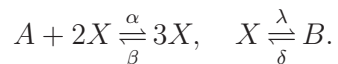


## Homework # 3 (Due Friday, Oct. 28)

1. We shall again consider the same nonlinear (bio)chemical reaction system as in Homework # 2:



But this time, the reaction system is in a very small system with volume  $V$ , and we shall consider the dynamics in terms of the number of  $X$  molecule at time  $t$ . The  $n_X(t)$  is integer-valued, continuous-time Markov process, defined on  $\mathbb{Z}$ . We keep the convention that  $A$  and  $B$  are chemical species with *maintained*, constant concentrations  $a$  and  $b$ . In these terminologies, the concentrations of  $X$  is also stochastic processes with discrete values  $\frac{n_X(t)}{V}$ .

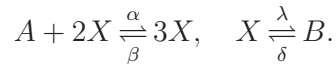
(a) Write the Chemical Master Equation (CME) for the reaction system: i.e., a system of ordinary differential equations for the probability  $p_m(t) = \Pr \{n_X(t) = m\}$ .

(b) Write a short computer code to carry out a Monte Carlo simulations of the stochastic dynamics. The method is widely known as the Gillespie algorithm. Let volume  $V = 100$ , using the set of parameters:  $\alpha = 3c^{-2}t^{-1}$  where  $c$  denotes the unit for “concentration” and  $t$  denotes the unit for “time”,  $\beta = 0.6c^{-2}t^{-1}$ ,  $\lambda = 2.95t^{-1}$ ,  $\delta = 0.25t^{-1}$ ;  $a = 1c$  and  $b = 1.5c$ , run a long simulation and collect the histogram for the time the system taking values  $N_X(t) = n$ . Note that Gillespie algorithm has non-uniform time steps.

(c) Repeat the (c) again using the parameter set  $\alpha = 3c^{-2}t^{-1}$ ,  $\beta = 0.6c^{-2}t^{-1}$ ,  $\lambda = 2.95t^{-1}$ ,  $\delta = 0.25t^{-1}$ ;  $a = 1c$  and  $b = 59c$ .

(d) Alternatively, you can obtain the stationary distributions for  $p_n(t)$  from the CME you obtained in (c). Compare the analytical results with the simulations.

2. We shall again consider the nonlinear chemical reaction system:



We keep the convention that  $X$  and  $Y$  are *dynamic chemical species* and  $A$  and  $B$  are chemical species with *maintained*, constant concentrations. Use lower cases  $x, y$  and  $a, b$  for the concentrations of the corresponding chemical species.

(a) Following the *Law of Mass Action*, write the ordinary differential equation (ODE) for the chemical reaction system.

(b) Show that if the concentrations of  $A$  and  $B$  are at their chemical equilibrium, i.e.,

$$\frac{a \times \alpha \times \lambda}{b \times \delta \times \beta} = 1,$$

then the chemical dynamics following the equation in (a) has only a single fixed point.

In fact, the fixed point is a chemical equilibrium.

(c) Even though there are six parameters in the model, show that if we use the non-dimensionalized concentration  $\hat{x} = (\lambda x / \delta b)$  and non-dimensionalized time  $\hat{t} = \lambda t$  as new variables, then the ODE from (a) can be simplified into

$$\frac{d\hat{x}}{d\hat{t}} = -\theta \hat{x}^3 + \mu \hat{x}^2 - \hat{x} + 1 = f(x, \mu, \theta), \quad (1.14)$$

where the two positive parameters are

$$\mu = \frac{\alpha \delta a b}{\lambda^2} \quad \text{and} \quad \theta = \frac{\beta (\delta b)^2}{\lambda^3}.$$

(d) Show that the curve in  $(\theta, \mu)$  plane, which divides it into regions where there are 1 or 3 positive steady states is given parametrically by

$$\theta = \frac{\xi - 2}{\xi^3}, \quad \mu = \frac{2\xi - 3}{\xi^2}.$$

Show that the two curves meet at a cusp, where  $d\theta/d\xi = d\mu/d\xi = 0$  at a critical  $\xi^*$ .

Sketch the curve in  $(\theta, \mu)$  plane.

3. This is an open-ended problem.

(a) Repeat the computations in Problem 1 with increasing number of molecules and increasing volume  $V$  such that  $n_X(t)/V$  becomes smoother and smoother. Convince yourself in the limit of  $V \rightarrow \infty$ , your simulation results agree with the predictions from the ODE in Problem 2.

(b) Try to obtain the stationary distribution of the problem with finite  $V$ , then let  $V \rightarrow \infty$ . In other words, taking the limit  $V \rightarrow \infty$  after  $t \rightarrow \infty$ . Do you think this will give you the same answer as  $t \rightarrow \infty$  after  $V \rightarrow \infty$ ? If not, what do you expect to see? Try to use analytical mathematics or computations as tools for this question.