW and Reg estimators under such a case. As in Table 6.12, DRSS has a larger RMSE than the IPW estimators and the Reg estimator when $r_1 = 1$ under Setting g1, $r_2 \in \{0.3, 0.2\}$ under Setting g2, and $r_3 = 0.3$ under Setting g3; it is also worse than the Reg estimator when $r_3 = 0.2$ under Setting g3. However, when at least one misspecification level is relatively small, the DRSS estimator outperforms all the others in the sense of the RMSEs (except the

Table 6.11. Simulation setting c' with $p = 500$. The rest of the caption details remain the same as those in Table 6.1.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length	Coverage	ESD	ASD
$s_m = 3, s_\pi = 15, N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
$\tilde{Y}_{\text{labeled}}$	0.755	0.760	0.351	0.000	0.090	0.090	
	oracle	0.000	0.074	0.284	0.944	0.074	0.072
constant	Lasso	0.089	0.103	0.184	0.522	0.052	0.047
	poly-Lasso	0.098	0.111	0.185	0.458	0.052	0.047
log-Lasso	Lasso	0.038	0.071	0.203	0.860	0.060	0.052
	poly-Lasso	0.042	0.073	0.204	0.848	0.060	0.052
$s_m = 3, s_\pi = 15, N = 200000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 2000)$							
$\tilde{Y}_{\text{labeled}}$	0.754	0.756	0.175	0.000	0.045	0.045	
	oracle	0.000	0.035	0.143	0.960	0.035	0.037
constant	Lasso	0.044	0.051	0.090	0.506	0.025	0.023
	poly-Lasso	0.049	0.055	0.090	0.430	0.025	0.023
log-Lasso	Lasso	0.012	0.032	0.111	0.914	0.030	0.028
	poly-Lasso	0.013	0.033	0.111	0.914	0.030	0.028
$s_m = 15, s_\pi = 3, N = 50000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 500)$							
$\tilde{Y}_{\text{labeled}}$	0.752	0.757	0.349	0.000	0.085	0.089	
	oracle	0.001	0.068	0.283	0.966	0.068	0.072
constant	Lasso	0.155	0.169	0.196	0.200	0.068	0.050
	poly-Lasso	0.184	0.197	0.201	0.118	0.070	0.051
log-Lasso	Lasso	0.049	0.086	0.237	0.824	0.071	0.060
	poly-Lasso	0.058	0.094	0.245	0.794	0.074	0.063
$s_m = 15, s_\pi = 3, N = 200000, \bar{\pi}_N = 0.01 (N\bar{\pi}_N = 2000)$							
$\tilde{Y}_{\text{labeled}}$	0.753	0.754	0.175	0.000	0.044	0.045	
	oracle	0.000	0.036	0.144	0.964	0.036	0.037
constant	Lasso	0.076	0.082	0.091	0.172	0.032	0.023
	poly-Lasso	0.089	0.094	0.091	0.094	0.032	0.023
log-Lasso	Lasso	0.013	0.037	0.124	0.912	0.034	0.032
	poly-Lasso	0.015	0.038	0.125	0.904	0.035	0.032

oracle one); see Setting g1 with $r_1 \in \{0.5, 0.3, 0.1\}$, Setting g2 with $r_2 \in \{0.1, 0.05\}$, and Setting g3 with $r_3 \in \{0.1, 0.05\}$. In addition, similarly to the DRSS estimator, the SS estimator can also be seen as a de-biased version of the Reg estimator. However, since the bias correction is constructed based on a constant PS model, which is far away from the truth, such a correction seems to move the estimate in the wrong direction – SS always performs worse than Reg in Table 6.12.

6.2 Application to the NHEFS data

We now apply our imbalanced ATE estimator proposed in Section 5 to assess the effect of smoking and alcohol drinking on weight gain, using a subset of data from the National Health and Nutrition Examination Survey Data I Epidemiologic Follow-up Study (NHEFS). As per Hernán and Robins [2023], the NHEFS was jointly initiated by the National Center for Health Statistics and the National Institute on Aging in collaboration with other agencies of the United States Public Health Service. The NHEFS dataset has been studied by Hernán and Robins [2023] and Ertefaie et al. [2022]. The subset of the NHEFS data we consider consists of $N = 1561$ cigarette smokers aged 25 – 74 years, who had a baseline visit and a follow-up visit approximately 10 years later. We consider two types of product (joint)

Table 6.12. Simulations under Settings g1, g2, and g3, with $p = 100, N = 2000$ and $\bar{\pi}_N = 0.05$ ($N\bar{\pi}_N = 100$). Bias: empirical bias; RMSE: root mean square error. The words “bad” and “good” indicate that the models have large and small misspecification errors, respectively. The symbol “ \rightarrow ” indicates a change in the model misspecification levels, which is characterized by r_1, r_2 , and r_3 – the smaller the values are, the smaller the misspecification error is. The green color denotes the smallest RMSE among the estimators (except the oracle one) for each setting.

$\hat{\theta}$	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Setting g1 (“bad” outcome + “bad” PS \rightarrow “good” outcome + “bad” PS)								
	$r_1 = 1$		$r_1 = 0.5$		$r_1 = 0.3$		$r_1 = 0.1$	
\bar{Y}_{labeled}	2.404	2.430	1.812	1.835	1.575	1.597	1.337	1.361
oracle	0.005	0.152	0.005	0.149	0.005	0.148	0.005	0.147
IPW-POP	0.767	0.861	0.371	0.526	0.212	0.426	0.054	0.373
IPW-NR	0.758	0.854	0.363	0.520	0.205	0.422	0.047	0.371
Reg	0.914	0.937	0.523	0.543	0.377	0.401	0.240	0.274
SS	0.965	0.988	0.561	0.581	0.408	0.432	0.263	0.297
DRSS	0.891	0.938	0.429	0.472	0.242	0.303	0.054	0.198
Setting g2 (“bad” outcome + “bad” PS \rightarrow “bad” outcome + “good” PS)								
	$r_2 = 0.3$		$r_2 = 0.2$		$r_2 = 0.1$		$r_2 = 0.05$	
\bar{Y}_{labeled}	1.405	1.415	1.147	1.160	0.853	0.867	0.709	0.727
oracle	-0.003	0.173	0.002	0.153	-0.005	0.143	-0.002	0.146
IPW-POP	0.735	0.797	0.477	0.531	0.295	0.337	0.228	0.278
IPW-NR	0.730	0.793	0.469	0.525	0.287	0.331	0.222	0.274
Reg	0.943	0.984	0.585	0.636	0.295	0.367	0.178	0.273
SS	1.027	1.061	0.664	0.709	0.362	0.421	0.246	0.317
DRSS	0.921	1.099	0.537	0.658	0.228	0.328	0.133	0.254
Setting g3 (“bad” outcome + “bad” PS \rightarrow “good” outcome + “good” PS)								
	$r_3 = 0.3$		$r_3 = 0.2$		$r_3 = 0.1$		$r_3 = 0.05$	
\bar{Y}_{labeled}	3.635	3.641	3.147	3.154	2.721	2.728	2.524	2.530
oracle	-0.007	0.349	0.008	0.362	-0.010	0.348	0.028	0.317
IPW-POP	0.959	1.313	0.779	1.048	0.661	0.896	0.768	0.920
IPW-NR	0.896	1.237	0.708	0.977	0.597	0.838	0.709	0.865
Reg	0.957	1.042	0.569	0.650	0.444	0.519	0.480	0.535
SS	1.102	1.176	0.693	0.758	0.527	0.589	0.557	0.603
DRSS	0.780	1.638	0.396	0.773	0.191	0.399	0.228	0.426

treatment indicators $R_1^{(1)}, R_1^{(2)} \in \{0, 1\}$: $R^{(1)} = 1$ denotes that the individual has not quit smoking before the follow-up visit and has not drunk alcohol before the baseline visit, and $R^{(1)} = 0$ otherwise; $R^{(2)} = 1$ denotes that the individual has quit smoking before the follow-up visit and has not drunk alcohol before the baseline visit, and $R^{(2)} = 0$ otherwise. We omit 5 individuals whose alcohol drinking information was missing. The weight gain (in kg), $Y \in \mathbb{R}$, was measured as the body weight at the follow-up visit minus the body weight at the baseline visit. Same as in Hernán and Robins [2023] and Ertefaie et al. [2022], the following 9 confounding variables, \mathbf{X} are considered: sex (0: male, 1: female), age (in years), race (0: white, 1: other), education (5 categories), intensity and duration of smoking (number of cigarettes per day and years of smoking), physical activity in daily life (3 categories), recreational exercise (3 categories), and weight (in kg).

We estimate the ATE of the product (joint) treatments $R^{(1)}$ and $R^{(2)}$ on weight gain. The ATE

estimators $\widehat{\theta}_{ATE}^{(1)}$ and $\widehat{\theta}_{ATE}^{(2)}$ are constructed using (5.2), based on samples $\mathbb{S}^{(1)} := (Y_i, R_i^{(1)}, \mathbf{X}_i)_{i=1}^N$ and $\mathbb{S}^{(2)} := (Y_i, R_i^{(2)}, \mathbf{X}_i)_{i=1}^N$, respectively. Recall that the sample size is $N = 1561$, and the dimension (after converting categorical variables) is $p = 12$. The estimated proportions of the treated groups are $\widehat{\pi}_N^{(1)} := N^{-1} \sum_{i=1}^N R_i^{(1)} = 0.088$ and $\widehat{\pi}_N^{(2)} := N^{-1} \sum_{i=1}^N R_i^{(2)} = 0.037$, which clearly indicates that the treatment groups are very unbalanced. We consider three PS estimators: a constant estimator, an offset based logistic estimator with a Lasso-type penalty (log-Lasso), and a random forest (RF). We consider four outcome models: a Lasso estimator, a degree-2 polynomial estimator without interactions and with a Lasso-type penalty (poly-Lasso), a random forest (RF), and a reproducing kernel Hilbert space (RKHS) estimator. We compare the proposed estimators with naive empirical difference (empdiff) estimators, $(\sum_{i=1}^N R_i^{(j)})^{-1} \sum_{i=1}^N R_i^{(j)} Y_i - \{\sum_{i=1}^N (1 - R_i^{(j)})\}^{-1} \sum_{i=1}^N (1 - R_i^{(j)}) Y_i$, for $j = 1, 2$. To reduce the randomness coming from the sample splitting, we repeat the sample splitting for $B = 10$ times and report the median of the ATE estimators based on each split. The asymptotic variance is then estimated by a plugged-in version using the mean estimators as well as the asymptotic variance estimators based on each split; see more details of this technique in Definition 3.3 of Chernozhukov et al. [2018].

We report the ATE estimators, the corresponding 95% confidence intervals, and the length of the confidence intervals in the Table 6.13. We can see negative estimated ATEs for $\theta_{ATE}^{(1)}$ and positive estimated ATEs for $\theta_{ATE}^{(2)}$. Moreover, our proposed ATE estimators are close to each other and fairly different from the empirical difference estimator, especially for $\theta_{ATE}^{(2)}$. Therein, all our confidence intervals do not include 0 while the one based on the empirical difference does. The difference between our proposed ATE estimators and the empirical difference estimator seems to suggest presence of substantial confounding via \mathbf{X} , and a significant causal effect of the treatment on the response after adjusting for the confounding.

7. Discussion

In this paper, we study the mean estimation problem in the semi-supervised setting with a decaying PS while allowing for selection bias in the labeling mechanism. To our knowledge this is one of the first full-hearted attempts in extending the SS inference literature to the case of selection bias, and that too in a very general way, as well as the MAR literature to the case of a (uniformly) decaying PS. The proposed DRSS mean estimator is based on estimators of the outcome and the decaying PS models. We establish estimation and inference results under different cases of the correctness of the models, while allowing flexible model choices, including high-dimensional and non-parametric methods. The subtleties of the problem setting and the non-standard asymptotics, among others, make the method and its analyses challenging and our results reveal several novel insights in the process. In particular, we find that the consistency rate of the proposed estimator depends on the (expected) size of the labeled sample and the tail of the PS distribution. Throughout the paper, Na_N (recall that $a_N = [\mathbb{E}\{\pi_N^{-1}(\mathbf{X})\}]^{-1}$) is a crucial value, in that it serves as the “effective sample size” in our MAR-SS setting with a decaying PS. This paper provides details as to why this happens.

As a necessary component of analyzing the MAR-SS setting, we further propose estimators of the decaying PS under three different models: MCAR, stratified labeling, and a novel offset logistic model, under both high and low dimensional settings. The consistency rates of the PS models are established, which are of independent interest. We also extend our methods to an ATE estimation problem where the treatment groups can be extremely imbalanced. We provide extensive numerical studies to illustrate our results in finite-sample simulations, as well as a real data analysis using the NHEFS data.

Table 6.13. **Real data analysis:** estimation and inference of $\theta_{ATE}^{(1)}$ and $\theta_{ATE}^{(2)}$. We compare a naive empirical difference (empdiff) estimator with our proposed estimators based on various choices of nuisance estimators. ATE: the estimated average treatment effect; CI: a 95% confidence interval; Length: length of the 95% confidence interval.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	$\theta_{ATE}^{(1)}$			$\theta_{ATE}^{(2)}$		
		ATE	CI	Length	ATE	CI	Length
	empdiff	-2.003	(-3.282,-0.725)	2.558	1.867	(-0.642,4.377)	5.019
constant	Lasso	-1.935	(-3.219,-0.651)	2.568	4.209	(1.743,6.676)	4.933
	poly-Lasso	-1.865	(-3.152,-0.578)	2.574	3.291	(0.719,5.864)	5.145
	RF	-1.729	(-2.992,-0.466)	2.526	3.095	(0.607,5.584)	4.977
	RHKS	-1.941	(-3.227,-0.655)	2.573	3.183	(0.642,5.723)	5.081
log-Lasso	Lasso	-1.967	(-3.518,-0.416)	3.102	4.780	(1.954,7.605)	5.650
	poly-Lasso	-1.717	(-3.321,-0.113)	3.207	3.890	(0.804,6.976)	6.172
	RF	-1.873	(-3.424,-0.322)	3.102	3.890	(0.955,6.825)	5.870
	RHKS	-2.051	(-3.663,-0.440)	3.223	4.221	(1.068,7.375)	6.307
RF	Lasso	-1.727	(-2.970,-0.484)	2.486	4.932	(1.518,8.345)	6.827
	poly-Lasso	-1.612	(-2.878,-0.346)	2.532	4.693	(0.763,8.622)	6.827
	RF	-1.608	(-2.845,-0.371)	2.474	4.456	(0.942,7.970)	7.027
	RHKS	-1.772	(-3.016,-0.528)	2.487	4.411	(0.621,8.200)	7.579

The semi-supervised decaying PS setting is an interesting scenario that occurs in numerous applications in the modern era, and yet has been largely under-studied so far. We provide a detailed analysis of the mean estimation problem under such setting. We hope it serves as a start towards understanding this practically very relevant, and yet technically challenging, scenario in all its subtleties, therefore opening doors to many new questions where inferential results need to be adjusted for the “effective sample size”.

Supplementary Material

Supplement to “Double Robust Semi-supervised Inference for the Mean: Selection Bias under MAR Labeling with Decaying Overlap”. In the supplement (Appendices A-D), we present additional results and discussions, as well as all proofs of the main theoretical results. We provide some further discussions on the offset based PS model in Appendix A. In Appendix B, we provide additional results on the MCAR PS model. We illustrate additional numerical simulation results for the adjusted confidence intervals in Appendix C. The proofs of the theoretical results are included in Appendix D.

Data Availability Statement

The NHEFS data are available at <https://www.hsph.harvard.edu/miguel-hernan/causal-inference-book/>. This data set, along with the R codes for all the simulation studies as well as real data analysis presented

in this article, are available separately at <https://github.com/yuqianruc/DRSS-code>.

Funding Acknowledgement

AC's research was supported in part by the National Science Foundation grant NSF DMS-2113768. JB's research was supported in part by the National Science Foundation grant NSF DMS-1712481.

The authors would also like to thank the Editor, the anonymous Associate Editor and the two Reviewers for their constructive comments and useful suggestions that helped significantly improve the article.

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Supplementary Material to “Double Robust Semi-Supervised Inference for the Mean: Selection Bias under MAR Labeling with Decaying Overlap”

This supplementary document (Appendices A–D) contains additional results and discussions that could not be accommodated in the main manuscript, as well as the proofs of the main theoretical results. All results and notation are numbered and used as in the main text unless stated otherwise.

ORGANIZATION The rest of the document is organized as follows. First, we introduce the (additional) notations we will use in the document. In Appendix A, we provide some further discussions on the offset based propensity score (PS) model. In Appendix B, we provide additional results on the Missing completely at random (MCAR) PS model. In Appendix C, we illustrate additional numerical simulation results for the adjusted confidence intervals. The proofs of the theoretical results are included in Appendix D.

NOTATION Constants $c, C > 0$, independent of N and p , may change values from one line to the other. For any $\tilde{\mathbb{S}} \subseteq \mathbb{S} = (\mathbf{Z}_i)_{i=1}^N$, define $\mathbb{P}_{\tilde{\mathbb{S}}}$ as the joint distribution of $\tilde{\mathbb{S}}$ and $\mathbb{E}_{\tilde{\mathbb{S}}}(f) = \int f d\mathbb{P}_{\tilde{\mathbb{S}}}$. For any $r > 0$, let $\|f(\cdot)\|_{r,\mathbb{P}} := \{\mathbb{E}|f(\mathbf{Z})|^r\}^{1/r}$. We abbreviate “with probability approaching one” and “almost surely” by “w.p.a. 1” and “a.s.”, respectively.

A. Further discussions on the offset based PS model

We provide further discussions here on the offset based PS model proposed in Section 4.1 of the main file.

REMARK A.1 (Rationale behind the offset model (4.1) and its connections with a diverging intercept model) Instead of an offset based model as in (4.1), let us now directly consider a standard logistic model for $\mathbb{P}(R = 1|\mathbf{X}) = \pi_N(\mathbf{X})$ but allowing (necessarily) for a diverging intercept, given by:

$$\pi_N(\mathbf{X}) = g(\vec{\mathbf{X}}^T \boldsymbol{\beta}) = \frac{\exp(\vec{\mathbf{X}}^T \boldsymbol{\beta})}{1 + \exp(\vec{\mathbf{X}}^T \boldsymbol{\beta})}, \text{ where} \quad (\text{A.1})$$

$\boldsymbol{\beta} = (\boldsymbol{\beta}(1), \boldsymbol{\beta}(-1)^T)^T \in \mathbb{R}^{p+1}$ is a vector allowed to depend on N ; e.g., see Owen [2007] and Wang [2020]. For further simplification, let us assume that the slope $\boldsymbol{\beta}(-1)$, while allowed to depend on N , is finite, i.e., $\|\boldsymbol{\beta}(-1)\|_2 < C < \infty$ for some C independent of N .

Under the model (A.1), the following holds. Let $\text{MGF}_{\mathbf{X}}(\mathbf{v}) := \mathbb{E}\{\exp(\mathbf{v}^T \mathbf{X})\}$ denote the moment generating function (MGF) of \mathbf{X} at $\mathbf{v} \in \mathbb{R}^p$ and assume $\text{MGF}_{\mathbf{X}}(\mathbf{v})$ exists (i.e., finite) at $\mathbf{v} = \boldsymbol{\beta}(-1)$ and $\mathbf{v} = -\boldsymbol{\beta}(-1)$. Then, the following holds for the intercept $\boldsymbol{\beta}(1)$:

$$\frac{1}{\bar{\pi}_N} \frac{1 - \bar{\pi}_N}{\text{MGF}_{\mathbf{X}}(-\boldsymbol{\beta}(-1))} \leq \exp(-\boldsymbol{\beta}(1)) \leq \frac{1}{\bar{\pi}_N} \text{MGF}_{\mathbf{X}}(\boldsymbol{\beta}(-1)), \text{ and consequently,} \quad (\text{A.2})$$

$$\frac{1}{\bar{\pi}_N} \frac{1 - \bar{\pi}_N}{\mathbb{E}\{\exp(\|\boldsymbol{\beta}(-1)\|_2 \|\mathbf{X}\|_2)\}} \leq \exp(-\boldsymbol{\beta}(1)) \leq \frac{1}{\bar{\pi}_N} \mathbb{E}\{\exp(\|\boldsymbol{\beta}(-1)\|_2 \|\mathbf{X}\|_2)\}. \quad (\text{A.3})$$

For the special case of a Gaussian \mathbf{X} , i.e., $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}, \Sigma)$,

$$\text{MGF}_{\mathbf{X}}(-\boldsymbol{\beta}(-1)) = \text{MGF}_{\mathbf{X}}(\boldsymbol{\beta}(-1)) \leq \exp\{\|\boldsymbol{\beta}(-1)\|_2^2 \lambda_{\max}(\Sigma)\}.$$

Hence, as long as $\|\boldsymbol{\beta}(-1)\|_2^2 < C < \infty$ and $\lambda_{\max}(\Sigma) < \infty$, then using (A.2), $\exp(-\boldsymbol{\beta}(1)) \asymp \bar{\pi}_N^{-1} \rightarrow \infty$. More generally, if $\|\boldsymbol{\beta}(-1)\|_2^2 < C < \infty$ and $\mathbb{E}\{\exp(C\|\mathbf{X}\|_2)\} < \infty$ (e.g., if \mathbf{X} is sub-Gaussian), then using (A.3), we will have $\exp(-\boldsymbol{\beta}(1)) \asymp \bar{\pi}_N^{-1} \rightarrow \infty$.

Rationale for the offset model (4.1). The result clearly shows that the intercept $\boldsymbol{\beta}(1)$ diverges to $-\infty$ and does so precisely at a rate of $\log(\bar{\pi}_N)$, i.e., $c_1 + \log(\bar{\pi}_N) \leq \boldsymbol{\beta}(1) \leq C_1 + \log(\bar{\pi}_N)$. This provides a clear justification for our offset based model (4.1) where we precisely extract out this $\log(\bar{\pi}_N)$ as an offset (to be estimated separately and plugged in apriori to the sample likelihood equation), and then treat the intercept α_0 and the slope parameter $\boldsymbol{\beta}_0$ to be well-behaved, i.e., finite and independent of N (or at least bounded in N).

This makes the parameter space more amenable to theoretical analysis where it is common practice to assume that the truths (the true *unconstrained* minimizers) lie as interior points of some compact set. Such assumptions are commonplace in most of empirical process and M -estimation theory, and these results won't be applicable without this assumption, something that has clear justification under the offset model but not under the diverging intercept model.

REMARK A.2 (Connections with density ratio estimation) It is interesting (though elementary) to note that the PS is also related to the density ratio of \mathbf{X} (given $R = 0$ or 1), in that

$$\Lambda_N(\mathbf{X}) := \frac{f(\mathbf{X}|R=0)}{f(\mathbf{X}|R=1)} = \frac{\mathbb{P}(R=0|\mathbf{X})\mathbb{P}(R=1)}{\mathbb{P}(R=1|\mathbf{X})\mathbb{P}(R=0)} = \frac{\{1 - \pi_N(\mathbf{X})\}\bar{\pi}_N}{\pi_N(\mathbf{X})(1 - \bar{\pi}_N)},$$

where $f(\cdot|R=\cdot)$ is the conditional density function of \mathbf{X} given R . The density ratio is usually used in the so-called “covariate shift” setting in semi-supervised learning (SSL) and missing data, where R_i 's are treated as fixed (or conditioned on) and $\mathbb{P}_{\mathbf{X}} \neq \mathbb{P}_{\mathbf{X}|R=1}$ is allowed; see for example Kawakita and Kanamori [2013], Liu et al. [2020], and Section 4 of Kallus and Mao [2020].

Here, we discuss a simple and fairly obvious connection of the offset model (4.1) to a corresponding model for density ratio estimation. The analysis here can actually be seen to be model-free and non-parametric. Observe that

$$\text{logit}\{\pi_N(\mathbf{X})\} = \log(\bar{\pi}_N) - \log(1 - \bar{\pi}_N) - \log\{\Lambda_N(\mathbf{X})\}.$$

The standard approach to modeling the density ratio is to model $\log\{\Lambda_N(\mathbf{X})\}$ through basis function expansion based on some basis functions $\{\phi_j(\mathbf{X})\}_{j=1}^d$ (e.g., the linear bases will lead to standard parametric forms). But this in general can be difficult to implement in practice. However, the above representation suggests that the *same* model can be fitted by simply using a logistic regression model for

$R|\mathbf{X}$ with covariates as the *same* basis functions, and further using $\log\{\bar{\pi}_N/(1-\bar{\pi}_N)\}$ as an *offset* (which can be estimated separately and plugged in apriori into the likelihood equation). This provides a simple and flexible regression modelling approach to estimate the density ratio. Our offset based model (4.1) precisely implements such a model (albeit we had different motivations to consider it), and therefore provides a way to estimate the density ratio as well. This is a key quantity involved (as a nuisance function) in the semi-parametric efficiency bound for our parameter; see Theorem 4.1 of [Kallus and Mao \[2020\]](#). Our approach provides an automated and agnostic way of bypassing its estimation through a theoretically equivalent but practically more flexible regression modeling approach.

A discussion similar to above can be found in Section 1 of [Qin \[1998\]](#) who further proves that the estimation approach as above corresponds to an optimal choice of the estimating equation for estimating the density ratio among the class of all such equations. It is also interesting to note that the semi-supervised (SS) setting actually bears a very close relation to so-called *case-control study designs* (which are *retrospective* designs, as opposed to the *prospective* cohort studies that we usually consider), since here the labeling indicator R is typically treated as *non-random (or conditioned)*, which is similar in spirit to case-control designs (with R being replaced by case/control status). For statistical analyses of these kind of studies, density ratio estimation models are often required and an estimation strategy via a logistic regression model of the PS, similar as above, is often employed; see Section 1 of [Qin \[1998\]](#) for more discussions.

REMARK A.3 (Connection with the maximum likelihood estimate (MLE) of the model (A.1)) In fact, there is an one-one correspondence between $\hat{\boldsymbol{\gamma}}$ (we suppress the dependence on k for a moment) and the MLE of the model (A.1): if $(\hat{\boldsymbol{\beta}}(1), \hat{\boldsymbol{\beta}}(-1))$ denotes a sample MLE, i.e., a solution (assuming it exists) to the (sample) likelihood equation for the model (A.1), then $\hat{\boldsymbol{\gamma}} = (\hat{\boldsymbol{\beta}}(1) - \log \hat{\pi}_N, \hat{\boldsymbol{\beta}}(-1))$ is a sample MLE for the model (4.1). Conversely, if $\hat{\boldsymbol{\gamma}}$ is a sample MLE for the model (4.1), then $(\hat{\boldsymbol{\beta}}(1), \hat{\boldsymbol{\beta}}(-1)) = (\hat{\boldsymbol{\gamma}}(1) + \log(\hat{\pi}_N), \hat{\boldsymbol{\gamma}}(-1))$ is a sample MLE for the model (A.1). All these claims are straightforward to show by means of direct verification.

REMARK A.4 (Existence and uniqueness of $\hat{\boldsymbol{\gamma}}$) The uniqueness of $\hat{\boldsymbol{\gamma}}$ is a direct consequence of the convexity of the sample log-likelihood. As for the existence, we appeal to the one-one correspondence between $\hat{\boldsymbol{\gamma}}$ and the sample MLE of the model (A.1). We further use the results of [Owen \[2007\]](#) who demonstrated the existence of the sample MLE for the model (A.1) under a fairly mild (sample) overlap condition; see Lemma 5 therein. Note that [Owen \[2007\]](#) shows this result for a slightly modified version of the log-likelihood wherein the empirical average over unlabeled data is replaced by an expectation (assuming N is very large). But the same proof technique could be applied to the actual log-likelihood along with a corresponding appropriate modification of the (sample) overlap condition to conclude the existence of the sample MLE for model (A.1). Consequently, this also establishes the existence of the sample MLE for the offset model (4.1).

B. Missing completely at random (MCAR) labeling: Theory and comparisons

Apart from the offset based model and the stratified labeling model discussed in Sections 4.1 and 4.2, a simple but commonly used PS model would be a MCAR mechanism. In this section, we consider this special MCAR mechanism with $\pi_N(\cdot) \equiv \bar{\pi}_N$, and derive the properties of $\hat{\theta}_{DRSS}$ including an adjusted regular and asymptotically linear (RAL) expansion allowing for misspecification of $\hat{m}(\cdot)$. In this case, a cross-fitted estimator of the PS is proposed as $\hat{\pi}_N(\mathbf{X}_i) = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i$ for any $i \in \mathcal{I}_k$, where $N_{-k} := |\mathcal{I}_{-k}|$ and $\mathcal{I}_{-k} := \mathcal{I} \setminus \mathcal{I}_k$. Based on such a MCAR PS estimator, we have the following result on the

conditions and conclusions in Theorem 3.2.

THEOREM A.1 Assume $\pi_N(\mathbf{X}) \equiv \bar{\pi}_N, N\bar{\pi}_N \rightarrow \infty$ as $N \rightarrow \infty$, $\|m(\cdot) - \mu(\cdot)\|_{2,\mathbb{P}} < \infty$ and $\|\widehat{m}(\cdot) - \mu(\cdot)\|_{2,\mathbb{P}} = o_p(1)$. Then, $a_N = \bar{\pi}_N$ and

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\bar{\pi}_N(\mathbf{X})} \right\}^2 \right] = \mathbb{E} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\bar{\pi}_N(\mathbf{X})} \right\}^2 = O_p((N\bar{\pi}_N)^{-1}).$$

Furthermore,

$$\begin{aligned} \widehat{\theta}_{\text{DRSS}} - \theta_0 &= N^{-1} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + o_p((Na_N)^{-1/2}), \quad \text{where } \Psi(\mathbf{Z}) := \psi_\mu(\mathbf{Z}) + \text{IF}_\pi(\mathbf{Z}), \quad (\text{B.1}) \\ \psi_\mu(\mathbf{Z}) &= \frac{R}{\bar{\pi}_N} \{Y - \mu(\mathbf{X})\} + \mu(\mathbf{X}) - \theta_0, \\ \text{IF}_\pi(\mathbf{Z}) &:= \left(\frac{R - \bar{\pi}_N}{\bar{\pi}_N} \right) \Delta_\mu, \quad \Delta_\mu := \mathbb{E}\{\mu(\mathbf{X}) - m(\mathbf{X})\}. \end{aligned}$$

Note that Theorem A.1 still holds if the PS is estimated without cross-fitting that $\widehat{\pi}_N(\mathbf{X}) \equiv \widehat{\pi}_N = n/N$, where $n = \sum_{i=1}^N R_i$.

REMARK B.1 The modification on the RAL expansion of the mean estimator is needed only when $\Delta_\mu = \mathbb{E}\{\mu(\mathbf{X}) - m(\mathbf{X})\} \neq 0$. Recall Remark 4.3; if the outcome model is fitted by a linear model that $\mu(\mathbf{X}) = \vec{\mathbf{X}}^T \boldsymbol{\beta}^*$, where $\vec{\mathbf{X}} = (1, \mathbf{X}^T)^T$, $\boldsymbol{\beta}^* = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^{p+1}} \mathbb{E}\{(Y - \vec{\mathbf{X}}^T \boldsymbol{\beta})^2\} = \{\mathbb{E}(\vec{\mathbf{X}} \vec{\mathbf{X}}^T)\}^{-1} \mathbb{E}(\vec{\mathbf{X}} Y)$ is the optimal population slope. Then, we have $\Delta_\mu = \mathbb{E}\{\mu(\mathbf{X}) - m(\mathbf{X})\} = \mathbb{E}(\vec{\mathbf{X}}^T) \{\mathbb{E}(\vec{\mathbf{X}} \vec{\mathbf{X}}^T)\}^{-1} \mathbb{E}(\vec{\mathbf{X}} Y) - \mathbb{E}(Y) = 0$. This suggests that, the RAL modification is unnecessary when $\widehat{m}(\cdot)$ converges to the linear projection. In other words, for such cases, the original asymptotic normality (3.9) still holds even if $\mu(\cdot) \neq m(\cdot)$ and it coincides with the results in Zhang et al. [2019], Zhang and Bradic [2022]. Classical examples for such $\widehat{m}(\cdot)$ include least squares (LS) estimator and regularized least squares such as Lasso and ridge under appropriate conditions.

RECONCILIATION WITH “TRADITIONAL” SS INFERENCE LITERATURE UNDER MCAR Now, we consider the “traditional” SS setting where all the R_i ’s are considered *deterministic* (or conditioned) apart from an underlying MCAR assumption. Under this SS setting, we consider the SS mean estimator proposed in Zhang and Bradic [2022]. In fact, their estimator is a special case of our double robust SS (DRSS) mean estimator $\widehat{\theta}_{\text{DRSS}}$ except that the PS is estimated without cross-fitting, i.e., $\widehat{\pi}_N(\mathbf{X}) \equiv \widehat{\pi}_N = n/N$. Moreover, the SS mean estimator proposed by Zhang et al. [2019] based on a linear outcome model estimated via least squares can be further seen as a special case of the SS estimator of Zhang and Bradic [2022] – the latter estimator allows the usage of flexible outcome models (including high-dimensional or non-parametric models), whereas the former one is restricted to a linear outcome nuisance model in low dimensions estimated via least squares. In addition, since Zhang et al. [2019] only focus on low-dimensional parametric estimators, their outcome function is also estimated without cross-fitting, whereas a cross-fitting technique is required in Zhang and Bradic [2022] to estimate the outcome function. In the following, we compare our proposed mean estimators with the latest SS mean estimator proposed by Zhang and Bradic [2022].

Under this SS setting, the semi-supervised mean estimator proposed by Zhang and Bradic [2022],

denoted as $\hat{\theta}_{\text{SS}}$, has the following RAL expansion:

$$\hat{\theta}_{\text{SS}} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,\text{SS}}(\mathbf{Z}_i) + o_p(n^{-1/2}), \text{ where} \quad (\text{B.2})$$

$$\psi_{\mu,\text{SS}}(\mathbf{Z}) = \frac{NR}{n} \{Y - \mu(\mathbf{X})\} + \mu(\mathbf{X}) - \theta_0. \quad (\text{B.3})$$

Here, conditional on R_i 's, $\{\psi_{\mu,\text{SS}}(\mathbf{Z}_i)\}_{i=1}^N$ are independent and identically distributed, with mean zero.

VARIANCE COMPARISON In the following, we present an interesting comparison of the mean estimators' asymptotic variances under the following three cases:

- (a) R_i 's are considered as random (MCAR), $\bar{\pi}_N$ is known, and the mean estimator $\tilde{\theta}$, defined as (3.1), is based on the true PS $\bar{\pi}_N$;
- (b) R_i 's are considered as random (MCAR), the mean estimator $\hat{\theta}_{\text{DRSS}}$, defined as (3.4) and studied in Theorem A.1, is based on the cross-fitted constant estimate that $\hat{\pi}_N(\mathbf{X}_i) = |\mathbb{S}_{-k}|^{-1} \sum_{i \notin \mathcal{I}_k} R_i$ for $i \in \mathcal{I}_k$; and
- (c) R_i 's are considered as fixed (SS) and the semi-supervised mean estimator $\hat{\theta}_{\text{SS}}$ is as defined in Zhang and Bradic [2022].

Recall that $\psi_{\mu,\pi}(\mathbf{Z})$, $\Psi(\mathbf{Z})$, and $\psi_{\mu,\text{SS}}(\mathbf{Z})$, defined in (3.2), (B.1), and (B.3), are the IFs for the estimators considered in the above cases (a), (b), and (c), respectively; see the RAL expansions provided in Theorems 3.1, A.1, and (B.2). Hence, the considered estimators have the following asymptotic variances:

- (a) $\text{Var}\{\psi_{\mu,\pi}(\mathbf{Z})\} = \frac{\mathbb{E}\{Y - \mu(\mathbf{X})\}^2}{\bar{\pi}_N} - \Delta_\mu^2 + \text{Var}\{\mu(\mathbf{X})\} + 2\text{Cov}\{Y - \mu(\mathbf{X}), \mu(\mathbf{X})\},$
- (b) $\text{Var}\{\Psi(\mathbf{Z})\} = \frac{\mathbb{E}\{Y - \mu(\mathbf{X})\}^2}{\bar{\pi}_N} - \frac{\Delta_\mu^2}{\bar{\pi}_N} + \text{Var}\{\mu(\mathbf{X})\} + 2\text{Cov}\{Y - \mu(\mathbf{X}), \mu(\mathbf{X})\}, \text{ and}$
- (c) $\text{Var}\{\psi_{\mu,\text{SS}}(\mathbf{Z})|(R_i)_{i \in \mathcal{I}}\} = \frac{\mathbb{E}\{Y - \mu(\mathbf{X})\}^2}{\hat{\pi}_N} - \frac{\Delta_\mu^2}{\hat{\pi}_N} + \text{Var}\{\mu(\mathbf{X})\} + 2\text{Cov}\{Y - \mu(\mathbf{X}), \mu(\mathbf{X})\}.$

We can see that, $\text{Var}\{\psi_{\mu,\pi}(\mathbf{Z})\} = \text{Var}\{\Psi(\mathbf{Z})\} + (\bar{\pi}_N^{-1} - 1)\Delta_\mu^2 \geq \text{Var}\{\Psi(\mathbf{Z})\}$, i.e., the asymptotic variance of $\tilde{\theta}$ is larger than (or equal to) the asymptotic variance of $\hat{\theta}_{\text{DRSS}}$. It suggests that, under the MCAR setting, even if $\bar{\pi}_N$ is known, it is still worth estimating $\bar{\pi}_N$ instead of directly plugging in the true value $\bar{\pi}_N$ as long as $\Delta_\mu \neq 0$. As for the asymptotic variance of $\hat{\theta}_{\text{SS}}$ under the SS setting, notice the fact that

$$\frac{\bar{\pi}_N}{\hat{\pi}_N} - 1 = O_p\left((N\bar{\pi}_N)^{-1/2}\right).$$

Hence, $\text{Var}\{\psi_{\mu,\text{SS}}(\mathbf{Z})|(R_i)_{i \in \mathcal{I}}\} = \text{Var}\{\Psi(\mathbf{Z})\}\{1 + O_p((N\bar{\pi}_N)^{-1/2})\} = \text{Var}\{\Psi(\mathbf{Z})\}\{1 + o_p(1)\}$.

C. Inference results based on adjusted confidence intervals

In Section 6, we illustrated the simulation performance of the proposed confidence interval (3.14), which requires both the outcome and PS models to be correctly specified, as in part (a) of Theorem 3.2. In

this section, we compare (3.14) with an adjusted version based on the asymptotic expansion in part (b) of Theorem 3.2 and the RAL expansion in Remark 3.9. The adjusted confidence interval allows inference via $\hat{\theta}_{\text{DRSS}}$ even under misspecified outcome models, and the adjusted RAL expansions based on different PS models are provided in Theorems 4.1, 4.3, and A.1. Here, we only focus on the results for the skewed offset logistic PS model as discussed in Theorem 4.1, and we present numerical results to validate the inference provided by the adjusted RAL expansion (4.10) of $\hat{\theta}_{\text{DRSS}}$ given therein.

Apart from the settings c and d in Section 6.1.1, an additional DGP, Setting f: P2+O3, is considered. Here, P2 is the offset logistic PS model as in Section 6.1.1, and O3 is a cubic outcome model defined as follows:

$$\text{O3. (Cubic outcome)} Y_i = \vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0 + \sum_{j=1}^{p+1} \boldsymbol{\alpha}_0(j) \vec{\mathbf{X}}_i(j)^2 + \sum_{j=1}^{p+1} \boldsymbol{\zeta}_0(j) \vec{\mathbf{X}}_i(j)^3 + \varepsilon_i.$$

The parameter value is defined as:

$$\boldsymbol{\zeta}_0 = (0, 0.2, 0.2, 0.2, \mathbf{0}_{1 \times (p-3)})^T.$$

We illustrate the behavior of the original confidence interval (3.14) and the adjusted confidence interval based on the RAL expansion (4.10). We consider the Settings c, d, and f, where the outcome models are polynomial (without interaction) with degrees 1, 2, and 3, respectively. Apart from the empirical estimator \bar{Y}_{labeled} and the oracle estimator as in Section 6.1.1, we also consider the proposed mean estimators $\hat{\theta}_{\text{DRSS}}$ based on an offset logistic model based PS estimator, and polynomial model based outcome regression estimators with degrees 1, 2, and 3. The simulation results are presented in Table A.1.

We can clearly see the improvement of the coverage based on the adjusted confidence intervals, especially for polynomial estimators $\hat{m}(\cdot)$ with degrees 2 and 3. As mentioned in Remark 4.3, a latent misspecification arises here since the effective sample size $N\bar{\pi}_N = 100$ is comparable with the dimension of the working model: for polynomial regression with degrees 2 and 3, the dimensions of the design matrix are 21 and 31, respectively. Under such a circumstance, $\hat{m}(\cdot)$ tends to be a biased estimate and a (latent) misspecification arises, in that its (effective) target (or limit) becomes some $\mu(\cdot) \neq m(\cdot)$.

Such an example suggests that, the adjusted confidence intervals, when $\pi_N(\cdot)$ is correctly specified, allow us to better capture the model complexity of $\hat{m}(\cdot)$ and improve the efficiency of the DRSS estimator. The modified confidence intervals can *still* provide valid inference even when a degree of freedom of the model becomes comparable with the effective sample size.

In Table A.1, one can see that neither of the averages of the estimated standard deviations (ASDs) or the adjusted ASDs are close to the empirical standard deviations (ESDs) for the DRSS mean estimators based on polynomial regressions with degrees 2 and 3, while we can still achieve fairly acceptable coverages for the confidence intervals. This is not contradicted with our theory: we only obtain asymptotic results in terms of convergence in distribution or probability, whereas $\text{ASD} = \text{ESD} + o(1)$ requires a convergence in mean (i.e., L_1 convergence). Such a difference is possibly related to the instability of the LS-type outcome estimator, when the dimension of the working model is comparable with the sample size.

D. Proofs of main results

D.1 Auxiliary lemmas

The following Lemmas will be useful in the proofs.

Table A.1. Simulations under Settings c, d and f, with $p = 10, N = 10000$ and $\bar{\pi}_N = 0.01$ ($N\bar{\pi}_N = 100$). Bias: empirical bias; RMSE: root mean square error; Length: average length of the 95% confidence intervals; Coverage: average coverage of the 95% confidence intervals; ESD: empirical standard deviation; ASD: average of estimated standard deviations. The results for adjusted (adj) confidence intervals based on the RAL expansion (4.10) in Theorem 4.1 are provided in parentheses.

$\hat{\pi}_N(\cdot)$	$\hat{m}(\cdot)$	Bias	RMSE	Length(adj)	Coverage(adj)	ESD	ASD(adj)
Setting c							
\bar{Y}_{labeled}		0.979	0.998	0.784	0.002	0.197	0.200
oracle		-0.007	0.164	0.607	0.972	0.164	0.155
logistic	poly1(LS)	-0.009	0.227	0.865(0.881)	0.952(0.964)	0.227	0.221(0.225)
	poly2	0.002	0.277	1.017(1.039)	0.940(0.964)	0.277	0.260(0.265)
	poly3	-0.013	0.491	1.436(1.465)	0.922(0.956)	0.492	0.366(0.374)
Setting d							
\bar{Y}_{labeled}		1.923	1.968	1.628	0.002	0.418	0.415
oracle		0.011	0.158	0.615	0.960	0.158	0.157
logistic	poly2	0.493	1.638	4.352(4.202)	0.908(0.936)	1.564	1.110(1.072)
	poly1(LS)	-0.006	0.401	1.058(1.080)	0.916(0.954)	0.401	0.270(0.275)
	poly3	-0.013	0.562	1.457(1.483)	0.918(0.942)	0.563	0.372(0.378)
Setting f							
\bar{Y}_{labeled}		2.613	2.670	2.182	0.000	0.549	0.557
oracle		-0.003	0.164	0.623	0.966	0.164	0.159
logistic	poly3	0.302	1.267	4.406(4.163)	0.914(0.918)	1.232	1.124(1.062)
	poly2	-0.018	0.584	1.752(1.800)	0.862(0.900)	0.584	0.447(0.459)
	poly1(LS)	-0.005	0.410	1.279(1.316)	0.894(0.926)	0.410	0.326(0.336)

LEMMA A.1 Let $(X_N)_{N \geq 1}$ and $(Y_N)_{N \geq 1}$ be sequences of random variables in \mathbb{R} . If $\mathbb{E}(|X_N|^r | Y_N) = O_p(1)$ for any $r \geq 1$, then $X_N = O_p(1)$.

Proof of Lemma A.1. For any $c > 0$, there exists $C > 0$ such that, for large enough N ,

$$\mathbb{P}\{\mathbb{E}(X_N^r | Y_N) > C\} < c/2.$$

Let $\delta = (2C/c)^{1/r}$, then

$$\begin{aligned} \mathbb{P}(|X_N| > \delta) &= \mathbb{E}(\mathbb{E}[1\{|X_N| > \delta\} | Y_N]) \\ &= \mathbb{E}[1\{\mathbb{E}(|X_N|^r | Y_N) \leq C\} \mathbb{E}(1\{|X_N| > \delta\} | Y_N)] \\ &\quad + \mathbb{E}[1\{\mathbb{E}(|X_N|^r | Y_N) > C\} \mathbb{E}(1\{|X_N| > \delta\} | Y_N)] \\ &\leq \mathbb{E}[1\{\mathbb{E}(|X_N|^r | Y_N) \leq C\} \mathbb{E}(\delta^{-r} |X_N|^r | Y_N)] + \mathbb{E}(1\{\mathbb{E}(|X_N|^r | Y_N) > C\}) \\ &= \delta^{-r} \mathbb{E}[1\{\mathbb{E}(|X_N|^r | Y_N) \leq C\} \mathbb{E}(|X_N|^r | Y_N)] + \mathbb{P}[\mathbb{E}(|X_N|^r | Y_N) > C] \\ &\leq c/2 + c/2 = c. \end{aligned}$$

That is, $X_N = O_p(1)$. \square

LEMMA A.2 (Lemma 6.1 of Chernozhukov et al. [2018]) Let $(X_N)_{N \geq 1}$ and $(Y_N)_{N \geq 1}$ be sequences of random variables in \mathbb{R} . If for any $c > 0$, $\mathbb{P}(|X_N| > c | Y_N) = o_p(1)$, then $X_N = o_p(1)$.

In particular, Lemma A.2 occurs if $\mathbb{E}(|X_N|^q | Y_N) = o_p(1)$ for some $q \geq 1$. A typical example we used in our proofs is $X_N = \sum_{i=1}^N Z_{N,i}/N$, where $(Z_{N,i})_{N \geq 1, i \leq N}$ is a row-wise independent and identically distributed triangular array with $\mathbb{E}(|Z_{N,i}| | Y_N) = o_p(1)$.

LEMMA A.3 Let $(Z_{N,i})_{N \geq 1, i \leq N}$ be a row-wise independent and identical distributed triangular array, suppose there exists a sequence b_N such that $N^{-r} b_N^{-1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = o(1)$ with $0 < r < 1$ and $b_N > 0$. Then,

$$N^{-1} \sum_{i=1}^N Z_{N,i} - \mathbb{E}(Z_{N,1}) = o_p(b_N).$$

Proof of Lemma A.3. Let $Y_{N,i} = Z_{N,i} \mathbf{1}\{|Z_{N,i}| \leq Nb_N\}$. For any $c > 0$,

$$\begin{aligned} & \mathbb{P}\left(\left|N^{-1} \sum_{i=1}^N Z_{N,i} - \mathbb{E}(Y_{N,1})\right| \geq cb_N\right) \\ & \leq \mathbb{P}\left(\bigcup_{i=1}^N [Z_{N,i} \neq Y_{N,i}] \cup \left[\left|\sum_{i=1}^N Y_{N,i} - \mathbb{E}(Y_{N,1})\right| \geq Ncb_N\right]\right) \\ & \leq \mathbb{P}\left(\bigcup_{i=1}^N [Z_{N,i} \neq Y_{N,i}]\right) + \mathbb{P}\left(\left|\sum_{i=1}^N Y_{N,i} - \mathbb{E}(Y_{N,1})\right| \geq Ncb_N\right), \end{aligned}$$

where with a slight abuse of notation, here \mathbb{P} denotes the joint distribution of $(Z_{N,i})_{N \geq 1, i \leq N}$. By Markov's inequality,

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i=1}^N [Z_{N,i} \neq Y_{N,i}]\right) & \leq N \mathbb{P}(Z_{N,1} \neq Y_{N,1}) = N \mathbb{P}(|Z_{N,1}| > Nb_N) \\ & \leq N(Nb_N)^{-1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = N^{-r} b_N^{-1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = o(1), \end{aligned}$$

where the last equality follows from the assumptions. Moreover, by Chebyshev's inequality

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{i=1}^N Y_{N,i} - \mathbb{E}(Y_{N,1})\right| \geq Ncb_N\right) & \leq (Ncb_N)^{-2} \mathbb{E}\left\{\left|\sum_{i=1}^N Y_{N,i} - \mathbb{E}(Y_{N,1})\right|^2\right\} \\ & = c^{-2} N^{-1} b_N^{-2} \mathbb{E}\{Y_{N,i} - \mathbb{E}(Y_{N,i})\}^2 \leq c^{-2} N^{-1} b_N^{-2} \mathbb{E}(Y_{N,i}^2) \\ & = c^{-2} N^{-1} b_N^{-2} \mathbb{E}[Z_{N,1}^2 \mathbf{1}\{|Z_{N,1}| \leq Nb_N\}] \leq c^{-2} N^{-1} b_N^{-2} (Nb_N)^{1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) \\ & = c^{-2} N^{-r} b_N^{-1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = o(1), \end{aligned}$$

where in the second to last inequality we used Markov's inequality on $Z_{N,1}$ s. Hence,

$$N^{-1} \sum_{i=1}^N Z_{N,i} - \mathbb{E}(Y_{N,1}) = o_p(b_N).$$

In addition, by similar arguments

$$\begin{aligned} \mathbb{E}(Z_{N,1}) - \mathbb{E}(Y_{N,1}) & = \mathbb{E}[Z_{N,1} \mathbf{1}\{|Z_{N,1}| > Nb_N\}] = \mathbb{E}[|Z_{N,1}|^{1+r} |Z_{N,1}|^{-r} \mathbf{1}\{|Z_{N,1}| > Nb_N\}] \\ & \leq (Nb_N)^{-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = b_N N^{-r} b_N^{-1-r} \mathbb{E}(|Z_{N,1}|^{1+r}) = o(b_N). \end{aligned}$$

Therefore,

$$N^{-1} \sum_{i=1}^N Z_{N,i} - \mathbb{E}(Z_{N,1}) = N^{-1} \sum_{i=1}^N Z_{N,i} - \mathbb{E}(Z_{N,1}) + \mathbb{E}(Z_{N,1}) - \mathbb{E}(Y_{N,1}) = o_p(b_N).$$

□

LEMMA A.4 For any function $g(\cdot)$ and $\theta \in \mathbb{R}$, define

$$\psi(\mathbf{Z}, \theta) := g(\mathbf{Z}) - \theta.$$

Let $\theta^0 := \mathbb{E}\{g(\mathbf{Z})\}$. Assume

$$\mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\} \asymp b_N^{-1}, \quad N^{-r} b_N^{r+1} \mathbb{E}\{|\psi(\mathbf{Z}, \theta^0)|^{2+2r}\} = o(1), \quad (\text{D.1})$$

for some sequence b_N and $0 < r < 1$. Moreover, let $\hat{\theta} \in \mathbb{R}$ be such that $\hat{\theta} - \theta^0 = o_p(b_N^{-1/2})$. Additionally, for $k \leq K$, and some (possibly random) function $g_{-k}(\cdot) \perp\!\!\!\perp \mathbb{S}_k$, define $\psi_{-k}(\mathbf{Z}, \theta) := g_{-k}(\mathbf{Z}) - \theta$ and suppose that

$$\mathbb{E}\{\psi_{-k}(\mathbf{Z}, \theta^0) - \psi(\mathbf{Z}, \theta^0)\}^2 = o_p(b_N^{-1}).$$

Then, as $N \rightarrow \infty$, we have

$$N^{-1} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} \psi_{-k}^2(\mathbf{Z}_i, \hat{\theta}) = \mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\}\{1 + o_p(1)\}.$$

Proof of Lemma A.4.

By Young's inequality with $(a+b)^2 \leq 2a^2 + 2b^2$,

$$\begin{aligned} |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \hat{\theta}) - \psi(\mathbf{Z}_i, \theta^0)\}^2 \\ \leq 2(\hat{\theta} - \theta^0)^2 + 2|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \theta^0) - \psi(\mathbf{Z}_i, \theta^0)\}^2 = o_p(b_N^{-1}). \end{aligned} \quad (\text{D.2})$$

In what follows we will use the following equality which is a consequence of Lemma A.3 and the condition in (D.1):

$$|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \theta^0) - \psi(\mathbf{Z}_i, \theta^0)\}^2 = o_p(b_N^{-1}). \quad (\text{D.3})$$

Using the fact that $a^2 - b^2 = (a+b)(a-b) = (a-b)^2 + 2b(a-b)$, and using the triangle and then Cauchy-Schwarz inequality

$$\begin{aligned} & \left| |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \psi_{-k}^2(\mathbf{Z}_i, \hat{\theta}) - |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \psi^2(\mathbf{Z}_i, \theta^0) \right| \\ &= \left| |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \hat{\theta}) - \psi(\mathbf{Z}_i, \theta^0)\}^2 + 2|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \hat{\theta}) - \psi(\mathbf{Z}_i, \theta^0)\} \psi(\mathbf{Z}_i, \theta^0) \right| \\ &\leq |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \hat{\theta}) - \psi(\mathbf{Z}_i, \theta^0)\}^2 \\ &\quad + 2|\mathcal{I}_k|^{-1} \left[\sum_{i \in \mathcal{I}_k} \{\psi_{-k}(\mathbf{Z}_i, \hat{\theta}) - \psi(\mathbf{Z}_i, \theta^0)\}^2 \sum_{i \in \mathcal{I}_k} \psi^2(\mathbf{Z}_i, \theta^0) \right]^{1/2} \\ &\stackrel{(i)}{=} o_p(a_N^{-1}) + o_p(a_N^{-1/2}) [\mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\}\{1 + o_p(1)\}]^{1/2} \stackrel{(ii)}{=} o_p(a_N^{-1}), \end{aligned}$$

where (i) follows by (D.2) and (D.3), and in (ii), we utilized the assumption $\mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\} \asymp a_N^{-1}$ to conclude the asymptotic order of the quantities of interest. Then, by utilizing the result of Lemma A.3, i.e., (D.3), we have

$$\begin{aligned} N^{-1} \sum_{k=1}^K \sum_{i \in \mathcal{J}_k} \psi_{-k}^2(\mathbf{Z}_i, \hat{\theta}) &= |\mathcal{J}_k|^{-1} \sum_{i \in \mathcal{J}_k} \psi^2(\mathbf{Z}_i, \theta^0) + o_p(a_N^{-1}) \\ &= \mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\}\{1 + o_p(1)\} + o_p(a_N^{-1}) = \mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\}\{1 + o_p(1)\}, \end{aligned}$$

since $\mathbb{E}\{\psi^2(\mathbf{Z}, \theta^0)\} \asymp a_N^{-1}$ by assumption. \square

LEMMA A.5 The following are some useful properties regarding sub-Gaussian variables.

- (a) If $|X| \leq |Y|$ a.s., then $\|X\|_{\psi_2} \leq \|Y\|_{\psi_2}$. If $|X| \leq M$ a.s. for some constant M , then $\|X\|_{\psi_2} \leq \{\log(2)\}^{-1/2}M$.
- (b) If $\|X\|_{\psi_2} \leq \sigma$, then $\mathbb{E}(|X|^m) \leq 2\sigma^m \Gamma(m/2 + 1)$, for all $m \geq 1$, where $\Gamma(a) := \int_0^\infty x^{a-1} \exp(-x) dx$ denotes the Gamma function. Hence, $\mathbb{E}(|X|) \leq \sigma\sqrt{\pi}$ and $\mathbb{E}(|X|^m) \leq 2\sigma^m (m/2)^{m/2}$ for $m \geq 2$.
- (c) If $\|X - \mathbb{E}(X)\|_{\psi_2} \leq \sigma$, then $\mathbb{E}(\exp[t\{X - \mathbb{E}(X)\}]) \leq \exp(2\sigma^2 t^2)$, for all $t \in \mathbb{R}$.
- (d) Let $\mathbf{X} \in \mathbb{R}^p$ be a random vector with $\sup_{1 \leq j \leq p} \|\mathbf{X}(j)\|_{\psi_2} \leq \sigma$. Then, $\|\|\mathbf{X}\|_\infty\|_{\psi_2} \leq \sigma\{\log(p) + 2\}^{1/2}$.
- (e) Let $(X_i)_{i=1}^N$ be independent random variables with means $(\mu_i)_{i=1}^N$ such that $\|X_i - \mu_i\|_{\psi_2} \leq \sigma$. Then, $\|N^{-1} \sum_{i=1}^N (X_i - \mu_i)\|_{\psi_2} \leq 4\sigma N^{-1/2}$.

Lemma A.5 is a simple consequence of Lemmas D.1 and D.2 of Chakrabortty et al. [2019].

LEMMA A.6 Assume $(\mathbf{X}_i)_{i=1}^N$ are independent and identically distributed, $\lambda_{\min}\{\mathbb{E}(\vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T)\} \geq c > 0$ and $\sup_{\|\mathbf{v}\|_2=1} \mathbb{E}\{(\vec{\mathbf{X}}_i^T \mathbf{v})^4\} < C < \infty$, with constants c and C . Assume $\boldsymbol{\gamma}_0 \in \mathbb{R}^{p+1}$ satisfies $\|\boldsymbol{\gamma}_0\|_2 < C < \infty$, $\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0$ is a sub-Gaussian random variable, and \mathbf{X}_i is a marginal sub-Gaussian random vector with

$$\begin{aligned} \|\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0\|_{\psi_2} &= \inf\left\{t > 0 : \mathbb{E}[\exp\{t^{-2}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0)^2\}] \leq 2\right\} < \infty, \\ \sup_{1 \leq j \leq p} \|\mathbf{X}_i(j)\|_{\psi_2} &= \inf\left\{t > 0 : \mathbb{E}[\exp\{t^{-2}\mathbf{X}_i^2(j)\}] \leq 2\right\} < \infty. \end{aligned}$$

Recall that

$$\begin{aligned} \ell_N^{\text{bal}}(\boldsymbol{\gamma}) &= -N^{-1} \sum_{i=1}^N [R_i^* \vec{\mathbf{X}}_i^T \boldsymbol{\gamma} - \log\{1 + \exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma})\}] \quad \forall \boldsymbol{\gamma} \in \mathbb{R}^{p+1}, \\ \delta\ell(\Delta; 1; \boldsymbol{\gamma}) &= \ell_N^{\text{bal}}(\boldsymbol{\gamma} + \Delta) - \ell_N^{\text{bal}}(\boldsymbol{\gamma}) - \Delta^T \nabla_{\boldsymbol{\gamma}} \ell_N^{\text{bal}}(\boldsymbol{\gamma}) \quad \forall \boldsymbol{\gamma}, \Delta \in \mathbb{R}^{p+1}. \end{aligned}$$

where $(R_i^*)_{i=1}^N$ are i.i.d. pseudo binary random variables satisfying $\mathbb{P}(R_i^* = 1 | \mathbf{X}) = g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0)$. Then, for some constants $c_1, c_2, c_3, c_4 > 0$,

$$\delta\ell(\Delta; 1; \boldsymbol{\gamma}_0) \geq c_1 \|\Delta\|_2 \left\{ \|\Delta\|_2 - c_2 \sqrt{\frac{\log(p+1)}{N}} \|\Delta\|_1 \right\} \quad \forall \Delta \in \mathbb{R}^{p+1}, \quad \|\Delta\|_2 \leq 1,$$

with probability at least $1 - c_3 \exp(-c_4 N)$.

Lemma A.6 is a slightly more general version of Proposition 2 of Negahban et al. [2010]: instead of assuming \mathbf{X} to be joint sub-Gaussian with mean zero, one can repeat their proof by only requiring \mathbf{X} to be a marginal sub-Gaussian vector and $\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0$ be a sub-Gaussian variable, as well as an additional 4-th moment condition that $\sup_{\|\mathbf{v}\|_2=1} \mathbb{E}\{(\vec{\mathbf{X}}^T \mathbf{v})^4\} < C < \infty$. Unlike Negahban et al. [2010], the intercept term is also considered here: since we do not require zero-mean covariates, the intercept term $\vec{\mathbf{X}}(1) = 1$ can be seen as a sub-gaussian variable.

LEMMA A.7 (Theorem 3.26 of Wainwright [2019]) Let \mathcal{F} be a class of functions of the form $f : \mathcal{X} \rightarrow \mathbb{R}$, and let $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ be drawn from a product distribution $\mathbb{P} = \bigotimes_{i=1}^N \mathbb{P}_i$, where each \mathbb{P}_i is supported on some set $\mathcal{X}_i \subseteq \mathcal{X}$. For each $f \in \mathcal{F}$ and $i = 1, \dots, N$, assume that there are real numbers $a_{i,f} \leq b_{i,f}$ such that $f(\mathbf{x}) \in [a_{i,f}, b_{i,f}]$ for all $\mathbf{x} \in \mathcal{X}_i$. Let $Z = \sup_{f \in \mathcal{F}} \{N^{-1} \sum_{i=1}^N f(\mathbf{X}_i)\}$. Then for all $t \geq 0$, we have $\mathbb{P}\{Z \geq \mathbb{E}(Z) + t\} \leq \exp(-Nt^2/4L^2)$, where $L^2 := \sup_{f \in \mathcal{F}} \{N^{-1} \sum_{i=1}^N (b_{i,f} - a_{i,f})^2\}$.

D.2 Proofs of the Main Statements

Proof of Theorem 3.1.

We prove Theorem 3.1 by decomposing the estimation error into two terms: $N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i)$ and $\widehat{\Delta}_{N,1,k}$ defined below in (D.4). We use Lemma A.1 and the Lindeberg-Feller theorem for self-normalized partial sums. Observe that

$$\tilde{\theta} - \theta_0 = N^{-1} \sum_{i=1}^N \left\{ \frac{R_i - \pi_N(\mathbf{X}_i)}{\pi_N(\mathbf{X}_i)} [Y_i - \widehat{m}(\mathbf{X}_i)] + Y_i - \theta_0 \right\} = N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i) + \sum_{k=1}^K \widehat{\Delta}_{N,1,k}, \quad (\text{D.4})$$

where

$$\widehat{\Delta}_{N,1,k} = -N^{-1} \sum_{i \in \mathcal{J}_k} \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - 1 \right\} \{\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X}_i)\}. \quad (\text{D.5})$$

Consider the remainder term $\widehat{\Delta}_{N,1,k}$. For each $k \leq K$, notice that $\widehat{\Delta}_{N,1,k}$ is a summation of independent random variables conditional on the training sample \mathbb{S}_{-k} :

$$\widehat{\Delta}_{N,1,k} = -N^{-1} \sum_{i \in \mathcal{J}_k} \xi_i, \quad \xi_i = \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - 1 \right\} \{\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X}_i)\},$$

with $\xi_i \perp \xi_j | \mathbb{S}_{-k}$ for $i, j \in \mathcal{J}_k$. Hence, with

$$\xi = \left\{ \frac{R}{\pi_N(\mathbf{X})} - 1 \right\} \{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})\},$$

and recall that $\mathbb{E}_{\mathbb{S}_k}$ denotes the expectation with respect to (w.r.t.) the samples in the k -th fold,

$$\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,1,k}) = -N^{-1} |\mathcal{J}_k| \mathbb{E}\{\mathbb{E}(\xi | \mathbf{X})\} = 0, \quad (\text{D.6})$$

$$\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,1,k}^2) = N^{-2} |\mathcal{J}_k| \mathbb{E}\left(\mathbb{E}\left[\left\{ \frac{R}{\pi_N(\mathbf{X})} - 1 \right\}^2 \{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})\}^2 | \mathbf{X}\right]\right) \quad (\text{D.7})$$

$$= N^{-2} |\mathcal{J}_k| \mathbb{E}\left\{ \left[\frac{1}{\pi_N(\mathbf{X})} - 1 \right] [\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})]^2 \right\} \quad (\text{D.8})$$

$$= O_p((Na_N)^{-1} c_{\mu, N}^2), \quad (\text{D.9})$$

In the above equations, (D.6) and (D.7) used the fact that $\xi_i \perp \xi_j | \mathbb{S}_{-k}$ for $i, j \in \mathcal{I}_k$; (D.8) used the fact that $R^2 = R$; (D.9) used the fact that $|\mathcal{I}_k| < N$ and the definition of $c_{\mu, N}$; the definition $\mathbb{E}(R|\mathbf{X}) = \pi_N(\mathbf{X})$ is also used in (D.6) and (D.8). These techniques will be used for multiple times throughout the proof, and we will not emphasize them again in the following proofs.

By Lemma A.1,

$$\widehat{\Delta}_{N,1,k} = O_p((Na_N)^{-1/2} c_{\mu, N}). \quad (\text{D.10})$$

As for the influence function $N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i)$,

$$\begin{aligned} \mathbb{E}_{\mathbb{S}} \left[N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i) \right] &= \mathbb{E} \left(\mathbb{E} \left\{ \mu(\mathbf{X}) - \theta_0 + \frac{R[Y - \mu(\mathbf{X})]}{\pi_N(\mathbf{X})} \mid \mathbf{X} \right\} \right) = 0, \\ \mathbb{E}_{\mathbb{S}} \left[N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i) \right]^2 &= N^{-1} \mathbb{E} \left[\mathbb{E} \left(\left\{ \mu(\mathbf{X}) - \theta_0 + \frac{R[Y - \mu(\mathbf{X})]}{\pi_N(\mathbf{X})} \right\}^2 \mid \mathbf{X} \right) \right] = N^{-1} V_N(\mu), \end{aligned}$$

where

$$\begin{aligned} V_N(\mu) &= \mathbb{E} \left[\mu(\mathbf{X}) - \theta_0 + \frac{R\{Y - \mu(\mathbf{X})\}}{\pi_N(\mathbf{X})} \right]^2 = \mathbb{E} \left[\frac{\{R - \pi_N(\mathbf{X})\}\{Y - \mu(\mathbf{X})\}}{\pi_N(\mathbf{X})} + Y - \theta_0 \right]^2 \\ &= \mathbb{E} \left[\left\{ \frac{1 - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \right\}^2 \{Y - \mu(\mathbf{X})\}^2 \right] + \text{Var}(Y). \end{aligned}$$

To control the order of $V_N(\mu)$, we enforce uniform lower and upper bounds for $\mathbb{E}[\{Y - \mu(\mathbf{X})\}^2 | \mathbf{X}]$ and $\text{Var}(Y)$. Under Assumption 3.1,

$$\mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X}] \geq \sigma_{\xi,1}^2, \quad \mathbb{E}[\{Y - \mu(\mathbf{X})\}^2 | \mathbf{X}] \leq \sigma_{\xi,2}^2, \quad \text{Var}(Y) \leq \sigma_{\xi,2}^2.$$

Additionally, we have the following lower bounds as $m(\mathbf{X}) = \mathbb{E}(Y | \mathbf{X})$,

$$\begin{aligned} \mathbb{E}[\{Y - \mu(\mathbf{X})\}^2 | \mathbf{X}] &= \mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X}] + \mathbb{E}[\{m - \mu(\mathbf{X})\}^2 | \mathbf{X}] \geq \sigma_{\xi,1}^2, \\ \text{Var}(Y) &= \mathbb{E}(\mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X}] + \mathbb{E}[\{m(\mathbf{X}) - \theta_0\}^2 | \mathbf{X}]) \geq \sigma_{\xi,1}^2. \end{aligned}$$

Recall that by definition, $a_N = \mathbb{E}\{\pi_N^{-1}(\mathbf{X})\}$. Therefore,

$$\begin{aligned} a_N V_N(\mu) &\geq a_N \left\{ \sigma_{\xi,1}^2 \mathbb{E} \left[\frac{1 - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \right] + \sigma_{\xi,1}^2 \right\} = \sigma_{\xi,1}^2 > 0, \\ a_N V_N(\mu) &\leq a_N \left\{ \sigma_{\xi,2}^2 \mathbb{E} \left[\frac{1 - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \right] + \sigma_{\xi,2}^2 \right\} = \sigma_{\xi,2}^2 < \infty, \end{aligned}$$

and $V_N(\mu) \asymp a_N^{-1}$. Since

$$\mathbb{E} \left\{ (Na_N)^{1/2} N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i) \right\}^2 = a_N V_N(\mu) = O(1), \quad (\text{D.11})$$

by Lemma A.1, $N^{-1} \sum_{i=1}^N \psi_{\mu, \pi}(\mathbf{Z}_i) = O_p((Na_N)^{-1/2})$. Therefore,

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = O_p((Na_N)^{-1/2}).$$

In addition, under Assumption 3.2, for any $c > 0$,

$$N^{-1} \sum_{i=1}^N \mathbb{E}[V_N^{-1}(\mu) \psi_{\mu,\pi}^2(\mathbf{Z}_i) 1\{V_N^{-1/2}(\mu) |\psi_{\mu,\pi}(\mathbf{Z}_i)| > cN^{1/2}\}] = o(1).$$

By Proposition 2.27 (Lindeberg-Feller theorem) of [Van der Vaart \[2000\]](#),

$$V_N^{-1/2}(\mu) N^{-1/2} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) \xrightarrow{d} \mathcal{N}(0, 1).$$

Recall that

$$\begin{aligned} N^{1/2} V_N^{-1/2}(\mu)(\tilde{\theta} - \theta_0) &= V_N^{-1/2}(\mu) N^{-1/2} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) + N^{1/2} V_N^{-1/2}(\mu) \sum_{k=1}^K \widehat{\Delta}_{N,1,k} \\ &= V_N^{-1/2}(\mu) N^{-1/2} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) + O_p(N^{1/2} a_N^{1/2} (Na_N)^{-1/2} c_{\mu,N}) \\ &= V_N^{-1/2}(\mu) N^{-1/2} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) + O_p(c_{\mu,N}) = V_N^{-1/2}(\mu) N^{-1/2} \sum_{i=1}^N \psi_{\mu,\pi}(\mathbf{Z}_i) + o_p(1). \end{aligned}$$

By Lemma 2.8 (Slutsky) of [Van der Vaart \[2000\]](#),

$$N^{1/2} V_N^{-1/2}(\mu)(\tilde{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1).$$

□

Proof of Theorem 3.2. We prove Theorem 3.2 by considering two cases: (a) the nuisance models are both correctly specified, and (b) only one of the nuisance models is correctly specified. For case (a), we design a suitable decomposition, (D.12), and apply Lemma A.1 and the Lindeberg-Feller theorem for self-normalized sums to obtain asymptotic normality. For case (b), we design two different decompositions of the estimation error: one suitable for the case when PS model is correct (D.19) and the other suitable for the case when the outcome model is correct (D.20).

Case (a): $\mu(\cdot) = m(\cdot)$ and $e_N(\cdot) = \pi_N(\cdot)$. Observe that

$$\begin{aligned} \widehat{\theta}_{\text{DRSS}} - \theta_0 &= N^{-1} \sum_{i=1}^N \left[\frac{R_i - \widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})}{\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} \{Y_i - \widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k})\} + Y_i - \theta_0 \right] \\ &= N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + \sum_{k=1}^K (\widehat{\Delta}_{N,1,k} + \widehat{\Delta}_{N,2,k} + \widehat{\Delta}_{N,3,k}), \end{aligned} \quad (\text{D.12})$$

where $\widehat{\Delta}_{N,1,k}$ is defined as (D.5) and we further define

$$\widehat{\Delta}_{N,2,k} = N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{R_i}{\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{Y_i - m(\mathbf{X}_i)\}, \quad (\text{D.13})$$

$$\widehat{\Delta}_{N,3,k} = -N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{R_i}{\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X}_i)\}. \quad (\text{D.14})$$

Recall from (D.10), $\widehat{\Delta}_{N,1,k} = O_p((Na_N)^{-1/2} c_{\mu,N})$. As for the remainder term $\widehat{\Delta}_{N,2,k}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,2,k}) &= N^{-1} |\mathcal{I}_k| \mathbb{E} \left(\mathbb{E} \left[\left\{ \frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})} \right\} \{Y - m(\mathbf{X})\} | \mathbf{X} \right] \right) = 0, \\ \mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,2,k}^2) &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left(\mathbb{E} \left[\left\{ \frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})} \right\}^2 \{Y - m(\mathbf{X})\}^2 | \mathbf{X} \right] \right) \\ &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left[\frac{\pi_N(\mathbf{X})}{e_N^2(\mathbf{X})} \left\{ 1 - \frac{e_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\}^2 \{Y - m(\mathbf{X})\}^2 \right] \quad (\text{D.15})\end{aligned}$$

$$\stackrel{(i)}{\leqslant} N^{-1} \sigma_{\xi,2}^2 c_{e,N}^2 a_N^{-1} = O_p((Na_N)^{-1} c_{e,N}^2), \quad (\text{D.16})$$

where (i) holds under the Assumption 3.1 and the condition in (3.8) with $e_N(\cdot) = \pi_N(\cdot)$ and also noting the fact that $|\mathcal{I}_k| \leqslant N$. By Lemma A.1,

$$\widehat{\Delta}_{N,2,k} = O_p((Na_N)^{-1/2} c_{e,N}). \quad (\text{D.17})$$

Now, consider the last remainder term $\widehat{\Delta}_{N,3,k}$, by the triangular inequality and the tower rule,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}(|\widehat{\Delta}_{N,3,k}|) &\leqslant N^{-1} |\mathcal{I}_k| \mathbb{E} \left[\mathbb{E} \left\{ \left| \frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})} \right| |\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X})| |\mathbf{X} \right\} \right] \\ &= N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \left| 1 - \frac{e_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right| |\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X})| \right\} = O_p(r_{\mu,N} r_{e,N}).\end{aligned}$$

By Lemma A.1,

$$\widehat{\Delta}_{N,3,k} = O_p(r_{\mu,N} r_{e,N}). \quad (\text{D.18})$$

Lastly, for the influence function $N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i)$,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}} \left\{ N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) \right\} &= \mathbb{E} \left(\mathbb{E} \left[\mu(\mathbf{X}) - \theta_0 + \frac{R\{Y - m(\mathbf{X})\}}{\pi_N(\mathbf{X})} | \mathbf{X} \right] \right) = 0, \\ \mathbb{E}_{\mathbb{S}} \left\{ N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) \right\}^2 &= N^{-1} \mathbb{E} \left\{ \mathbb{E} \left(\left[\mu(\mathbf{X}) - \theta_0 + \frac{R\{Y - m(\mathbf{X})\}}{\pi_N(\mathbf{X})} \right]^2 | \mathbf{X} \right) \right\} = N^{-1} V_N(\mu, e).\end{aligned}$$

Now we control the rate of the variance, $V_N(\mu, e)$. Under Assumption 3.1, $\text{Var}(Y) \geqslant \mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X}] \geqslant \sigma_{\xi,1}^2$, $\mathbb{E}[\{Y - m(\mathbf{X})\}^2 | \mathbf{X}] \leqslant \sigma_{\xi,2}^2$ and $\text{Var}(Y) \leqslant \sigma_{\xi,2}^2$. Hence,

$$\begin{aligned}a_N V_N(\mu, e) &\geqslant a_N \left[\sigma_{\xi,1}^2 \mathbb{E} \left\{ \frac{1 - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \right\} + \sigma_{\xi,1}^2 \right] \geqslant \sigma_{\xi,1}^2 > 0, \\ a_N V_N(\mu, e) &\leqslant a_N \left[\sigma_{\xi,2}^2 \mathbb{E} \left\{ \frac{1 - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \right\} + \sigma_{\xi,2}^2 \right] \leqslant \sigma_{\xi,2}^2 < \infty.\end{aligned}$$

It follows that $V_N(\mu, e) \asymp a_N^{-1}$.

Recall the definition of $\psi_{\mu,e}$ in (3.5). By Lemma A.1, $N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) = O_p((Na_N)^{-1/2})$. Therefore, $\widehat{\theta}_{\text{DRSS}} - \theta_0 = O_p((Na_N)^{-1/2})$. Moreover, by Proposition 2.27 (Lindeberg-Feller theorem) of Van der Vaart [2000],

$$N^{1/2} V_N^{1/2}(\mu, e) N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) \xrightarrow{d} \mathcal{N}(0, 1).$$

By Lemma 2.8 (Slutsky) of Van der Vaart [2000],

$$N^{1/2} V_N^{-1/2}(\mu, e) (\widehat{\theta}_{\text{DRSS}} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1).$$

Case (b.i): $e_N(\cdot) = \pi_N(\cdot)$. Observe that

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + \sum_{k=1}^{\mathbb{K}} (\widehat{\Delta}_{N,1,k} + \widehat{\Delta}_{N,2,k} + \widehat{\Delta}_{N,3,k} + \widehat{\Delta}_{N,4,k}), \quad (\text{D.19})$$

where $\widehat{\Delta}_{N,1,k}$, $\widehat{\Delta}_{N,2,k}$, and $\widehat{\Delta}_{N,3,k}$ are defined as (D.5), (D.13), and (D.14), respectively, and we further define

$$\widehat{\Delta}_{N,4,k} = N^{-1} \sum_{i \in \mathcal{J}_k} \left\{ \frac{R_i}{\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{m(\mathbf{X}_i) - \mu(\mathbf{X}_i)\}.$$

As shown in (D.10), (D.17), and (D.18), we have $\widehat{\Delta}_{N,1,k} = O_p((Na_N)^{-1/2} c_{\mu,N})$, $\widehat{\Delta}_{N,2,k} = O_p((Na_N)^{-1/2} c_{e,N})$ and $\widehat{\Delta}_{N,3,k} = O_p(r_{\mu,N} r_{e,N})$. In addition, for the remainder term $\widehat{\Delta}_{N,4,k}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{S}_k}(|\widehat{\Delta}_{N,4,k}|) &\leq N^{-1} |\mathcal{J}_k| \mathbb{E} \left[\mathbb{E} \left\{ \left| \frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})} \right| |m(\mathbf{X}_i) - \mu(\mathbf{X})| \middle| \mathbf{X} \right\} \right] \\ &= N^{-1} |\mathcal{J}_k| \mathbb{E} \left\{ 1 - \frac{e_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \middle| |m(\mathbf{X}_i) - \mu(\mathbf{X})| \right\} \\ &\leq \left\| 1 - \frac{e_N(\cdot)}{\widehat{\pi}_N(\cdot)} \right\|_{2, \mathbb{P}_{\mathbf{X}}} \|m(\cdot) - \mu(\cdot)\|_{2, \mathbb{P}_{\mathbf{X}}} = O_p(r_{e,N}). \end{aligned}$$

By Lemma A.1,

$$\widehat{\Delta}_{N,4,k} = O_p(r_{e,N}).$$

Lastly, for the influence function $N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i)$, similarly as in (D.11) and by Lemma A.1,

$$N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) = O_p((Na_N)^{-1/2}).$$

Case (b.ii): $\mu(\cdot) = m(\cdot)$. Observe that

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = N^{-1} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + \sum_{k=1}^{\mathbb{K}} (\widehat{\Delta}_{N,1,k} + \widehat{\Delta}_{N,2,k} + \widehat{\Delta}_{N,3,k} + \widehat{\Delta}_{N,4,k}), \quad (\text{D.20})$$

where $\widehat{\Delta}_{N,1,k}$, $\widehat{\Delta}_{N,2,k}$, and $\widehat{\Delta}_{N,3,k}$ are defined as (D.5), (D.13), and (D.14), respectively, and we further define

$$\widehat{\Delta}_{N,5,k} = N^{-1} \sum_{i \in \mathcal{J}_k} \left\{ \frac{R_i}{\pi_N(\mathbf{X}_i)} - \frac{R_i}{e_N(\mathbf{X}_i)} \right\} \{\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X}_i)\}.$$

Similarly as shown in (D.10), (D.17), and (D.18), we have $\widehat{\Delta}_{N,1,k} = O_p((Na_N)^{-1/2}c_{\mu,N})$, $\widehat{\Delta}_{N,2,k} = O_p((Na_N)^{-1/2}c_{e,N})$ and $\widehat{\Delta}_{N,2,k} = O_p(r_{\mu,N}r_{e,N})$. Here, the only difference from the previous proofs is that, in (D.15), instead of obtaining $\pi_N(\mathbf{X})/e_N^2(\mathbf{X}) = \pi_N^{-1}(\mathbf{X})$ using $e_N(\cdot) = \pi_N(\cdot)$, here we bound $\pi_N(\mathbf{X})/e_N^2(\mathbf{X}) \leq c^{-2}\pi_N^{-1}(\mathbf{X})$ by assuming that, a.s., $\pi_N(\mathbf{X})/e_N(\mathbf{X}) \geq c$. For the remainder term $\widehat{\Delta}_{N,5,k}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}(|\widehat{\Delta}_{N,5,k}|) &\leq N^{-1}|\mathcal{I}_k|\mathbb{E}\left[\mathbb{E}\left\{\left|\frac{R}{\pi_N(\mathbf{X})} - \frac{R}{e_N(\mathbf{X})}\right| |\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X})| \mid \mathbf{X}\right\}\right] \\ &= N^{-1}|\mathcal{I}_k|\mathbb{E}\left\{1 - \frac{\pi_N(\mathbf{X})}{e_N(\mathbf{X})} \mid |\widehat{m}(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu(\mathbf{X})|\right\} \\ &\leq \|1 - \pi_N(\cdot)/e_N(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} \|\widehat{m}(\cdot) - \mu(\cdot)\|_{2,\mathbb{P}_{\mathbf{X}}} = O_p(r_{\mu,N}).\end{aligned}$$

By Lemma A.1,

$$\widehat{\Delta}_{N,5,k} = O_p(r_{\mu,N}).$$

Lastly, for the influence function $N^{-1}\sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i)$, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{S}}\left\{N^{-1}\sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i)\right\} &= \mathbb{E}\left(\mathbb{E}\left[m(\mathbf{X}) - \theta_0 + \frac{R\{Y - m(\mathbf{X})\}}{e_N(\mathbf{X})} \mid \mathbf{X}\right]\right) = 0, \\ \mathbb{E}_{\mathbb{S}}\left\{N^{-1}\sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i)\right\}^2 &= N^{-1}\mathbb{E}\left\{\mathbb{E}\left(\left[m(\mathbf{X}) - \theta_0 + \frac{R\{Y - m(\mathbf{X})\}}{e_N(\mathbf{X})}\right]^2 \mid \mathbf{X}\right)\right\} = N^{-1}V_N(\mu, e).\end{aligned}$$

Here,

$$\begin{aligned}V_N(\mu, e) &= \text{Var}\{m(\mathbf{X})\} + \mathbb{E}\{\pi_N(\mathbf{X})\{Y - m(\mathbf{X})\}^2 / \{\pi_N(\mathbf{X})\}^2\} \\ &\leq \sigma_{\zeta,2}^2(1 + \mathbb{E}[\pi_N(\mathbf{X})/\{\pi_N(\mathbf{X})\}^2]) \leq \sigma_{\zeta,2}^2(1 + C^2a_N^{-1}) = O(a_N^{-1}).\end{aligned}$$

By Lemma A.1,

$$N^{-1}\sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) = O_p((Na_N)^{-1/2}).$$

□

Proof of Theorem 3.3. We prove the consistency results of the asymptotic variance estimators for the two cases (known PS and unknown PS). The results follows from Lemma A.4 after we validate the conditions therein.

Case (a). By Lemma A.4, it is sufficient to show $a_N\mathbb{E}(\delta_{N,1,k}^2) = o_p(1)$, where

$$\delta_{N,1,k} = -\left\{\frac{R}{\pi_N(\mathbf{X})} - 1\right\} \{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})\}.$$

Recall from (D.9), we have

$$a_N\mathbb{E}(\widehat{\delta}_{N,1,k}^2) = O_p(c_{\mu,N}^2) = o_p(1).$$

Case (b). By Lemma A.4, it is sufficient to show $a_N \mathbb{E}(\delta_{N,1,k} + \delta_{N,2,k} + \delta_{N,3,k})^2 = o_p(1)$, where

$$\begin{aligned}\delta_{N,1,k} &= -\left\{\frac{R}{\pi_N(\mathbf{X})} - 1\right\}\{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})\}, \\ \delta_{N,2,k} &= \left\{\frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})}\right\}\{Y - m(\mathbf{X})\}, \\ \delta_{N,3,k} &= -\left\{\frac{R}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{R}{e_N(\mathbf{X})}\right\}\{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - \mu(\mathbf{X})\}.\end{aligned}$$

Recall (D.9) and (D.16), we have

$$a_N \mathbb{E}(\widehat{\delta}_{N,1,k}^2) = O_p(c_{\mu,N}^2) = o_p(1), \quad a_N \mathbb{E}(\widehat{\delta}_{N,2,k}^2) = O_p(c_{e,N}^2) = o_p(1).$$

Besides, by the condition (3.13),

$$a_N \mathbb{E}(\widehat{\delta}_{N,3,k}^2) = \mathbb{E}\left[\frac{a_N}{\pi_N(\mathbf{X})}\left\{1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}(\mathbf{X}; \mathbb{S}_{-k})}\right\}^2\{\widehat{m}(\mathbf{X}; \mathbb{S}_{-k}) - m(\mathbf{X})\}^2\right] = o_p(1).$$

Therefore,

$$a_N \mathbb{E}(\widehat{\Delta}_{N,1,k} + \widehat{\Delta}_{N,2,k} + \widehat{\Delta}_{N,3,k})^2 \leq 3a_N \mathbb{E}(\widehat{\Delta}_{N,1,k}^2 + \widehat{\Delta}_{N,2,k}^2 + \widehat{\Delta}_{N,3,k}^2) = o_p(1).$$

□

Proof of Theorem 4.1. In the proof of Theorem 4.1, we work directly on the cross-fitted version of $\widehat{\boldsymbol{\gamma}}$. The results for a non cross-fitted $\widehat{\boldsymbol{\gamma}}$ can be obtained analogously by repeating the procedure using the full sample \mathbb{S} . Here, we first obtain an RAL expansion of the offset logistic regression estimator. Then, we establish the RAL expansion of the DRSS estimator.

For any $k \leq \mathbb{K}$, $a \in (0, 1]$, and $\boldsymbol{\gamma} \in \mathbb{R}^{p+1}$, let

$$\ell_N(\boldsymbol{\gamma}; a) = -N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \left[R_i \vec{\mathbf{X}}_i^T \boldsymbol{\gamma} - \log\{1 + a \exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma})\} \right].$$

Define $g(u) = \exp(u)/\{1 + \exp(u)\}$, then $\dot{g}(u) = g(u)\{1 - g(u)\}$ and $\ddot{g}(u) = g(u)\{1 - g(u)\}\{1 - 2g(u)\}$. We have

$$\begin{aligned}g(u + \log(a)) &= \frac{a \exp(u)}{1 + a \exp(u)} \geq \frac{a \exp(u)}{1 + \exp(u)} = ag(u), \quad \forall u \in \mathbb{R}, a \in (0, 1], \\ g(u) &\leq \exp(u), \quad \dot{g}(u) \leq g(u) \leq \exp(u), \quad |\ddot{g}(u)| \leq g(u) \leq \exp(u), \quad \forall u \in \mathbb{R}.\end{aligned}\tag{D.21}$$

For any $\mathbf{u} \in \mathbb{R}^{p+1}$, define

$$\ell_N(\mathbf{u}) = N_{-k} \left\{ \ell_N(\boldsymbol{\gamma}_0 + (N_{-k} \bar{\pi}_N)^{-1/2} \mathbf{u}, \widehat{\pi}_N) - \ell_N(\boldsymbol{\gamma}_0, \bar{\pi}_N) \right\} - N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i \log(\widehat{\pi}_N / \bar{\pi}_N).$$

Since $\boldsymbol{\gamma} = \widehat{\boldsymbol{\gamma}}$ minimizes $\ell_N(\boldsymbol{\gamma}; \widehat{\pi}_N)$, the terms $\ell_N(\boldsymbol{\gamma}_0, \bar{\pi}_N)$ and $N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i \log(\widehat{\pi}_N / \bar{\pi}_N)$ are both independent of $\boldsymbol{\gamma}$, we know that $\mathbf{u}_N = (N_{-k} \bar{\pi}_N)^{1/2} (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)$ minimizes $\ell_N(\mathbf{u})$. Here, $\widehat{\pi}_N = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i$ is the

cross-fitted estimate of $\bar{\pi}_N$. By Taylor's Theorem, where some $(\tilde{\gamma}_1, \log(\tilde{\pi}_{N,1}))$ lies between $(\gamma_0, \log(\bar{\pi}_N))$ and $(\gamma_0 + (N_{-k}\bar{\pi}_N)^{-1/2}\mathbf{u}, \log(\hat{\pi}_N))$,

$$\ell_N(\mathbf{u}) = \frac{1}{2} \mathbf{u}^T \mathbf{A}_N(\tilde{\gamma}_1, \tilde{\pi}_{N,1}) \mathbf{u} + \mathbf{B}_{N,1}^T(\tilde{\gamma}_1, \tilde{\pi}_{N,1}) \mathbf{u} + C_N(\tilde{\gamma}_1, \tilde{\pi}_{N,1}),$$

where

$$\begin{aligned} \mathbf{A}_N(\tilde{\gamma}_1, \tilde{\pi}_{N,1}) &= (N_{-k}\bar{\pi}_N)^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\gamma}_1 + \log(\tilde{\pi}_{N,1})) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T, \\ \mathbf{B}_{N,1}(\tilde{\gamma}_1, \tilde{\pi}_{N,1}) &= -(N_{-k}\bar{\pi}_N)^{-1/2} \sum_{i \in \mathcal{I}_{-k}} \left\{ R_i - g(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \right. \\ &\quad \left. - \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\gamma}_1 + \log(\tilde{\pi}_{N,1})) \log(\hat{\pi}_N / \bar{\pi}_N) \right\} \vec{\mathbf{X}}_i, \\ C_N(\tilde{\gamma}_1, \tilde{\pi}_{N,1}) &= \frac{1}{2} \sum_{i \in \mathcal{I}_{-k}} \left\{ \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\gamma}_1 + \log(\tilde{\pi}_{N,1})) - \dot{g}(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \right\} \{\log(\hat{\pi}_N / \bar{\pi}_N)\}^2. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{J}(\gamma_0, \bar{\pi}_N) &= \mathbb{E} \left\{ \vec{\mathbf{X}} \vec{\mathbf{X}}^T \dot{g}(\vec{\mathbf{X}}^T \gamma_0 + \log(\bar{\pi}_N)) \right\}, \\ \mathbf{B}_{N,2} &= -(N_{-k}\bar{\pi}_N)^{-1/2} \sum_{i \in \mathcal{I}_{-k}} \left\{ R_i - g(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \right. \\ &\quad \left. - \dot{g}(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \log(\hat{\pi}_N / \bar{\pi}_N) \right\} \vec{\mathbf{X}}_i, \\ \boldsymbol{\zeta}_N &= (N_{-k}\bar{\pi}_N)^{1/2} \mathcal{J}^{-1}(\gamma_0, \bar{\pi}_N) N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \left\{ R_i - g(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \right. \\ &\quad \left. - \dot{g}(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N)) \log(\hat{\pi}_N / \bar{\pi}_N) \right\} \vec{\mathbf{X}}_i. \end{aligned} \tag{D.22}$$

Then, $\boldsymbol{\zeta}_N$ is the unique minimizer of

$$Z_N(\mathbf{u}) = \mathbf{u}^T \bar{\pi}_N^{-1} \mathcal{J}(\gamma_0, \bar{\pi}_N) \mathbf{u} / 2 + \mathbf{B}_{N,2}^T \mathbf{u}.$$

By Lemma 2 of [Hjort and Pollard \[2011\]](#), for each $\delta > 0$,

$$\mathbb{P}_{\mathbb{S}_{-k}}(\|\mathbf{u}_N - \boldsymbol{\zeta}_N\|_2 \geq \delta) \leq \mathbb{P} \left\{ \Delta_N(\delta) \geq \frac{1}{2} h_N(\delta) \right\},$$

where $\mathbb{S}_{-k} = \mathbb{S} \setminus \mathbb{S}_k$ and

$$\Delta_N(\delta) = \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} |\ell_N(\mathbf{u}) - Z_N(\mathbf{u})|, \quad h_N(\delta) = \inf_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 = \delta} Z_N(\mathbf{u}) - Z_N(\boldsymbol{\zeta}_N).$$

Hence, to prove

$$\left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 - (N_{-k}\bar{\pi}_N)^{-1/2} \boldsymbol{\zeta}_N \right\|_2 = (N_{-k}\bar{\pi}_N)^{-1/2} \|\mathbf{u}_N - \boldsymbol{\zeta}_N\|_2 = o_p \left((N\bar{\pi}_N)^{-1/2} \right), \tag{D.23}$$

it suffices to show that, for each $\delta > 0$, $\Delta_N(\delta) = o_p(1)$ and $h_N(\delta) > c(\delta)$ with some constant $c(\delta) > 0$ independent of N . First notice that

$$\begin{aligned} h_N(\delta) &= \inf_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 = \delta} \frac{1}{2} (\mathbf{u} - \boldsymbol{\zeta}_N)^T \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)(\mathbf{u} - \boldsymbol{\zeta}_N) \\ &\geq \frac{1}{2} \delta^2 \bar{\pi}_N^{-1} \lambda_{\min}\{\mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\} \geq \frac{1}{2} \delta^2 \lambda_{\min}[\mathbb{E}\{\vec{\mathbf{X}} \vec{\mathbf{X}}^T \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\}]. \end{aligned}$$

Now, it remains to show $\Delta_N(\delta) = o_p(1)$. A sufficient condition we would like to show is the following:

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} |\mathbf{u}^T \{\mathbf{A}_N(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1}) - \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\} \mathbf{u}| = o_p(1), \quad (\text{D.24})$$

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} |(\mathbf{B}_{N,1}(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1}) - \mathbf{B}_{N,2})^T \mathbf{u}| = o_p(1), \quad (\text{D.25})$$

$$|C_N(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1})| = o_p(1). \quad (\text{D.26})$$

To prove (D.24)-(D.26), we first analyze some basic properties of $\tilde{\pi}_{N,1}$ and $\boldsymbol{\zeta}_N$. With (D.65), we have

$$\tilde{\pi}_N = \bar{\pi}_N \left\{ 1 + O_p\left((N\bar{\pi}_N)^{-1/2}\right) \right\}, \quad \text{for any } \tilde{\pi}_N \text{ lies between } \bar{\pi}_N \text{ and } \hat{\pi}_N. \quad (\text{D.27})$$

In addition, by the fact that $\log(u) \leq 1 - u$ for all $u > 0$ and (D.27) we have

$$|\log(\tilde{\pi}_N/\bar{\pi}_N)| \leq |1 - \tilde{\pi}_N/\bar{\pi}_N| = O_p\left((N\bar{\pi}_N)^{-1/2}\right), \quad \text{for any } \tilde{\pi}_N \text{ lies between } \bar{\pi}_N \text{ and } \hat{\pi}_N. \quad (\text{D.28})$$

For any $t < \infty$ and $r < \infty$,

$$\begin{aligned} \mathbb{E}\left\{\exp(t\|\vec{\mathbf{X}}\|_2)\|\vec{\mathbf{X}}\|_2^r\right\} &\leq r!\mathbb{E}\left[\exp\{(t+1)\|\vec{\mathbf{X}}\|_2\}\right] \\ &\leq r!\exp(t+1)\mathbb{E}[\exp\{(t+1)\|\mathbf{X}\|_2\}] < \infty. \end{aligned} \quad (\text{D.29})$$

Now, to control the supremum over $\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta$ in (D.24) and (D.25), we analyse asymptotic properties for $\boldsymbol{\zeta}_N$ defined in (D.22). We consider the following representation:

$$\boldsymbol{\zeta}_N = \boldsymbol{\zeta}_{N,1} - \boldsymbol{\zeta}_{N,2}, \quad \text{where} \quad (\text{D.30})$$

$$\boldsymbol{\zeta}_{N,1} = (N_{-k}\bar{\pi}_N)^{1/2} \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \left\{ R_i - g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \right\} \vec{\mathbf{X}}_i, \quad (\text{D.31})$$

$$\boldsymbol{\zeta}_{N,2} = \log(\hat{\pi}_N/\bar{\pi}_N) (N_{-k}\bar{\pi}_N)^{1/2} \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}_i.$$

Moreover, define

$$\begin{aligned} \boldsymbol{\zeta}_{N,3} &= \log(\hat{\pi}_N/\bar{\pi}_N) (N_{-k}\bar{\pi}_N)^{1/2} \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \mathbb{E}\left\{\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}\right\} \\ &= (N_{-k}\bar{\pi}_N)^{1/2} \log(\hat{\pi}_N/\bar{\pi}_N) \mathbf{e}_1. \end{aligned} \quad (\text{D.32})$$

In the above we used $\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \mathbb{E}\{\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}\} = \mathbf{e}_1$. Note that,

$$\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) = \frac{\bar{\pi}_N \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)}{\{1 + \bar{\pi}_N \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\}^2} \geq \frac{\bar{\pi}_N \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)}{\{1 + \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\}^2} = \bar{\pi}_N \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0).$$

Hence,

$$\|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2 \leq \bar{\pi}_N^{-1} \left\| \left[\mathbb{E}\{\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \vec{\mathbf{X}} \vec{\mathbf{X}}^T\} \right]^{-1} \right\|_2 = O(\bar{\pi}_N^{-1}). \quad (\text{D.33})$$

Then,

$$\begin{aligned} \mathbb{E}_{S_{-k}} \|\boldsymbol{\zeta}_{N,1}\|_2^2 &\stackrel{(i)}{=} \bar{\pi}_N \mathbb{E} \left\{ \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \vec{\mathbf{X}}\|_2^2 \right\} \\ &\stackrel{(ii)}{\leq} \bar{\pi}_N \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2^2 \mathbb{E} \left\{ \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\vec{\mathbf{X}}\|_2^2 \right\} \\ &\stackrel{(iii)}{\leq} \bar{\pi}_N^2 \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2^2 \mathbb{E} \left\{ \exp(\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2) \|\vec{\mathbf{X}}\|_2^2 \right\} \stackrel{(iv)}{=} O(1), \end{aligned}$$

where (i) holds by the tower rule with the fact $\mathbb{E}[\{R - g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\}^2 | \mathbf{X}] = \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))$, (ii) holds by the fact that $|\mathbf{A}\mathbf{a}| \leq \|\mathbf{A}\|_2 \|\mathbf{a}\|_2$ for any $\mathbf{a} \in \mathbb{R}^{p+1}$ and $\mathbf{A} \in \mathbb{R}^{(p+1) \times (p+1)}$, (iii) follows by the fact that $\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \leq g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \leq \bar{\pi}_N \exp(\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2)$, and (iv) holds by (D.29) and (D.33). Besides,

$$\begin{aligned} \mathbb{E}_{S_{-k}} \|\{\log(\hat{\pi}_N/\bar{\pi}_N)\}^{-1}(\boldsymbol{\zeta}_{N,2} - \boldsymbol{\zeta}_{N,3})\|_2^2 &= \bar{\pi}_N \text{Var} \left\{ \dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \vec{\mathbf{X}}\|_2 \right\} \\ &\leq \bar{\pi}_N \mathbb{E} \left\{ \dot{g}^2(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \vec{\mathbf{X}}\|_2^2 \right\} \\ &\stackrel{(i)}{\leq} \bar{\pi}_N \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2^2 \mathbb{E} \left\{ \dot{g}^2(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\vec{\mathbf{X}}\|_2^2 \right\} \\ &\stackrel{(ii)}{\leq} \bar{\pi}_N^3 \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2^2 \mathbb{E} \left\{ \exp(2\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2) \|\vec{\mathbf{X}}\|_2^2 \right\} \stackrel{(iii)}{=} O(\bar{\pi}_N), \end{aligned}$$

where (i) holds by the fact that $|\mathbf{A}\mathbf{a}| \leq \|\mathbf{A}\|_2 \|\mathbf{a}\|_2$ for any $\mathbf{a} \in \mathbb{R}^{p+1}$ and $\mathbf{A} \in \mathbb{R}^{(p+1) \times (p+1)}$, (ii) follows by the fact that $\dot{g}(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \leq \bar{\pi}_N \exp(\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2)$, and (iii) holds by (D.29) and (D.33). By Chebyshev's Inequality,

$$\|\boldsymbol{\zeta}_{N,1}\| = O_p(1), \quad \|\{\log(\hat{\pi}_N/\bar{\pi}_N)\}^{-1}(\boldsymbol{\zeta}_{N,2} - \boldsymbol{\zeta}_{N,3})\|_2 = O_p(\bar{\pi}_N^{1/2}).$$

Hence, by (D.28),

$$\|(\boldsymbol{\zeta}_{N,2} - \boldsymbol{\zeta}_{N,3})\|_2 = \{\log(\hat{\pi}_N/\bar{\pi}_N)\} O_p(\bar{\pi}_N^{1/2}) = O_p(N^{-1/2}), \quad (\text{D.34})$$

with

$$\|\boldsymbol{\zeta}_{N,3}\|_2 \leq |\log(\hat{\pi}_N/\bar{\pi}_N)| (N_{-k} \bar{\pi}_N)^{1/2} \|\mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2 \bar{\pi}_N \mathbb{E} \left\{ \exp(\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2) \|\vec{\mathbf{X}}\|_2 \right\} = O_p(1).$$

Therefore,

$$\|\boldsymbol{\zeta}_N\|_2 \leq \|\boldsymbol{\zeta}_{N,1}\|_2 + \|\boldsymbol{\zeta}_{N,2} - \boldsymbol{\zeta}_{N,3}\|_2 + \|\boldsymbol{\zeta}_{N,3}\|_2 = O_p(1).$$

It follows that,

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\mathbf{u}\|_2 \leq \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 + \|\boldsymbol{\zeta}_N\|_2 \leq \delta + \|\boldsymbol{\zeta}_N\|_2 = O_p(1), \quad (\text{D.35})$$

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 \leq \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} (N_{-k} \bar{\pi}_N)^{-1/2} \|\mathbf{u}\|_2 = O_p((N_{-k} \bar{\pi}_N)^{-1/2}), \quad (\text{D.36})$$

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\tilde{\boldsymbol{\gamma}}_1\|_2 \leq \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 + \|\boldsymbol{\gamma}_0\|_2 < M, \quad \text{w.p.a. 1}, \quad (\text{D.37})$$

where $M > 0$ is a constant independent of N .

Now, we prove (D.24). For any \mathbf{u} satisfying $\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta$,

$$\begin{aligned} & |\mathbf{u}^T \{\mathbf{A}_N(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1}) - \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\} \mathbf{u}| \\ & \leq \left| (N_{-k} \bar{\pi}_N)^{-1} \|\mathbf{u}\|_2^2 \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_1 + \log(\tilde{\pi}_{N,1})) - \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\vec{\mathbf{X}}_i\|_2^2 \right| \\ & \quad + \|\mathbf{u}\|_2^2 \left\| (N_{-k} \bar{\pi}_N)^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T - \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \right\|_2. \end{aligned}$$

By Taylor's Theorem, with some $(\tilde{\boldsymbol{\gamma}}_2, \tilde{\pi}_{N,2})$ lies between $(\boldsymbol{\gamma}_0, \bar{\pi}_N)$ and $(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1})$, uniformly on $\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta$,

$$\begin{aligned} & \left| N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_1 + \log(\tilde{\pi}_{N,1})) \|\vec{\mathbf{X}}_i\|_2^2 - N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\vec{\mathbf{X}}_i\|_2^2 \right| \\ & \stackrel{(i)}{=} \left| N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_2 + \log(\tilde{\pi}_{N,2})) \left\{ \vec{\mathbf{X}}_i^T (\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0) + \log(\tilde{\pi}_{N,1}/\bar{\pi}_N) \right\} \|\vec{\mathbf{X}}_i\|_2^2 \right| \\ & \stackrel{(ii)}{\leq} \tilde{\pi}_{N,2} N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \exp(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_2) \left| \vec{\mathbf{X}}_i^T (\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0) + \log(\tilde{\pi}_{N,1}/\bar{\pi}_N) \right| \|\vec{\mathbf{X}}_i\|_2^2 \\ & \stackrel{(iii)}{\leq} \tilde{\pi}_{N,2} N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \exp(\|\vec{\mathbf{X}}_i\|_2 M) \left\{ \|\vec{\mathbf{X}}_i\|_2 \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 + |\log(\tilde{\pi}_{N,1}/\bar{\pi}_N)| \right\} \|\vec{\mathbf{X}}_i\|_2^2, \quad (\text{D.38}) \end{aligned}$$

with probability approaching 1. Here, (i) holds by Taylor's Theorem, (ii) holds by (D.21), (iii) holds by (D.37). Recall (D.29), by Markov's Inequality,

$$N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \exp(\|\vec{\mathbf{X}}_i\|_2 M) \|\vec{\mathbf{X}}_i\|_2^r = O_p(1). \quad (\text{D.39})$$

Hence,

$$\begin{aligned} & \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} (N_{-k} \bar{\pi}_N)^{-1} \|\mathbf{u}\|_2^2 \sum_{i \in \mathcal{I}_{-k}} \left| \dot{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_1 + \log(\tilde{\pi}_{N,1})) - \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \right| \|\vec{\mathbf{X}}_i\|_2^2 \\ & \stackrel{(i)}{\leq} \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \bar{\pi}_N^{-1} \|\mathbf{u}\|_2^2 \tilde{\pi}_{N,2} \left\{ \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 O_p(1) + |\log(\tilde{\pi}_{N,1}/\bar{\pi}_N)| O_p(1) \right\} \\ & \stackrel{(ii)}{=} O_p \left((N \bar{\pi}_N)^{-1/2} \right) = o_p(1). \quad (\text{D.40}) \end{aligned}$$

where (i) holds by (D.38) and (D.39), (ii) holds by (D.27), (D.28), (D.35) and (D.36). Notice that

$$\begin{aligned} & \mathbb{E}\{\dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \|\vec{\mathbf{X}}_i\|_2^2\} \leq \bar{\pi}_N \mathbb{E}\{\exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0) \|\vec{\mathbf{X}}_i\|_2^2\}. \\ & \|\mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2 \leq \bar{\pi}_N \|\mathbb{E}\{\exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T\}\|_2 \leq \bar{\pi}_N \mathbb{E}\left\{\exp(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0) \|\vec{\mathbf{X}}_i\|_2^2\right\}. \end{aligned}$$

Recall that p is fixed, by Theorem 5.48 of Vershynin [2010], with some constant $C > 0$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{S}_{-k}} \left\| N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T - \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \right\|_2 \\ & \leq \max \left[\|\mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N)\|_2^{1/2} C \sqrt{\frac{\bar{\pi}_N \log\{\min(N, p+1)\}}{N}}, \frac{C^2 \bar{\pi}_N \log\{\min(N, p+1)\}}{N} \right] \\ & = O \left(\max \left(N^{-1/2} \bar{\pi}_N, N^{-1} \bar{\pi}_N \right) \right) = O \left(N^{-1/2} \bar{\pi}_N \right). \end{aligned}$$

By Markov's Inequality,

$$\left\| N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T - \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \right\|_2 = O_p \left(N^{-1/2} \bar{\pi}_N \right).$$

It follows that

$$\begin{aligned} & \sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} \|\mathbf{u}\|_2^2 \left\| (N_{-k} \bar{\pi}_N)^{-1} \sum_{i \in \mathcal{I}_{-k}} \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T - \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \right\|_2 \\ & = O_p \left((N \bar{\pi}_N)^{-1/2} \right) = o_p(1). \end{aligned} \tag{D.41}$$

Hence, by (D.40) and (D.41),

$$\sup_{\|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta} |\mathbf{u}^T \{ \mathbf{A}_N - \bar{\pi}_N^{-1} \mathcal{J}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \} \mathbf{u}| = O_p \left((N \bar{\pi}_N)^{-1/2} \right) = o_p(1). \tag{D.42}$$

Now, we show (D.25). By Taylor's Theorem, where some $(\tilde{\boldsymbol{\gamma}}_3, \tilde{\pi}_{N,3})$ lies between $(\boldsymbol{\gamma}_0, \bar{\pi}_N)$ and $(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1})$,

$$\begin{aligned} & |(\mathbf{B}_{N,1} - \mathbf{B}_{N,2})^T \mathbf{u}| \\ & \stackrel{(i)}{=} \left| \log \left(\frac{\hat{\pi}_N}{\bar{\pi}_N} \right) (N_{-k} \bar{\pi}_N)^{-1/2} \sum_{i \in \mathcal{I}_{-k}} \ddot{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_3 + \log(\tilde{\pi}_{N,3})) \left\{ \vec{\mathbf{X}}_i^T (\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0) + \log \left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N} \right) \right\} \vec{\mathbf{X}}_i^T \mathbf{u} \right| \\ & \stackrel{(ii)}{\leq} \left| \log \left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N} \right) \right| (N_{-k} \bar{\pi}_N)^{-1/2} \tilde{\pi}_{N,3} \sum_{i \in \mathcal{I}_{-k}} \exp(\|\vec{\mathbf{X}}_i\|_2 M) \|\vec{\mathbf{X}}_i\|_2^2 \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 \|\mathbf{u}\|_2 \\ & \quad + \left| \log \left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N} \right) \right| (N_{-k} \bar{\pi}_N)^{-1/2} \tilde{\pi}_{N,3} \sum_{i \in \mathcal{I}_{-k}} \exp(\|\vec{\mathbf{X}}_i\|_2 M) \left| \log \left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N} \right) \right| \|\vec{\mathbf{X}}_i\|_2 \|\mathbf{u}\|_2 \quad \text{w.p.a. 1} \\ & \stackrel{(iii)}{\leq} \left| \log \left(\frac{\hat{\pi}_N}{\bar{\pi}_N} \right) \right| N_{-k}^{1/2} \bar{\pi}_N^{-1/2} \|\mathbf{u}\|_2 \tilde{\pi}_{N,3} \left\{ \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 O_p(1) + \left| \log \left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N} \right) \right| O_p(1) \right\} \quad \text{w.p.a. 1} \\ & \stackrel{(iv)}{=} O_p \left((N \bar{\pi}_N)^{-1/2} \right) = o_p(1), \quad \text{uniformly on } \|\mathbf{u} - \boldsymbol{\zeta}_N\|_2 \leq \delta, \end{aligned} \tag{D.43}$$

where (i) holds by Taylor's Theorem, (ii) holds by (D.21) and (D.37), (iii) holds by (D.39), (iv) holds by (D.27), (D.28) (D.35) and (D.36).

As for (D.26), by Taylor's Theorem, with some $(\tilde{\boldsymbol{\gamma}}_4, \tilde{\pi}_{N,4})$ lies between $(\boldsymbol{\gamma}_0, \bar{\pi}_N)$ and $(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1})$,

$$\begin{aligned}
|C_N(\tilde{\boldsymbol{\gamma}}_1, \tilde{\pi}_{N,1})| &\stackrel{(i)}{=} \left| \frac{1}{2} \sum_{i \in \mathcal{J}_{-k}} \tilde{g}(\vec{\mathbf{X}}_i^T \tilde{\boldsymbol{\gamma}}_4 + \log(\tilde{\pi}_{N,4})) \left\{ \vec{\mathbf{X}}^T (\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0) + \log\left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N}\right) \right\} \left\{ \log\left(\frac{\hat{\pi}_N}{\bar{\pi}_N}\right) \right\}^2 \right| \\
&\stackrel{(ii)}{\leq} \frac{1}{2} \tilde{\pi}_{N,4} \sum_{i \in \mathcal{J}_{-k}} \exp(\|\vec{\mathbf{X}}_i\| M) \left\{ \|\vec{\mathbf{X}}\|_2 \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 + \left| \log\left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N}\right) \right| \right\} \left\{ \log\left(\frac{\hat{\pi}_N}{\bar{\pi}_N}\right) \right\}^2 \text{ w.p.a. 1} \\
&\stackrel{(iii)}{\leq} \frac{1}{2} \left| \log\left(\frac{\hat{\pi}_N}{\bar{\pi}_N}\right) \right|^2 \tilde{\pi}_{N,4} N_{-k} \left\{ \|\tilde{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_0\|_2 O_p(1) + \left| \log\left(\frac{\tilde{\pi}_{N,1}}{\bar{\pi}_N}\right) \right| O_p(1) \right\} \text{ w.p.a. 1} \\
&\stackrel{(iv)}{=} O_p\left((N\bar{\pi}_N)^{-1/2}\right) = o_p(1). \tag{D.44}
\end{aligned}$$

where (i) holds by Taylor's Theorem, (ii) holds by (D.21) and (D.37), (iii) holds by (D.39), (iv) holds by (D.27), (D.28) (D.35) and (D.36).

Combining (D.42), (D.43) and (D.44), we have

$$\Delta_N(\delta) = o_p(1), \quad \text{for any } \delta > 0,$$

and hence (D.23) holds. Recall the definition of $\zeta_{N,3}$ in (D.32), we have

$$\begin{aligned}
\left\| \zeta_{N,3} - N_{-k}^{-1/2} \bar{\pi}_N^{1/2} \sum_{i \in \mathcal{J}_{-k}} (\bar{\pi}_N^{-1} R_i - 1) \mathbf{e}_1 \right\|_2 &= \left\| \zeta_{N,3} - (N_{-k} \bar{\pi}_N)^{1/2} \frac{\hat{\pi}_N - \bar{\pi}_N}{\bar{\pi}_N} \mathbf{e}_1 \right\|_2 \\
&\stackrel{(i)}{=} \left\| (N_{-k} \bar{\pi}_N)^{1/2} \frac{(\hat{\pi}_N - \bar{\pi}_N)^2}{\bar{\pi}_{N,5}^2} \mathbf{e}_1 \right\|_2 = (N_{-k} \bar{\pi}_N)^{1/2} \frac{(\hat{\pi}_N - \bar{\pi}_N)^2}{\bar{\pi}_{N,5}^2} = O_p\left((N\bar{\pi}_N)^{-1/2}\right) = o_p(1).
\end{aligned}$$

where (i) follows from the Taylor's Theorem with some $\tilde{\pi}_{N,5}$ lying between $\bar{\pi}_N$ and $\hat{\pi}_N$. Hence, with $\text{IF}_{\boldsymbol{\gamma}}(\mathbf{Z}) = \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \{R - g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\} \vec{\mathbf{X}} - (\bar{\pi}_N^{-1} R - 1) \mathbf{e}_1$,

$$\begin{aligned}
\left\| \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 - N_{-k}^{-1} \sum_{i \in \mathcal{J}_{-k}} \text{IF}_{\boldsymbol{\gamma}}(\mathbf{Z}_i) \right\|_2 &= (N_{-k} \bar{\pi}_N)^{-1/2} \left\| \zeta_{N,3} - N_{-k}^{-1/2} \bar{\pi}_N^{1/2} \sum_{i \in \mathcal{J}_{-k}} \text{IF}_{\boldsymbol{\gamma}}(\mathbf{Z}_i) \right\|_2 \\
&= (N_{-k} \bar{\pi}_N)^{-1/2} \left\| N_{-k}^{-1/2} \bar{\pi}_N^{1/2} \sum_{i \in \mathcal{J}_{-k}} (\bar{\pi}_N^{-1} R_i - 1) \mathbf{e}_1 - \zeta_{N,3} + (\zeta_{N,3} - \zeta_{N,2}) \right\|_2 \\
&\leq (N_{-k} \bar{\pi}_N)^{-1/2} \left\| N_{-k}^{-1/2} \bar{\pi}_N^{1/2} \sum_{i \in \mathcal{J}_{-k}} (\bar{\pi}_N^{-1} R_i - 1) \mathbf{e}_1 - \zeta_{N,3} \right\|_2 + (N_{-k} \bar{\pi}_N)^{-1/2} \|\zeta_{N,3} - \zeta_{N,2}\|_2 \\
&= (N_{-k} \bar{\pi}_N)^{-1/2} O_p\left((N\bar{\pi}_N)^{-1/2} + N^{-1/2}\right) = o_p\left((N_{-k} \bar{\pi}_N)^{-1/2}\right).
\end{aligned}$$

Now, it remains to analyze the IF of the PS $\hat{\pi}_N(\mathbf{X}) = g(\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}} + \log(\hat{\pi}_N))$. For this, define

$$\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\gamma}} + \log(\hat{\pi}_N) \mathbf{e}_1, \quad \boldsymbol{\beta}_0 = \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N) \mathbf{e}_1, \tag{D.45}$$

$$\text{IF}_{\boldsymbol{\beta}}(\mathbf{Z}) = \mathcal{J}^{-1}(\boldsymbol{\gamma}_0, \bar{\pi}_N) \{R - g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\} \vec{\mathbf{X}}. \tag{D.46}$$

Then,

$$\begin{aligned}
& \left\| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 - N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \text{IF}_{\boldsymbol{\beta}}(\mathbf{Z}_i) \right\|_2 \stackrel{(i)}{=} \left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 + \log(\widehat{\pi}_N / \bar{\pi}_N) \mathbf{e}_1 - (N_{-k} \bar{\pi}_N)^{-1/2} \boldsymbol{\zeta}_{N,1} \right\|_2 \\
& \stackrel{(ii)}{=} \left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 - (N_{-k} \bar{\pi}_N)^{-1/2} \boldsymbol{\zeta}_N + (N_{-k} \bar{\pi}_N)^{-1/2} (\boldsymbol{\zeta}_{N,3} - \boldsymbol{\zeta}_{N,1} - \boldsymbol{\zeta}_N) \right\|_2 \\
& \stackrel{(iii)}{\leq} \left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 - (N_{-k} \bar{\pi}_N)^{-1/2} \boldsymbol{\zeta}_N \right\|_2 + (N_{-k} \bar{\pi}_N)^{-1/2} \left\| \boldsymbol{\zeta}_{N,3} - \boldsymbol{\zeta}_{N,2} \right\|_2 \\
& \stackrel{(iv)}{=} o_p((N \bar{\pi}_N)^{-1/2}) + (N_{-k} \bar{\pi}_N)^{-1/2} O_p(N^{-1/2}) = o_p((N \bar{\pi}_N)^{-1/2}), \tag{D.47}
\end{aligned}$$

where (i) holds by (D.31), (D.45) and (D.46), (ii) holds by (D.32), (iii) holds by (D.30) and the triangular inequality, (iv) holds by (D.23) and (D.34). It follows that

$$\begin{aligned}
\left\| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 \right\|_2 & \stackrel{(i)}{\leq} \left\| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 - N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \text{IF}_{\boldsymbol{\beta}}(\mathbf{Z}_i) \right\|_2 + (N_{-k} \bar{\pi}_N)^{-1/2} \left\| \boldsymbol{\zeta}_{N,1} \right\|_2 \\
& \stackrel{(ii)}{=} o_p((N \bar{\pi}_N)^{-1/2}) + O_p((N \bar{\pi}_N)^{-1/2}) = O_p((N \bar{\pi}_N)^{-1/2}),
\end{aligned}$$

where (i) holds by (D.31) and the triangular inequality, (ii) holds by (D.47) and (D.34). Furthermore,

$$\begin{aligned}
\left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 \right\|_2 & \leq \left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 - (N_{-k} \bar{\pi}_N)^{-1/2} \boldsymbol{\zeta}_N \right\|_2 + (N_{-k} \bar{\pi}_N)^{-1/2} \left\| \boldsymbol{\zeta}_N \right\|_2 \\
& = o_p((N \bar{\pi}_N)^{-1/2}) + O_p((N \bar{\pi}_N)^{-1/2}) = O_p((N \bar{\pi}_N)^{-1/2}) = o_p(1),
\end{aligned}$$

and hence $\left\| \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 \right\|_2 < 1$ w.p.a. 1. By Taylor's Theorem, for any $\mathbf{x} \in \mathcal{X}$, where some $(\tilde{\boldsymbol{\gamma}}_6, \tilde{\pi}_{N,6})$ (depending on \mathbf{x}) lies between $(\boldsymbol{\gamma}_0, \bar{\pi}_N)$ and $(\widehat{\boldsymbol{\gamma}}, \widehat{\pi}_N)$ and $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\gamma}}_6 + \log(\tilde{\pi}_{N,6}) \mathbf{e}_1$,

$$\begin{aligned}
& 1 - \frac{g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))}{g(\overrightarrow{\mathbf{x}}^T \widehat{\boldsymbol{\gamma}} + \log(\widehat{\pi}_N))} - \{1 - g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0)\} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \\
& = g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0) \left\{ g^{-1}(\overrightarrow{\mathbf{x}}^T \tilde{\boldsymbol{\beta}}) - 1 \right\} \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \right\}^2 \\
& = g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0) \exp(-\overrightarrow{\mathbf{x}}^T \tilde{\boldsymbol{\beta}}) \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \right\}^2 \\
& = \tilde{\pi}_{N,6}^{-1} g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0) \exp(-\overrightarrow{\mathbf{x}}^T \tilde{\boldsymbol{\gamma}}) \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \right\}^2 \\
& \leq \max(\bar{\pi}_N^{-1}, \widehat{\pi}_N^{-1}) g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0) \exp\{\|\overrightarrow{\mathbf{x}}\|_2 (\|\boldsymbol{\gamma}_0\|_2 + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2)\} \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \right\}^2 \\
& \leq \max(\bar{\pi}_N^{-1}, \widehat{\pi}_N^{-1}) g(\overrightarrow{\mathbf{x}}^T \boldsymbol{\beta}_0) \exp\{\|\overrightarrow{\mathbf{x}}\|_2 (\|\boldsymbol{\gamma}_0\|_2 + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2)\} \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \overrightarrow{\mathbf{x}} \right\}^2.
\end{aligned}$$

Therefore, for any fixed $r > 0$, with \mathbf{X} independent of $\widehat{\boldsymbol{\gamma}}$ and $\widehat{\pi}_N$,

$$\begin{aligned}
& \left\| 1 - \frac{g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}^T \widehat{\boldsymbol{\beta}})} - \left\{ 1 - g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \right\} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \vec{\mathbf{X}} \right\|_{r,\mathbb{P}} \\
& \leq \max(\bar{\pi}_N^{-1}, \widehat{\pi}_N^{-1}) \left\| g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \exp \left\{ \|\vec{\mathbf{X}}\|_2 (\|\boldsymbol{\gamma}_0\|_2 + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2) \right\} \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \vec{\mathbf{X}} \right\}^2 \right\|_{r,\mathbb{P}} \\
& \leq \max(\bar{\pi}_N^{-1}, \widehat{\pi}_N^{-1}) \left\| g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \exp \left\{ \|\vec{\mathbf{X}}\|_2 (\|\boldsymbol{\gamma}_0\|_2 + 1) \right\} \left\{ (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \vec{\mathbf{X}} \right\}^2 \right\|_{r,\mathbb{P}} \text{ w.p.a. 1} \\
& \leq \max(1, \bar{\pi}_N \widehat{\pi}_N^{-1}) \|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|_2^2 \left\| \exp \left\{ \|\vec{\mathbf{X}}\|_2 (2\|\boldsymbol{\gamma}_0\|_2 + 1) \right\} \|\vec{\mathbf{X}}\|_2^2 \right\|_{r,\mathbb{P}} \text{ w.p.a. 1} \\
& = \{1 + o_p(1)\} O_p((N\bar{\pi}_N)^{-1}) O(1) = O_p((N\bar{\pi}_N)^{-1}) = o_p((N\bar{\pi}_N)^{-1/2}). \tag{D.48}
\end{aligned}$$

Define

$$\text{IF}_{\pi}(\mathbf{Z}; S_{-k}) = \left\{ 1 - g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \right\} \vec{\mathbf{X}}^T N_{-k}^{-1} \sum_{i \in \mathcal{I}_k} \text{IF}_{\boldsymbol{\beta}}(\mathbf{Z}_i), \tag{D.49}$$

where \mathbf{Z} is independent of $(\mathbf{Z}_i)_{i \in \mathcal{I}_k}$. Then,

$$\begin{aligned}
& \left\| \left\{ 1 - g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \right\} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \vec{\mathbf{X}} - \text{IF}_{\pi}(\mathbf{Z}; S_{-k}) \right\|_{r,\mathbb{P}} \\
& \leq \left\| \|\vec{\mathbf{X}}\|_2 \right\|_{r,\mathbb{P}} \left\| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 - N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \text{IF}_{\boldsymbol{\beta}}(\mathbf{Z}_i) \right\|_2 = o_p((N\bar{\pi}_N)^{-1/2}), \tag{D.50}
\end{aligned}$$

and

$$\left\| \left\{ 1 - g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0) \right\} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T \vec{\mathbf{X}} \right\|_{r,\mathbb{P}} \leq \|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|_2 \left\| \|\vec{\mathbf{X}}\|_2 \right\|_{r,\mathbb{P}} = O_p((N\bar{\pi}_N)^{-1/2}).$$

Hence,

$$\left\| 1 - \frac{g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}^T \widehat{\boldsymbol{\beta}})} \right\|_{r,\mathbb{P}} = O_p((N\bar{\pi}_N)^{-1/2}).$$

For any fixed $r > 0$,

$$\|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} = \left\| 1 + \bar{\pi}_N^{-1} \exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \right\|_{r,\mathbb{P}} \leq 1 + \bar{\pi}_N^{-1} \left\| \exp(\|\vec{\mathbf{X}}\|_2 \|\boldsymbol{\gamma}_0\|_2) \right\|_{r,\mathbb{P}} = O(\bar{\pi}_N^{-1}).$$

Additionally, by Jensen's Inequality,

$$\|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} = [\mathbb{E}\{\bar{\pi}_N^{-r}(\mathbf{X})\}]^{1/r} \geq \mathbb{E}\{\pi_N^{-1}(\mathbf{X})\} = \bar{\pi}_N^{-1}, \tag{D.51}$$

and hence $\|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} \asymp \bar{\pi}_N^{-1}$, which implies that $a_N \asymp \bar{\pi}_N$. It follows that, with $r,s > 0$ satisfying $1/r + 1/s = 1$ and $2s = 2 + c$,

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \{\widehat{m}(\mathbf{X}) - \mu(\mathbf{X})\}^2 \right] \leq a_N \|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} \|\widehat{m}(\cdot) - \mu(\cdot)\|_{2s,\mathbb{P}}^2 = o_p(1), \tag{D.52}$$

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\bar{\pi}_N(\mathbf{X})} \right\}^2 \right] \leq a_N \|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} \left\| 1 - \frac{g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}^T \widehat{\boldsymbol{\beta}})} \right\|_{2s,\mathbb{P}}^2 = O_p((N\bar{\pi}_N)^{-1}) = o_p(1),$$

where (D.52) requires an additional assumption that $\|\hat{m}(\cdot) - \mu(\cdot)\|_{2+c,\mathbb{P}} = o_p(1)$.

Now, we analyze $\hat{\theta}_{\text{DRSS}} - \theta_0$, where we further assume that $\|m(\cdot) - \mu(\cdot)\|_{2+c,\mathbb{P}} < \infty$. Applying part (b) of Theorem 3.2, we have

$$(\hat{\theta}_{\text{DRSS}} - \theta_0) = \frac{1}{N} \sum_{i=1}^N \psi_\mu(\mathbf{Z}_i) + o_p\left((N\bar{\pi}_N)^{-1/2}\right) + \hat{\Delta}_N,$$

where $\psi_{\mu,e}(\mathbf{Z}) = \mu(\mathbf{X}) - \theta_0 + R/\pi_N(\mathbf{X})\{Y - \mu(\mathbf{X})\}$ and

$$\begin{aligned} \hat{\Delta}_N &= N^{-1} \sum_{k=1}^{\mathbb{K}} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \left\{ 1 - \frac{\pi_N(\mathbf{X}_i)}{\hat{\pi}_N(\mathbf{X}_i)} \right\} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \\ &= N^{-1} \sum_{k=1}^{\mathbb{K}} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \text{IF}_\pi(\mathbf{Z}; S_{-k}) \\ &\quad + N^{-1} \sum_{k=1}^{\mathbb{K}} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \left\{ 1 - \frac{g(\vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}_i^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}; S_{-k}) \right\}. \end{aligned}$$

For each $k \leq \mathbb{K}$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{S}_k} \left| |\mathcal{J}_k|^{-1} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \left\{ 1 - \frac{g(\vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}_i^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}; S_{-k}) \right\} \right| \\ &\leq \mathbb{E} \left| \{\mu(\mathbf{X}) - m(\mathbf{X})\} \left\{ 1 - \frac{g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}; S_{-k}) \right\} \right| \\ &\leq \|\mu(\cdot) - m(\cdot)\|_{2,\mathbb{P}} \left\| 1 - \frac{g(\vec{\mathbf{X}}^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}; S_{-k}) \right\|_{2,\mathbb{P}} \stackrel{(i)}{=} o_p\left((N\bar{\pi}_N)^{-1/2}\right), \end{aligned}$$

where (i) holds by (D.48) and (D.50). By Lemma A.2,

$$|\mathcal{J}_k|^{-1} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \left\{ 1 - \frac{g(\vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}_i^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}_i; S_{-k}) \right\} = o_p\left((N\bar{\pi}_N)^{-1/2}\right),$$

and hence

$$N^{-1} \sum_{k=1}^{\mathbb{K}} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \left\{ 1 - \frac{g(\vec{\mathbf{X}}_i^T \boldsymbol{\beta}_0)}{g(\vec{\mathbf{X}}_i^T \hat{\boldsymbol{\beta}})} - \text{IF}_\pi(\mathbf{Z}_i; S_{-k}) \right\} = o_p\left((N\bar{\pi}_N)^{-1/2}\right).$$

Besides, for each $k \leq \mathbb{K}$, with $r, s > 0$ satisfying $1/r + 1/s = 1$ and $2s = 2 + c$, and recalling the definition of $\text{IF}_\pi(\mathbf{Z}; S_{-k})$ in (D.49), we have

$$\begin{aligned} &\text{Var}_{\mathbb{S}_k} \left[|\mathcal{J}_k|^{-1} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \text{IF}_\pi(\mathbf{Z}_i; S_{-k}) \right] \\ &\stackrel{(i)}{\leq} |\mathcal{J}_k|^{-1} \left\| \mathbb{E} \left[\pi_N^{-1}(\mathbf{X}) \{\mu(\mathbf{X}) - m(\mathbf{X})\}^2 \|\vec{\mathbf{X}}\|_2 \right] \right\|_2 \left\| N_{-k}^{-1} \sum_{j \in \mathcal{J}_{-k}} \text{IF}_\pi(\mathbf{Z}_j) \right\|_2^2 \\ &\leq |\mathcal{J}_k|^{-1} \|\pi_N^{-1}(\mathbf{X})\|_{2r,\mathbb{P}} \|\mu(\mathbf{X}) - m(\mathbf{X})\|_{2s,\mathbb{P}}^2 \left\| \|\vec{\mathbf{X}}\|_2 \right\|_{2r,\mathbb{P}} \left\| (N_{-k} \bar{\pi}_N)^{-1} \|\zeta_{N,1}\|_2^2 \right\|_{2r,\mathbb{P}} = O\left((N\bar{\pi}_N)^{-2}\right). \end{aligned}$$

By Lemma A.1 and recalling the definition (D.49),

$$\begin{aligned} |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\} \text{IF}_{\pi}(\mathbf{Z}_i; S_{-k}) &= \mathbb{E}_{\mathbf{X}} [\{\mu(\mathbf{X}) - m(\mathbf{X})\} \text{IF}_{\pi}(\mathbf{Z}; S_{-k})] + O_p((N\bar{\pi}_N)^{-1}) \\ &= N_{-k}^{-1} \sum_{j \in \mathcal{I}_{-k}} \text{IF}_{\pi}(\mathbf{Z}_j) + O_p((N\bar{\pi}_N)^{-1}) = N_{-k}^{-1} \sum_{j \in \mathcal{I}_{-k}} \text{IF}_{\pi}(\mathbf{Z}_j) + o_p((N\bar{\pi}_N)^{-1/2}), \end{aligned}$$

where $\text{IF}_{\pi}(\mathbf{Z}) = \mathbb{E} \left[\{1 - \pi_N(\mathbf{X})\} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \vec{\mathbf{X}}^T \right] J^{-1}(\bar{\pi}_N, \boldsymbol{\gamma}_0) \vec{\mathbf{X}} \{R - \pi_N(\mathbf{X})\}$. Therefore,

$$\hat{\Delta}_N = \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} N_{-k}^{-1} \sum_{j \in \mathcal{I}_{-k}} \text{IF}_{\pi}(\mathbf{Z}_j) + o_p((N\bar{\pi}_N)^{-1/2}) = N^{-1} \sum_{i=1}^N \text{IF}_{\pi}(\mathbf{Z}_i) + o_p((N\bar{\pi}_N)^{-1/2}).$$

□

Proof of Lemma 4.1. For any $a \in (0, 1]$, we have the corresponding Jacobian (or Hessian) matrices of $\ell_N(\boldsymbol{\gamma}; a)$ and $\ell_N(\boldsymbol{\gamma}, 1)$ w.r.t. $\boldsymbol{\gamma} \in \mathbb{R}^{p+1}$ satisfy the following (analytical) inequality:

$$\begin{aligned} \frac{\partial^2}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\gamma}^T} \{\ell_N(\boldsymbol{\gamma}; a)\} &= N^{-1} \sum_{i=1}^N \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma} + \log(a)) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T \\ &\succeq a N^{-1} \sum_{i=1}^N \dot{g}(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}) \vec{\mathbf{X}}_i \vec{\mathbf{X}}_i^T = a \frac{\partial^2}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\gamma}^T} \{\ell_N(\boldsymbol{\gamma}; 1)\}, \end{aligned}$$

since for any $a \in (0, 1]$ and $u \in \mathbb{R}$,

$$\dot{g}(u + \log(a)) = \frac{a \exp(u)}{\{1 + a \exp(u)\}^2} \geq \frac{a \exp(u)}{\{1 + \exp(u)\}^2} = a \dot{g}(u).$$

Let $\mathcal{G}(\boldsymbol{\gamma}; a) := \ell_N(\boldsymbol{\gamma}; a) - a \ell(\boldsymbol{\gamma}; 1)$. Then,

$$\frac{\partial^2}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\gamma}^T} \{\mathcal{G}(\boldsymbol{\gamma}; a)\} \succeq \mathbf{0}, \quad \forall \boldsymbol{\gamma} \in \mathbb{R}^{p+1}.$$

That is, the function $\mathcal{G}(\boldsymbol{\gamma}, a)$ is convex in $\boldsymbol{\gamma} \in \mathbb{R}^{p+1}$, and hence by the basic properties of convex functions, we have: for any $\boldsymbol{\gamma}, \Delta \in \mathbb{R}^{p+1}$,

$$\mathcal{G}(\boldsymbol{\gamma} + \Delta; a) - \mathcal{G}(\boldsymbol{\gamma}; a) - \Delta^T \{\nabla_{\boldsymbol{\gamma}} \mathcal{G}(\boldsymbol{\gamma}; a)\} \geq 0$$

and hence

$$\begin{aligned} \delta \ell(\Delta; a; \boldsymbol{\gamma}) &= \ell_N(\boldsymbol{\gamma} + \Delta; a) - \ell_N(\boldsymbol{\gamma}; a) - \Delta^T \{\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}; a)\} \\ &\geq a [\ell_N(\boldsymbol{\gamma} + \Delta; 1) - \ell_N(\boldsymbol{\gamma}; 1) - \Delta^T \{\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}; 1)\}] = a \{\delta \ell(\Delta; 1; \boldsymbol{\gamma})\}. \end{aligned}$$

Therefore,

$$\delta \ell(\Delta; a; \boldsymbol{\gamma}) \geq a \kappa \|\Delta\|_2^2, \quad \forall \Delta \in A,$$

if $\delta \ell(\Delta; 1; \boldsymbol{\gamma}) \geq \kappa \|\Delta\|_2^2$ for all $\Delta \in A$.

□

Proof of Lemma 4.2. We first derive the following useful properties: define $\mu_{\gamma_0} = \mathbb{E}(\vec{\mathbf{X}}^T \gamma_0)$, then by Lemma A.5 and some calculation, for all $t \in \mathbb{R}$, $j \leq p+1$, and $r \geq 1$,

$$\begin{aligned} |\mu_{\gamma_0}| &\leq \mathbb{E}(|\vec{\mathbf{X}}^T \gamma_0|) \leq \sigma_{\gamma_0} \sqrt{\pi} < 2\sigma_{\gamma_0}, \\ \|\vec{\mathbf{X}}^T \gamma_0 - \mu_{\gamma_0}\|_{\psi_2} &\leq \|\vec{\mathbf{X}}^T \gamma_0\|_{\psi_2} + \|\mu_{\gamma_0}\|_{\psi_2} \leq \sigma_{\gamma_0} + \{\log(2)\}^{-1/2} |\mu_{\gamma_0}| < 4\sigma_{\gamma_0}, \\ \mathbb{E}\{\exp(t \vec{\mathbf{X}}^T \gamma_0)\} &= \exp(t \mu_{\gamma_0}) \mathbb{E}[\exp\{t(\vec{\mathbf{X}}^T \gamma_0 - \mu_{\gamma_0})\}] \leq \exp\{2\sigma_{\gamma_0}|t| + 20\sigma_{\gamma_0}^2 t^2\}, \quad (\text{D.53}) \\ \|\vec{\mathbf{X}}(j) - \mathbb{E}\{\vec{\mathbf{X}}(j)\}\|_{\psi_2} &\leq \|\vec{\mathbf{X}}(j)\|_{\psi_2} + \|\mathbb{E}\{\vec{\mathbf{X}}(j)\}\|_{\psi_2} \leq \sigma + \{\log(2)\}^{-1/2} \sqrt{\pi} \sigma, \\ \max_{1 \leq j \leq p+1} \mathbb{E}\{|\vec{\mathbf{X}}(j)|^r\} &\leq r! \max_{1 \leq j \leq p+1} \mathbb{E}[\exp\{|\vec{\mathbf{X}}(j)|\}] \leq r! \exp(20\sigma^2). \end{aligned}$$

Notice that $\|\cdot\|_{\psi_2}$ is a monotone increasing function leading to $\|X_2\|_{\psi_2} \geq \|X_1\|_{\psi_2}$ if $|X_2| \geq |X_1|$. Hence,

$$\max_{1 \leq j \leq p+1} \|\{R_i - \pi_N(\mathbf{X}_i)\} \vec{\mathbf{X}}_{ij}\|_{\psi_2} \leq \max_{1 \leq j \leq p+1} \|\vec{\mathbf{X}}_{ij}\|_{\psi_2} \leq \sigma.$$

In addition,

$$\begin{aligned} \max_{1 \leq j \leq p+1} \mathbb{E}\left[\{R - \pi_N(\mathbf{X})\}^2 \vec{\mathbf{X}}^2(j)\right] &= \max_{1 \leq j \leq p+1} \mathbb{E}\left[\pi_N(\mathbf{X})\{1 - \pi_N(\mathbf{X})\} \vec{\mathbf{X}}^2(j)\right] \leq \max_{1 \leq j \leq p+1} \bar{\pi}_N \mathbb{E}\left\{\exp(\vec{\mathbf{X}}^T \gamma_0) \vec{\mathbf{X}}^2(j)\right\} \\ &\leq \bar{\pi}_N \left[\mathbb{E}\{\exp(2 \vec{\mathbf{X}}^T \gamma_0)\} \max_{1 \leq j \leq p+1} \mathbb{E}\{\vec{\mathbf{X}}^4(j)\} \right]^{1/2} \leq 2 \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2) \bar{\pi}_N. \end{aligned}$$

Now, apply Theorem 3.4 of [Kuchibhotla and Chakraborty \[2022\]](#), for any $t_1 \geq 0$, with probability at least $1 - 3 \exp(-t_1)$,

$$\begin{aligned} \|\nabla_{\gamma} \ell_N(\gamma_0; \bar{\pi}_N)\|_{\infty} &= \left\| N_{-k} \sum_{i \in \mathcal{I}_{-k}} \{R_i - \pi_N(\mathbf{X}_i)\} \vec{\mathbf{X}}_i \right\|_{\infty} \\ &\leq 7 \sqrt{\frac{2 \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2) \bar{\pi}_N \{t_1 + \log(p+1)\}}{N}} \\ &\quad + \frac{c_6 \sigma \sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N}, \quad (\text{D.54}) \end{aligned}$$

with some constant c_6 independent of N . Define $\mathcal{B} = \mathcal{B}(t_1)$ to be an event that (D.54) holds, then $\mathbb{P}(\mathcal{B}) \geq 1 - 3 \exp(-t_1)$.

Now, we consider the error that originated from the first step estimation $\hat{\pi}_N$. By Taylor's Theorem, for each $i \leq N$, there exists $\tilde{\pi}'_N$ (depends on i) lies between $\bar{\pi}_N$ and $\hat{\pi}_N$, such that

$$\begin{aligned} |g(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\hat{\pi}_N)) - g(\vec{\mathbf{X}}_i^T \gamma_0 + \log(\bar{\pi}_N))| &= \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\tilde{\pi}'_N} |\phi(\vec{\mathbf{X}}_i^T \gamma + \log(\tilde{\pi}'_N))| \\ &\leq \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\tilde{\pi}'_N} g(\vec{\mathbf{X}}_i^T \gamma + \log(\tilde{\pi}'_N)) \leq \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\min(\bar{\pi}_N, \hat{\pi}_N)} g(\vec{\mathbf{X}}_i^T \gamma + \log(\max\{\bar{\pi}_N, \hat{\pi}_N\})), \quad (\text{D.55}) \end{aligned}$$

since function $g(\cdot)$ is monotone increasing. Observe that, for each $r \geq 2$,

$$\begin{aligned}\mathbb{E}|R - \bar{\pi}_N|^r &= \mathbb{E}[|1 - \bar{\pi}_N|^r \pi_N(\mathbf{X}) + |- \bar{\pi}_N|^r \{1 - \pi_N(\mathbf{X})\}] \\ &= (1 - \bar{\pi}_N)^r \bar{\pi}_N + \bar{\pi}_N^r (1 - \bar{\pi}_N) \leq 2\bar{\pi}_N \leq \frac{r!}{2} 1^{r-2} \cdot 2\bar{\pi}_N.\end{aligned}$$

By Theorem 1 of [van de Geer and Lederer \[2013\]](#), for any $t_2 > 0$,

$$\mathbb{P}_{\mathbb{S}} \left(|\hat{\pi}_N - \bar{\pi}_N| \geq 2 \sqrt{\frac{t_2 \bar{\pi}_N}{N}} + \frac{t_2}{N} \right) \leq 2 \exp(-t_2).$$

Define event

$$\mathcal{A} = \mathcal{A}(t_2) := \{|\hat{\pi}_N - \bar{\pi}_N| < 2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N\}. \quad (\text{D.56})$$

Then, $\mathbb{P}_{\mathbb{S}}(\mathcal{A}) \geq 1 - 2 \exp(-t_2)$. Define

$$\pi_{N,\min} = \bar{\pi}_N - 2\sqrt{t_2 \bar{\pi}_N / N} - t_2 / N, \quad \pi_{N,\max} = \bar{\pi}_N + 2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N.$$

Suppose $t_2 < N\bar{\pi}_N/9$, then $2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N < 7\bar{\pi}_N/9$, $\pi_{N,\min} > 2\bar{\pi}_N/9 > 0$ and $\pi_{N,\max} < 16\bar{\pi}_N/9 < 16/9$. Recall (D.55), on event \mathcal{A} , we have for each $i \leq N$,

$$\left| g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\hat{\pi}_N)) - g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N)) \right| \leq \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\pi_{N,\min}} g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})),$$

and

$$\begin{aligned}\left\| \nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \hat{\pi}_N) - \nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \bar{\pi}_N) \right\|_{\infty} &= \left\| N^{-1} \sum_{i=1}^N \{g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\hat{\pi}_N)) - g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\bar{\pi}_N))\} \vec{\mathbf{X}}_i \right\|_{\infty} \\ &\leq \frac{\hat{\pi}_N - \bar{\pi}_N}{\pi_{N,\min}} \left\| N^{-1} \sum_{i=1}^N g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}_i \right\|_{\infty} \\ &\leq \frac{\hat{\pi}_N - \bar{\pi}_N}{\pi_{N,\min}} \left\| N^{-1} \sum_{i=1}^N \mathbf{V}_i \right\|_{\infty} + \frac{\hat{\pi}_N - \bar{\pi}_N}{\pi_{N,\min}} \left\| \mathbb{E} \left\{ g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}} \right\} \right\|_{\infty},\end{aligned}$$

where

$$\mathbf{V}_i = g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}_i - \mathbb{E} \left\{ g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}} \right\}.$$

For any vector \mathbf{v} , let $\mathbf{v}(j)$ denotes the j -th element of the vector \mathbf{v} . Notice that, on event \mathcal{A} ,

$$\begin{aligned}\max_{1 \leq j \leq p+1} \left| \mathbb{E} \left\{ g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}(j) \right\} \right| &\leq \max_{1 \leq j \leq p+1} \pi_{N,\max} \mathbb{E} \{ \exp(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) | \vec{\mathbf{X}}(j) \} \\ &\leq \pi_{N,\max} \left[\mathbb{E} \{ \exp(2\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \} \max_{1 \leq j \leq p+1} \mathbb{E} \{ \vec{\mathbf{X}}^2(j) \} \right]^{1/2} \leq \pi_{N,\max} \sqrt{2} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 40\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2),\end{aligned}$$

and hence

$$\begin{aligned} \max_{1 \leq j \leq p+1} \|\mathbf{V}(j)\|_{\psi_2} &\leq \max_{1 \leq j \leq p+1} \left\| g(\vec{\mathbf{X}}_i^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}_i(j) \right\|_{\psi_2} \\ &\quad + \max_{1 \leq j \leq p+1} \left\| \mathbb{E} \left\{ g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}(j) \right\} \right\|_{\psi_2} \\ &\leq \max_{1 \leq j \leq p+1} \left\| \vec{\mathbf{X}}_i(j) \right\|_{\psi_2} + \pi_{N,\max} \sqrt{2} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 40\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2) \\ &\leq \sigma + \frac{16\bar{\pi}_N}{9} \sqrt{2} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 40\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2). \end{aligned}$$

Additionally,

$$\begin{aligned} \max_{1 \leq j \leq p+1} \mathbb{E}(\mathbf{V}_i^2) &\leq \max_{1 \leq j \leq p+1} \mathbb{E} \left\{ g^2(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}}^2(j) \right\} \\ &\leq \pi_{N,\max}^2 \max_{1 \leq j \leq p+1} \mathbb{E} \left\{ \exp(2\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \vec{\mathbf{X}}^2(j) \right\} \\ &\leq \pi_{N,\max}^2 \left[\mathbb{E} \left\{ \exp(4\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \right\} \max_{1 \leq j \leq p+1} \mathbb{E} \left\{ \vec{\mathbf{X}}^4(j) \right\} \right]^{1/2} \\ &\leq \pi_{N,\max}^2 2 \exp(4\sigma_{\boldsymbol{\gamma}_0} + 160\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2). \end{aligned}$$

Define

$$\mathcal{C} = \left\{ \left\| N^{-1} \sum_{i=1}^N \mathbf{V}_i \right\|_{\infty} \leq 7c_8 \pi_{N,\max} \sqrt{\frac{t_1 + \log(p+1)}{N}} + \frac{c_9 \sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N} \right\},$$

where $c_8 = \sqrt{2} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 80\sigma_{\boldsymbol{\gamma}_0}^2 + 5\sigma^2)$, $c_9 = c_6 \{ \sigma + 16\bar{\pi}_N \exp(2\sigma_{\boldsymbol{\gamma}_0} + 40\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2)/9 \}$. By Theorem 3.4 of [Kuchibhotla and Chakrabortty \[2022\]](#), $\mathbb{P}(\mathcal{C}) \geq 1 - 3 \exp(-t_1)$. It follows that, on events \mathcal{A} and \mathcal{C} ,

$$\begin{aligned} &\|\nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \hat{\pi}_N) - \nabla_{\boldsymbol{\gamma}} \ell_N(\boldsymbol{\gamma}_0; \bar{\pi}_N)\|_{\infty} \\ &\leq \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\pi_{N,\min}} \left\| N \sum_{i=1}^N \mathbf{V}_i \right\|_{\infty} + \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\pi_{N,\min}} \left\| \mathbb{E} \left\{ g(\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0 + \log(\pi_{N,\max})) \vec{\mathbf{X}} \right\} \right\|_{\infty} \\ &\leq \frac{2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N}{\pi_{N,\min}} \left\{ 7c_8 \pi_{N,\max} \sqrt{\frac{t_1 + \log(p+1)}{N}} + \frac{c_9 \sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N} \right\} \\ &\quad + \frac{2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N}{\pi_{N,\min}} \pi_{N,\max} \sqrt{2} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 40\sigma_{\boldsymbol{\gamma}_0}^2 + 10\sigma^2). \end{aligned}$$

Recall that, when $t_2 < N\bar{\pi}_N/9$,

$$\frac{2\sqrt{t_2 \bar{\pi}_N / N} + t_2 / N}{\pi_{N,\min}} < \frac{7}{2}, \quad \pi_{N,\min} > \frac{2}{9} \bar{\pi}_N, \quad \pi_{N,\max} < \frac{16}{9} \bar{\pi}_N.$$

Hence, when $t_2 < N\bar{\pi}_N/9$, on events \mathcal{A} , \mathcal{B} and \mathcal{C} ,

$$\begin{aligned} \|\nabla_{\gamma} \ell_N(\gamma_0; \hat{\pi}_N)\|_{\infty} &\leq \|\nabla_{\gamma} \ell_N(\gamma_0; \bar{\pi}_N)\|_{\infty} + \|\nabla_{\gamma} \ell_N(\gamma_0; \hat{\pi}_N) - \nabla_{\gamma} \ell_N(\gamma_0; \bar{\pi}_N)\|_{\infty} \\ &\leq 7 \sqrt{\frac{2 \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2) \bar{\pi}_N \{t_1 + \log(p+1)\}}{N}} + \frac{c_6 \sigma \sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N} \\ &\quad + \frac{2\sqrt{t_2 \bar{\pi}_N/N} + t_2/N}{\pi_{N,\min}} \left\{ 7c_8 \pi_{N,\max} \sqrt{\frac{t_1 + \log(p+1)}{N}} + \frac{c_9 \sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N} \right\} \\ &\quad + \frac{2\sqrt{t_2 \bar{\pi}_N/N} + t_2/N}{\pi_{N,\min}} \pi_{N,\max} \sqrt{2} \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2) \\ &\leq C_1 (\bar{\pi}_N + \bar{\pi}_N^{1/2}) \sqrt{\frac{\{t_1 + \log(p+1)\}}{N}} + (C_2 + C_3 \bar{\pi}_N) \frac{\sqrt{\log(2N)} \{t_1 + \log(p+1)\}}{N} \\ &\quad + C_4 \left\{ \sqrt{\frac{t_2 \bar{\pi}_N}{N}} + \frac{t_2}{N} \right\}. \end{aligned}$$

where $\mathbb{P}_{\mathbf{S}}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \geq 1 - 6\exp(-t_1) - 2\exp(-t_2)$,

$$C_1 = 62 \exp(2\sigma_{\gamma_0} + 80\sigma_{\gamma_0}^2 + 5\sigma^2), \quad C_2 = \frac{9}{2} c_6 \sigma, \quad (\text{D.57})$$

$$C_3 = \frac{56}{9} \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2), \quad C_4 = 16\sqrt{2} \exp(2\sigma_{\gamma_0} + 40\sigma_{\gamma_0}^2 + 10\sigma^2). \quad (\text{D.58})$$

□

Proof of Theorem 4.2. Here, we establish a non-asymptotic property of the offset logistic regression estimator based on the full sample \mathbf{S} . The result follows from the Lemmas 4.1 and 4.2, where we obtained the RSC property and controlled the gradient $\|\nabla_{\gamma} \ell_N(\gamma_0; \hat{\pi}_N)\|_{\infty}$, respectively. After that, we validate the conditions required in Theorem 3.2 for the proposed offset logistic PS estimator.

For any $t \in \mathbb{R}$, set $t_1 = t_2 = t \log(p+1)$. By Lemma 4.2, on events \mathcal{A} , \mathcal{B} and \mathcal{C} , with $\mathbb{P}_{\mathbf{S}}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \geq 1 - 8(p+1)^{-t}$,

$$\begin{aligned} \|\nabla_{\gamma} \ell_N(\gamma_0; \hat{\pi}_N)\|_{\infty} &\leq (t+1) \left\{ C_1 (\bar{\pi}_N + \bar{\pi}_N^{1/2}) \sqrt{\frac{\log(p+1)}{N}} + (C_2 + C_3 \bar{\pi}_N) \frac{\sqrt{\log(2N)} \log(p+1)}{N} \right\} \\ &\quad + C_4 t^{1/2} \bar{\pi}_N^{1/2} \sqrt{\frac{\log(p+1)}{N}} + C_4 t \frac{\log(p+1)}{N} \\ &\leq (t+1) \left\{ (C_1 + C_4) (\bar{\pi}_N + \bar{\pi}_N^{1/2}) \sqrt{\frac{\log(p+1)}{N}} + (C_2 + C_4 + C_3 \bar{\pi}_N) \frac{\sqrt{\log(2N)} \log(p+1)}{N} \right\} \\ &\leq (t+1) M_N, \end{aligned}$$

where

$$M_N = 2(C_1 + C_4) \bar{\pi}_N^{1/2} \sqrt{\frac{\log(p+1)}{N}} + (C_2 + C_3 + C_4) \frac{\sqrt{\log(2N)} \log(p+1)}{N}.$$

Hence, for any $\lambda_N \geq 2(1+c)M_N$ with constant $c > 0$,

$$2\|\nabla_{\gamma}\ell_N(\gamma_0; \widehat{\pi}_N)\|_{\infty} \leq \lambda_N, \quad \text{on events } \mathcal{A}, \mathcal{B} \text{ and } \mathcal{C}.$$

Define event

$$\mathcal{D} := \left\{ \delta\ell(\Delta; \widehat{\pi}_N; \gamma) \geq \widehat{\pi}_N \kappa \|\Delta\|_2^2, \quad \forall \Delta \in \mathbb{C}_{\delta}(S; 3) \text{ and } \delta \leq 1 \right\}.$$

By Lemma 4.1, $\mathbb{P}(\mathcal{D}) \geq 1 - \alpha_N$. Let $\delta_N^* = 2\lambda_N s^{1/2}(\widehat{\pi}_N \kappa)^{-1}$. Then, the RSC condition holds for $\ell_N(\cdot; \widehat{\pi}_N)$ with parameter $\widehat{\pi}_N \kappa$ over $\mathbb{C}_{\delta_N^*}(S; 3)$. By Theorem 1 of Negahban et al. [2010], when $\lambda_N \geq 2\|\nabla_{\gamma}\ell_N(\gamma_0; \widehat{\pi}_N)\|_{\infty}, 2\lambda_N s^{1/2}(\widehat{\pi}_N \kappa)^{-1} \leq 1$ and on event \mathcal{D} ,

$$\|\widehat{\gamma} - \gamma_0\|_2 \leq \delta_N^* \leq 2\lambda_N s^{1/2}(\widehat{\pi}_N \kappa)^{-1}.$$

Recall (D.56), for any $t > 0$, on event $\mathcal{A} = \mathcal{A}(t)$,

$$\begin{aligned} \widehat{\pi}_N &\geq \bar{\pi}_N - 2\sqrt{\frac{t \log(p+1) \bar{\pi}_N}{N}} - \frac{t \log(p+1)}{N} \geq \frac{2}{9} \bar{\pi}_N, \\ 2\lambda_N s^{1/2}(\widehat{\pi}_N \kappa)^{-1} &\leq \frac{1}{9} \lambda_N s^{1/2} \bar{\pi}_N^{-1} \kappa^{-1} \leq 1, \end{aligned}$$

when $t < N\bar{\pi}_N\{\log(p+1)\}^{-1}/9$ and $\lambda_N \leq 9\kappa\bar{\pi}_N s^{-1/2}$. Hence, if $N\bar{\pi}_N > 9c\log(p+1)$,

$$\|\widehat{\gamma} - \gamma_0\|_2 \leq \frac{1}{9} \lambda_N s^{1/2} \bar{\pi}_N^{-1} \kappa^{-1}, \quad \text{on events } \mathcal{A}, \mathcal{B}, \mathcal{C} \text{ and } \mathcal{D},$$

where $\mathbb{P}_{\mathbb{S}}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{D}) \geq 1 - 8(p+1)^{-c} - \alpha_N$.

Now, consider the asymptotic performance that as $N \rightarrow \infty$, $\log(p)\log(N) = O(N\bar{\pi}_N)$ and $s\log(p) = o(N\bar{\pi}_N)$. Then,

$$M_N \asymp \bar{\pi}_N^{1/2} \sqrt{\frac{\log(p)}{N}}.$$

Hence, with some $\lambda_N \asymp \{N^{-1}\bar{\pi}_N \log(p)\}^{1/2}$,

$$\|\widehat{\gamma} - \gamma_0\|_2 = O_p \left(\sqrt{\frac{s \log(p)}{N \bar{\pi}_N}} \right) = o_p(1). \quad (\text{D.59})$$

Now we analyze the consistency rate of the PS estimator $\widehat{\pi}_N(\cdot)$. For any $r > 0$,

$$\left\| 1 - \frac{\pi_N(\cdot)}{\widehat{\pi}_N(\cdot)} \right\|_{r, \mathbb{P}} \leq \|\widehat{\pi}_N(\cdot) - \pi_N(\cdot)\|_{2r, \mathbb{P}} \|\widehat{\pi}_N^{-1}(\cdot)\|_{2r, \mathbb{P}}.$$

Let $u_0 = \vec{x}^T \gamma_0 + \log(\bar{\pi}_N)$ and $\Delta_u = \vec{x}^T \widehat{\gamma} + \log(\widehat{\pi}_N) - \{\vec{x}^T \gamma_0 + \log(\bar{\pi}_N)\}$. By mean value theorem, and notice that $g'(u) = g(u)\{1 - g(u)\}$, for some $v' \in (0, 1)$,

$$\begin{aligned} |g(u_0 + \Delta_u) - g(u_0)| &= g'(u_0 + v'\Delta_u)|\Delta_u| \leq g(u_0 + v'\Delta_u)|\Delta_u| \\ &\leq \max\{g(u_0), g(u_0 + \Delta_u)\}|\Delta_u| \leq \{g(u_0) + g(u_0 + \Delta_u)\}|\Delta_u|, \end{aligned}$$

since the function $g(\cdot) > 0$ is monotone increasing. Besides, notice that, on \mathcal{A} and $\mathcal{E} := \{\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 \leq 1\}$,

$$\begin{aligned} |\log(\hat{\pi}_N) - \log(\bar{\pi}_N)| &\leq \frac{|\hat{\pi}_N - \bar{\pi}_N|}{\min(\hat{\pi}_N, \bar{\pi}_N)} \leq \frac{2\sqrt{t_2\bar{\pi}_N/N} + t_2/N}{2\bar{\pi}_N/9} \leq \frac{21}{2}\sqrt{\frac{t_2}{N\bar{\pi}_N}}, \\ g(u_0) &= \frac{\bar{\pi}_N \exp(-\vec{\mathbf{x}}^T \boldsymbol{\gamma}_0)}{1 + \bar{\pi}_N \exp(-\vec{\mathbf{x}}^T \boldsymbol{\gamma}_0)} \leq \bar{\pi}_N \exp(-\vec{\mathbf{x}}^T \boldsymbol{\gamma}_0), \\ g(u_0 + \Delta_u) &= \frac{\hat{\pi}_N \exp(-\vec{\mathbf{x}}^T \hat{\boldsymbol{\gamma}})}{1 + \hat{\pi}_N \exp(-\vec{\mathbf{x}}^T \hat{\boldsymbol{\gamma}})} \leq \hat{\pi}_N \exp(-\vec{\mathbf{x}}^T \hat{\boldsymbol{\gamma}}) \leq \frac{16}{9}\bar{\pi}_N \exp(-\vec{\mathbf{x}}^T \hat{\boldsymbol{\gamma}}), \end{aligned}$$

when $t_2 < N\bar{\pi}_N/9$. Hence, on $\mathcal{A} \cap \mathcal{E}$,

$$\begin{aligned} \|\hat{\pi}_N(\cdot) - \pi_N(\cdot)\|_{2r, \mathbb{P}} &\leq \bar{\pi}_N \left\| \left\{ \exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) + \frac{16}{9} \exp(-\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}}) \right\} \left\{ |\vec{\mathbf{X}}^T (\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)| + \frac{21}{2} \sqrt{\frac{t_2}{N\bar{\pi}_N}} \right\} \right\|_{2r, \mathbb{P}} \\ &\leq \left[\left\| \exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \right\|_{4r, \mathbb{P}} + \frac{16}{9} \left\| \exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0) \right\|_{8r, \mathbb{P}} \left\| \exp\{-\vec{\mathbf{X}}^T (\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)\} \right\|_{8r, \mathbb{P}} \right] \\ &\quad \cdot \left\{ \left\| \vec{\mathbf{X}}^T (\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) \right\|_{4r, \mathbb{P}} + \frac{21}{2} \sqrt{\frac{t_2}{N\bar{\pi}_N}} \right\} \\ &\leq C \left\{ \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 + \sqrt{\frac{t_2}{N\bar{\pi}_N}} \right\}, \end{aligned}$$

with some constant $C > 0$. Here, $\mathbb{P}(\mathcal{A}) \geq 1 - \exp(-t_2)$ and recall (D.59). Hence,

$$\|\hat{\pi}_N(\cdot) - \pi_N(\cdot)\|_{2r, \mathbb{P}} = O_p \left(\sqrt{\frac{s \log(p)}{N\bar{\pi}_N}} \right). \quad (\text{D.60})$$

Additionally, observe that

$$\begin{aligned} \|\hat{\pi}_N^{-1}(\cdot)\|_{2r, \mathbb{P}} &= \|1 + \hat{\pi}_N^{-1} \exp(-\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}})\|_{2r, \mathbb{P}} \leq 1 + \hat{\pi}_N^{-1} \|\exp(-\vec{\mathbf{X}}^T \hat{\boldsymbol{\gamma}})\|_{2r, \mathbb{P}} \\ &\leq 1 + \hat{\pi}_N^{-1} \|\exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\|_{4r, \mathbb{P}} \|\exp(-U)\|_{4r, \mathbb{P}}. \end{aligned}$$

By (D.53), $\|\exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\|_{4r, \mathbb{P}} = O(1)$. By (D.65), $\hat{\pi}_N^{-1} = \bar{\pi}_N^{-1}\{1 + o_p(1)\}$. By Lemma part (b) of A.5,

$$|\mathbb{E}(U)| \leq \mathbb{E}(|U|) \leq \sigma\sqrt{\pi}\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2.$$

Hence, by triangular inequality and part (a) of A.5,

$$\|U - \mathbb{E}(U)\|_{\psi_2} \leq \|U\|_{\psi_2} + \|\mathbb{E}(U)\|_{\psi_2} \leq \{1 + \sqrt{\pi/\log(2)}\}\sigma\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2 \leq 4\sigma\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2.$$

By part (c) of Lemma A.5,

$$\|\exp\{-U + \mathbb{E}(U)\}\|_{4r, \mathbb{P}} = (\mathbb{E}\exp[-4r\{U - \mathbb{E}(U)\}])^{1/(4r)} \leq \exp(128r\sigma^2\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2^2).$$

Hence,

$$\begin{aligned} \|\exp(-U)\|_{4r, \mathbb{P}} &= \|\exp\{-U + \mathbb{E}(U)\}\|_{4r, \mathbb{P}} \exp\{-\mathbb{E}(U)\} \\ &\leq \exp(128r\sigma^2\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2^2) \exp(\sigma\sqrt{\pi}\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|_2) = 1 + o_p(1). \end{aligned}$$

It follows that

$$\|\widehat{\pi}_N^{-1}(\cdot)\|_{2r,\mathbb{P}} \leq 1 + \bar{\pi}_N^{-1}\{1 + o_p(1)\} \cdot O(1) \cdot \{1 + o_p(1)\} = O_p(\bar{\pi}_N^{-1}). \quad (\text{D.61})$$

Therefore, by (D.60) and (D.61),

$$\left\| 1 - \frac{\pi_N(\cdot)}{\widehat{\pi}_N(\cdot)} \right\|_{r,\mathbb{P}} \leq \|\pi_N(\cdot) - \widehat{\pi}_N(\cdot)\|_{2r,\mathbb{P}} \|\widehat{\pi}_N^{-1}(\cdot)\|_{2r,\mathbb{P}} = O_p \left(\sqrt{\frac{s \log(p)}{N \bar{\pi}_N}} \right).$$

Besides, recall (D.51) and notice that

$$\|\pi_N^{-1}(\mathbf{X})\|_{r,\mathbb{P}} \leq 1 + \bar{\pi}_N^{-1} \|\exp(-\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0)\|_{r,\mathbb{P}} \leq 1 + \bar{\pi}_N^{-1} \exp(2\sigma_{\boldsymbol{\gamma}_0} + 20\sigma_{\boldsymbol{\gamma}_0} r) = O(\bar{\pi}_N^{-1}).$$

Therefore,

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X})} \right\}^2 \right] \leq a_N \|\pi_N^{-1}(\mathbf{X})\|_{2,\mathbb{P}} \left\| 1 - \frac{\pi_N(\cdot)}{\widehat{\pi}_N(\cdot)} \right\|_{4,\mathbb{P}}^2 = O_p \left(\frac{s \log(p)}{N \bar{\pi}_N} \right).$$

If further assume $\|\widehat{m}(\cdot) - m(\cdot)\|_{2+c,\mathbb{P}} = o_p(1)$, then

$$\mathbb{E} \left[\frac{a_N}{\pi_N(\mathbf{X})} \{\widehat{m}(\mathbf{X}) - m(\mathbf{X})\}^2 \right] \leq a_N \|\pi_N^{-1}(\mathbf{X})\|_{1+c/2,\mathbb{P}} \|\widehat{m}(\cdot) - m(\cdot)\|_{2+c,\mathbb{P}}^2 = o_p(1). \quad \square$$

Proof of Lemma 4.3. The proof of Lemma 4.3 is based on the proof of Proposition 2 in Negahban et al. [2010]. Here, we only provide the details that are different from their proof and we will use our notations in the following proof. As a reminder, N denotes the number of samples, $\vec{\mathbf{X}} \in \mathbb{R}^{p+1}$ is the covariate containing the intercept term and $\boldsymbol{\gamma}_0 \in \mathbb{R}^{p+1}$ is the coefficient of the balanced logistic model (the dimension p in their proof will be replaced by $p+1$ everywhere because of the usage of the intercept term).

The proof consists of 3 main steps: 1) show that (71) of Negahban et al. [2010] holds under our assumptions and the parameter K_3 we choose, 2) prove a slightly different version of (72) in Negahban et al. [2010], 3) conclude the RSC property result.

Step 1. For the inequality (71), similarly as in their proof, we notice that $\mathbb{E}\{(\vec{\mathbf{X}}^T \Delta)^2\} \geq \kappa_l \|\Delta\|_2^2 = \kappa_l$ for any $\|\Delta\|_2 = 1$. Hence, it suffices to show their inequality (73). Instead of assuming $\vec{\mathbf{X}}$ to be a zero-mean jointly sub-Gaussian random vector (which is not true since we have the intercept term here), we only assume a $(2+c)$ -th moment condition that $\sup_{\|\mathbf{v}\|_2 \leq 1} \|\vec{\mathbf{X}}^T \mathbf{v}\|_{2+c,\mathbb{P}} \leq M < \infty$ and a c -th moment condition that $\|\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0\|_{c,\mathbb{P}} \leq \mu_c < \infty$, with our choice on the constant K_3 . We have

$$\begin{aligned} \sup_{\|\Delta\|_2 \leq 1} \mathbb{P}(|\vec{\mathbf{X}}^T \Delta| \geq \tau/2) &\leq (\tau/2)^{-2-c} \sup_{\|\mathbf{v}\|_2 \leq 1} \mathbb{E} |\vec{\mathbf{X}}^T \Delta|^{2+c} \leq M^{2+c} (\tau/2)^{-2-c}, \\ \mathbb{P}(|\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0| \geq T) &\leq \mathbb{E} |\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0|^c T^{-c} = \mu_c^c T^{-c}. \end{aligned}$$

Hence, by Hölder's Inequality, for any $\|\Delta\|_2 \leq 1$,

$$\begin{aligned} \mathbb{E} \left\{ (\vec{\mathbf{X}}^T \Delta)^2 \mathbf{1}_{|\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0| \geq T} \right\} &\leq \|\vec{\mathbf{X}}^T \Delta\|_{2+c,\mathbb{P}}^2 \{ \mathbb{P}(|\vec{\mathbf{X}}^T \boldsymbol{\gamma}_0| \geq T) \}^{\frac{c}{2+c}} \leq M^2 \mu_c^{\frac{c^2}{2+c}} T^{-\frac{c^2}{2+c}}, \\ \mathbb{E} \left\{ (\vec{\mathbf{X}}^T \Delta)^2 \mathbf{1}_{|\vec{\mathbf{X}}^T \Delta| \geq \tau/2} \right\} &\leq \|\vec{\mathbf{X}}^T \Delta\|_{2+c,\mathbb{P}}^2 \{ \mathbb{P}(|\vec{\mathbf{X}}^T \Delta| \geq \tau/2) \}^{\frac{c}{2+c}} \leq M^{2+c} (\tau/2)^{-c}. \end{aligned}$$

It follows that, for $\tau^2 = T^2 = K_3 \geq 1$,

$$\begin{aligned} \mathbb{E}\left\{(\vec{\mathbf{X}}^T \Delta)^2 - g_\Delta(\mathbf{X})\right\} &\leq \mathbb{E}\left\{(\vec{\mathbf{X}}^T \Delta)^2 1_{|\vec{\mathbf{X}}^T \gamma_0| \geq T}\right\} + \mathbb{E}\left\{(\vec{\mathbf{X}}^T \Delta)^2 1_{|\vec{\mathbf{X}}^T \Delta| \geq \tau/2}\right\} \\ &\leq M^2 \mu_c^{\frac{c^2}{2+c}} T^{-\frac{c^2}{2+c}} + M^{2+c} (\tau/2)^{-c} \leq (M^2 \mu_c^{\frac{c^2}{2+c}} + M^{2+c} 2^c) K_3^{-\frac{c^2}{4+2c}}. \end{aligned}$$

Hence, (73) of [Negahban et al. \[2010\]](#) holds when we set

$$K_3 = \max \left[1, \left\{ 2\kappa_l^{-1} (M^2 \mu_c^{\frac{c^2}{2+c}} + M^{2+c} 2^c) \right\}^{\frac{4+2c}{c}} \right].$$

Step 2. We will demonstrate a slightly different version of (72) in [Negahban et al. \[2010\]](#) that

$$\mathbb{P}_{\mathbb{S}} \left\{ Z(t) \geq \frac{\kappa_l}{4} + 66K_3\sigma \sqrt{\frac{\log(p+1)}{N}} t \right\} \leq \exp \left\{ -\frac{N\kappa_l^2}{64K_3^2} - \sigma^2 t^2 \log(p+1) \right\}. \quad (\text{D.62})$$

Set $z^*(t) = \kappa_l/4 + 2K_3\sigma \sqrt{\log(p+1)/N}t$ and let $\mathcal{F} := \{\pm f(\cdot) : f(\mathbf{u}) = g_\Delta(\mathbf{u}) - \mathbb{E}\{g_\Delta(\mathbf{X})\}, \|\Delta\|_2 = 1, \|\Delta\|_1 = t\}$. Since $0 \leq g_\Delta(\mathbf{u}) \leq K_3$ for all \mathbf{u} , we have $|f(\mathbf{X}_i)| \leq K_3$ for all $f \in \mathcal{F}$. By Lemma A.7, we have a slightly different version of their (76):

$$\mathbb{P}_{\mathbb{S}}[Z(t) \geq \mathbb{E}\{Z(t)\} + z^*(t)] \leq \exp \left[-\frac{N\{z^*(t)\}^2}{4K_3^2} \right] \leq \exp \left\{ -\frac{N\kappa_l^2}{64K_3^2} - \sigma^2 t^2 \log(p+1) \right\}. \quad (\text{D.63})$$

Now, we need to obtain an upper bound for $\mathbb{E}_{\mathbb{S}}\|N^{-1} \sum_{i=1}^N \varepsilon_i \mathbf{u}_i\|_\infty$ only using the marginal sub-Gaussianity of $\vec{\mathbf{X}}$. Firstly, since $|\varepsilon_i \mathbf{u}_i(j)| \leq |X_i(j)|$, by part (a) of Lemma A.5,

$$\sup_{1 \leq j \leq p+1} \|\varepsilon_i \mathbf{u}_i(j)\|_{\psi_2} \leq \sup_{1 \leq j \leq p+1} \|\vec{\mathbf{X}}_i(j)\|_{\psi_2} \leq \sigma.$$

Notice that $\mathbb{E}(\varepsilon \mathbf{u}) = 0$ since ε is independent with \mathbf{u} and $\mathbb{E}(\varepsilon) = 0$, by part (e) of Lemma A.5, for any $1 \leq j \leq p+1$,

$$\left\| N^{-1} \sum_{i=1}^N \varepsilon_i \mathbf{u}_i(j) \right\|_{\psi_2} \leq 4\sigma/\sqrt{N}.$$

By part (d) of Lemma A.5,

$$\left\| \left\| N^{-1} \sum_{i=1}^N \varepsilon_i \mathbf{u}_i \right\|_\infty \right\|_{\psi_2} \leq 4\{\log(p+1) + 2\}^{1/2} \sigma/\sqrt{N} \leq 8\{\log(p+1)\}^{1/2} \sigma/\sqrt{N},$$

for any $p \geq 1$. Hence, by part (b) of Lemma A.5,

$$\mathbb{E}_{\mathbb{S}} \left\| N^{-1} \sum_{i=1}^N \varepsilon_i \mathbf{u}_i \right\|_\infty \leq 8\sqrt{\pi} \sigma \sqrt{\frac{\log(p+1)}{N}}.$$

Combining the upper bound with (78) of [Negahban et al. \[2010\]](#), we have a slightly different version of their inequality (77):

$$\mathbb{E}_{\mathbb{S}}\{Z(t)\} \leq 64K_3 t \sqrt{\pi} \sigma \sqrt{\frac{\log(p+1)}{N}}.$$

and recall (D.63), hence (D.62) follows. Notice that the statements in Step 2 are all independent of the choice of K_3 , so our choice on the constant K_3 does not affect the validity of the results.

Step 3. By inequality (71) of Negahban et al. [2010] and our (D.62), we conclude that: for any $t > 0$,

$$\begin{aligned} \mathbb{P}_{\mathbb{S}} \left[\widehat{\mathbb{E}}_{\mathbb{S}} \{ g_{\Delta}(\mathbf{X}) \} < \frac{\kappa_l}{4} - 66K_3\sigma \sqrt{\frac{\log(p+1)}{N}} t, \exists \Delta \in \mathbb{R}^{p+1}, \text{ with } \|\Delta\|_1 = t, \|\Delta\|_2 = 1 \right] \\ &\leq \exp \left\{ -\frac{N\kappa_l^2}{64K_3^2} - \sigma^2 t^2 \log(p+1) \right\}. \end{aligned}$$

Let $\mathbb{S}(1, t) = \{\Delta \in \mathbb{R}^{p+1} : \|\Delta\|_2 \leq 1, \|\Delta\|_1 / \|\Delta\|_2 = t\}$. By their inequality (66) and the technique in (69),

$$\begin{aligned} \mathbb{P}_{\mathbb{S}} \left[\delta\ell(\Delta; 1; \gamma_0) < L_{\psi}(K_3^{1/2}) \left\{ \frac{\kappa_l}{4} \|\Delta\|_2^2 - 66K_3\sigma \sqrt{\frac{\log(p+1)}{N}} \|\Delta\|_2 t \right\}, \exists \Delta \in \mathbb{S}(1, t) \right] \\ &\leq \exp \left\{ -\frac{N\kappa_l^2}{64K_3^2} - \sigma^2 t^2 \log(p+1) \right\}, \end{aligned}$$

where for a logistic model, $L_{\psi}(K_3^{1/2}) = \dot{g}(2K_3^{1/2}) > 0$.

By a peeling argument as in Raskutti et al. [2010], (4.15) holds. If further assume that $s \log(p) = o(N)$. For any $\Delta \in \mathcal{C}(S, 3)$,

$$\|\Delta\|_1 = \|\Delta_S\|_1 + \|\Delta_{S^c}\|_1 \leq 4\|\Delta_S\|_1 \leq 4\sqrt{s}\|\Delta_S\|_2 \leq 4\sqrt{s}\|\Delta\|_2.$$

and hence (4.16) holds. \square

Proof of Theorem 4.3. We establish the asymptotic properties of the stratified PS estimator and the DRSS estimator based on the stratified PS estimator.

Let $\bar{\pi}_N = \mathbb{E}(R)$, then $\pi_{1,N} + \pi_{0,N} \in (\bar{\pi}_N/(1-C), \bar{\pi}_N/C)$ and $\pi(\mathbf{X}) \in (C\bar{\pi}_N/(1-C), (1-C)\bar{\pi}_N/C)$ for all $\mathbf{X} \in \mathcal{X}$. Let $N_1 = \sum_{i \in \mathcal{I}_{-k}} \delta_i$ and $N_0 = \sum_{i \in \mathcal{I}_{-k}} (1 - \delta_i)$. Similarly as in (D.65), for $j \in \{0, 1\}$, $N_j^{-1} = O_p(N^{-1})$, $\hat{\pi}_j(\mathbb{S}_{-k}) - \pi_{j,N} = O_p(\sqrt{\bar{\pi}_N/N})$ and $1 - \pi_{j,N}/\hat{\pi}_j(\mathbb{S}_{-k}) = O_p(1/\sqrt{N\bar{\pi}_N})$. Hence, $\hat{\pi}_j^{-1}(\mathbb{S}_{-k}) = \pi_{j,N}^{-1}\{1 + O_p(1/\sqrt{N\bar{\pi}_N})\}$. It follows that there exists $c > 0$ such that

$$\mathbb{P}_{\mathbb{S}_{-k}}(\hat{\pi}_j(\mathbb{S}_{-k}) > c\bar{\pi}_N) \rightarrow 1, \quad \text{for } j \in \{0, 1\}.$$

Hence, with probability approaching 1,

$$\begin{aligned} \hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) &= \hat{\pi}_1(\mathbb{S}_{-k})\hat{p}_{\delta}(\mathbf{X}; \mathbb{S}_{-k}) + \hat{\pi}_0(\mathbb{S}_{-k})\{1 - \hat{p}_{\delta}(\mathbf{X}; \mathbb{S}_{-k})\} \\ &\geq c\bar{\pi}_N\hat{p}_{\delta}(\mathbf{X}; \mathbb{S}_{-k}) + c\bar{\pi}_N\{1 - \hat{p}_{\delta}(\mathbf{X}; \mathbb{S}_{-k})\} = c\bar{\pi}_N. \end{aligned}$$

Observe that

$$\begin{aligned} \hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X}) &= \{\hat{\pi}_1(\mathbb{S}_{-k}) - \hat{\pi}_0(\mathbb{S}_{-k})\}\{\hat{p}_{\delta}(\mathbf{X}; \mathbb{S}_{-k}) - p_{\delta}(\mathbf{X})\} \\ &\quad + \{\hat{\pi}_1(\mathbb{S}_{-k}) - \pi_{1,N}\}p_{\delta}(\mathbf{X}) + \{\hat{\pi}_0(\mathbb{S}_{-k}) - \pi_{0,N}\}\{1 - p_{\delta}(\mathbf{X})\}. \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \frac{\hat{\pi}_N(\cdot; \mathbb{S}_{-k}) - \pi_N(\cdot)}{\pi_N(\cdot)} \right\|_{2, \mathbb{P}_{\mathbf{X}}} &= O_p\left(r_{p_{\delta}, N} + (N\bar{\pi}_N)^{-1/2}\right), \\ \left\| \frac{\hat{\pi}_N(\cdot; \mathbb{S}_{-k}) - \pi_N(\cdot)}{\hat{\pi}_N(\cdot; \mathbb{S}_{-k})} \right\|_{2, \mathbb{P}_{\mathbf{X}}} &= O_p\left(r_{p_{\delta}, N} + (N\bar{\pi}_N)^{-1/2}\right). \end{aligned}$$

Following the case (b) in Theorem 3.2 that $\pi_N(\cdot) = e_N(\cdot)$ being correctly specified,

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = \frac{1}{N} \sum_{i=1}^N \psi_{\mu,e}(\mathbf{Z}_i) + \widehat{\Delta}_N + O_p \left(\frac{c_{\mu,N}}{\sqrt{Na_N}} + \frac{c_{e,N}}{\sqrt{Na_N}} + r_{\pi,m,N} \right),$$

where $\psi_{\mu,e}(\mathbf{Z}) = \mu(X) - \theta_0 + R/\bar{\pi}_N(X)[Y - \mu(X)]$, $\widehat{\Delta}_N = \sum_{k=1}^K \widehat{\Delta}_{N,k}$ and

$$\widehat{\Delta}_{N,k} = N^{-1} \sum_{i \in \mathcal{J}_k} \frac{R_i}{\pi_N(\mathbf{X}_i)} \left\{ 1 - \frac{\pi_N(\mathbf{X}_i)}{\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} \right\} \{\mu(\mathbf{X}_i) - m(\mathbf{X}_i)\}.$$

With a slight abuse of notation, let $\mathbf{Z} = (\delta, R, \mathbf{X})$, $\mathbf{Z}_i = (\delta_i, R_i, \mathbf{X}_i)$ and $\mathbb{S}_k = \{\mathbf{Z}_i : i \in \mathcal{J}_k\}$. Then,

$$\begin{aligned} \text{Var}_{\mathbb{S}_k}(\widehat{\Delta}_{N,k}) &= N^{-2} |\mathcal{J}_k| \text{Var} \left[\frac{R}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &\leq N^{-1} \mathbb{E} \left[\frac{1}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\}^2 \{\mu(\mathbf{X}) - m(\mathbf{X})\}^2 \right] \\ &\leq \frac{1-C}{C} (N\bar{\pi}_N)^{-1} \left\| \frac{\widehat{\pi}_N(\cdot; \mathbb{S}_{-k}) - \pi_N(\cdot)}{\widehat{\pi}_N(\cdot; \mathbb{S}_{-k})} \right\|_{2, \mathbb{P}_{\mathbf{X}}}^2 \|\mu(\cdot) - m(\cdot)\|_{\infty, \mathbb{P}_{\mathbf{X}}}^2 \\ &= O_p \left((N\bar{\pi}_N)^{-1} r_{p_\delta, N}^2 + (N\bar{\pi}_N)^{-2} \right), \end{aligned}$$

and by Lemma A.1,

$$\widehat{\Delta}_{N,k} = \mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,k}) + O_p \left((N\bar{\pi}_N)^{-1/2} r_{p_\delta, N} + (N\bar{\pi}_N)^{-1} \right).$$

In addition,

$$\begin{aligned} N|\mathcal{J}_k|^{-1} \mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,k}) &= \mathbb{E} \left[\frac{R}{\pi_N(\mathbf{X})} \left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &= \mathbb{E} \left[\left\{ 1 - \frac{\pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \right\} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &= \mathbb{E} \left[\frac{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &\quad - \mathbb{E} \left[\frac{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \cdot \frac{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &= \mathbb{E} \left[\frac{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] \\ &\quad + O_p \left(\left\| \frac{\widehat{\pi}_N(\cdot; \mathbb{S}_{-k}) - \pi_N(\cdot)}{\pi_N(\cdot)} \right\|_{2, \mathbb{P}_{\mathbf{X}}} \left\| \frac{\widehat{\pi}_N(\cdot; \mathbb{S}_{-k}) - \pi_N(\cdot)}{\widehat{\pi}_N(\cdot; \mathbb{S}_{-k})} \right\|_{2, \mathbb{P}_{\mathbf{X}}} \|\mu(\cdot) - m(\cdot)\|_{\infty} \right) \\ &= \mathbb{E} \left[\frac{\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{\mu(\mathbf{X}) - m(\mathbf{X})\} \right] + O_p \left(\|\widehat{p}_\delta(\cdot) - p_\delta(\cdot)\|_{2, \mathbb{P}_{\mathbf{X}}}^2 + (N\bar{\pi}_N)^{-1} \right). \end{aligned}$$

Let $\tilde{\pi}_N(\cdot; \mathbb{S}_{-k}) = \hat{\pi}_1(\mathbb{S}_{-k})p_\delta(\cdot) + \hat{\pi}_0(\mathbb{S}_{-k})\{1 - p_\delta(\cdot)\}$. Then,

$$\begin{aligned} & \mathbb{E} \left[\frac{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] \\ &= \mathbb{E} \left[\frac{\tilde{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] + \mathbb{E} \left[\frac{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \tilde{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] \\ &= \mathbb{E} \left[\frac{\tilde{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] + O_p(r_{p_\delta, N} \|\mu(\cdot) - m(\cdot)\|_{2, \mathbb{P}_X}) \\ &= \{\hat{\pi}_1(\mathbb{S}_{-k}) - \pi_{1,N}\} \mathbb{E} \left[\frac{p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] \\ &\quad + \{\hat{\pi}_0(\mathbb{S}_{-k}) - \pi_{0,N}\} \mathbb{E} \left[\frac{1 - p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] + O_p(r_{p_\delta, N}). \end{aligned}$$

Let $\hat{p}_\delta(\mathbb{S}_{-k}) = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \delta_i$ and $p_\delta = \mathbb{E}[p_\delta(\mathbf{X})]$. Then, similarly as (D.65), $\hat{p}_\delta^{-1}(\mathbb{S}_{-k}) = p_\delta^{-1}\{1 + O_p(1/\sqrt{N})\}$. Hence,

$$\begin{aligned} \hat{\pi}_1(\mathbb{S}_{-k}) - \pi_{1,N} &= N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{\delta_i R_i}{\hat{p}_\delta(\mathbb{S}_{-k})} - \pi_{1,N} \\ &= N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{\delta_i R_i}{p_\delta} - \pi_{1,N} + O_p(N^{-1/2}) N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{\delta_i R_i}{p_\delta} = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{\delta_i R_i}{p_\delta} - \pi_{1,N} + O_p(N^{-1/2} \bar{\pi}_N). \end{aligned}$$

Similarly,

$$\hat{\pi}_0(\mathbb{S}_{-k}) - \pi_{0,N} = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{(1 - \delta_i) R_i}{1 - p_\delta} - \pi_{0,N} + O_p(N^{-1/2} \bar{\pi}_N).$$

Let

$$\begin{aligned} \text{IF}_\pi(\mathbf{Z}) &= \left\{ \frac{\delta R}{p_\delta} - \pi_{1,N} \right\} \mathbb{E} \left[\frac{p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] \\ &\quad + \left\{ \frac{(1 - \delta) R}{1 - p_\delta} - \pi_{0,N} \right\} \mathbb{E} \left[\frac{1 - p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right]. \end{aligned}$$

Then,

$$\mathbb{E} \left[\frac{\hat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})}{\pi_N(\mathbf{X})} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \} \right] = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \text{IF}_\pi(\mathbf{Z}_i) + O_p(N^{-1/2}).$$

Hence,

$$\begin{aligned} \hat{\Delta}_N &= \sum_{k=1}^K \hat{\Delta}_{N,k} = (\mathbb{K} N_{-k})^{-1} \sum_{k=1}^K \sum_{i \in \mathcal{I}_{-k}} \text{IF}_\pi(\mathbf{Z}_i) + O_p(\|\hat{p}_\delta(\cdot) - p_\delta(\cdot)\|_{2, \mathbb{P}_X}^2 + (N \bar{\pi}_N)^{-1}) \\ &\quad + O_p(r_{p_\delta, N}) + O_p(N^{-1/2}) + O_p((N \bar{\pi}_N)^{-1/2} r_{p_\delta, N} + (N \bar{\pi}_N)^{-1}) \\ &= N^{-1} \sum_{i=1}^N \text{IF}_\pi(\mathbf{Z}_i) + O_p(r_{p_\delta, N} + (N \bar{\pi}_N)^{-1} + N^{-1/2}). \end{aligned}$$

By part (b) of Theorem 3.2,

$$\begin{aligned}\widehat{\theta}_{\text{DRSS}} - \theta_0 &= \frac{1}{N} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + O_p\left(r_{p_\delta, N} + (N\bar{\pi}_N)^{-1} + N^{-1/2}\right) \\ &\quad + O_p\left(r_{\mu, N}(N\bar{\pi}_N)^{-1/2} + \left\{r_{p_\delta, N} + (N\bar{\pi}_N)^{-1/2}\right\} \left\{(N\bar{\pi}_N)^{-1/2} + r_{\mu, N}\right\}\right), \\ &= \frac{1}{N} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + O_p\left((N\bar{\pi}_N)^{-1} + N^{-1/2} + r_{\mu, N}(N\bar{\pi}_N)^{-1/2} + r_{p_\delta, N}\right),\end{aligned}$$

where $\Psi(\mathbf{Z}) := \psi_\mu(\mathbf{Z}) + \text{IF}_\pi(\mathbf{Z})$ and $\mathbb{E}\{\Psi(\mathbf{Z})\} = 0$ with

$$\begin{aligned}\psi_\mu(\mathbf{Z}) &= \frac{R}{\pi_N(\mathbf{X})}\{Y - \mu(\mathbf{X})\} + \mu(\mathbf{X}) - \theta_0, \\ \text{IF}_\pi(\mathbf{Z}) &= \left\{\frac{\delta R}{p_\delta} - \pi_{1,N}\right\} \mathbb{E}\left[\frac{p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})}\{\mu(\mathbf{X}) - m(\mathbf{X})\}\right] \\ &\quad + \left\{\frac{(1-\delta)R}{1-p_\delta} - \pi_{0,N}\right\} \mathbb{E}\left[\frac{1-p_\delta(\mathbf{X})}{\pi_N(\mathbf{X})}\{\mu(\mathbf{X}) - m(\mathbf{X})\}\right].\end{aligned}$$

If further $r_{p_\delta, N} = o_p((N\bar{\pi}_N)^{-1/2})$,

$$\widehat{\theta}_{\text{DRSS}} - \theta_0 = \frac{1}{N} \sum_{i=1}^N \Psi(\mathbf{Z}_i) + o_p\left((N\bar{\pi}_N)^{-1/2}\right).$$

□

Proof of Theorem 5.1. Since $\pi_N(\mathbf{X}) > c\bar{\pi}_N$, we have

$$a_N = 1/\mathbb{E}\{\pi_N^{-1}(\mathbf{X})\} \geq c^{-1}\bar{\pi}_N.$$

Additionally, by Jensen's inequality, $a_N \leq \bar{\pi}_N$. Hence, $a_N \asymp \bar{\pi}_N$. For each $k \leq \mathbb{K}$, define the following event

$$\mathcal{E}_{-k} := \{\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) < 2C\bar{\pi}_N, \forall \mathbf{x} \in \mathcal{X}\}.$$

Then, under conditions $e_N(\mathbf{X}) < C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$, (5.3), and $r_{e,N} = o(1)$, we have

$$\mathbb{P}_{\mathbb{S}_{-k}}(\mathcal{E}_{-k}) \geq \mathbb{P}\left(\sup_{\mathbf{x} \in \mathcal{X}} \left|\frac{\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) - e_N(\mathbf{X})}{\bar{\pi}_N}\right| > C\right) = 1 - o(1). \quad (\text{D.64})$$

Recall that $\varepsilon = Y - Rm_1(\mathbf{X}) - (1-R)m_0(\mathbf{X})$. We have $\mathbb{E}(\varepsilon|R, \mathbf{X}) = 0$. Observe that

$$\widehat{\theta}^0 - \theta^0 = N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i) + \sum_{k=1}^{\mathbb{K}} (\widehat{\Delta}'_{N,1,k} + \widehat{\Delta}'_{N,2,k} + \widehat{\Delta}'_{N,3,k} + \widehat{\Delta}'_{N,4,k} + \widehat{\Delta}'_{N,5,k}),$$

where

$$\begin{aligned}
\psi_0(\mathbf{Z}) &= \mu_0(\mathbf{X}) - \theta^0 + \frac{1-R}{1-e_N(\mathbf{X})} \{Y - \mu_0(\mathbf{X})\} \\
&= \frac{e_N(\mathbf{X})-R}{1-e_N(\mathbf{X})} \{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\} + m_0(\mathbf{X}) - \theta^0 + \frac{\varepsilon(1-R)}{1-e_N(\mathbf{X})}, \\
\widehat{\Delta}'_{N,1,k} &= -N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{1-R_i}{1-\pi_N(\mathbf{X}_i)} - 1 \right\} \{\widehat{m}_0(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}_i)\}, \\
\widehat{\Delta}'_{N,2,k} &= N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{1-R_i}{1-\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{1-R_i}{1-e_N(\mathbf{X}_i)} \right\} \{Y_i - m_0(\mathbf{X}_i)\}, \\
\widehat{\Delta}'_{N,3,k} &= -N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{1-R_i}{1-\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{1-R_i}{1-e_N(\mathbf{X}_i)} \right\} \{\widehat{m}_0(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}_i)\}, \\
\widehat{\Delta}'_{N,4,k} &= N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{1-R_i}{1-\widehat{\pi}_N(\mathbf{X}_i; \mathbb{S}_{-k})} - \frac{1-R_i}{1-e_N(\mathbf{X}_i)} \right\} \{m_0(\mathbf{X}_i) - \mu_0(\mathbf{X}_i)\}, \\
\widehat{\Delta}'_{N,5,k} &= N^{-1} \sum_{i \in \mathcal{I}_k} \left\{ \frac{1-R_i}{1-\pi_N(\mathbf{X}_i)} - \frac{1-R_i}{1-e_N(\mathbf{X}_i)} \right\} \{\widehat{m}_0(\mathbf{X}_i; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}_i)\}.
\end{aligned}$$

We first obtain the rates for the terms $N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i)$, $\widehat{\Delta}'_{N,1,k}$, and $\widehat{\Delta}'_{N,2,k}$ for each $k \leq \mathbb{K}$. Observe the following properties for the first moments:

$$\begin{aligned}
\mathbb{E}\{\psi_0(\mathbf{Z})\} &= \mathbb{E}\left[\frac{e_N(\mathbf{X}) - \pi_N(\mathbf{X})}{1 - e_N(\mathbf{X})} \{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}\right] = 1\{e_N(\cdot) \neq \pi_N(\cdot), \mu(\cdot) \neq m(\cdot)\} O_p(\bar{\pi}_N), \\
\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}'_{N,1,k}) &= \mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}'_{N,2,k}) = 0.
\end{aligned}$$

For the second moments, we have

$$\begin{aligned}
\text{Var}\{\psi_0(\mathbf{Z})\} &= \text{Var}\left[\frac{\{e_N(\mathbf{X}) - R\}\{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}}{1 - e_N(\mathbf{X})} + m_0(\mathbf{X}) - \theta^0 + \frac{\varepsilon(1-R)}{1-e_N(\mathbf{X})}\right] \\
&\stackrel{(i)}{=} \text{Var}\left[\frac{\{e_N(\mathbf{X}) - R\}\{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}}{1 - e_N(\mathbf{X})} + m_0(\mathbf{X}) - \theta^0\right] + \text{Var}\left\{\frac{\varepsilon(1-R)}{1 - e_N(\mathbf{X})}\right\} \\
&\leq \left\|\frac{\{e_N(\mathbf{X}) - R\}\{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}}{1 - e_N(\mathbf{X})} + m_0(\mathbf{X}) - \theta^0\right\|_{2,\mathbb{P}}^2 + \left\|\frac{\varepsilon(1-R)}{1 - e_N(\mathbf{X})}\right\|_{2,\mathbb{P}}^2 \\
&\leq 2 \left\|\frac{\{e_N(\mathbf{X}) - R\}\{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}}{1 - e_N(\mathbf{X})}\right\|_{2,\mathbb{P}}^2 + 2 \|m_0(\mathbf{X}) - \theta^0\|_{2,\mathbb{P}}^2 + \left\|\frac{\varepsilon(1-R)}{1 - e_N(\mathbf{X})}\right\|_{2,\mathbb{P}}^2 \\
&\stackrel{(ii)}{=} 2\mathbb{E}\left(\frac{[(e_N(\mathbf{X}) - \pi_N(\mathbf{X}))^2 + \pi_N(\mathbf{X})\{1 - \pi_N(\mathbf{X})\}]\{m_0(\mathbf{X}) - \mu_0(\mathbf{X})\}^2}{\{1 - e_N(\mathbf{X})\}^2}\right) + 2 \|m_0(\mathbf{X}) - \theta^0\|_{2,\mathbb{P}}^2 \\
&\quad + \left\|\frac{\varepsilon(1-R)}{1 - e_N(\mathbf{X})}\right\|_{2,\mathbb{P}}^2 \\
&\stackrel{(iii)}{\leq} 2 [(1 - C\bar{\pi}_N)^{-2} \{(2C\bar{\pi}_N)^2 + C\bar{\pi}_N\} + 1] \|m_0(\mathbf{X}) - \theta^0\|_{2,\mathbb{P}}^2 + (1 - C\bar{\pi}_N)^{-2} \|\varepsilon\|_{2,\mathbb{P}}^2 = O(1).
\end{aligned}$$

where (i) holds by the fact that $\mathbb{E}(\varepsilon|R, \mathbf{X}) = 0$, (ii) holds by the tower rule with $\mathbb{E}(R|\mathbf{X}) = \pi_N(\mathbf{X})$, and (iii) follows by the assumption that $\bar{\pi}_N(\mathbf{x}), e_N(\mathbf{x}) < C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$. Besides,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,1,k}^2) &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left[\left\{ \frac{\pi_N(\mathbf{X}) - R}{1 - \pi_N(\mathbf{X})} \right\}^2 \{ \widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}) \}^2 \right] \\ &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left[\frac{\pi_N(\mathbf{X})}{1 - \pi_N(\mathbf{X})} \{ \widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}) \}^2 \right] \\ &\stackrel{(i)}{\leqslant} N^{-1} (1 - C\bar{\pi}_N)^{-1} C\bar{\pi}_N \| \widehat{m}_0(\cdot; \mathbb{S}_{-k}) - \mu_0(\cdot) \|_{2,\mathbb{P}_X}^2 = O_p(N^{-1} \bar{\pi}_N r_{\mu,0,N}^2),\end{aligned}$$

where (i) holds by the fact that $\bar{\pi}_N(\mathbf{x}) < C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$. On the event \mathcal{E}_{-k} , with (D.64), we have

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,2,k}^2) &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left[\left\{ \frac{1-R}{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{1-R}{1-e_N(\mathbf{X})} \right\}^2 \{ Y - m_0(\mathbf{X}) \}^2 \right] \\ &= N^{-2} |\mathcal{I}_k| \mathbb{E} \left[\frac{\{1-\pi_N(\mathbf{X})\} \{ \widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - e_N(\mathbf{X}) \}^2 \varepsilon^2}{\{1-e_N(\mathbf{X})\}^2 \{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})\}^2} \right] \\ &\stackrel{(i)}{\leqslant} N^{-1} (1 - 2C\bar{\pi}_N)^{-4} (1 - c\bar{\pi}_N) \bar{\pi}_N^2 \sup_{\mathbf{x} \in \mathcal{X}} \left| \frac{\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) - e_N(\mathbf{x})}{\bar{\pi}_N} \right|^2 \| \varepsilon \|_{2,\mathbb{P}}^2 = O_p(N^{-1} \bar{\pi}_N^2 r_{e,N}^2),\end{aligned}$$

where (i) holds by the fact that $c\bar{\pi}_N < \bar{\pi}_N(\mathbf{x}), e_N(\mathbf{x}) < C\bar{\pi}_N$ and $\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) < 2C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$ on \mathcal{E}_{-k} . Here, if we fix (or conditional on) \mathbb{S}_{-k} , on the event \mathcal{E}_{-k} , the inequality $\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) < 2C\bar{\pi}_N$ holds almost surely, w.r.t. the probability measure \mathbb{P} ; if \mathbb{S}_{-k} is treated as random, recall (D.64), the inequality holds w.p.a. 1, w.r.t. the joint probability measure of \mathbb{P} and $\mathbb{P}_{\mathbb{S}_{-k}}$. As a result, we have $\mathbb{E}_{\mathbb{S}_k}(\widehat{\Delta}_{N,2,k}^2) = O_p(N^{-1} \bar{\pi}_N^2 r_{e,N}^2)$ w.r.t. the joint probability measure of \mathbb{P} and $\mathbb{P}_{\mathbb{S}_{-k}}$. By Lemma A.1,

$$\begin{aligned}N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i) &= 1\{e_N(\cdot) \neq \pi_N(\cdot), \mu(\cdot) \neq m(\cdot)\} O_p(\bar{\pi}_N) + O_p(N^{-1/2}), \\ \widehat{\Delta}'_{N,1,k} &= O_p(N^{-1/2} \bar{\pi}_N^{1/2} r_{\mu,0,N}), \\ \widehat{\Delta}'_{N,2,k} &= O_p(N^{-1/2} \bar{\pi}_N r_{e,N}).\end{aligned}$$

Now we consider the terms $\widehat{\Delta}_{N,3,k}, \widehat{\Delta}_{N,4,k}$, and $\widehat{\Delta}_{N,5,k}$. On the event \mathcal{E}_{-k} , we have

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k}[\widehat{\Delta}_{N,3,k}] &\stackrel{(i)}{\leqslant} N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \left| \frac{1-R}{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{1-R}{1-e_N(\mathbf{X})} \right| | \widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}) | \right\} \\ &\stackrel{(ii)}{=} N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \frac{\{1-\pi_N(\mathbf{X})\} |\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - e_N(\mathbf{X})|}{\{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})\} \{1-e_N(\mathbf{X})\}} | \widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X}) | \right\} \\ &\stackrel{(iii)}{\leqslant} \frac{1-c\bar{\pi}_N}{(1-2C\bar{\pi}_N)^2} \bar{\pi}_N \sup_{\mathbf{x} \in \mathcal{X}} \left| \frac{\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) - e_N(\mathbf{x})}{\bar{\pi}_N} \right| \| \widehat{m}_0(\cdot; \mathbb{S}_{-k}) - \mu_0(\cdot) \|_{2,\mathbb{P}_X} = O_p(\bar{\pi}_N r_{e,N} r_{\mu,0,N}),\end{aligned}$$

where (i) holds by the triangular inequality, (ii) follows by the tower rule with the fact that $E(R|\mathbf{X}) = \pi_N(\mathbf{X})$, and (iii) holds by the fact that $c\bar{\pi}_N < \bar{\pi}_N(\mathbf{x}), e_N(\mathbf{x}) < C\bar{\pi}_N$ and $\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) < 2C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$

on \mathcal{E}_{-k} . Similarly, on the event \mathcal{E}_{-k} ,

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k} |\widehat{\Delta}_{N,4,k}| &\leq N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \left| \frac{1-R}{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{1-R}{1-\pi_N(\mathbf{X})} \right| |m_0(\mathbf{X}) - \mu_0(\mathbf{X})| \right\} \\ &= N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \frac{\{1-\pi_N(\mathbf{X})\} |\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) - \pi_N(\mathbf{X})|}{\{1-\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})\} \{1-e_N(\mathbf{X})\}} |m_0(\mathbf{X}) - \mu_0(\mathbf{X})| \right\} \\ &\stackrel{(i)}{\leq} \frac{1-c\bar{\pi}_N}{(1-2C\bar{\pi}_N)^2} \bar{\pi}_N \sup_{\mathbf{x} \in \mathcal{X}} \left| \frac{\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) - e_N(\mathbf{x})}{\bar{\pi}_N} \right| \|m_0(\cdot) - \mu_0(\cdot)\|_{2,\mathbb{P}_X} \\ &= 1\{m_0(\cdot) \neq \mu_0(\cdot)\} O_p(\bar{\pi}_N r_{e,N}),\end{aligned}$$

where (i) holds by the assumption that $c\bar{\pi}_N < \bar{\pi}_N(\mathbf{x})$, $e_N(\mathbf{x}) < C\bar{\pi}_N$ and $\widehat{\pi}_N(\mathbf{x}; \mathbb{S}_{-k}) < 2C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$ on \mathcal{E}_{-k} . Additionally, we also have

$$\begin{aligned}\mathbb{E}_{\mathbb{S}_k} |\widehat{\Delta}_{N,5,k}| &\leq N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \left| \frac{1-R}{1-\bar{\pi}_N(\mathbf{X}; \mathbb{S}_{-k})} - \frac{1-R}{1-\pi_N(\mathbf{X})} \right| |\widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X})| \right\} \\ &= N^{-1} |\mathcal{I}_k| \mathbb{E} \left\{ \frac{|\pi_N(\mathbf{X}) - e_N(\mathbf{X})|}{1-e_N(\mathbf{X})} |\widehat{m}_0(\mathbf{X}; \mathbb{S}_{-k}) - \mu_0(\mathbf{X})| \right\} \\ &\leq (1-C\bar{\pi}_N)^{-1} \sup_{\mathbf{x} \in \mathcal{X}} |\bar{\pi}_N(\mathbf{x}) - e_N(\mathbf{x})| \|\widehat{m}_0(\cdot; \mathbb{S}_{-k}) - \mu_0(\cdot)\|_{2,\mathbb{P}_X} \\ &= 1\{e_N(\cdot) \neq \pi_N(\cdot)\} O_p(\bar{\pi}_N r_{\mu,0,N}),\end{aligned}$$

since $\bar{\pi}_N(\mathbf{x}), e_N(\mathbf{x}) < C\bar{\pi}_N$ for all $\mathbf{x} \in \mathcal{X}$ by assumption. By Lemma A.1,

$$\begin{aligned}\widehat{\Delta}'_{N,3,k} &= O_p(\bar{\pi}_N r_{e,N} r_{\mu,N}), \\ \widehat{\Delta}'_{N,4,k} &= 1\{m_0(\cdot) \neq \mu_0(\cdot)\} O_p(\bar{\pi}_N r_{e,N}), \\ \widehat{\Delta}'_{N,5,k} &= 1\{e_N(\cdot) \neq \pi_N(\cdot)\} O_p(\bar{\pi}_N r_{\mu,0,N}).\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\theta}_{\text{DRSS}}^0 - \theta^0 &= N^{-1} \sum_{i=1}^N \psi_0(\mathbf{Z}_i) + O_p(N^{-1/2} \bar{\pi}_N^{1/2} r_{\mu,0,N} + N^{-1/2} \bar{\pi}_N r_{e,N} + \bar{\pi}_N r_{e,N} r_{\mu,0,N}) \\ &\quad + 1\{m_0(\cdot) \neq \mu_0(\cdot)\} O_p(\bar{\pi}_N r_{e,N}) + 1\{e_N(\cdot) \neq \pi_N(\cdot)\} O_p(\bar{\pi}_N r_{\mu,0,N}).\end{aligned}$$

□

Corollary 5.1 is a direct consequence of Theorems 3.2 and 5.1.

D.3 Proof of Theorem A.1

Proof of Theorem A.1. Here we provide asymptotic results of the DRSS estimator based on a MCAR PS.

Under MCAR, neither $\pi_N(\mathbf{X}) \equiv \bar{\pi}_N$ nor $\widehat{\pi}_N(\mathbf{X}; \mathbb{S}_{-k}) \equiv \widehat{\pi}_N(\mathbb{S}_{-k})$ depend on \mathbf{X} , and we recall that $\bar{\pi}_N = \mathbb{P}(R = 1)$. For each $k \leq \mathbb{K}$, $\widehat{\pi}_N(\mathbb{S}_{-k}) = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i$, where $N_{-k} = N - N/\mathbb{K}$. Notice that

$$\mathbb{E}_{\mathbb{S}_{-k}} \left[\left\{ \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} \right\}^2 \right] = \bar{\pi}_N^{-2} N_{-k}^{-1} \mathbb{E}(R - \bar{\pi}_N)^2 = N_{-k}^{-1} \bar{\pi}_N^{-1} (1 - \bar{\pi}_N) = O((N\bar{\pi}_N)^{-1}).$$

By Lemma A.1,

$$\frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} = O_p \left((N\bar{\pi}_N)^{-1/2} \right).$$

By the fact that

$$\frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \left\{ 1 + \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} \right\} = \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N},$$

we have

$$\frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} = \left\{ 1 + \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} \right\}^{-1} \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} = O_p \left((N\bar{\pi}_N)^{-1/2} \right). \quad (\text{D.65})$$

Hence,

$$\frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} - \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} = \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} \cdot \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} = O_p \left((N\bar{\pi}_N)^{-1} \right).$$

Additionally, notice that

$$\begin{aligned} \mathbb{E}_{\mathbb{S}_k} \left[|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} \right] &= \mathbb{E} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \}, \\ \mathbb{E}_{\mathbb{S}_k} \left[|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} \right]^2 &= |\mathcal{I}_k|^{-1} \bar{\pi}_N^{-1} \mathbb{E} [\{ \mu(\mathbf{X}) - m(\mathbf{X}) \}^2] = O \left((N\bar{\pi}_N)^{-1} \right). \end{aligned}$$

Let $\Delta_\mu = \mathbb{E} \{ \mu(\mathbf{X}) - m(\mathbf{X}) \}$. By Lemma A.1,

$$|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} - \Delta_\mu = O_p \left((N\bar{\pi}_N)^{-1/2} \right). \quad (\text{D.66})$$

Using the definition of $\widehat{\Delta}_N$ from Theorem 3.2 and adapting it to the MCAR setting,

$$\begin{aligned} \widehat{\Delta}_N &= N^{-1} \sum_{i=1}^N \frac{R_i}{\pi_N(\mathbf{X}_i)} \left\{ \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \right\} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} \\ &= \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \left[|\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} \right]. \end{aligned}$$

Recall (D.65) and (D.66), since $\mathbb{K} < \infty$ is a fixed number, we have

$$\begin{aligned} \sup_{k \leq \mathbb{K}} \left| \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \right| &= O_p \left((N\bar{\pi}_N)^{-1/2} \right), \\ \sup_{k \leq \mathbb{K}} \left| |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} - \Delta_\mu \right| &= O_p \left((N\bar{\pi}_N)^{-1/2} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \left| \widehat{\Delta}_N - \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \Delta_\mu \right| &\leq \sup_{k \in \mathbb{K}} \left| \frac{\widehat{\pi}_N(\mathbb{S}_{-k}) - \bar{\pi}_N}{\widehat{\pi}_N(\mathbb{S}_{-k})} \right| \sup_{k \in \mathbb{K}} \left| |\mathcal{I}_k|^{-1} \sum_{i \in \mathcal{I}_k} \frac{R_i}{\bar{\pi}_N} \{ \mu(\mathbf{X}_i) - m(\mathbf{X}_i) \} - \Delta_\mu \right| \\ &= O_p((N\bar{\pi}_N)^{-1}). \end{aligned}$$

It follows that,

$$\begin{aligned} \widehat{\Delta}_N &= \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} \left(\frac{\widehat{\pi}_N(S_{-k}) - \bar{\pi}_N}{\bar{\pi}_N} \right) \Delta_\mu + O_p((N\bar{\pi}_N)^{-1}) \\ &\stackrel{(i)}{=} \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} \frac{R_i - \bar{\pi}_N}{\bar{\pi}_N} \Delta_\mu + O_p((N\bar{\pi}_N)^{-1}) \\ &= \mathbb{K}^{-1} \sum_{k=1}^{\mathbb{K}} N_{-k}^{-1} \left(\sum_{i=1}^N \frac{R_i - \bar{\pi}_N}{\bar{\pi}_N} - \sum_{i \in \mathcal{I}_k} \frac{R_i - \bar{\pi}_N}{\bar{\pi}_N} \right) \Delta_\mu + O_p((N\bar{\pi}_N)^{-1}) \\ &= \{ N_{-k}^{-1} - (\mathbb{K}N_{-k})^{-1} \} \sum_{i=1}^N \frac{R_i - \bar{\pi}_N}{\bar{\pi}_N} \Delta_\mu + O_p((N\bar{\pi}_N)^{-1}) \\ &\stackrel{(ii)}{=} N^{-1} \sum_{i=1}^N \text{IF}_\pi(\mathbf{Z}_i) + O_p((N\bar{\pi}_N)^{-1}), \end{aligned}$$

where $\text{IF}_\pi(\mathbf{Z}) = (\bar{\pi}_N^{-1} R - 1) \Delta_\mu$. Here, (i) holds by the definition that $\widehat{\pi}_N(S_{-k}) = N_{-k}^{-1} \sum_{i \in \mathcal{I}_{-k}} R_i$ and (ii) follows by the fact that $N_{-k}^{-1} - (\mathbb{K}N_{-k})^{-1} = \mathbb{K}/\{(\mathbb{K}-1)N\} - 1/\{(\mathbb{K}-1)N\} = N^{-1}$. \square