The Ivy Cube - Rough Draft

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Abstract

This paper introduces a notation for the Ivy Cube. It is then shown that the Ivy Cube has 29,160 possible configurations and has a God's number of 12.

1 Introduction

The Ivy Cube is a three dimensional puzzle similar to the Rubik's cube or Pyraminx puzzle. The shapes of the center pieces of the puzzle resemble poison ivy leaves which are moved in pairs of three. The Ivy Cube was created by mechanical engineer and puzzle designer Eitan Cher by modifying a Pyraminx puzzle [2, 3]. Compared to the Rubik's cube the Ivy cube is easy to solve. An explanation for why will be given in the last section of this paper after going through the necessary concepts from group theory and graph theory.

For permutation puzzles there are these concepts of God's algorithm and God's number. God's algorithm is an algorithm such that using it always produces the optimal solution on the puzzle. The idea is that an omniscient being can always perform the optimal solution on a permutation puzzle to solve it. God's number is a related concept. It is the minimum number of moves needed to solve any configuration of a given permutation puzzle. If a permutation puzzle, like the Ivy Cube, can always be solved in 12 moves or less then its God's number is 12.

2 Background

2.1 Permutation Groups

Given n elements we can rearrange them and create a one-to-one function that defines their permutation. For example the three letters in the word ***hat*** can be jumbled to ***tha*** and then a one-to-one function can be created to define this permutation. Mathematically this can be done by representing ***h*** as 1, ***a*** as 2, ***t*** as 3 and saying that there is a function f such that $f(1) = 2, f(2) = 3, f(3) = 1$. In other words 1 goes to 2, 2 goes to 3, and 3 goes to 1.

More concisely we could write this permutation as $(1, 2, 3)$. This notation is known as cyclic notation. If this permutation is applied twice we define a new permutation

$$(1, 2, 3)(1, 2, 3) = (1, 3, 2)$$

This can be read as "1 goes to 2 which in the next permutation goes to 3 so in the end 1 goes to 3." If the permutation $(1, 3, 2)$ were applied to the letters in ***hat*** then it would become ***ath***.

Composing (1,2,3) three times sends every element to itself.

$$(1, 2, 3)(1, 2, 3)(1, 2, 3) = (1, 2, 3)(1, 3, 2) = ()$$

So 1 goes back to 1, 2 to 2, 3 to 3, which means in the end nothing is permuted. This is represented by () and means nothing moves. The number of times a permutation must be applied to itself to produce () is known as its order. The permutation $(1, 2, 3)$ has order 3. Note that this is distinct from the order of a group which is the size of a set. Both meanings of the word are used in this paper.

It must be noted that in this notation we read compositions from right to left. This means that if permutation $(2, 4, 1, 3)$ is applied before $(4, 3, 2, 1)$ then this composition should be written as

$$(4, 3, 2, 1)(2, 4, 1, 3) = (1, 2, 3, 4)$$

2.1.1 The Symmetric Group

The group formed by the permutations on n elements is called the symmetric group S_n. For the set {1,2,3} we can form the symmetric group S_3 by taking as each element of the group a permutation of the set. The size of the group formed by S_3 is 3!. In general the number of elements in S_n is $n!$ [1]. The six elements of this group are $\{(), (1, 2, 3), (1, 2), (1, 3), (1, 3, 2), (2, 3)\}$.

To be clear about this notation notice that when a number is omitted from a permutation that means it is unaffected. For example the composition

$$(1, 2, 3)(1, 2) = (1, 3)$$

has 2 go to 2 and so it is left out of the result.

2.1.2 Even and Odd Permutations

A permutation can be defined as being even or odd. The highly technical definition for even and odd will not be necessary for this paper. The only thing to note will be that a cyclic permutation is even if and only if the length of its cycle is odd [4]. The length of a cyclic permutation is the number of elements it permutes. For example $(1, 2, 3)$ is an even permutation since it has an odd length. A cyclic permutation is odd if it is not even. So permutations like $(1, 3)$ and $(2, 4, 1, 3)$ are both odd.

2.1.3 The Alternating Group

If we consider only the even elements of S_n we form a subgroup known as the Alternating group A_n which has an order that is half the order of S_n [1]. Taking only the even permutations of S_3 produces $A_3 = \{(), (1, 2, 3), (1, 3, 2)\}$. Composing two permutations of opposite parity produces an odd permutation and two of the same parity produces an even function [1]. In other words applying two odd or two even permutations produces an even permutation. Composing an odd and even permutation produces an odd permutation.

It has been shown earlier in this paper that $(1, 2, 3)$ produces both $(1, 3, 2)$ and () so we can say that $(1, 2, 3)$ generates A_3. Using the notion of generators A_3 can be written as $\langle(1, 2, 3)\rangle$. In general if a set of permutations $\{a_1, a_2, \ldots, a_n\}$ generates a group then that group can be expressed as $\langle a_1, a_2, \ldots, a_n \rangle$.

It should be clear that if the generating set of a permutation group has only even elements then every element of that group is even.

2.2 Cayley Graphs

A Cayley graphs is a useful type of graph for representing permutation groups. As an example the set of two permutations $\{(1,2),(2,3)\}$ can be used to generate a permutation group

$$\langle (1,2),(2,3) \rangle = \{(1,2),(2,3),(1,2,3),(1,3),(1,3,2),()\}$$

This group is S_3 and can be represented by a Cayley graph. In figure 1 the nodes of the graph represent the elements of S_3. A directed edge between two nodes can be thought of as the permutation applied to the first node that results in the node that the arrow points two. In figure 1 the node $(1,2)$ has a directed edge that points to $(1,3,2)$. This directed edge represents some permutation in the group S_3 that when applied to $(1,2)$ produces $(1,3,2)$.

A path is a finite sequence of edges that connect a finite sequence of nodes. Every node in figure 1 has to only follow a path 3 edges or less to reach any other node. This property is known as the graph's diameter. The shortest path between two nodes is a path that minimizes the number of edges. The longest path from the set made up of all the minimizing paths between all the nodes is known as the graph's diameter. The diameter of a Cayley graph created from the permutation group of a permutation puzzle is that puzzle's God's number [4].

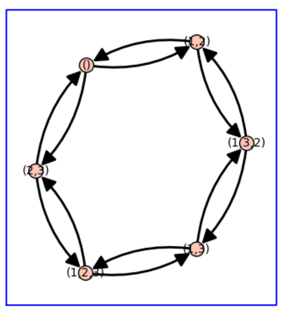


Figure 1: Cayley Graph of S_3

3 Notation and Moves

The Ivy cube puzzle is a cube that uses the standard color pattern for cubical Rubik's like puzzles. This particular cube has 4 corner pieces and 6 center pieces. A corner piece cannot exchange places with another corner piece and can only spin in place. The center pieces can exchange places with other centers. Clearly the centers and corners cannot exchange places.

To define a move it is useful to consider any move as performing two permutations. In this paper the permutation δ acts on the 4 corners of the cube and the permutation σ acts on the 6 centers.

To keep consistency with the names of the faces of the Ivy Cube it is necessary to not allow any rotations of the cube. Hold the cube fixed such that a corner piece is on the top left hand side of the user. The side facing the user is denoted as F. The top side of the cube is T. The right and left faces from the perspective of the user of the cube are denoted as L and R. The side the cube is resting on, which is opposite of T, is denoted as D. The side opposite of F is B.

The center pieces of the cube are referenced by which side of the cube they are on. The four corner pieces are each part of three different faces and are denoted by the three faces they share. For example the corner piece that shares the T, F, and L faces is denoted as T-F-L. See figure 2.

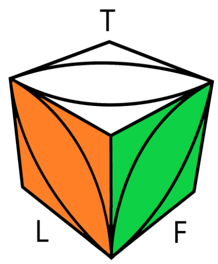


Figure 2: The F, L, and T faces

To define the permutations using cycle notation it is necessary to associate with each face of each piece a number.

For the center pieces number their faces as such

$$F \to 3 \quad L \to 6 \quad T \to 4 \quad D \to 2 \quad R \to 1 \quad B \to 5$$

Then the permutation σ is defined as

$$\sigma_0 = ()$$
$$\sigma_1 = (4,3,6)$$
$$\sigma_2 = (4,5,1)$$
$$\sigma_3 = (1,2,3)$$
$$\sigma_4 = (2,5,6)$$

With $\sigma_i \cdot \sigma_i = \sigma_i^{-1}$ for $i \in \{0,1,2,3,4\}$

For the corner pieces the order of the faces that defines the piece is also the order of the numbering of their faces, which is

$$FLT \to 1,2,3, \quad TBR \to 4,5,6, \quad RDF \to 7,8,9, \quad DBL \to 10,11,12$$

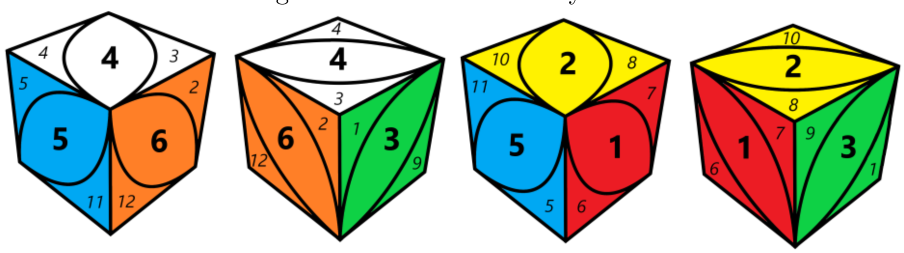
The permutation δ is defined as

$$\delta_0 = ()$$

$$\delta_1 = (1, 2, 3)$$
$$\delta_2 = (4, 5, 6)$$
$$\delta_3 = (7, 8, 9)$$
$$\delta_4 = (10, 11, 12)$$

With $\delta_i \cdot \delta_i = \delta_i^{-1}$ for $i \in \{0, 1, 2, 3, 4\}$

Figure 3: The numbered Ivy Cube



The set of basic moves M are defined as

$$M_0 = (\delta_0, \sigma_0)$$
$$M_1 = (\delta_1, \sigma_1)$$
$$M_2 = (\delta_2, \sigma_2)$$
$$M_3 = (\delta_3, \sigma_3)$$
$$M_4 = (\delta_4, \sigma_4)$$

The operation $\cdot$ is defined as

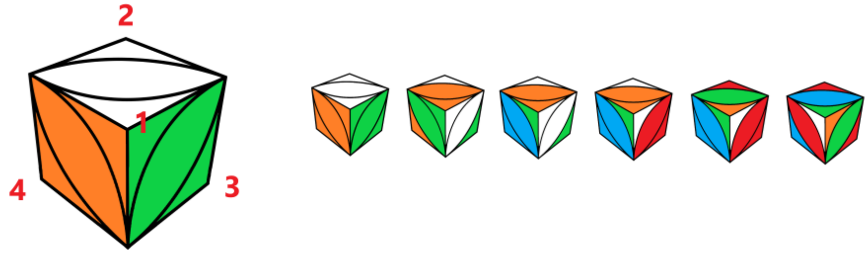
$$M_i \cdot M_j = (\delta_i \cdot \delta_j, \sigma_i \cdot \sigma_j) \qquad i, j \in \{0, 1, 2, 3, 4\}$$

And interestingly
$$M_i \cdot M_i = M_i^{-1}$$

In layman's terms a move is a 120° turn of one of the corners in a clockwise direction with the corner upright and pointing toward the user. The inverse of a move is a 120° turn in a counter-clockwise direction. There are only four corners and no two corners can be spun at the same time. Turning a corner also exchanges the centers of the three connecting faces.

Figure 4 numbers the corner pieces according to the i associated with M_i and demonstrates the effect of applying the sequence of moves $M_1, M_4, M_3^{-1}, M_2, M_1$ to the unscrambled cube. Remember though that in this paper the moves should be read from right to left. So this sequence should be written as $M_1 M_2 M_3^{-1} M_4 M_1$

Figure 4: The corners numbered and a sequence of moves



4 The Ivy Group

The arrangements of the Ivy Cube do not themselves form a group. Instead a group is formed once the faces of the pieces have been associated with a number and the basic movements of the puzzle cube are encoded into a permutation. The set of finite sequences from M with the operation $\cdot$ generate a group I, the Ivy Group.

Theorem 1. $\langle M \rangle = I$ *is a group.*

Proof.

The two permutations δ, σ that make up M and by extension I are using an associative operation. Therefore I is associative. It is also the case that the composition of a finite sequence with a finite sequence produces a finite sequence which is by definition an element of I. Therefore I is closed.

For $m \in I$, $m = (\delta_i \cdot \delta_j \cdots \delta_k, \sigma_i \cdot \sigma_j \cdots \sigma_k)$, with $i, \cdots, j, k \in \{0, 1, 2, 3, 4\}$

$$m \cdot M_0 = (\delta_i \cdot \delta_j \cdots \delta_k \cdot \delta_0, \sigma_i \cdot \sigma_j \cdots \sigma_k \cdot \sigma_0) = (\delta_i \cdot \delta_j \cdots \delta_k, \sigma_i \cdot \sigma_j \cdots \sigma_k) = m$$

Then I has the identity M_0. Taking the same m

$$m \cdot (\delta_k^{-1} \cdots \delta_j^{-1} \cdot \delta_i^{-1}, \sigma_k^{-1} \cdots \sigma_i^{-1} \cdot \sigma_j^{-1})$$

$$(\delta_i \cdot \delta_j \cdots \delta_k \delta_k^{-1} \cdots \delta_j^{-1} \cdot \delta_i^{-1}, \sigma_i \cdot \sigma_j \cdots \sigma_k \sigma_k^{-1} \cdots \sigma_i^{-1} \cdot \sigma_j^{-1})$$

$$(\delta_0 \cdot \delta_0 \cdots \delta_0, \sigma_0 \cdot \sigma_0 \cdots \sigma_0) = (\delta_0, \sigma_0) = M_0$$

So for any m there exists an m^{-1}.

Since I is associative, closed under $\cdot$, has an identity, and every element has an inverse it is therefore a group.

□

Each element of I is a sequence of moves from M. Starting with the cube in the solved state associate with each element $m \in I$ the resulting configuration of the cube when m acts on the solved cube. Statements made about the configurations of the cube are really statements about the group formed by the permutations defined by the possible sequence of moves on the cube.

5 The Possible Arrangements

5.1 An Upper Limit

By a cursory glance of the Ivy cube it is easy to calculate an upper limit to the number of configurations. The 4 corner pieces cannot be exchanged and can only be spun in place. Each corner has only 3 possible configurations. So the number of possible arrangements of all the corners is $3^4 = 81$. The number of possible configurations of the 6 centers is a permutation of 6 elements which is $6! = 720$. The number of maximum conceivable arrangements is therefore

$$3^4 \cdot 6! = 81 \cdot 720 = 58320$$

5.2 The Centers form A_6

For each move M ignore for a moment δ and only consider σ. This permutation is a subset of S_6 since it defines a permutation of six elements. The order of S_6 is 720. Half of the elements of S_6 are even permutations and half are odd [1].

Each element of σ is of odd length which implies that every element is an even permutation. So σ is a proper subset of S_6. It does not follow automatically that σ will form A_6 but it does follow that $\sigma \subset A_6$.

The set {(1,2,3), (1,2,4), (1,2,6), (1,2,5)} generates A_6 [4]. If σ can generate each element from the set that generates A_6 then σ will generate A_6.

Theorem 2. *The elements of σ form A_6.*

Proof.

$$(1,2,3) = \sigma_3$$
$$(1,2,4) = (6,3,4)(1,2,3)(4,3,6) = \sigma_1^{-1} \cdot \sigma_3 \cdot \sigma_1$$
$$(1,2,6) = (4,3,6)(123)(6,3,4) = \sigma_1 \cdot \sigma_3 \cdot \sigma_1^{-1}$$
$$(1,2,5) = (6,5,2)(436)(2,5,6)(1,2,3)(6,5,2)(6,3,4)(2,5,6) = \sigma_4^{-1} \cdot \sigma_1 \cdot \sigma_4 \cdot \sigma_3 \cdot \sigma_4^{-1} \cdot \sigma_1^{-1} \cdot \sigma_4$$

Therefore σ generates A_6. As stated earlier σ is a subset of A_6. The fact that it also generates A_6 implies that $\langle\sigma\rangle = A_6$. □

The earlier estimate of 58320 arrangements was too high. The permutation group $\langle\sigma\rangle$ has the same order as A_6, which is half of the order of S_6. Therefore the new upper limit is

$$3^4 \cdot \frac{6!}{2} = 81 \cdot 360 = 29160$$

This upper limit assumes that the elements of the group generated by δ can be attained independent of the group generated by σ. It also assumes that the elements of δ are independent of each other. Since each move M has a δ and a σ it is not clear whether $\langle M\rangle = (\langle\delta\rangle, \langle\sigma\rangle)$.

5.3 Independent Corners

Theorem 3. $\langle M\rangle = (\langle\delta\rangle, \langle\sigma\rangle)$

Proof.

The elements of δ are disjoint so they are independent of each other. For $m \in \langle M\rangle$ let $m = (p, q)$ where $q \in \langle\sigma\rangle$ and $p \in \langle\delta\rangle$

$$M_2^{-1}M_1^{-1}M_2M_3^{-1}M_4M_3M_4^{-1}M_3M_4^{-1}M_3^{-1}M_4M_4M_3^{-1}M_4^{-1}M_3 \cdot m = (p \cdot \delta_1^{-1}, q)$$

$$M_1^{-1}M_2^{-1}M_1M_4^{-1}M_3M_4M_3^{-1}M_4M_3^{-1}M_4^{-1}M_3M_3M_4^{-1}M_3^{-1}M_4 \cdot m = (p \cdot \delta_2^{-1}, q)$$

$$M_2^{-1}M_3^{-1}M_2M_4^{-1}M_1M_4M_1^{-1}M_4M_1^{-1}M_4^{-1}M_1M_1M_4^{-1}M_1^{-1}M_4 \cdot m = (p \cdot \delta_3^{-1}, q)$$

$$M_3^{-1}M_4^{-1}M_3M_2^{-1}M_1M_2M_1^{-1}M_2M_1^{-1}M_2^{-1}M_1M_1M_2^{-1}M_1^{-1}M_2 \cdot m = (p \cdot \delta_4^{-1}, q)$$

Therefore all the elements of $\langle\delta\rangle$ can be produced for any fixed element of $\langle\sigma\rangle$. □

It is important to point out that the elements of σ are of order 3. Applying one of the sequences of moves given in the previous proof twice to m will result in $(p \cdot \delta_i, q)$. On the physical Ivy cube only one sequence of moves is necessary to demonstrate since the cube can be rotated in such a way that the names of the corners are swapped. To avoid confusion in the proof the cube was assumed fixed from the outset of this paper.

5.4 There are 29160 Possible Arrangements

Theorem 4. *There are 29160 Possible Arrangements*

Proof.

Every element from $\langle\delta\rangle$ can be attained for any given element from $\langle\sigma\rangle$. So to count the order of $\langle M\rangle$ multiply the orders of $\langle\delta\rangle$ and $\langle\sigma\rangle$. It has been shown that $\langle\sigma\rangle$ has the same order as A_6 which is $\frac{6!}{2}$. The order of the elements from δ are each 3. Since each element of δ is independent from each other the order of $\langle\delta\rangle$ is 3^4. Therefore the total number of elements of $\langle M\rangle$ is

$$3^4 \cdot \frac{6!}{2} = 29160$$

□

The size of the Ivy Group was calculated with arguments about the structure of the group. It is also possible to calculate the group size with the computer algebra system SageMath. Here are the commands for calculating the group's size. In this code every face of the pieces of the cube is given a different number, from 1 through 18, instead of the 1 through 6 and 1 through 12 used earlier.

```
I = PermutationGroup(['(1,2,3)(13,14,15)', '(4,5,6)(15,18,17)','(7,8,9)(13,17,16)',
    '(10,11,12)(18,14,16)'])
I.order()
```

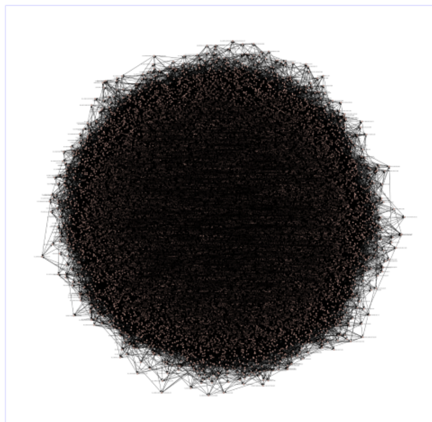
This returns the size of the group formed, 29160.

6 God's Number is 12

The Ivy Group is small enough to calculate God's number directly on a desktop computer. The code to calculate God's number follows.

```
I = PermutationGroup(['(1,2,3)(13,14,15)', '(4,5,6)(15,18,17)', '(7,8,9)(13,17,16)',
    '(10,11,12)(18,14,16)'])
C = I.cayley_graph()
C.diameter()
```

Figure 5: Cayley Graph of the Ivy Group



First the Ivy Group is created, then a Cayley graph is created from the Ivy Group. Then the diameter of the Cayley graph is calculated. The result is 12. SageMath was also used to produce figure 5, which is a Cayley graph of I. It is difficult to tell what is going on in the graph. Every node is an element of the group. Each node has degree eight, meaning it has eight edges touching it. The eight edges come from the four moves of M and their inverses. The identity move M_0 is not represented since it just takes a node to itself. A sequence of edges can be thought of as moves which go from one configuration to another. This, again, is a path. A path of length 12 or less can get from any element in the group to any other.

Calculating God's number for permutation puzzles is difficult when the size of the permutation group formed is too large. For example the Rubik's cube has more than 43 Quintilian configurations [6]. Finding God's number for the Rubik's cube took decades of work and was finally calculated in 2010 through a combined effort of hobbyists using some clever arguments about symmetry, specific cases that required a minimum of 20 moves, and Google's vast computing power [5].

For the Rubik's cube there are two common metrics as to what counts as a turn, the half turn metric and quarter turn metric. The half turn metric counts any turn of the face as a turn. The quarter metric counts turns in 90 degree increments. The God's number of 20 for the cube was first calculated using the half turn metric. The team that had originally found God's number for the half turn metric found it using the quarter turn metric in 2014 and it was found to be 26 [5]. For the Ivy Cube there is no distinction since any move is a 120 degree turn. Turning clockwise twice is the same as turning counter-clockwise once. Any configuration of the Ivy Cube can be solved in 12 moves or less under this definition of a move.

7 A rough metric of difficulty

Learning to solve the Ivy cube does not require a complicated algorithm. Still, the question of difficulty remains and God's number can act as a rough guide to a puzzle's difficulty. There is not a one to one correspondence between a puzzle's God's number and its difficulty (using different metrics on the same puzzle produces different numbers). For puzzles of similar type and using the same metric God's number can act as a rough indicator of difficulty. The very concept of difficulty has not been defined. Instead it is possible for a person to say "I think this puzzle is more, less, or as difficult as that puzzle." The Ivy cube is considered easier than the Rubik's cube to solve simply because most users think it so. Comparisons of difficulty become simple though when we can assign numbers that correlate to difficulty in some way.

When first trying to understand the Ivy cube I borrowed a concept called the variegation of the Rubik's cube [6]. Looking at the average variegation of the Rubik's cube is meant to measure the disorder of the puzzle. The technique used to calculate variegation on the Rubik's cube would not work on the Ivy cube. Instead to calculate disorder I took the number of distinct colors appearing on each side of the cube and averaged them. The minimum is 0, which only occurs in the solved state. The maximum is 3, which only occurs if each of the six sides each have three distinct colors. This measure, let's call it nariegation, may also act as a rough measure of difficulty. The benefit of nariegation is that it can be applied to any Rubik's like puzzle. Average nariegation should be higher for a more difficult difficult puzzle when compared to a puzzle with the same number of faces.

The metrics mentioned so far are related to the number of possible combinations a Rubik's like puzzle has. Of course they do not directly correlate to difficulty. It is easy to think up a puzzle that has more possible arrangements than the Rubik's cube but is trivial to solve. What may hint at difficultly even more so than possible arrangements, nariegation, or a puzzle's God's number is the number of pieces of the same type that can exchange place multiplied together.

This metric is again a rough measure of difficulty but it is a measure that is not trivially modified. Let us call this quantity a puzzle's morpho. The morpho of the Rubik's cube is 48. In the standard cube the centers do not exchange places so they do not count. There are left two other types of pieces: edges and corners. There are 12 edges and 4 corners. An edge can switch with any other edge. Not every combination of edges is possible but that is not the point [6]. So assign the cube a morpho of 48. The morpho of the Ivy cube is 6. There are four corners but they do not move so only the centers count. The morpho of the Pyraminx is also 6. Taking this idea to other puzzles not mentioned before, the morpho of the 2 by 2 magic cube is 8. The morpho of the Megaminx is 600 (it has 20 corner pieces, and 30 edge pieces). Considering only whether a morpho is greater, lesser, or equal we arrive at an ordering that is pretty good in terms of difficulty.

Measures of difficulty that are easy to use and calculate allow puzzlers to more quickly understand a new puzzle on a scale with other puzzles. The mistake that is easy to make when looking at these numbers is thinking in absolute terms. A morpho of 600 does not mean that the Megaminx is 12.5 times harder than the Rubik's cube. It only means that it is greater. The morpho scale should be thought of as analogous to the Mohs scale of mineral hardness. In the Mohs scale the hardness of a mineral is not identified to some absolute standard but relative to the other minerals on the scale.

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