1. Let G be a graph with adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Use the fact that the ij-th entry of A^k gives the number of walks (defined in problem 2, below) of length k from vertex i to vertex j in G to argue whether or not G is connected. Drawing a picture of the graph is not sufficient.

I wrote following MATLAB codes for this question:

```
--------------------------------------------------------------------------------
% @Author: Baihan Lin, @Date: Oct 2016

% initiate adjacency matrix A.
A = [0 0 1 0 0 0 0 0 1;
    0 0 0 1 1 0 0 0 0;
    1 0 0 0 0 0 1 0 0;
    0 1 0 0 1 0 1 0 0;
    0 1 0 1 0 1 0 0 0;
    0 0 0 0 1 0 0 1 1;
    0 0 1 1 0 0 0 0 0;
    0 0 0 0 0 1 0 0 0;
    1 0 0 0 0 1 0 0 0;];

% find minimum steps for every vertex can connect to each other (complete).
n = whenComplete(A, 1);

n  % minimum steps to reach a complete graph (if n = -1, not connected).
An = A^n % the final complete adjacency graph in n steps.

--------------------------------------------------------------------------------
% to find out in how many steps can each vertex be connected to another one.
function n = whenComplete(A, n)
% A is the input adjacency matrix
% n is the starting steps (set to -1 if it is not connected at all)

if n ~= -1
    % only in connected graphs, we find minimum steps towards complete.
    if prod(sum(A)) == 0 % if one column of 0, not connected, set n to -1.
        n = -1;
    else
        if prod(prod(A^n)) == 0 % not complete graph yet within n step.
            n = whenComplete(A, n+1);
        end
    end
end
end
```

Here is the output on MATLAB console:

```
n =

     6


An =

    21    15     1    11    10    24    17     1     1
    15    41    10    43    43    35    27     9    11
     1    10    21    25    17     9     3     7    17
    11    43    25    65    46    42    19    10    17
    10    43    17    46    73    21    34    25    33
    24    35     9    42    21    54    17     2     3
    17    27     3    19    34    17    29     8     9
     1     9     7    10    25     2     8    12    17
     1    11    17    17    33     3     9    17    28
```

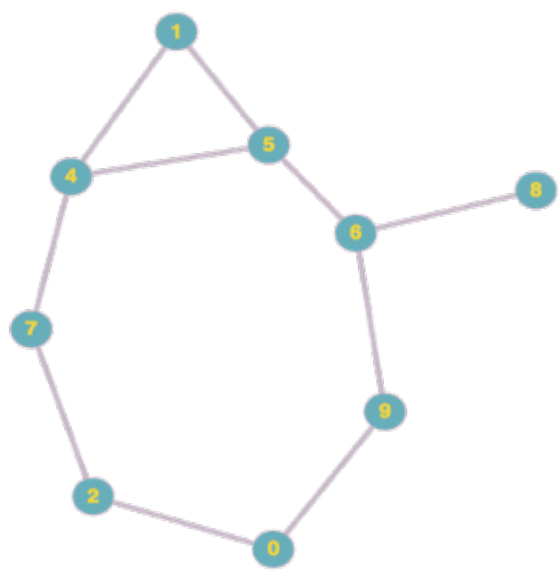


Figure 1. Graph based on adjacency matrix

As shown in Figure 1, based on the adjacency matrix, this graph is connected.

But we can further prove this by definition of this adjacency matrix: A^k gives the number of walks of length k from vertex i to vertex j in graph G.

In another word, if G is connected, there must be a number k of walks to satisfy the path from any vertex i to any vertex j in G.

If so, it means from every vertex, it is possible to travel to another vertex in certain steps, which suggests the connectivity of the graph.

Therefore, I wrote a program to find the minimum steps it takes, to let every vertex in this graph to travel towards any other vertices (more details in comments).

I found that when n = 6, every vertex has at least one possible ways to travel to any other vertex in 6 steps. Since A^6 has no zero entries, we know there is a path (of length 6) between every pair of vertices.

Thus, this proves that the graph is connected.

2. Let a and b be positive integers. We say that a divides b iff $b = ak$ for some integer k.

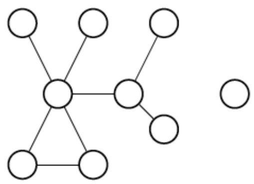
   So 2 divides 10, and 3 does not divide 8.

   Let $V = \{2, 3, 4, \ldots, 22\} = \{j \in \mathbb{Z} : 2 \leq j \leq 22\}$.

   Define a graph H with V as its vertex set and edge set E defined by $(v_1, v_2) \in E$ iff $v_1 \neq v_2$ and v_1 divides v_2 or v_2 divides v_1. So $(2, 6)$ is an edge in H; $(3, 4)$ is not.

   Recall that the edge $(2, 6)$ is the same as the edge $(6, 2)$.

   (a) The *degree sequence* of a graph G is the sequence of the degrees of all vertices in G. For example, the graph below has degree sequence $5, 3, 2, 2, 1, 1, 1, 1, 0$ (in decreasing order).

   Give the degree sequence of H in decreasing order.

Based on the definition of E, we can enumerate out all combinations of (v_1, v_2):

For vertex 2, we have 10 edges:
  (2, 4), (2, 6), (2, 8), (2, 10), (2, 12), (2, 14), (2, 16), (2, 18), (2, 20), (2, 22)
  in another word, vertex 2 has degree 10

For vertex 3, we have 6 edges:
  (3, 6), (3, 9), (3, 12), (3, 15), (3, 18), (3, 21)
  in another word, vertex 3 has degree 6

For vertex 4, we have 5 edges:
  (2, 4), (4, 8), (4, 12), (4, 16), (4, 20)
  in another word, vertex 4 has degree 5

For vertex 5, we have 3 edges:
  (5, 10), (5, 15), (5, 20)
  in another word, vertex 5 has degree 3

For vertex 6, we have 4 edges:
  (2, 6), (3, 6), (6, 12), (6, 18)
  in another word, vertex 6 has degree 4

For vertex 7, we have 2 edges:
  (7, 14), (7, 21)
  in another word, vertex 7 has degree 2

For vertex 8, we have 3 edges:
  (2, 8), (4, 8), (8, 16)
  in another word, vertex 8 has degree 3

For vertex 9, we have 2 edges:
  (3, 9), (9, 18)
  in another word, vertex 9 has degree 2

For vertex 10, we have 2 edges:
(5, 10), (10, 20)
in another word, vertex 10 has degree 2

For vertex 11, we have 1 edges:
(11, 22)
in another word, vertex 11 has degree 1

For vertex 12, we have 4 edges:
(2, 12), (3, 12), (4, 12), (6, 12)
in another word, vertex 12 has degree 4

For vertex 13, we have 0 edges:
in another word, vertex 13 has degree 0

For vertex 14, we have 2 edges:
(2, 14), (7, 14)
in another word, vertex 14 has degree 2

For vertex 15, we have 2 edges:
(3, 15), (5, 15)
in another word, vertex 15 has degree 2

For vertex 16, we have 3 edges:
(2, 16), (4, 16), (8, 16)
in another word, vertex 16 has degree 3

For vertex 17, we have 0 edges:
in another word, vertex 17 has degree 0

For vertex 18, we have 4 edges:
(2, 18), (3, 18), (6, 18), (9, 18)
in another word, vertex 18 has degree 4

For vertex 19, we have 0 edges:
in another word, vertex 19 has degree 0

For vertex 20, we have 4 edges:
(2, 20), (4, 20), (5, 20), (10, 20)
in another word, vertex 20 has degree 4

For vertex 21, we have 3 edges:
(3, 21), (7, 21)
in another word, vertex 21 has degree 3

For vertex 22, we have 2 edges:
(2, 22), (11, 22)
in another word, vertex 22 has degree 2

As these vertices are vertices of graph H, the degree sequence of H is:
10, 6, 5, 4, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1, 0, 0, 0 (in decreasing order)

(b) A *path* from vertex u to vertex v in a graph G is an alternating sequence of vertices and edges

$$u = u_0, e_1, u_1, e_2, \ldots, u_{n-1}, e_n, u_n = v$$

where $e_i = (u_{i-1}, u_i)$ and none of the vertices are repeated.

(A *walk* is the same as a path except that we allow repetition of edges and vertices.)

The *length* of a path is the number of edges in the path.

The *distance* from vertex u to vertex v in a graph G is the length of the shortest path from u to v.

A graph G is connected if, for all pairs of vertices u and v in G, there is a path from u to v.

Remove the vertices that have degree zero from H (along with their incident edges), to get the *subgraph* H'. Is H' connected? Support your claim fully.

What two vertices in H' are farthest apart? Support your claim fully.

From (a), we find that vertices 13, 17 and 19 have degree 0. Thus we remove them from H, H' consists of vertices 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, and 22.

The adjacency matrix for H' is

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 14 & 15 & 16 & 18 & 20 & 21 & 22 \end{array} \\
\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 14 \\ 15 \\ 16 \\ 18 \\ 20 \\ 21 \\ 22 \end{array} &
\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

So as question 2, I wrote following MATLAB codes to approach this question:

```
------------------------------------------------------------------------------
% @Author: Baihan Lin, @Date: Oct 2016

Hp = [0 0    1    0    1    0    1    0    1    0    1    1    0    1    1    1    0    1;
     0    0    0    0    1    0    0    1    0    0    1    0    1    0    1    0    1    0;
     1    0    0    0    0    0    1    0    0    0    1    0    0    1    0    1    0    0;
     0    0    0    0    0    0    0    0    1    0    0    0    1    0    0    1    0    0;
     1    1    0    0    0    0    0    0    0    0    1    0    0    0    1    0    0    0;
     0    0    0    0    0    0    0    0    0    0    0    1    0    0    0    0    1    0;
     1    0    1    0    0    0    0    0    0    0    0    0    0    1    0    0    0    0;
     0    1    0    0    0    0    0    0    0    0    0    0    0    0    1    0    0    0;
     0    0    0    1    0    0    0    0    0    0    0    0    0    0    0    1    0    0;
     0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    1;
     1    1    1    0    1    0    0    0    0    0    0    0    0    0    0    0    0    0;
     1    0    0    0    0    1    0    0    0    0    0    0    0    0    0    0    0    0;
     0    1    0    1    0    0    0    0    0    0    0    0    0    0    0    0    0    0;
     1    0    1    0    0    0    1    0    0    0    0    0    0    0    0    0    0    0;
     1    1    0    0    1    0    0    1    0    0    0    0    0    0    0    0    0    0;
     1    0    1    1    0    0    0    0    1    0    0    0    0    0    0    0    0    0;
     0    1    0    0    0    1    0    0    0    0    0    0    0    0    0    0    0    0;
     1    0    0    0    0    0    0    0    0    1    0    0    0    0    0    0    0    0;];

% find minimum steps for every vertex can connect to each other (complete).
nH = whenComplete(Hp, 1);

nH  % minimum steps to reach a complete graph (if n = -1, not connected).
Hn = Hp^nH % the final complete adjacency graph in n steps.

------------------------------------------------------------------------------
% to find out in how many steps can each vertex be connected to another one.
function n = whenComplete(A, n)
% A is the input adjacency matrix
% n is the starting steps (set to -1 if it is not connected at all)

if n ~= -1
    % only in connected graphs, we find minimum steps towards complete.
    if prod(sum(A)) == 0 % if one column of 0, not connected, set n to -1.
        n = -1;
    else
        if prod(prod(A^n)) == 0 % not complete graph yet within n step.
            n = whenComplete(A, n+1);
        end
    end
end
end
------------------------------------------------------------------------------
```

Here is the output on MATLAB console:

```
nH =

     6


Hn =

    2281    1148    1515     586    1173     220     981     450     767     141    1192     419     274     981     967    1028     204     413
    1127     768     694     282     664     140     416     298     330      63     615     193     157     416     575     432     138     187
    1478     691    1065     375     807     119     711     287     574      82     837     332     201     711     673     740     150     327
     384     205     261     136     181      38     152      65     144      22     177      62      45     152     145     180      24      62
    1168     668     829     265     798      98     582     338     476      56     811     314     245     582     708     619     210     299
     219     141     120      56      98      37      65      43      47      15      87      21      15      65      78      66      12      21
     963     417     729     227     580      64     551     200     434      48     660     276     169     550     520     585     127     271
     447     299     291      94     337      43     200     165     164      22     326     114     113     200     301     216     104     104
     307     132     219     101     151      24     138      54     133      18     158      59      51     138     121     166      27      58
     141      64      83      36      56      15      49      22      35      12      55      15       9      49      41      50       6      15
    1172     616     858     261     809      86     660     326     528      54     897     357     266     660     757     714     220     342
     419     195     349      94     316      21     278     115     232      15     359     165     109     278     298     301      92     156
     241     148     180      54     223      14     147     108     137       8     241      94     102     147     215     179      84      85
     963     417     729     227     580      64     550     200     434      48     660     276     169     551     520     585     127     271
     962     578     695     211     709      78     522     302     427      41     759     297     236     522     673     564     204     282
     824     357     646     214     537      48     512     188     430      36     632     269     188     512     499     580     132     263
     202     138     153      34     209      12     126     104     109       6     219      91      88     126     203     141      86      78
     413     189     344      94     301      21     273     105     227      15     344     156     100     273     283     295      79     153
```

If H' is connected, there must be a number k of walks to satisfy the path from any vertex i to any vertex j in H'. If so, it means from every vertex, it is possible to travel to another vertex in certain steps, which suggests the connectivity of the graph.

Therefore, I wrote a program to find the minimum steps it takes, to let every vertex in this graph to travel towards any other vertices (more details in comments).

I found that when n = 6, every vertex has at least one possible ways to travel to any other vertex in 6 steps. Since H'^6 has no zero entries, we know there is a path (of length 6) between every pair of vertices in this graph H'.

Thus, this proves that the graph H' is connected.

For the next question, I wrote another MATLAB function to find the farthest vertices.

```
% find minimum distance between every two vertices.
Hd = minD(Hp, nH)
```

Here in part (b), the code also shares part (a)'s other MATLAB code as inputs of this function:

```
--------------------------------------------------------------------------------
% to minimum distance between every two vertices (min path length).

function d = minD(H, n)
% H is the input adjacency matrix
% n is the minimum steps to make the graph complete.

fin = [H ~= 0]; % finished connecting pairs
d = 1*[H ~= 0]; % the vertex pairs with distance of 1.
for k = 2:n
    d = d + k.*((H^k ~= 0) - fin); % add
    fin = [H^k ~= 0]; % update the already connected pairs
end
--------------------------------------------------------------------------------
```

Here is the output on MATLAB console:

```
--------------------------------------------------------------------------------

 Hn =

     2     2     1     2     1     3     1     2     1     3     1     2     3     1     1     1     3     2
     2     2     2     3     1     3     3     1     3     4     1     3     2     3     1     3     2     3
     1     2     2     2     2     3     1     3     2     3     1     2     3     1     2     1     3     2
     2     3     2     2     3     4     3     3     1     4     3     3     2     3     3     1     4     3
     1     1     2     3     2     3     2     2     2     3     1     2     2     2     1     2     2     2
     3     3     3     4     3     4     3     3     3     5     3     3     4     3     3     3     3     4
     1     3     1     3     2     3     2     3     2     3     2     2     4     1     2     2     4     2
     2     1     3     3     2     3     3     2     3     4     2     3     2     3     1     3     2     3
     2     3     2     1     3     4     3     4     2     4     3     3     2     3     3     1     4     3
     3     4     3     4     3     5     3     4     3     4     3     4     5     3     3     3     5     3
     1     1     1     3     1     3     2     2     2     3     2     2     2     2     2     2     2     2
     2     3     2     3     2     3     2     3     2     4     2     3     4     2     2     2     3     3
     3     2     3     2     2     4     4     2     2     5     2     4     3     4     2     2     3     4
     1     3     1     3     2     3     1     3     2     3     2     2     4     2     2     2     4     2
     1     1     2     3     1     3     2     1     2     3     2     2     2     2     2     2     2     2
     1     3     1     1     2     3     2     3     1     3     2     2     2     2     2     2     4     2
     3     2     3     4     2     3     4     2     4     5     2     3     3     4     2     4     3     4
     2     3     2     3     2     4     2     3     2     3     2     3     4     2     2     2     4     3

--------------------------------------------------------------------------------
```

As shown in this matrix, which shows the minimum distance of path between each two vertices, we can see that the maximum distance is 5 (next page).

	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**14**	**15**	**16**	**18**	**20**	**21**	**22**
2	2	2	1	2	1	3	1	2	1	3	1	2	3	1	1	1	3	2
3	2	2	2	3	1	3	3	1	3	4	1	3	2	3	1	3	2	3
4	1	2	2	2	2	3	1	3	2	3	1	2	3	1	2	1	3	2
5	2	3	2	2	3	4	3	3	1	4	3	3	2	3	3	1	4	3
6	1	1	2	3	2	3	2	2	2	3	1	2	2	2	1	2	2	2
7	3	3	3	4	3	4	3	3	3	**5**	3	3	4	3	3	3	3	4
8	1	3	1	3	2	3	2	3	2	3	2	2	4	1	2	2	4	2
9	2	1	3	3	2	3	3	2	3	4	2	3	2	3	1	3	2	3
10	2	3	2	1	3	4	3	4	2	4	3	3	2	3	3	1	4	3
11	3	4	3	4	3	**5**	3	4	3	4	3	4	**5**	3	3	3	**5**	3
12	1	1	1	3	1	3	2	2	2	3	2	2	2	2	2	2	2	2
14	2	3	2	3	2	3	2	3	2	4	2	3	4	2	2	2	3	3
15	3	2	3	2	2	4	4	2	2	**5**	2	4	3	4	2	2	3	4
16	1	3	1	3	2	3	1	3	2	3	2	2	4	2	2	2	4	2
18	1	1	2	3	1	3	2	1	2	3	2	2	2	2	2	2	2	2
20	1	3	1	1	2	3	2	3	1	3	2	2	2	2	2	2	4	2
21	3	2	3	4	2	3	4	2	4	**5**	2	3	3	4	2	4	3	4
22	2	3	2	3	2	4	2	3	2	3	2	3	4	2	2	2	4	3

And the farthest paths (or vertices pairs) are:

(7 ←→ 11)
(11 ←→15)
(11 ←→21)

They all needs at least 5-degree distance apart in *H'*.