[MAT3]-28-Wektory i wartości własne

Zad. 1.

wielomian charakterystyczny

$$\chi_A(x) = \det(A - xI_d)$$

Znaleźć wielomian charakterystyczny macierzy:

(a) $\begin{pmatrix} 2+i & 1 \\ 2 & 2-i \end{pmatrix}$,

$$x \cdot I_d = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

$$A - xI_d = \begin{bmatrix} 2+i-x & 1 \\ 2 & 2-i-x \end{bmatrix}$$

$$\chi_A(x) = \begin{vmatrix} 2+i-x & 1 \\ 2 & 2-i-x \end{vmatrix} = (2+i-x)(2-i-x) - 2$$

$$\chi_A(x) = (2-x)^2 - i^2 - 2 = x^2 - 4x + 4 + 1 - 2 = x^2 - 4x + 3$$

$$\chi_A(x) = (x-3)(x-1)$$

(b) $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

$$\chi_A(x) = \det(A - xI_d)$$

$$\chi_A(x) = \begin{vmatrix} 2-x & -1 & 2 \\ 5 & -3-x & 3 \\ -1 & 0 & -2-x \end{vmatrix} = (-1)\cdot(-1)^{3+1}\cdot\begin{vmatrix} -1 & 2 \\ -3-x & 3 \end{vmatrix} +$$

$$+ (-2-x)\cdot(-1)^{3+3}\cdot\begin{vmatrix} 2-x & -1 \\ 5 & -3-x \end{vmatrix} = (-1)\cdot(-3+6+2x) -$$

$$(x+2)\cdot\left[(2-x)(-3-x)+5\right] = -2x-3-(x+2)(x^2+x-6+5)$$

$$= -2x-3-(x+2)(x^2+x-1) = -x^3-x^2+x-2x^2-2x+2-2x-3$$

$$= -x^3-3x^2-3x-1 = -(x^3+3x^2+3x+1) = \underline{-(x+1)^3}$$

Zad. 2.

Znaleźć wartości własne i wektory własne podanych liniowych przekształceń przestrzeni liniowych. Dla każdej wartości własnej λ znaleźć podprzestrzeń N_λ wektorów własnych.

(a) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F([x_1, x_2]) = [x_1, x_1 + x_2]$,

baza standardowa: $[1,0], [0,1]$

$F([1,0]) = [1,1]$

$F([0,1]) = [0,1]$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\chi_A(x) = \det(A - xI_d) = \begin{vmatrix} 1-x & 0 \\ 1 & 1-x \end{vmatrix} = (x-1)^2$$

$x = 1 \to$ wartość własna

wektory własne: $(A - xI_d)\cdot v = 0$

$$\begin{bmatrix} 1-1 & 0 \\ 1 & 1-1 \end{bmatrix} \cdot v = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot v = 0 \qquad v = [x_1, x_2]$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 = 0 \quad x_2 \in \mathbb{R} \qquad v = [0, x_2]$$

$v = x_2 \cdot [0,1]$

N_λ:

dla $\lambda=1$: $\quad v=[0,1]$

baza $N_1=\mathcal{L}([0,1])$

(b) $F:\mathbb{R}^3\to\mathbb{R}^3$, $F([x_1,x_2,x_3])=[x_1-x_3,2x_2,x_1+x_3]$.

baza standardowa: $[1,0,0],[0,1,0],[0,0,1]$

$F([1,0,0])=[1,0,1]$

$F([0,1,0])=[0,2,0]$

$F([0,0,1])=[-1,0,1]$

$$A=\begin{bmatrix}1&0&-1\\0&2&0\\1&0&1\end{bmatrix}$$

$$\chi_A(x)=\det(A-xI_d)=\begin{vmatrix}1-x&0&-1\\0&2-x&0\\1&0&1-x\end{vmatrix}=(2-x)\cdot(-1)^{2+2}\cdot\begin{vmatrix}1-x&-1\\1&1-x\end{vmatrix}$$

$$=(2-x)\cdot\left[(x-1)^2+1\right]=(2-x)\underset{\Delta<0}{(x^2-2x+2)}$$

$x=2\to$ wartość własna

$(A-xI_d)\cdot v=0 \qquad v=[x_1,x_2,x_3]$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\begin{bmatrix}-1&0&-1\\0&0&0\\1&0&-1\end{bmatrix}\begin{bmatrix}-x_1-x_3\\0\\x_1-x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases} \quad \Big| -$$

$2x_1 = 0 \Rightarrow x_1 = 0 \qquad x_3 = 0 \qquad x_2 = t \qquad , t \in \mathbb{R}$

$$v = [x_1, x_2, x_3] = t[0,1,0]$$

$$N_2 = \mathcal{L}([0,1,0])$$

(c) $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F([x_1, x_2]) = [-x_2, x_1]$,

baza standardowa: $[1,0], [0,1]$

$F([1,0]) = [0,1]$

$F([0,1]) = [-1,0]$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\chi_A(\lambda) = \det(A - \lambda I_d) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

$\lambda_1 = i \quad \wedge \quad \lambda_2 = -i$

$(A - \lambda I_d) \cdot v = 0$

$(A - i I_d) \cdot v = 0 \qquad v = [x_1, x_2]$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} -ix_1 - x_2 \\ x_1 - ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{r_2 - ir_1} \begin{array}{ccc} & t & \\ \left[\begin{array}{c} -i \\ 0 \end{array}\right. & \begin{array}{c} -1 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \end{array}\right] \end{array}$$

$x_2 = t$

$-ix_1 - t = 0 \implies x_1 = ti \quad , t \in \mathbb{C}$

$$v = [x_1, x_2] = [it, t] = t[i, 1]$$

$$N_i = \mathcal{L}([i, 1])$$

$$(A + iI_d) \cdot u = 0$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} ix_1 - x_2 \\ x_1 + ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow{r_2 + ir_1} \begin{array}{ccc} & t & \\ \left[\begin{array}{c} i \\ 0 \end{array}\right. & \begin{array}{c} -1 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \end{array}\right] \end{array}$$

$x_2 = t$

$ix_1 - t = 0 \Rightarrow x_1 = -it$

$$u = [x_1, x_2] = [-it, t] = t[-i, 1]$$

$$N_{-i} = \mathcal{L}([-i, 1])$$

(d) $F : \mathbb{C}^3 \to \mathbb{C}^3$, $F([x_1, x_2, x_3]) = [x_1 - x_3, 2x_2, x_1 + x_3]$.

baza standardowa: $[1,0,0], [0,1,0], [0,0,1]$

$F([1,0,0]) = [1,0,1]$

$F([0,1,0]) = [0,2,0]$

$F([0,0,1]) = [-1,0,1]$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\chi_A = \det(A - \lambda I_d) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\cdot(-1)^{2+2} \cdot \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \cdot [(1-\lambda)^2 + 1] = (2-\lambda)\cdot(\lambda^2 - 2\lambda + 2)$$

$\Delta = 4 - 8 = -4 \qquad \sqrt{\Delta} = 2i$

$$\lambda_1 = \frac{2-2i}{2} = 1 - i \quad \wedge \quad \lambda_2 = 1 + i \quad \wedge \quad \lambda_3 = 2$$

$$(A - \lambda I_d) \cdot v = 0$$

dla $\lambda_1 = 1-i$

$$\begin{bmatrix} i & 0 & -1 \\ 0 & 1+i & 0 \\ 1 & 0 & i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} i & 0 & -1 & 0 \\ 0 & 1+i & 0 & 0 \\ 1 & 0 & i & 0 \end{array}\right] \xrightarrow{r_3 + i r_1} \begin{array}{c} \begin{array}{cccc} & & t & \end{array} \\ \left[\begin{array}{ccc|c} i & 0 & -1 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$(1+i)x_2 = 0 \implies x_2 = 0$

$i x_1 - t = 0 \implies x_1 = -it$

$x_3 = t \qquad , \; t \in \mathbb{C}$

$$v = [-it, 0, t] = t[-i, 0, 1]$$

$$N_{1-i} = \mathcal{L}[-i, 0, 1]$$

dla $\lambda_2 = 1+i$:

$$\begin{bmatrix} -i & 0 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & 1-i & 0 & 0 \\ 1 & 0 & -i & 0 \end{array}\right] \xrightarrow{r_3 - i r_1} \begin{array}{c} \begin{array}{cccc} & & t & \end{array} \\ \left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_3 = t \qquad , \; t \in \mathbb{C}$

$(1-i)x_2 = 0 \Rightarrow x_2 = 0$

$-ix_1 - t = 0 \Rightarrow x_1 = it$

$v = [it, 0, t] = t[i, 0, 1]$

$N_{i,1} = \mathcal{L}([i, 0, 1])$

dla $\lambda = 2$ tak jak w podpunkcie b)

Zad. 3.

Niech $F\colon V \to V$ będzie przekształceniem liniowym o podanej macierzy $A = M_{\mathcal{B}}^{\mathcal{B}}(F)$ w bazie standardowej $\mathcal{B}$ przestrzeni $V(\mathbb{C})$. Sprawdzić, czy istnieje baza $\mathcal{C}$ odpowiedniej przestrzeni wektorowej, w której $M_{\mathcal{C}}^{\mathcal{C}}(F)$ jest macierzą diagonalną. Jeśli tak, znaleźć macierze zmiany bazy: $M_{\mathcal{C}}^{\mathcal{B}}(Id_V)$ oraz $M_{\mathcal{B}}^{\mathcal{C}}(Id_V)$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I_d) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

$$\lambda = -1 \quad \vee \quad \lambda = 2$$

$$(A - \lambda I_d) \cdot v = 0$$

dla $\lambda = -1$:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} & t \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + t = 0 \Rightarrow x_1 = -t$

$$v = [-t, t] = t[-1, 1]$$

$$N_{-1} = \mathcal{L}([-1,1])$$

dla $\lambda = 2$:

$$\begin{array}{cc} & t \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ \cancel{1} & \cancel{-2} & \cancel{0} \end{array}\right]$$

$x_2 = t$

$-x_1 + 2t = 0 \Rightarrow x_1 = 2t$

$u = t[2,1]$ $\qquad N_2 = \mathcal{L}([2,1])$

baza $C = \{v_1, v_2\}$ - baza złożona z wektorów własnych

istnieje: $C = \{[2,1], [-1,1]\}$

$$M_C^C(F) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_2 - \frac{1}{2}r_1} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \xrightarrow{r_1 + \frac{1}{3}r_2} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\dim N_{-1} + \dim N_2 = 1 + 1 = 2 = n$$

$\Longrightarrow F$ jest diagonalna

$M_C^B(I_dV) = ?$ $\qquad$ $M_B^C(I_dV) = ?$

$$M_C^B(I_dV) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$M_C^C(I_dV) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M_B^C(I_dV) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - \frac{1}{2}r_1} \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{r_1 + \frac{2}{3}\cdot r_2} \left[\begin{array}{cc|cc} 2 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[r_2 \cdot \frac{2}{3}]{r_1 \cdot \frac{1}{2}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$M_B^C(I_dV) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I_d) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \cdot (1-\lambda)(2-\lambda) = (1-\lambda)^2 (2-\lambda)$$

$$\lambda_1 = 1 \quad \wedge \quad \lambda_2 = 2$$

$$(A - \lambda I_d) \cdot v = 0$$

dla $\lambda = 1$

$$\begin{array}{c} t \quad\quad\quad\quad\quad \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$x_2 = 0$

$x_3 = 0$

$x_1 = t \quad , t \in \mathbb{C}$

$v = [t, 0, 0] = t[1, 0, 0]$

$N_1 = \mathcal{L}([1, 0, 0])$

dla $\lambda = 2$

$$\begin{array}{c} \quad\quad\quad\quad t \quad\quad \\ \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$-x_2 = 0 \Rightarrow x_2 = 0$

$-x_1 = 0 \Rightarrow x_1 = 0$

$x_3 = t \quad , t \in \mathbb{C}$

$u = [0, 0, t] = t[0, 0, 1]$

$N_2 = \mathcal{L}([0, 0, 1])$

$$C = \{[1, 0, 0], [0, 0, 1]\}$$

$n = 3$

$$\dim N_1 + \dim N_2 = 2 \neq n$$