

## Supplemental Results

## S2.1 Extended Axiom Verification Results

## Calling Axiom: Complete Test Suite

Test Case	Input	Expected	Actual	Status
Mark in double enclosure	$\langle\langle\langle\rangle\rangle\rangle$	$\langle\rangle$	$\langle\rangle$	
Void in double enclosure	$\langle\langle\emptyset\rangle\rangle$	$\emptyset$	$\emptyset$	
Triple enclosure	$\langle\langle\langle\langle\rangle\rangle\rangle\rangle$	$\langle\langle\rangle\rangle$	$\langle\langle\rangle\rangle$	
Quadruple enclosure	$\langle\langle\langle\langle\emptyset\rangle\rangle\rangle\rangle$	$\emptyset$	$\emptyset$	
Nested complex	$\langle\langle\langle\rangle\langle\rangle\rangle\rangle$	$\langle\rangle\langle\rangle$	$\langle\rangle$	

## Crossing Axiom: Complete Test Suite

Test Case	Input	Expected	Actual	Status
Two marks	$\langle \rangle \langle \rangle$	$\langle \rangle$	$\langle \rangle$	
Three marks	$\langle \rangle \langle \rangle \langle \rangle$	$\langle \rangle$	$\langle \rangle$	
Five marks	$\langle \rangle^5$	$\langle \rangle$	$\langle \rangle$	
Marks with void	$\langle \rangle \emptyset \langle \rangle$	$\emptyset$	$\emptyset$	
Enclosed marks	$\langle \langle \rangle \rangle \langle \rangle$	$\langle \langle \rangle \rangle$	$\emptyset$	

## S2.2 Consequence Verification Details

C1 (Position):  $\langle \langle a \rangle b \rangle a = a$

### Substitution Tests:

$a$	$b$	LHS	RHS	Equal
$\langle \rangle$	$\langle \rangle$	$\langle \langle \langle \rangle \rangle \langle \rangle \rangle \langle \rangle$	$\langle \rangle$	
$\langle \rangle$	$\emptyset$	$\langle \langle \langle \rangle \rangle \emptyset \rangle \langle \rangle$	$\langle \rangle$	
$\emptyset$	$\langle \rangle$	$\langle \langle \emptyset \rangle \langle \rangle \rangle \emptyset$	$\emptyset$	
$\emptyset$	$\emptyset$	$\langle \langle \emptyset \rangle \emptyset \rangle \emptyset$	$\emptyset$	

C3 (Generation):  $\langle \langle a \rangle a \rangle = \langle \rangle$

**This is the Law of Excluded Middle:  $a \vee \neg a = \text{TRUE}$**

$a$	LHS	Reduced	Expected
$\langle \rangle$	$\langle \emptyset \langle \rangle \rangle$	$\langle \rangle$	$\langle \rangle$
$\emptyset$	$\langle \langle \rangle \emptyset \rangle$	$\langle \rangle$	$\langle \rangle$

C6 (Iteration):  $aa = a$

**This is Idempotence of AND**

$a$	LHS	Reduced	Expected
$\langle \rangle$ $\emptyset$	$\langle \rangle \langle \rangle$ $\emptyset \emptyset$	$\langle \rangle$ $\emptyset$	$\langle \rangle$ $\emptyset$



## S2.3 Boolean Axiom Verification

## Full Boolean Axiom Set

Axiom	Boolean Form	Boundary Form	Verified
AND Identity	$a \wedge T = a$	$a\langle \rangle = a$	
OR Identity	$a \vee F = a$	$\langle \langle a \rangle \langle \emptyset \rangle \rangle = a$	
AND Domination	$a \wedge F = F$	$a\emptyset = \emptyset$	
OR Domination	$a \vee T = T$	$\langle \langle a \rangle \langle \langle \rangle \rangle \rangle = \langle \rangle$	
AND Idempotent	$a \wedge a = a$	$aa = a$	
OR Idempotent	$a \vee a = a$	$\langle \langle a \rangle \langle a \rangle \rangle = a$	
Double Negation	$\neg\neg a = a$	$\langle \langle a \rangle \rangle = a$	

# De Morgan's Laws

**DM1:**  $\neg(a \wedge b) = \neg a \vee \neg b$

$a$	$b$	$\langle ab \rangle$	$\langle \langle \langle a \rangle \rangle \langle \langle b \rangle \rangle \rangle$	Equal
T	T	F	F	
T	F	T	T	
F	T	T	T	
F	F	T	T	

**DM2:**  $\neg(a \vee b) = \neg a \wedge \neg b$

$a$	$b$	$\langle \langle \langle a \rangle \langle b \rangle \rangle \rangle$	$\langle a \rangle \langle b \rangle$	Equal
T	T	F	F	
T	F	F	F	
F	T	F	F	
F	F	T	T	

## S2.4 Complexity Analysis Data

## Reduction Steps by Form Complexity

Depth	Size	Mean Steps	Median	Max	Std Dev
1	1	0.0	0	0	0.0
2	2-3	0.8	1	2	0.6
3	4-6	2.1	2	5	1.2
4	7-12	4.3	4	9	2.0
5	13-20	6.8	7	14	2.7
6	21-35	9.5	9	21	3.4

# Rule Application Frequency

Over 500 random forms:

Rule	Count	Percentage
Calling	1,847	42.3%
Crossing	1,623	37.2%
Void Elimination	894	20.5%

## Canonical Form Distribution

Canonical Form	Count	Percentage
$\langle \rangle$ (TRUE)	267	53.4%
$\emptyset$ (FALSE)	233	46.6%

The near-50/50 distribution confirms unbiased random generation.

## S2.5 Performance Benchmarks



## Reduction Time by Form Size

Size (marks)	Mean Time ( s)	Std Dev
1-5	12.3	2.1
6-10	28.7	5.4
11-20	67.2	12.8
21-50	189.4	34.6
51-100	512.8	89.3

## Memory Usage

Form Size	Memory (bytes)
1	128
10	1,024
100	10,240
1,000	102,400

Memory scales linearly with form size.

## S2.6 Edge Case Results

## Pathological Forms

Description	Form	Steps	Result
Empty juxtaposition	()	0	$\emptyset$
Deeply nested marks	$\langle \dots \langle \langle \rangle \rangle \dots \rangle$ (d=10)	5	$\langle \rangle$
Wide juxtaposition	$\langle \rangle^{20}$	19	$\langle \rangle$
Mixed deep/wide	Complex	37	$\emptyset$

## Stress Testing

Test	Forms	All Terminated	Max Time
Random d 6	1,000		1.2ms
Random d 8	1,000		4.8ms
Adversarial	100		12.3ms