1 Cross Correlations

Imagine a field x drawn from a gaussian distribution with mean zero and dispersion σ_x. We know that the error on σ will be

$$Var(\sigma_x^2) = 2(\sigma_x^2 + \sigma_{nx}^2)^2 \tag{1}$$

Therefore, the signal to noise of this is

$$\left(\frac{S}{N}\right)_0 = \left(\frac{N}{2}\right)^{1/2} \frac{\sigma_x^2}{(\sigma_x^2 + \sigma_{nx}^2)} \tag{2}$$

where N is the number of measurements.

Now suppose there is another field y with its own variance σ_y and suppose it has no noise. What will be the variance of the cross-dispersion:

$$\sigma_{xy}^2 = \langle xy \rangle = r\sigma_x\sigma_y. \tag{3}$$

$$Var(\sigma_{xy}^2) = (\sigma_x^2 + \sigma_{nx}^2)\sigma_y^2 \tag{4}$$

So the signal to noise of the cross power spectrum is

$$\left(\frac{S}{N}\right)_c = N^{1/2} r \frac{\sigma_x\sigma_y}{(\sigma_x^2 + \sigma_{nx}^2)^{1/2}\sigma_y} = (2N)^{1/4}\, r \left(\frac{S}{N}\right)_0^{1/2}. \tag{5}$$

Another way of doing this is via Fisher is to put an A in front of the cross spectrum, so the likelihood is

$$2\ln(L) = -\ln\det(C) - dC^{-1}d \tag{6}$$

with

$$C = \begin{pmatrix} \sigma_x^2 + \sigma_{nx}^2 & Ar\sigma_x\sigma_y \\ Ar\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}. \tag{7}$$

So

$$\det(C) = (\sigma_x^2 + \sigma_{nx}^2)\sigma_y^2 - A^2r^2\sigma_x^2\sigma_y^2 \tag{8}$$

and

$$C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} \sigma_y^2 & -Ar\sigma_x\sigma_y \\ -Ar\sigma_x\sigma_y & \sigma_x^2 + \sigma_{nx}^2 \end{pmatrix}. \tag{9}$$

If we write $D \equiv \det(C) = \alpha - A^2\beta$, then

$$\begin{aligned} D' &= -2A\beta \\ D'' &= -2\beta \end{aligned} \tag{10}$$

so

$$\begin{aligned} (\ln D)' &= \frac{-2A\beta}{D} \\ (\ln D)'' &= \frac{-2\beta}{D} + \frac{2D'\beta}{D^2} = \frac{-2\beta}{D} + \frac{-4A\beta^2}{D^2}. \end{aligned} \tag{11}$$

So, when $A = 1$, the first term in the Fisher matrix is

$$(\ln D)'' = -\frac{2\beta}{D^2}\left[\alpha + \beta\right]. \tag{12}$$

To evaluate the second term, use

$$\begin{aligned}(D^{-1})' &= \frac{-D'}{D^2} = \frac{2A\beta}{D^2} \\ (D^{-1})'' &= \frac{2\beta}{D^2} + \frac{8A\beta^2}{D^3}.\end{aligned} \tag{13}$$

We want

$$\frac{\partial^2}{\partial A^2}\frac{1}{2}dC^{-1}d = \frac{\langle d_i d_j\rangle}{2}\frac{\partial^2}{\partial A^2}\left[D^{-1}\begin{pmatrix}\sigma_y^2 & -Ar\sigma_x\sigma_y \\ -Ar\sigma_x\sigma_y & \sigma_x^2 + \sigma_{nx}^2\end{pmatrix}.\right]. \tag{14}$$

There are two terms: one in which the derivative acts on D^{-1} once and the matrix once:

$$2 \times \frac{\langle d_i d_j\rangle}{2}(D^{-1})'\begin{pmatrix}0 & -r\sigma_x\sigma_y \\ -r\sigma_x\sigma_y & 0\end{pmatrix} = -\sum_{i\neq j}\frac{\beta^{1/2}}{2}\frac{4\beta}{D^2}\beta^{1/2} = -\frac{4\beta^2}{D^2}. \tag{15}$$

Then the second term, in which the derivative acts on the denominator twice:

$$\frac{\langle d_i d_j\rangle}{2}\left[\frac{2\beta}{D^2} + \frac{8A\beta^2}{D^3}\right]\begin{pmatrix}\sigma_y^2 & -Ar\sigma_x\sigma_y \\ -Ar\sigma_x\sigma_y & \sigma_x^2 + \sigma_{nx}^2\end{pmatrix} = \sum_{ij}\frac{C_{ij}}{2}\left[\frac{2\beta}{D} + \frac{8A\beta^2}{D^2}\right]C_{ij}^{-1}. \tag{16}$$

After contraction, this becomes

$$\left[\frac{2\beta}{D} + \frac{8\beta^2}{D^2}\right]. \tag{17}$$

So, we are left with

$$\begin{aligned}\frac{-\partial^2 L}{\partial A^2} &= -\frac{\beta}{D^2}\left[\alpha + \beta\right] + \left[\frac{2\beta}{D} + \frac{4\beta^2}{D^2}\right] \\ &= \frac{\beta}{D^2}\left[-\alpha - \beta + 2(\alpha - \beta) + 4\beta\right] \\ &= \frac{\beta(\alpha + \beta)}{(\alpha - \beta)}.\end{aligned} \tag{18}$$

If there were N such measurements, then the Fisher matrix would be

$$\frac{1}{\Delta A^2} = N\frac{r^2\sigma_x^2\sigma_y^2((\sigma_x^2 + \sigma_{nx}^2)\sigma_y^2 + r^2\sigma_x^2\sigma_y^2)}{((\sigma_x^2 + \sigma_{nx}^2)\sigma_y^2 - r^2\sigma_x^2\sigma_y^2)^2}. \tag{19}$$

If you neglect r everywhere except in the prefactor, we get

$$(\Delta A) = \frac{(\sigma_x^2 + \sigma_{nx}^2)^{1/2}}{N^{1/2}r\sigma_x}, \tag{20}$$

which agrees with Eq. (5).