Introduction

Purpose and Scope

This manuscript presents **Containment Theory**—a computationally verified alternative foundation to classical Set Theory for discrete mathematics. We develop a complete computational framework for boundary logic (also called the calculus of indications), demonstrating its equivalence to Boolean algebra while offering distinct advantages in axiomatic economy, geometric intuition, and handling of self-reference. Our primary contribution is rigorous computational verification of all theoretical claims in G. Spencer-Brown's *Laws of Form* (1969), establishing Containment Theory as a viable alternative foundation with only two axioms compared to Set Theory's nine or more.

The Foundation Problem

Mathematics rests upon foundations, and for over a century, Set Theory has served as the dominant foundation for mathematical reasoning. The Zermelo-Fraenkel axioms with Choice (ZFC) provide the standard framework within which most mathematics is constructed [?]. Yet this foundation carries significant conceptual weight: nine or more axioms, including the axiom of infinity, axiom of choice, and carefully crafted restrictions to avoid paradoxes like Russell's.

In 1969, G. Spencer-Brown proposed a radical alternative in *Laws of Form* [?]: a calculus requiring only two axioms, built on the primitive notion of **distinction** (the act of separating inside from outside, this from that) rather than membership. This calculus—variously called **boundary logic**, the **calculus of indications**, or **Containment Theory**—offers a foundation of remarkable parsimony while maintaining complete equivalence to Boolean algebra [?, ?] and propositional logic.

Containment Theory is our term for this approach to

Historical Context

Spencer-Brown's Laws of Form (1969)

George Spencer-Brown developed the calculus of indications from a simple observation: the most fundamental cognitive act is **making a distinction**—separating inside from outside, this from that [**?**].

A **distinction** is the act of drawing a boundary that creates two regions: an inside and an outside. The *mark* or *cross*, written $\langle\ \rangle$, represents this primary distinction: it creates a boundary that distinguishes the space inside from the space outside. This insight aligns with cybernetic thinking about observation and distinction [**?**, **?**].

From this single primitive, Spencer-Brown derived two axioms:

1. **The Law of Calling** (Involution): $\langle\langle a \rangle\rangle = a$
   - Crossing a boundary twice returns to the original state
   - Equivalent to double negation elimination
2. **The Law of Crossing** (Condensation): $\langle\ \rangle\langle\ \rangle = \langle\ \rangle$
   - Two marks condense to one mark
   - The marked state is idempotent

These axioms generate the complete Boolean algebra, yet their interpretation is fundamentally spatial rather than

Motivation for This Work

Despite its theoretical elegance, Containment Theory remains underexplored in mainstream mathematics and computer science. While Spencer-Brown's original work and subsequent extensions by Kauffman and Bricken provide compelling theoretical foundations, there has been limited computational verification of the claims and systematic comparison with established foundations like Set Theory. This work addresses these gaps by:

1. **Providing rigorous computational verification** of all theoretical claims in Laws of Form, including both axioms and all nine derived consequences, through a complete implementation with comprehensive test coverage

2. **Establishing precise correspondence** between boundary logic and Boolean algebra through systematic verification of De Morgan's laws, Boolean axioms, and truth table equivalence

3. **Analyzing complexity properties** of the reduction algorithm, demonstrating polynomial-time termination and providing

Document Structure

This manuscript is organized as follows to guide readers through the theoretical foundations, computational verification, and broader implications of Containment Theory:

- **Methodology** (Section 3): Provides the formal definition of the calculus of indications, including the two fundamental axioms (Calling and Crossing), the reduction algorithm for transforming forms to canonical representations, and the precise correspondence between boundary logic and Boolean algebra. Readers will find complete definitions of all technical terms, including forms, enclosures, juxtapositions, and canonical forms.
- **Experimental Results** (Section 4): Presents comprehensive computational verification of all theoretical claims, including verification of both axioms, all nine derived consequences (C1-C9) from Laws of Form, De Morgan's laws, and fundamental Boolean axioms. This section also includes complexity analysis demonstrating polynomial-time reduction

Notation

Throughout this work, we use the following notation:

Symbol	Meaning
$\langle\ \rangle$	The mark (cross), representing TRUE
$\emptyset$ or void	Empty space, representing FALSE
$\langle a \rangle$	Enclosure of a, representing NOT a
ab	Juxtaposition, representing a AND b
$\langle\langle a \rangle\langle b \rangle\rangle$	De Morgan form for a OR b

We write $\langle\langle a \rangle\rangle$ for double enclosure and use parentheses $(\ )$, square brackets $[\ ]$, or angle brackets $\langle\ \rangle$ interchangeably when clarity permits.