Time Delays in the Cosmic Microwave Background

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We consider the possibility of detecting the time delay field from cosmic microwave background experiments.

I. TIME DELAY ESTIMATOR

A. Flat Sky Approximation

We would like to obtain an expression for the change in the observed temperature due to the time delay akin to the one used for deflection. Recall that the impact of deflection can be traced by Taylor expanding

$$T(\vec{\theta}) = T^u(\vec{\theta}) - \vec{\alpha} \cdot \frac{\partial T^u}{\partial \vec{\theta}}. \tag{1}$$

The Fourier transform of the deflection term is then

$$\tilde{T}^\phi(\vec{l}) = \int \frac{d^2 l'}{(2\pi)^2} \vec{l'} \cdot (\vec{l} - \vec{l'}) \tilde{\phi}(\vec{l'}) \tilde{T}^u(\vec{l} - \vec{l'}) \tag{2}$$

We want to derive a similar expression for the impact of time delay.

A first guess at the Fourier space is that the unlensed temperature is simply

$$\tilde{T}^u(\vec{l}) = \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot[\chi_*\vec{\theta},\chi_*]} [\tilde{T}_0 + \tilde{\Psi}](\vec{k}) \tag{3}$$

which uses the small sky approximation to express the position from whence the CMB photons are emitted as $[\chi_*\vec{\theta}, \chi_*]$ and recognizes that the temperature is due to the sum of the monopole and the gravitational potential.Carrying out the $\vec{\theta}$ integral leads to a delta function in $\vec{k}_\perp$, so that

$$\tilde{T}^u(\vec{l};\chi_*) = \frac{1}{\chi_*^2} \int \frac{dk_z}{2\pi} [\tilde{T}_0 + \tilde{\Psi}](\vec{l}/\chi_*, k_z)\, e^{ik_z\chi_*} \tag{4}$$

where the fact that T^u depends on the distance to the last scattering surface is encoded in the argument on the left hand side. This is important, as the temperature due to the time delay requires us to take the derivative with respect to χ_*.

In particular,

$$T^d(\vec{\theta}) = d(\vec{\theta})\, \frac{\partial}{\partial \ln \chi_*}\, T^u(\vec{\theta}; \chi_*). \tag{5}$$

This is simply a Taylor expansion in the deviation from χ_* accounting for the fact that d is the fractional change in the distance to the last scattering surface. In Fourier space, this becomes

$$\tilde{T}^d(\vec{l}) = \int \frac{d^2 l'}{(2\pi)^2}\, \tilde{d}(\vec{l'}) \tilde{T}^{u'}(\vec{l} - \vec{l'}; \chi_*) \tag{6}$$

where the prime on T^u denote the derivative with respect to $\ln \chi_*$. Comparing Eq. (6) with Eq. (2) gives us a sense of the difficulty of extracting this signal. The contribution to the temperature due to deflection has two derivatives compared to the one due to time delay. These show up in Fourier space as extra factors of $\vec{l'} \cdot \vec{l}$, where $\vec{l'}$ is small (of order 50 for deflection, but $l \sim 10^3$. So the signal is nominally down by a factor of at order 10^4. Figure 5 in [1] suggests that the power spectrum is only down by a factor of 10^4 implying that the RMS is down by only 100, so things are apparently not as bad as they appear.

When I roughly carry this through to follow the deflection estimator, I get something roughly

$$\hat{\tilde{d}}(\vec{L}) \simeq \frac{\tilde{T}(\vec{l})\tilde{T}(\vec{l} - \vec{L})}{lC_l} \tag{7}$$

and if we wanted to construct the optimal estimator,we would sum these over all $\vec{l}$ with appropriate weighting.

Although this is a very rough derivation, we can already glimpse several points about the time delay estimator:

- Just like the deflection estimator, this will be a quadratic estimator pairing temperatures on small scales separated by a small Fourier space difference $\vec{L}$
- Because Hu and Cooray showed that the time delay power spectrum peaks on the largest of scales, instead of $\Delta L \sim 50$ used for the deflection potential, here we will be using very small ΔL's, perhaps as small as 1. This means an all-sky treatment of the time delay field is essential.
- While the deflection estimator upweights modes with small $\vec{L} \cdot \vec{l}$, i.e., it uses modes with lensing wavevector perpendicular to the anistropy wavevector, here there is no such angular preference. This should make it slightly easier to disentangle the two estimators.

B. All-sky Expressions

A starting point is

$$T(\hat{n}) = \sum_{lm} Y_{lm}(\hat{n}) a_{lm}(\chi_*) \tag{8}$$

where (Eq. (21) in [2])

$$a_{lm}(\chi_*) = (-i)^l \int \frac{d^3k}{(2\pi)^3} Y^*_{lm}(\hat{k})[\tilde{T}_0 + \tilde{\Psi}](\vec{k};\eta_*) j_l(k\chi_*) \tag{9}$$

where T_0 is the monopole; Ψ one of the scalar potentials; and χ_* the distance to the last scattering surface. Therefore, the small change due to the time delay is:

$$T^d(\hat{n}) = \sum_{lm} Y_{lm}(\hat{n}) \frac{\partial a_{lm}(\chi_*)}{\partial \chi_*} \chi_* d(\hat{n}) \tag{10}$$

where d is the fractional change in the distance to the last scattering surface as in [1]. We can go a step further using some of the formalism developed in [3] to express $d(\hat{n})$ in terms of its spherical harmonics; then,

$$T^d(\hat{n}) = \sum_{LM;L'M'} \frac{\partial a_{LM}(\chi_*)}{\partial \ln \chi_*} d_{L'M'} Y_{LM}(\hat{n}) Y_{L'M'}(\hat{n}). \tag{11}$$

Writing this as

$$T^d(\hat{n}) = \sum_{lm} D_{lm} Y_{lm}(\hat{n}) \tag{12}$$

where – to be clear – d_{lm} are the coefficients of the spherical harmonics of the time delay field while D_{lm} are the coefficients of the contribution to the change in the CMB anisotropies due to this time delay. Multiplying by Y^*_{lm} and integrating over $d\Omega_n$ leads to

$$D_{lm} = \sum_{LM;L'M'} \frac{\partial a_{LM}(\chi_*)}{\partial \ln \chi_*} d_{L'M'} \int d\Omega_{\hat{n}} Y_{LM}(\hat{n}) Y_{L'M'}(\hat{n}) Y^*_{lm}(\hat{n}). \tag{13}$$

The integral can be done by using the formula for the product of spherical harmonics and then orthogonality leading to

$$D_{lm} = \sum_{LM;L'M'} \frac{\partial a_{LM}(\chi_*)}{\partial \ln \chi_*} d_{L'M'} I(lm; LM; L'M') \tag{14}$$

where

$$I(lm; LM; L'M') = (-1)^m \left[\frac{(2L+1)(2L'+1)(2l+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l & L & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & L & L' \\ -m & M & M' \end{pmatrix}. \tag{15}$$

II. ESTIMATORS

The observed temperature field has coefficients

$$a^{\rm obs}_{lm} = a_{lm} + D_{lm}. \tag{16}$$

When we take quadratic estimators the perturbative term will have an expectation value:

$$\langle a^{\rm obs}_{l_1m_1} a^{\rm obs}_{l_2m_2} \rangle = \delta_{l_1l_2}\delta_{m_1-m_2} C_{l_1} + \langle a_{l_1m_1} D_{l_2m_2}\rangle + \langle a_{l_2m_2} D_{l_1m_1}\rangle. \tag{17}$$

Consider the middle term here and use Eq. (9) to get

$$\langle a_{l_1m_1}D_{l_2m_2}\rangle = \sum_{LM;L'M'} d_{L'M'}I(l_2m_2;LM;L'M')(-i)^{l_1+L}\int\frac{d^3k}{(2\pi)^3}Y^*_{l_1m_1}(\hat{k})j_{l_1}(k\chi_*)\int\frac{d^3k'}{(2\pi)^3}Y^*_{LM}(\hat{k}')\frac{\partial j_L(k'\chi_*)}{\partial\ln(\chi_*)}$$
$$\times\langle[\tilde{T}_0+\tilde{\Psi}](\vec{k};\eta_*)[\tilde{T}_0+\tilde{\Psi}](\vec{k}';\eta_*)\rangle. \tag{18}$$

But the expectation value leads to the 3D power spectrum multiplied by a delta function in $\vec{k}+\vec{k}'$, after which the angular integral over the spherical harmonics can be performed using $(Y^*_{LM}(-\hat{k}) = (-1)^{L+M}Y_{LM}(\hat{k})$, and we are left with

$$\langle a_{l_1m_1}D_{l_2m_2}\rangle = (i)^{2l_1}\sum_{L'M'} d_{L'M'}I(l_2m_2;l_1m_1;L'M')(-1)^{m_1}\int\frac{dkk^2}{(2\pi)^3}j_{l_1}(k\chi_*)\frac{\partial j_{l_1}(k\chi_*)}{\partial\ln(\chi_*)}P_{T_0+\Psi}(k). \tag{19}$$

We can use the identity (Eq 31 in [3])

$$\sum_{m_1m_2}\begin{pmatrix} l_2 & l_1 & L' \\ -m_2 & m_1 & M'\end{pmatrix}\begin{pmatrix} l_2 & l_1 & L \\ -m_2 & m_1 & M\end{pmatrix} = \frac{1}{2L+1}\delta_{LL'}\delta_{MM"} \tag{20}$$

so that

$$\sum_{m_1m_2}\begin{pmatrix} l_2 & l_1 & L \\ -m_2 & m_1 & M\end{pmatrix}\langle a_{l_1m_1}D_{l_2m_2}\rangle = \frac{(i)^{2l_1}}{2L+1}d_{LM}\left[\frac{(2l_1+1)(2L+1)(2l_2+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l_2 & l_1 & L \\ 0 & 0 & 0\end{pmatrix}C^{(1)}_{l_1} \tag{21}$$

where

$$C^{(1)}_{l_1} \equiv \int\frac{dkk^2}{(2\pi)^3}j_{l_1}(k\chi_*)\frac{\partial j_{l_1}(k\chi_*)}{\partial\ln(\chi_*)}P_{T_0+\Psi}(k). \tag{22}$$

So, neglecting the last term in Eq. (17), an estimator for the time delay field is

$$\hat{D}_{LM} = \mathcal{A}\sum_{m_1m_2}\begin{pmatrix} l_2 & l_1 & L \\ -m_2 & m_1 & M\end{pmatrix}a^{\rm obs}_{l_1m_1}a^{\rm obs}_{l_2m_2} \tag{23}$$

where the normalization is

$$\mathcal{A} \equiv (-1)^{l_1}\left[\frac{4\pi(2L+1)}{(2l_1+1)(2l_2+1)}\right]^{1/2}\left[\begin{pmatrix} l_2 & l_1 & L \\ 0 & 0 & 0\end{pmatrix}C^{(1)}_{l_1}\right]^{-1} \tag{24}$$

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