1.2 Homework # 2 (Due Friday, Oct. 21)

1. In a set of n i.i.d. random times $\left\{T^{(1)}, T^{(2)}, \cdots, T^{(n)}\right\}$ each and everyone has probability density function $f_T(t)$, $t \geq 0$, with $f_T(0) = 0$, show that

$$T_* = \min\left\{T^{(1)}, T^{(2)}, \cdots, T^{(n)}\right\}$$

approaches to zero when n tends to infinity. Find an appropriate ν such that $\hat{T}_* = n^{\nu} T^*$ has a meaningful probability density function in the limit of $n \to \infty$.

2. Consider a population consisting of identical and independent individual organisms, each with an exponentially distributed time for giving "birth", with rate λ, and going "death", with rate μ.

(a) Now when the population has exactly n individuals, what is the probability distribution for the waiting time to the next birth? What is the probability distribution for the waiting time to the next death? What is the probability distribution for the waiting time to the next birth or death event?

(b) Let $p_n(t)$ be the probability of having exactly n individuals in the population at time t:

$$\sum_{n=0}^{\infty} p_n(t) = 1.$$

What is the system of differential equations that $p_n(t)$ should satisfy?

(c) The mean population at time t is defined as

$$\langle n \rangle(t) = \sum_{n=0}^{\infty} n p_n(t).$$

Based on the system of differential equations you obtained in (b), show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n \rangle = \Big(\lambda - \mu\Big)\langle n \rangle.$$

3. Consider the following nonlinear chemical/biochemical reaction system which is

similar to the one you studied in Homework # 1:

$$A + X \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} 2X, \quad X \underset{\delta}{\overset{\lambda}{\rightleftharpoons}} B. \tag{1.8}$$

We again keep the convention that X is *dynamic chemical species* and A, B are chemical species with *maintained*, constant concentrations. Use n_X for the copy numbers of X, and lower cases a, b for the concentrations of A, B. We assume that the reaction (1.8) is in a small system with volume V. Thus, n_X are random variables taking non-negative integer values, and $n_X(t)$ is a Markov process.

(a) Draw a diagram which illustrates the states and the transitions between the states of this stochastic chemical reaction system. Label the transitions with transition rates. Note α, β, γ, and δ are *rate constants*, not the rates for transition probability.

(b) Let $p_n(t) = \Pr\{n_X(t) = n\}$ be the probability of the reaction system (1.8) having n number of X molecules at time t. Following the *Chemical Master Equation*, write the system of differential equations for $p_n(t)$ that describes the *stochastic dynamics* of the chemcial/biochemical reaction system.

(c) Can this system be written in term of a matrix form? If yes, what is the size of the matrix? Please give the matrix.

(d) Find the *stationary probability distribution* to the chemical master equation you obtained in (a). Give a rough sketch of the distribution for several different parameter sets. Any significant changes? [Hint: study the ODE for the same system (1.8). Is there a bifurcation?]

(e) If a and b, the concentrations of A and B, satisfy

$$\frac{a\ \alpha\ \lambda}{b\ \delta\ \beta} = 1,$$

show that the stationary distribution is Poissonian. What is the chemical/biochemical significance of this particular condition?