2018-05-22

Revising derivation of noise model likelihood function

Consider M measurements of an unknown, band-limited signal $\mu(t)$ subject to amplitude drift and temporal drift, so that the signal associated with measurement $j = 0, 1, \ldots, M-1$ is

$$S(t; A_j, \eta_j) = A_j \mu(t - \eta_j)$$

For each signal measurement we obtain N noisy samples at nominal times $t_n = nT$, $n = 0, 1, \ldots, N-1$, which we arrange in a matrix $\underset{\sim}{X}$ with size $N \times M$:

2018-05-22

$$x_{ij} = (1+\beta_{ij})\,\xi(t_i + \tau_{ij}; A_j, \eta_j) + \alpha_{ij}$$
$$= A_j(1+\beta_{ij})\,\mu(t_i - \eta_j + \tau_{ij}) + \alpha_{ij},$$

where the random variables $\alpha_{ij} \sim \mathcal{N}(0, \sigma_\alpha^2)$, $\beta_{ij} \sim \mathcal{N}(0, \sigma_\beta^2)$, and $\tau_{ij} \sim \mathcal{N}(0, \sigma_\tau^2)$ are fully independent and account for additive, multiplicative, and timebase noise, respectively. To fix the scale and location of $\mu(t)$, we set $A_0 = 1$ and $\eta_0 = 0$.

Expanding around the ideal value to first order in the random variables and introducing the notation $\xi_{ij} = \xi(t_i; A_j, \eta_j)$ and

$\dot{\xi}_{ij} = \dot{\xi}(t_i; A_j, \eta_j)$ gives

$$x_{ij} \simeq (1+\beta_{ij})(\xi_{ij} + \tau_{ij}\dot{\xi}_{ij}) + \alpha_{ij}$$
$$\simeq \xi_{ij} + \alpha_{ij} + \beta_{ij}\xi_{ij} + \tau_{ij}\dot{\xi}_{ij}.$$

We can express this in terms of the sequence $\mu_n = \mu(nT)$, $n = 0, 1, \ldots, N-1$. Let $\tilde{\underline{\mu}}$ denote the discrete Fourier transform of $\underline{\mu}$, i.e.,

$$\boxed{\tilde{\mu}_k = \sum_{j=0}^{N-1} \mu_j \exp(-2\pi i jk/N).}$$

The inverse transform is then

$$\mu_j = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{\mu}_k \exp(2\pi i jk/N).$$

2015-05-24

From the Fourier decomposition of $\underset{\sim}{\mu}$ we can construct the trigonometric interpolation polynomial $\hat{\mu}(t)$, which we can use to compute $\underset{\sim}{S}(t; \underset{\sim}{A}, \underset{\sim}{\eta})$. (For details, see https://en.m.wikipedia.org/wiki/Discrete_Fourier_transform#Trigonometric_interpolation_polynomial)

For even N,

$$\hat{\mu}(t) = \frac{1}{N}\left\{\sum_{k=0}^{N/2-1} \tilde{\mu}_k \exp(2\pi i k t/NT) + \tilde{\mu}_{N/2}\cos(\pi t/T) + \sum_{k=N/2+1}^{N-1} \tilde{\mu}_k \exp[2\pi i (k-N) t/NT]\right\},$$

and for odd N,

$$\hat{\mu}(t) = \frac{1}{N}\left\{\sum_{k=0}^{\lfloor N/2\rfloor} \tilde{\mu}_k \exp(2\pi i k t/NT) + \sum_{k=\lfloor N/2\rfloor+1}^{N-1} \tilde{\mu}_k \exp[2\pi i (k-N) t/NT]\right\}.$$

2018-05-24

We can simplify these expressions by defining discrete angular frequencies

$$\boxed{\omega_k = \begin{cases} 2\pi k/NT, & k < \lfloor N/2 \rfloor \\ 2\pi (k-N)/NT, & k \geq \lfloor N/2 \rfloor . \end{cases}}$$

The expressions for $\hat{\mu}(t)$ then simplify to

$$\boxed{\begin{aligned} \hat{\mu}(t) &= r_N(t) + \frac{1}{N}\sum_{k=0}^{N-1} \tilde{\mu}_k \exp(i\omega_k t), \\ \text{where} \quad r_N(t) &= \begin{cases} -i\dfrac{\tilde{\mu}_{N/2}}{N}\sin(\omega_{N/2}\, t) & \text{even } N, \\ 0 & \text{odd } N. \end{cases} \end{aligned}}$$

The functions $\xi(t; A_j, \eta_j)$ may then be written as

$$\boxed{\begin{aligned} \xi(t; A_j, \eta_j) &= A_j\, r_N(t-\eta_j) + \frac{A_j}{N}\sum_{k=0}^{N-1} \tilde{\mu}_k \exp[i\omega_k (t-\eta_j)] \\ &= A_j\, r_N(t-\eta_j) + \frac{A_j}{N}\sum_{k=0}^{N-1} \tilde{\mu}_k \exp(-i\omega_k \eta_j)\exp(i\omega_k t), \end{aligned}}$$

and their time derivatives as

$$\boxed{\dot{\varsigma}(t; A_j, \eta_j) = A_j \dot{r}_N (t - \eta_j) + \frac{A_j}{N} \sum_{k=0}^{N-1} i\omega_k \tilde{\mu}_k \exp(-i\omega_k \eta_j) \exp(i\omega_k t).}$$

Evaluating the interpolated functions at the discrete times $t_j = jT$ (changing to a different index convention to avoid confusion with the complex number i), we have

$$\boxed{\begin{aligned} \varsigma_{jk} &= \varsigma(t_j; A_k, \eta_k) \\ &= A_k r_N (t_j - \eta_k) + \frac{A_k}{N} \sum_{\ell=0}^{N-1} \tilde{\mu}_\ell \exp(-i\omega_\ell \eta_k) \exp(i\omega_\ell t_j), \\ \dot{\varsigma}_{jk} &= A_k \dot{r}_N (t_j - \eta_k) + \frac{A_k}{N} \sum_{\ell=0}^{N-1} i\omega_\ell \tilde{\mu}_\ell \exp(-i\omega_\ell \eta_k) \exp(i\omega_\ell t_j). \end{aligned}}$$

2015-05-24

For even N we can simplify these expressions further.

$$
\begin{aligned}
N\varsigma_{jk} &= -iA_k\tilde{\mu}_{N/2}\sin[\omega_{N/2}(t_j-\eta_k)] \\
&\quad + A_k\sum_{\ell=0}^{N-1}\tilde{\mu}_\ell\exp(-i\omega_\ell\eta_k)\exp(i\omega_\ell t_j), \\
&= -iA_k\tilde{\mu}_{N/2}\sin[\omega_{N/2}(t_j-\eta_k)] \\
&\quad + A_k\tilde{\mu}_{N/2}\exp[i\omega_{N/2}(t_j-\eta_k)] \\
&\quad + A_k\sum_{\ell=0}^{N/2-1}\tilde{\mu}_\ell\exp[i\omega_\ell(t_j-\eta_k)] \\
&\quad + A_k\sum_{\ell=N/2+1}^{N-1}\tilde{\mu}_\ell\exp[i\omega_\ell(t_j-\eta_k)] \\
&= -iA_k\tilde{\mu}_{N/2}\sin[\omega_{N/2}(t_j-\eta_k)] \\
&\quad + A_k\tilde{\mu}_{N/2}\exp[i\omega_{N/2}(t_j-\eta_k)] \\
&\quad + A_k\sum_{\ell=0}^{N/2-1}\tilde{\mu}_\ell\exp[i\omega_\ell(t_j-\eta_k)] \\
&\quad + A_k\sum_{\ell=N/2+1}^{N-1}\tilde{\mu}^*_{N-\ell}\exp[-i\omega_{N-\ell}(t_j-\eta_k)] \\
&= A_k\tilde{\mu}_{N/2}\cos[\omega_{N/2}(t_j-\eta_k)] \\
&\quad + 2A_k\sum_{\ell=0}^{N/2-1}\mathrm{Re}\{\tilde{\mu}_\ell\exp[i\omega_\ell(t_j-\eta_k)]\} \in \mathbb{R},
\end{aligned}
$$

2015-05-24

and since r_N is purely imaginary, we can write

$$\varsigma_{jk} = \mathrm{Re}\left\{ \frac{A_k}{N} \sum_{\ell=0}^{N-1} \tilde{\mu}_\ell \exp[i\omega_\ell (t_j - \eta_k)] \right\}$$

For $\dot{\underset{\sim}{\varsigma}}$ with even N, we have

$$\begin{aligned}
N \dot{\varsigma}_{jk} &= -i\omega_{N/2} A_k \tilde{\mu}_{N/2} \cos[\omega_{N/2}(t_j - \eta_k)] \\
&\quad + A_k \sum_{\ell=0}^{N-1} i\omega_\ell \tilde{\mu}_\ell \exp[i\omega_\ell (t_j - \eta_k)] \\
&= -i\omega_{N/2} A_k \tilde{\mu}_{N/2} \cos[\omega_{N/2}(t_j - \eta_k)] \\
&\quad + i\omega_{N/2} A_k \tilde{\mu}_{N/2} \exp[\omega_{N/2}(t_j - \eta_k)] \\
&\quad + A_k \sum_{\ell=0}^{N/2-1} i\omega_\ell \tilde{\mu}_\ell \exp[i\omega_\ell (t_j - \eta_k)] \\
&\quad + A_k \sum_{\ell=N/2+1}^{N-1} i\omega_\ell \tilde{\mu}_\ell \exp[i\omega_\ell (t_j - \eta_k)] \\
&= -\omega_{N/2} A_k \tilde{\mu}_{N/2} \sin[\omega_{N/2}(t_j - \eta_k)] \\
&\quad + A_k \sum_{\ell=0}^{N/2-1} i\omega_\ell \tilde{\mu}_\ell \exp[i\omega_\ell (t_j - \eta_k)] \\
&\quad + A_k \sum_{\ell=N/2+1}^{N-1} -i\omega_{N-\ell} \tilde{\mu}^*_{N-\ell} \exp[-i\omega_{N-\ell}(t_j - \eta_k)]
\end{aligned}$$

$$\Rightarrow N\dot{\varsigma}_{jk} = -\omega_{N/2} A_k \tilde{\mu}_{N/2} \sin[\omega_{N/2}(t_j - \cdot\,)]$$
$$+ 2A_k \sum_{\ell=0}^{N/2-1} \mathcal{Re}\{i\omega_\ell \tilde{\mu}_\ell \exp[i\omega_\ell(t_j - \eta_k)]\} \in \mathbb{R},$$

and since $\dot{r}_N$ is als pure, imaginary, we can write

$$\dot{\varsigma}_{jk} = \mathcal{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1} i\omega_\ell \tilde{\mu}_\ell e^{[i\omega_\ell(t_j-\eta_k)]}\right\}.$$

Writing the sequences $\{\,\}$, $\{\omega_j\}$, and $\{\tilde{\mu}_j\}$ in vector notation $\underset{\sim}{t}$, $\underset{\sim}{\omega}$, and $\underset{\sim}{\tilde{\mu}}$, respectively, and denoting the Hadamard product as $\{\omega_j \tilde{\mu}_j\} = \mathrm{diag}(\underset{\sim}{\omega})\underset{\sim}{\tilde{\mu}} = \underset{\sim}{\omega} \circ \underset{\sim}{\tilde{\mu}}$, this expression may be simplified further to:

$$\boxed{\begin{aligned} \underset{\sim}{\varsigma}_{\cdot k} &= A_k \mathcal{Re}\{\mathcal{F}^{-1}[\underset{\sim}{\tilde{\mu}} \circ \exp(-i\underset{\sim}{\omega}\eta_k)]\}, \\ \dot{\underset{\sim}{\varsigma}}_{\cdot k} &= A_k \mathcal{Re}\{\mathcal{F}^{-1}[i\underset{\sim}{\omega} \circ \underset{\sim}{\tilde{\mu}} \circ \exp(-i\underset{\sim}{\omega}\eta_k)]\}. \end{aligned}}$$

2018-05-24

Given a $N \times M$ data matrix $\underset{\sim}{x}$ and parameters $\underset{\sim}{\mu}$, σ_α^2, σ_β^2, σ_τ^2, $\underset{\sim}{A}$, and η, the likelihood function is

$$\mathcal{L}(\sigma_\alpha^2, \sigma_\beta^2, \sigma_\tau^2, \underset{\sim}{\mu}, \underset{\sim}{A}, \eta; \underset{\sim}{x})$$
$$= \prod_{jk} (2\pi \sigma_{jk}^2)^{-1/2} \exp\left[- \frac{(x_{jk} - \xi_{jk})^2}{2\sigma_{jk}^2} \right],$$

with

$$\sigma_{jk}^2 = \sigma_\alpha^2 + \sigma_\beta^2 \xi_{jk}^2 + \sigma_\tau^2 \dot{\xi}_{jk}^2 .$$

The log-likelihood is then

$$l(\sigma_\alpha^2, \sigma_\beta^2, \sigma_\tau^2, \underset{\sim}{\mu}, \underset{\sim}{A}, \eta; \underset{\sim}{x})$$
$$= \log \mathcal{L}(\sigma_\alpha^2, \sigma_\beta^2, \sigma_\tau^2, \underset{\sim}{\mu}, \underset{\sim}{A}, \eta; \underset{\sim}{x})$$
$$= -\frac{MN}{2} \log(2\pi) - \frac{1}{2} \sum_{jk} \log(\sigma_{jk}^2)$$
$$- \frac{1}{2} \sum_{jk} \frac{(x_{jk} - \xi_{jk})^2}{\sigma_{jk}^2} ,$$

2015-05-24

and the negative log-likelihood cost function is

$$\boxed{\begin{aligned} C(\sigma_\alpha^2, \sigma_\beta^2, \sigma_\tau^2, \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\eta}; \mathbf{x}) &= -\ell(\sigma_\alpha^2, \sigma_\beta^2, \sigma_\tau^2, \boldsymbol{\mu}, \mathbf{A}, \boldsymbol{\eta}; \mathbf{x}) \\ &= \frac{MN}{2}\log(2\pi) + \frac{1}{2}\sum_{jk}\log(\sigma_{jk}^2) \\ &\quad + \frac{1}{2}\sum_{jk}\frac{(x_{jk} - \xi_{jk})^2}{\sigma_{jk}^2}. \end{aligned}}$$

The variance parameters σ_α^2, σ_β^2, and σ_τ^2 are all nonnegative. To enforce this bound, it is convenient to define the auxiliary parameters $\theta_\alpha = \log \sigma_\alpha^2$, $\theta_\beta = \log \sigma_\beta^2$, and $\theta_\tau = \log \sigma_\tau^2$, which may remain unbounded.

2015-05-24

In this case the structure of the likelihood, log-likelihood, and negative log-likelihood cost functions remains unchanged, while the estimated point variance becomes

$$\boxed{\sigma_{ij}^2 = \exp\theta_\alpha + \xi_{ij}^2 \exp\theta_\beta + \dot{\xi}_{ij}^2 \exp\theta_\tau .}$$

To simulate the distribution given by l or minimize C, it is helpful to have analytic expressions for their derivatives w.r.t. the parameters. To simplify this calculation, define the additional auxiliary variable

$$\boxed{d_{jk} = x_{jk} - \xi_{jk}.} \leftarrow \text{res}$$

2018-05-24 (revised 2018-08-29, changed ℓ to C)

$$\frac{\partial C}{\partial \theta_\alpha} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial \theta_\alpha},$$

$$\frac{\partial C}{\partial \theta_\beta} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial \theta_\beta},$$

$$\frac{\partial C}{\partial \theta_\tau} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial \theta_\tau},$$

$$\frac{\partial C}{\partial \mu_p} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial \mu_p} + \sum_{jk} \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \varsigma_{jk}}{\partial \mu_p},$$

$$\frac{\partial C}{\partial A_p} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial A_p} + \sum_{jk} \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \varsigma_{jk}}{\partial A_p},$$

and

$$\frac{\partial C}{\partial \eta_p} = \frac{1}{2} \sum_{jk} \left[\frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2} \right] \frac{\partial \sigma_{jk}^2}{\partial \eta_p} + \sum_{jk} \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \varsigma_{jk}}{\partial \eta_p}.$$

2018-05-24

From the noise model we have

$$\frac{\partial \sigma_{jk}^2}{\partial \theta_\alpha} = \exp \theta_\alpha = \sigma_\alpha^2 ,$$

$$\frac{\partial \sigma_{jk}^2}{\partial \theta_\beta} = \zeta_{jk}^2 \exp \theta_\beta = \sigma_\beta^2 \, \zeta_{jk}^2 , \quad \text{and}$$

$$\frac{\partial \sigma_{jk}^2}{\partial \theta_\tau} = \dot{\zeta}_{jk}^2 \exp \theta_\tau = \sigma_\tau^2 \, \dot{\zeta}_{jk}^2 .$$

Defining $\boxed{\Delta_{jk} = \frac{1}{\sigma_{jk}^2} - \frac{d_{jk}^2}{(\sigma_{jk}^2)^2}}$ ← draw ,

$$\boxed{\begin{aligned} \frac{\partial C}{\partial \theta_\alpha} &= \frac{\exp \theta_\alpha}{2} \sum_{jk} \Delta_{jk} \\ \frac{\partial C}{\partial \theta_\beta} &= \frac{\exp \theta_\beta}{2} \sum_{jk} \Delta_{jk} \, \zeta_{jk}^2 , \quad \text{and} \\ \frac{\partial C}{\partial \theta_\tau} &= \frac{\exp \theta_\tau}{2} \sum_{jk} \Delta_{jk} \, \dot{\zeta}_{jk}^2 . \end{aligned}}$$

To determine $\partial C / \partial \mu_p$, we will need to calculate $\partial \zeta_{jk} / \partial \mu_p$ and $\partial \dot{\zeta}_{jk} / \partial \mu_p$.

2018-05-25

Starting with the expressions for ς_{jk}, $\dot{\varsigma}_{jk}$ in terms of the Fourier coefficients $\tilde{\underset{\sim}{\mu}}$,

$$\varsigma_{jk} = \operatorname{Re}\left\{ \frac{A_k}{N} \sum_{\ell=0}^{N-1} \tilde{\mu}_\ell \exp\left[i\omega_\ell (t_j - \eta_k)\right] \right\},$$

$$\dot{\varsigma}_{jk} = \operatorname{Re}\left\{ \frac{A_k}{N} \sum_{\ell=0}^{N-1} i\omega_\ell \tilde{\mu}_\ell \exp\left[i\omega_\ell (t_j - \eta_k)\right] \right\},$$

we see that both $\partial \varsigma_{jk}/\partial \mu_p$ and $\partial \dot{\varsigma}_{jk}/\partial \mu_p$ will depend on $\partial \tilde{\mu}_\ell / \partial \mu_p$. Returning to the definition of $\tilde{\underset{\sim}{\mu}}$,

$$\tilde{\mu}_k = \sum_{j=0}^{N-1} \mu_j \exp(-2\pi i jk/N),$$

we get

$$\frac{\partial \tilde{\mu}_\ell}{\partial \mu_p} = \sum_{j=0}^{N-1} \delta_{jp} \exp(-2\pi i j\ell/N) = \exp(-2\pi i \ell p/N)$$

$$= \exp\left(-i \frac{2\pi \ell}{NT} pT\right) = \exp\left[-i \frac{2\pi(\ell - N)}{NT} pT\right]$$

$$\Rightarrow \boxed{\frac{\partial \tilde{\mu}_\ell}{\partial \mu_p} = \exp(-i\omega_\ell t_p).}$$

2018-05-24

The derivatives of $\underset{\sim}{\varsigma}$, $\dot{\underset{\sim}{\varsigma}}$ w.r.t. $\underset{\sim}{\mu}$ are then

$$\frac{\partial \varsigma_{jk}}{\partial \mu_p} = \mathrm{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1}\frac{\partial \tilde{\mu}_\ell}{\partial \mu_p}\exp\left[i\omega_\ell(t_j-\eta_k)\right]\right\}$$

$$= \mathrm{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1}\exp\left[i\omega_\ell(t_j-t_p-\eta_k)\right]\right\}$$

and

$$\frac{\partial \dot{\varsigma}_{jk}}{\partial \mu_p} = \mathrm{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1} i\omega_\ell\frac{\partial \tilde{\mu}_\ell}{\partial \mu_p}\exp\left[i\omega_\ell(t_j-\eta_k)\right]\right\}$$

$$= \mathrm{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1} i\omega_\ell\exp\left[i\omega_\ell(t_j-t_p-\eta_k)\right]\right\}.$$

From the noise model, we also have

$$\frac{\partial \sigma_{jk}^2}{\partial \mu_p} = 2\sigma_\beta^2\,\varsigma_{jk}\frac{\partial \varsigma_{jk}}{\partial \mu_p} + 2\sigma_\tau^2\,\dot{\varsigma}_{jk}\frac{\partial \dot{\varsigma}_{jk}}{\partial \mu_p}$$

$$= 2\,\mathrm{Re}\left\{\frac{A_k}{N}\sum_{\ell=0}^{N-1}\left(\sigma_\beta^2\,\varsigma_{jk} + i\omega_\ell\,\sigma_\tau^2\,\dot{\varsigma}_{jk}\right)\right.$$

$$\left.\times\exp\left[i\omega_\ell(t_j-t_p-\eta_k)\right]\right\}.$$

2018-05-25

Putting it all together, we have

$$\frac{\partial C}{\partial \mu_p} = \frac{1}{2} \sum_{jk} \left(\Delta_{jk} \frac{\partial \sigma_{jk}^2}{\partial \mu_p} - 2 \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \xi_{jk}}{\partial \mu_p} \right)$$

$$= \mathrm{Re}\left\{ \frac{1}{N} \sum_{jkl} A_k \left[\Delta_{jk} \left(\sigma_\beta^2 \xi_{jk} + i\omega_l \sigma_\tau^2 \dot{\xi}_{jk} \right) - \frac{d_{jk}}{\sigma_{jk}^2} \right] \times \exp\left[i\omega_l (t_j - t_p - \eta_k) \right] \right\}$$

$$= \sum_k A_k \, \mathrm{Re}\left\{ \frac{1}{N} \sum_l \exp\left[+i\omega_l (t_p + \eta_k) \right] \times \sum_j \left[\sigma_\beta^2 (\underset{\sim}{\Delta} \circ \underset{\sim}{\xi})_{jk} - i\omega_l \sigma_\tau^2 (\underset{\sim}{\Delta} \circ \underset{\sim}{\dot{\xi}})_{jk} - (\underset{\sim}{d} \circ \underset{\sim}{\sigma}^{-2})_{jk} \right] \times \exp(-i\omega_l t_j) \right\}^*$$

$$= \sum_k A_k \, \mathrm{Re}\left\{ \frac{1}{N} \sum_l \exp\left[+i\omega_l (t_p + \eta_k) \right] \times \left[\sigma_\beta^2 \widetilde{(\underset{\sim}{\Delta} \circ \underset{\sim}{\xi})}_{lk} - i\omega_l \sigma_\tau^2 \widetilde{(\underset{\sim}{\Delta} \circ \underset{\sim}{\dot{\xi}})}_{lk} - \widetilde{(\underset{\sim}{d} \circ \underset{\sim}{\sigma}^{-2})}_{lk} \right]^* \right\},$$

where the notation $\tilde{\underset{\sim}{M}}$ denotes the columnwise discrete Fourier transform of a 2D matrix $\underset{\sim}{M}$.

Letting $\boldsymbol{M}_{\bullet k}$ denote the k-th column vector of $\boldsymbol{M}$, we may simplify further to get

$$\nabla_{\boldsymbol{\mu}} C = \sum_k A_k \, \mathcal{R}e\left\{ \mathcal{F}^{-1}\left[\exp(+i\boldsymbol{\omega}\,\eta_k)\, \tilde{\boldsymbol{P}}_{\bullet k} \right] \right\},$$

with

$$\tilde{\boldsymbol{P}}_{\bullet k} = \left[\sigma_\beta^2 \,\widehat{(\boldsymbol{\Delta} \circ \boldsymbol{\xi})}_{\bullet k} - i\boldsymbol{\omega}\, \sigma_\tau^2 \,\widehat{(\boldsymbol{\Delta} \circ \dot{\boldsymbol{\xi}})}_{\bullet k} - \widehat{(\boldsymbol{d} \circ \boldsymbol{\sigma}^{-2})}_{\bullet k} \right].$$

(Note that we have dropped the complex conjugation, since we are taking the real part anyway.)

2018-08-29

Moving on to $\partial C / \partial A_p$, we begin with the expression from p. 13,

$$\frac{\partial C}{\partial A_p} = \sum_{jk} \left(\frac{1}{2} \Delta_{jk} \frac{\partial \sigma_{jk}^2}{\partial A_p} + \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \xi_{jk}}{\partial A_p} \right).$$

From the expression for σ_{jk} on p. 12,

$$\frac{\partial \sigma_{jk}^2}{\partial A_p} = 2 \xi_{jk} \frac{\partial \xi_{jk}}{\partial A_p} \exp \theta_\beta + 2 \dot{\xi}_{jk} \frac{\partial \dot{\xi}_{jk}}{\partial A_p} \exp \theta_\tau .$$

From the expressions for $\underset{\sim}{\xi}$ and $\underset{\sim}{\dot{\xi}}$ on p. 9,

$$\frac{\partial \xi_{jk}}{\partial A_p} = \delta_{kp} \frac{\xi_{jk}}{A_k} , \qquad \frac{\partial \dot{\xi}_{jk}}{\partial A_p} = \delta_{kp} \frac{\dot{\xi}_{jk}}{A_k} .$$

Substituting,

$$\frac{\partial C}{\partial A_p} = \frac{1}{A_p} \sum_j \left[\Delta_{jp} \left(\xi_{jp}^2 \exp \theta_\beta + \dot{\xi}_{jp} \exp \theta_\tau \right) + \frac{d_{jp}}{\sigma_{jp}^2} \xi_{jp} \right]$$

$$\Rightarrow \boxed{\frac{\partial C}{\partial A_p} = \frac{1}{A_p} \sum_j \left[\Delta_{jp} \left(\sigma_{jp}^2 - \exp \theta_\alpha \right) + \frac{d_{jp}}{\sigma_{jp}^2} \xi_{jp} \right].}$$

2018-08-29

Finally, simplify the expression for $\partial C / \partial \eta_p$ on p. 13.

$$\frac{\partial C}{\partial \eta_p} = \sum_{jk} \left(\frac{1}{2} \Delta_{jk} \frac{\partial \sigma_{jk}^2}{\partial \eta_p} + \frac{d_{jk}}{\sigma_{jk}^2} \frac{\partial \xi_{jk}}{\partial \eta_p} \right)$$

Again, from the expression for σ_{jk}^2 on p. 13,

$$\frac{\partial \sigma_{jk}^2}{\partial \eta_p} = 2 \xi_{jk} \frac{\partial \xi_{jk}}{\partial \eta_p} \exp \theta_\beta + 2 \dot{\xi}_{jk} \frac{\partial \dot{\xi}_{jk}}{\partial \eta_p} \exp \theta_\tau .$$

From the expressions for $\underset{\sim}{\xi}$, $\underset{\sim}{\dot{\xi}}$ on p. 6,

$$\frac{\partial \xi_{jk}}{\partial \eta_p} = - \delta_{kp} \left[A_k \dot{r}_N (t_j - \eta_k) + \frac{A_k}{N} \sum_{\ell=0}^{N-1} i \omega_\ell \tilde{\mu}_\ell \exp(-i \omega_\ell \eta_k) \exp(i \omega_\ell t_j) \right]$$

$$= - \delta_{kp} \dot{\xi}_{jk} ,$$

$$\frac{\partial \dot{\xi}_{jk}}{\partial \eta_p} = - A_k \delta_{kp} \left[\ddot{r}_N (t_j - \eta_k) - \frac{1}{N} \sum_{\ell=0}^{M} \omega_\ell^2 \tilde{\mu}_\ell \exp(-i \omega_\ell \eta_k) \exp(i \omega_\ell t_j) \right].$$

$$= - \delta_{kp} \ddot{\xi}_{jk} .$$

Substituting gives

$$\boxed{\frac{\partial C}{\partial \eta_p} = - \sum_{j} \left[\Delta_{jp} \left(\xi_{jp} \dot{\xi}_{jp} \exp \theta_\beta + \dot{\xi}_{jp} \ddot{\xi}_{jp} \exp \theta_\tau \right) + \frac{d_{jp}}{\sigma_{jp}^2} \dot{\xi}_{jp} \right].}$$