THE THREE-BODY PROBLEM AND SPECIAL SOLUTIONS

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ABSTRACT

The three-body problem, a classical problem in celestial mechanics, studies the behavior and motion of three gravitationally-interacting bodies based on their initial positions and velocities. We first discuss the well-known closed-form solution of the two-body problem, then provide an overview of the equations of motion for the three-body problem. We introduce the restricted three-body problem, and finally, we give a mathematical explanation for three selected families of periodic solutions to the three-body problem.

CONTENTS

1 INTRODUCTION

The three-body problem was first introduced by Isaac Newton upon attempting to study the effects of the Sun's gravitational force on the Moon's orbit around Earth. He solved the two-body problem for the orbit of two bodies around each other when acted upon by gravity, but realized that adding a third body greatly complicated the problem. Since then, many of the greatest minds, including Euler, Lagrange, and Poincaré have studied this problem in great detail.

1.1 Two-Body Problem

First, let us review the two-body problem, in which two bodies act upon each other with a gravitational force. This problem was first solved by Newton in his *Principia Mathematica* (1687) and is also known as the Kepler Problem.

EQUATIONS OF MOTION Let the two bodies have masses m_1, m_2 and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, respectively. Using Newton's Law of Gravitation, we find that the force acting on body 1 from body 2 is given by

$$\mathbf{F} = \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \cdot \mathbf{u} = -Gm_1m_2 \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}.$$

The force acting on body 2 from body 1 has the same magnitude but opposite direction, as according to Newton's Third Law. Then, Newton's Second Law yields the system of differential equations

$$\ddot{\mathbf{r}}_i = -Gm_j \cdot \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

where i, j are the two permutations of $1, 2$. This corresponds to a system of 6 scalar second-order ordinary differential equations (ODEs), which in turn correspond to 12 first-order ODEs [1].

CENTER OF MASS Since the two-body system is not acted on by external forces, its center of mass moves with constant velocity. Setting a frame that moves with center of mass reduces its motion to zero. Thus it is convenient to switch to finding

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \text{ and } \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

the position vectors of the center of mass and the relative distance. Letting $M = m_1 + m_2$, the second-order differential equations then become

$$M\ddot{\mathbf{R}} = 0$$
$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}$$

The first equation can be reduced to $\mathbf{R} = 0$ by setting $\dot{\mathbf{R}}(0) = 0$, i.e. setting a frame that moves with the center of mass as its origin.

CONSERVATION OF ANGULAR MOMENTUM To solve the second equation, we use the conservation of angular momentum $\mathbf{L}$ of the system due to a lack of external torque:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L} = \frac{\mathrm{d}}{\mathrm{d}t}(M\mathbf{r} \times \dot{\mathbf{r}}) = M\dot{\mathbf{r}} \times \dot{\mathbf{r}} + M\mathbf{r} \times \ddot{\mathbf{r}} = 0.$$

1 An ordinary differential equation is a differential equation that involves only one variable. Its order is the degree of the highest derivative. For example, an ODE relating a variable to its second derivative would be a second-order ODE.

The vector between the two bodies moves in a plane and thus so do the bodies. Taking the xy plane to be the plane of motion of $\mathbf{r}$, we can use polar coordinates. Through analytic techniques [2], we can find that the motion is of the form

$$r = \frac{L^2}{GM^3} \cdot \frac{1}{1 + e\cos\theta}$$

where $e = \sqrt{1 + \frac{2EL^2}{G^2M^5}}$ is the eccentricity and E is the total energy of the system. When $E < 0$, the bodies orbit in an elliptical shape. Otherwise, when $E = 0$ exactly, the bodies travel in a parabolic shape, and when $E > 0$, they travel in a hyperbolic shape. We can then find the trajectories for each body as

$$\mathbf{r}_i = \mathbf{R} + \frac{m_j}{M}\mathbf{r}.$$

Thus, we can provide a closed-form expression to describe the motion of two bodies acting on each other with gravity. A similar expression holds for other two-particle systems with an inverse-squared force relations such as Coulomb's Law.

1.2 General Three-Body Problem

Now we can study the equations of motion for the three-body problem, which is significantly more difficult.

EQUATIONS OF MOTION Once again, let the three bodies have masses m_1, m_2, m_3, and positions $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively. Each body in our system is acted upon by the other two by a gravitational force, yielding, in vector form,

$$\mathbf{F}_i = -Gm_im_j \cdot \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - Gm_im_k \cdot \frac{\mathbf{r}_i - \mathbf{r}_k}{|\mathbf{r}_i - \mathbf{r}_k|^3}$$

where i, j, k are the three ordered permutations of 1, 2, 3. Using Newton's Second Law again, we get a system of three second-order differential equations:

$$\ddot{\mathbf{r}}_i = -Gm_j \cdot \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - Gm_k \cdot \frac{\mathbf{r}_i - \mathbf{r}_k}{|\mathbf{r}_i - \mathbf{r}_k|^3}.$$

This is equivalent to a system of 18 first order ODEs. The conservation momentum and energy can reduce this to a system of 6 first order ODEs, which still does not have a general solution. The equations of motions can be written more symmetrically as $\mathbf{s}_i = \mathbf{r}_j - \mathbf{r}_k$ such that $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 = 0$ and

$$\ddot{\mathbf{s}}_i = -\frac{G(m_j + m_k)\mathbf{s}_i}{|\mathbf{s}_i|^3} + Gm_i\left(\frac{\mathbf{s}_j}{|\mathbf{s}_j|^3} + \frac{\mathbf{s}_k}{|\mathbf{s}_k|^3}\right) = -\frac{GM\mathbf{s}_i}{|\mathbf{s}_i|^3} + Gm_i\sum_{i=1}^{3}\frac{\mathbf{s}_i}{|\mathbf{s}_i|^3}.$$

The first term of the right-hand side is identical to the equation of motion for the two-body problem; it is the second term that makes the problem incredible complex and impossible to solve due to it involving all three relative position vectors.

STABILITY Except for a set of special solutions, most initial conditions of the motion of the bodies in a three-body system are chaotic. This means that a small perturbation to the initial condition can cause a drastically different path of motion for the bodies. Figure 1 displays this. The two images have the same frame and number of steps, but their initial conditions differ by $< 1\%$, resulting in increasingly different paths as they move.

2 This is identical to the derivation of Kepler's First Law.

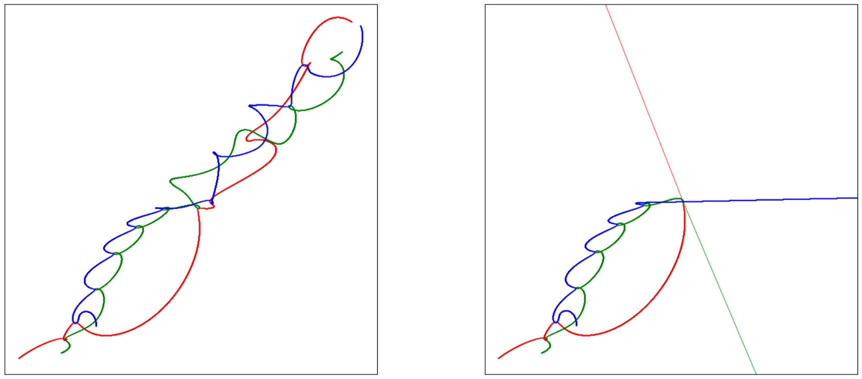
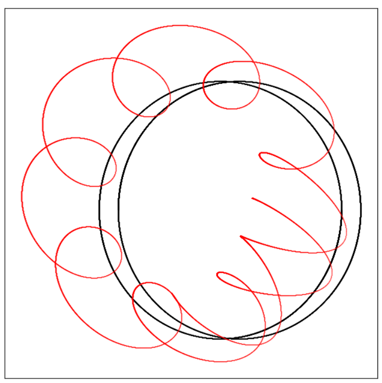
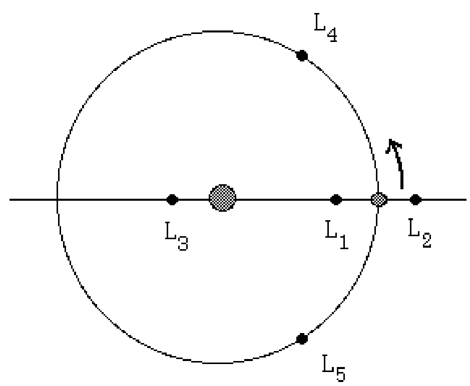


Figure 1: Trajectories of similar initial conditions



(a) Trajectory of the small mass



(b) The five Lagrange points

Figure 2: Restricted Three-Body Problem

1.3 Restricted Three-Body Problem

In this special case of the three-body problem, we take m_3 to be sufficiently small so that the force exerted by it on m_1 and m_2 is negligible to their acceleration. Then, m_1 and m_2 interact as they would in a two-body system (in elliptical orbits around each other) while we can determine the position of m_3. An example of such a trajectory can be seen in Figure 2, where the black paths are the massive bodies and the red path is the trajectory of the smaller body.

LAGRANGE POINTS In the rotating frame of the two massive bodies, there are 5 points that are stationary in reference to the massive bodies. In particular, at these points, the gravitational forces balance out the centrifugal force. These are called the Lagrange points and are denoted L_i with $1 \leq i \leq 5$. Three of these are collinear with the two bodies and were found by Euler. The remaining two rest at equilateral triangles to the two bodies and were found by Lagrange. The positions of the five lagrange points are illustrated in Figure 2.

2 SOLUTIONS

While there does not exist a general solution for all initial conditions of the three-body problem, certain configurations yield periodic orbits. We will study three of these; Euler's solutions, Lagrange's solutions, and the Figure-8 solution.

2.1 Euler's Solution

If our three bodies are collinear, then ideally, their trajectories will be periodic ellipses.

EQUATIONS OF MOTION Since $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are collinear, $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ as defined earlier are parallel or antiparallel. Without loss of generality, let m_3 be between m_1 and m_2. Then $\mathbf{s}_1$ and $\mathbf{s}_2$ are parallel, and $\mathbf{s}_3$ is antiparallel to them. We can then substitute

$$\mathbf{s}_2 = \lambda \mathbf{s}_1 \text{ and } \mathbf{s}_3 = -(1+\lambda)\mathbf{s}_1$$

into our previously obtained second-order differential equations to get, after some algebra,

$$\ddot{\mathbf{s}}_1 = -\frac{GM\mathbf{s}_1}{|\mathbf{s}_1|^3} \cdot \frac{m_1 + m_3(1+\lambda)^{-2}}{m_1 + m_3(1+\lambda)}.$$

Note that this is a two-body equation of motion, so thus, the bodies move along ellipses. Additionally, through manipulating the equations of motion, we get a fifth degree polynomial that restricts the value of λ:

$$\frac{m_3 + m_1(1+\lambda)}{m_2 - m_1\lambda} = \frac{m_3 + m_1(1+\lambda)^{-2}}{m_2 - m_1\lambda^{-2}}$$
$$\Rightarrow (m_2+m_3)\lambda^5 + (3m_2+2m_3)\lambda^4 + (3m_2+m_3)\lambda^3 - (3m_1+m_3)\lambda^2 - (3m_1+2m_3)\lambda - (m_1+m_2) = 0$$

When $\lambda = 0$, this is negative, but when $\lambda \to \infty$, this is positive. Combined with Descartes' Rule of Signs, we see that this polynomial has exactly one positive real root. Thus, for any set of masses, there is exactly three families of collinear solutions; one with each mass in the middle.

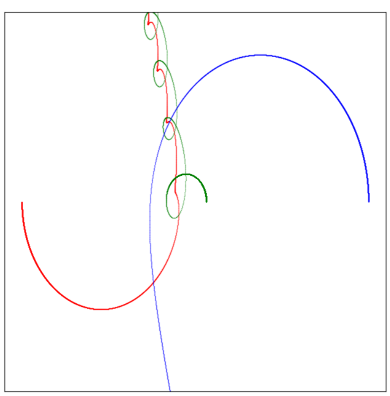
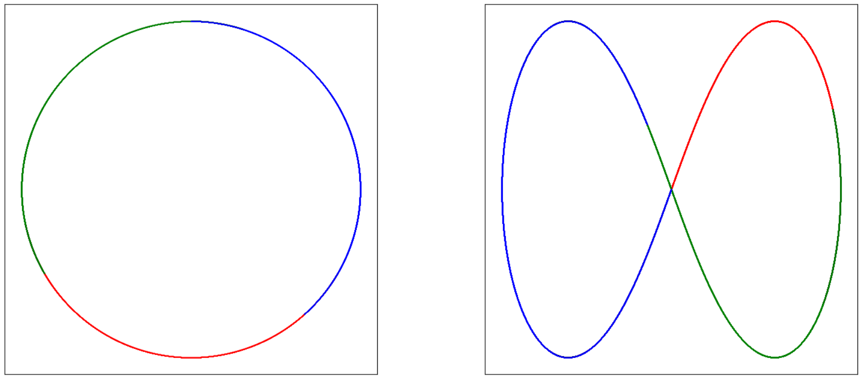


Figure 3: Euler's collinear solution

INSTABILITY However, these solutions are extremely unstable, i.e. any small perturbation will cause a deviation from the elliptical orbit, so they virtually impossible to realize in nature. In fact, this instability can be seen in Figure 3 as the simulated trajectory deviates from the elliptical path after half of a period due to an imperfect and approximated initial condition.

2.2 Lagrange's Solution

If our three bodies are in an equilateral triangle and have equal masses, then the bodies will move along confocal ellipses.



(a) Circular case of Lagrange's solution

(b) Figure 8 solution

Figure 4: Lagrange's and the Figure 8 Solutions

EQUATIONS OF MOTION Since the three bodies are always at vertices of an equilateral triangle, the forces applied by m_2 and m_3 on m_1 are equivalent to a single gravitational force from a more massive body at their midpoint. In particular, $|\mathbf{s}_1| = |\mathbf{s}_2| = |\mathbf{s}_3|$, so

$$\ddot{\mathbf{s}}_i = -G(M - 3m_i) \cdot \frac{\mathbf{s}_i}{|\mathbf{s}_i|^3}.$$

Then, if $\mathbf{r}$ is a solution to the two-body problem, Lagrange's solution can be written as $\mathbf{s}_i = s_{i_0}\mathbf{r}$. The special case for this solution is if $m_1 = m_2 = m_3$ and

$$\dot{r}_i^2 = \ddot{r}_i|\mathbf{r}_i - \mathbf{R}|$$

in which case the bodies satisfy uniform circular motion and will move along the same circle centered at the center of mass. The circular solution is illustrated in Figure 4.

2.3 Figure 8 Solution

We study another configuration in which the initial positions of the three bodies are collinear. Additionally, the masses are once again equal. Let the period of this solution be T. Then, $\mathbf{r}_2(t) = \mathbf{r}_1(t - T/3)$ and $\mathbf{r}_3(t) = \mathbf{r}_1(t - 2T/3)$. The initial configurations for the figure 8 are

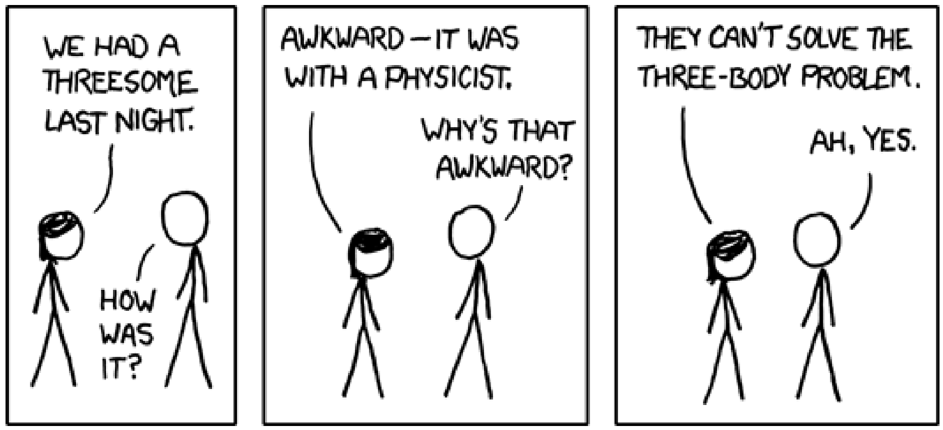
$$\mathbf{r}_1(0) = -\mathbf{r}_2(0), \mathbf{r}_3(0) = 0$$
$$2\dot{\mathbf{r}}_1(0) = 2\dot{\mathbf{r}}_2(0) = -\dot{\mathbf{r}}_3(0).$$

The configuration begins with the bodies collinear, which we can call an Euler configuration. It returns to an Euler configuration with the middle body cycling every $T/6$. Between two Euler configurations, the triangle formed by the masses is isosceles. It is one of the rare stable periodic solutions to the problem, meaning that small perturbations in its initial state will not cause a large divergence in its orbit. The Figure 8 solution is illustrated in Figure 4.

3 CONCLUSION

We studied the statement and special solutions to the three-body problem. We looked at three periodic solutions with different properties: Euler's solution, Lagrange's solution, and the Figure 8 solution. The three-body problem can be applied to real-world celestial motion, notably the Earth-Moon-Sun system. In addition to astronomy, the mathematical equations behind the gravitational force can be applied to other inverse-squared particle relations, such as the helium atom, in which the helium nucleus interacts with two electrons through Coulomb's Law. The three-body problem is a special case of the n-body problem, and is an age-old challenge that has been deemed unsolvable.

NOTE Every graph of the trajectories of bodies in this paper was created using Python and Matplotlib. My code for this simulation can be found here.



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