Central Limit Theorem

Eli Gurarie

StatR 101 - Lecture 8a
November 15, 2012

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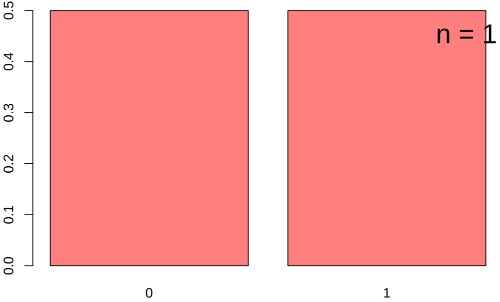
Coinflips

What is the distribution of a Bernoulli trial, repeated many, many times?



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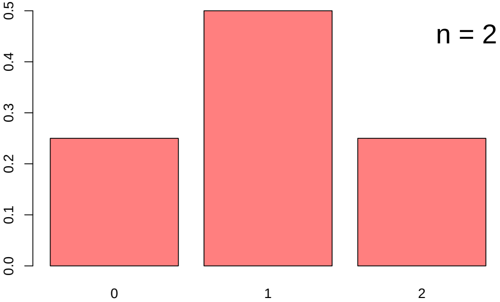

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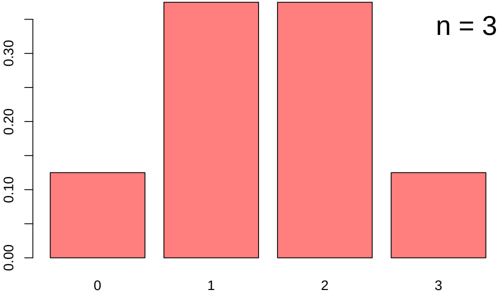

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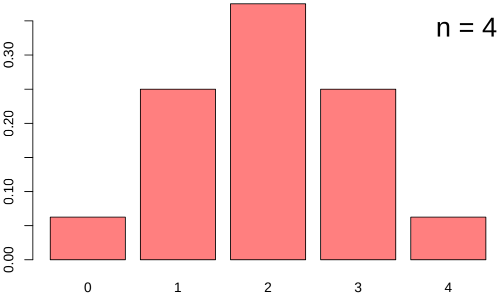

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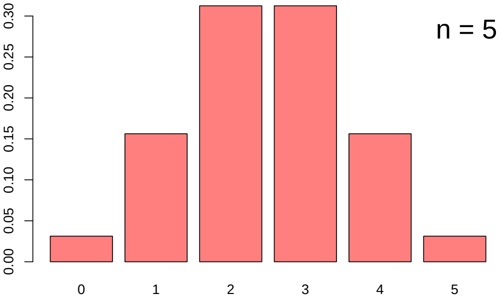

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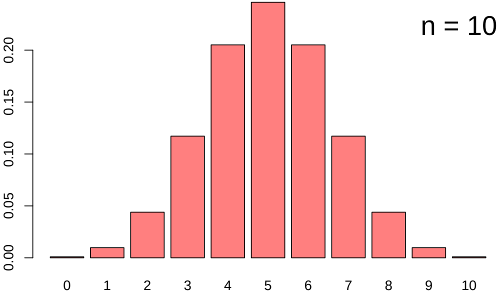

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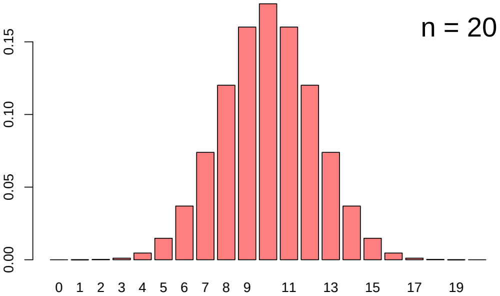

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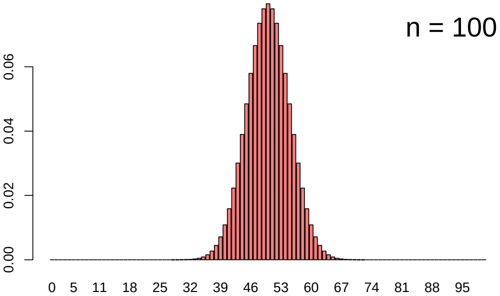

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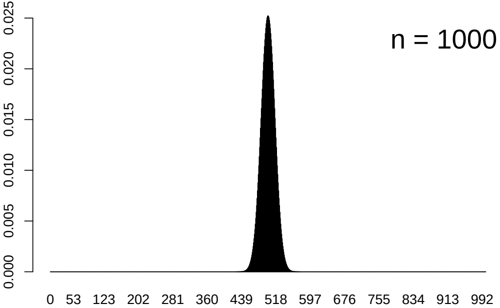
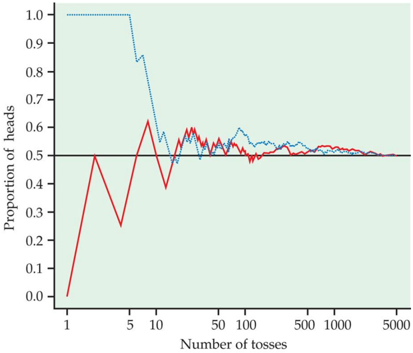


Note: Symmetric, bell-shaped!

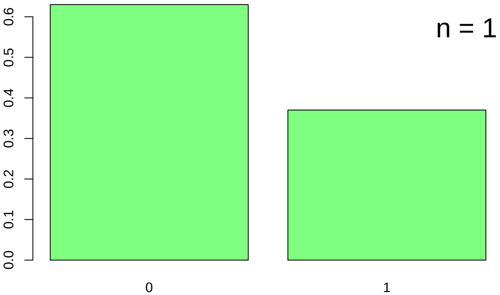
Law of Large Numbers

If you repeat an experiment X many, many, many times ($i = (1, 2, 3, ..., n)$, the average of X will asymptotically aproach $\mathrm{E}(X)$.



Shaq shoots

What about an assymetric distribution ($p = 0.37$)?



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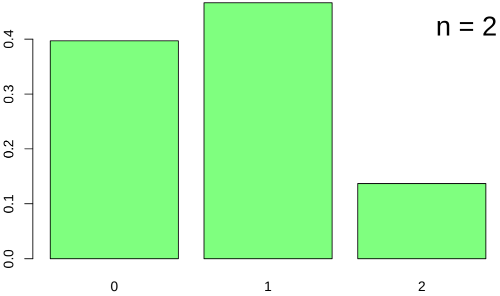

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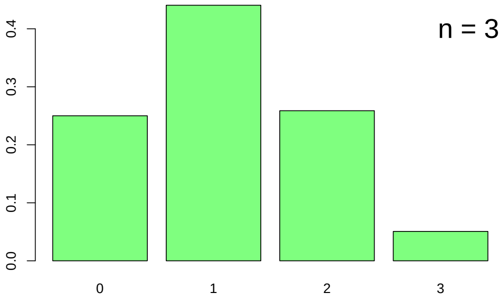

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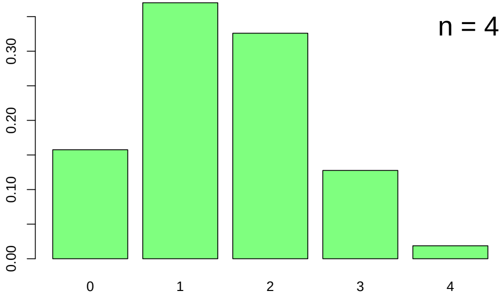

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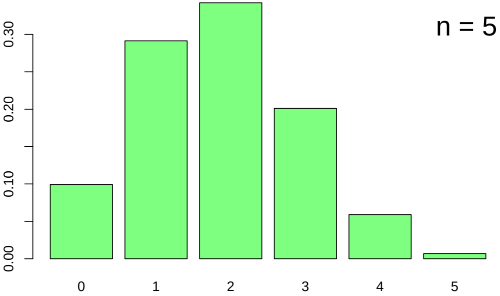

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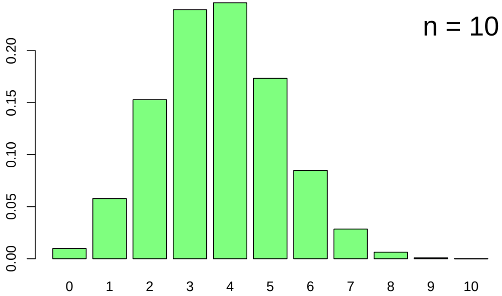

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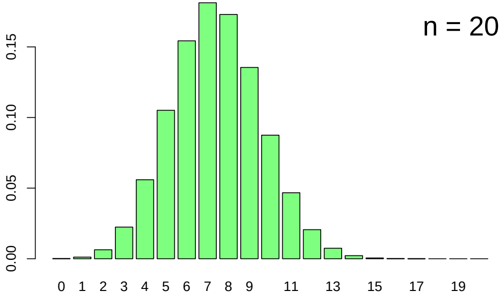

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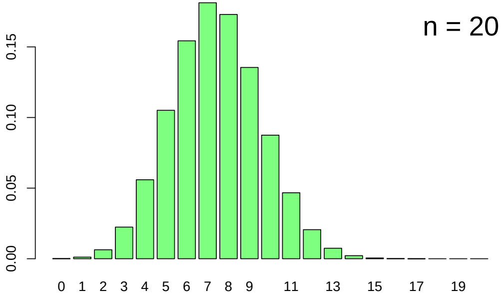
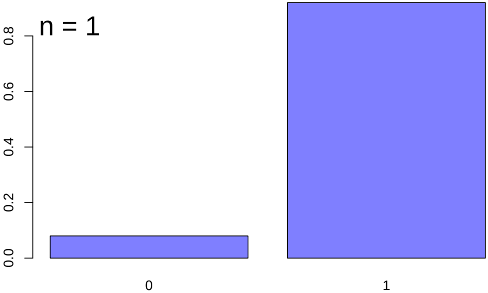


It ALSO looks symmetric, and bell-shaped!

Ray Allen shoots

Fine! What about an *extremely* assymetric distribution ($p = 0.92$)?



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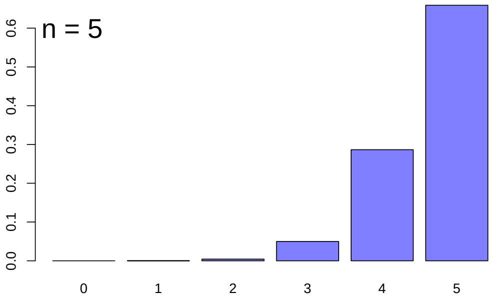

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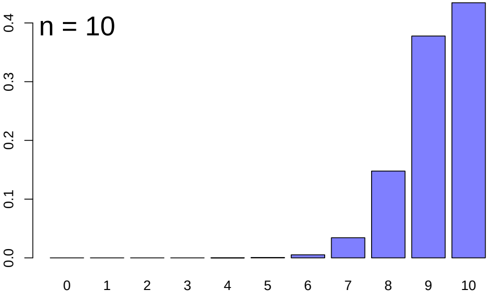

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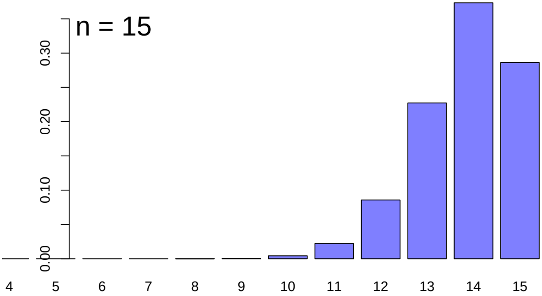

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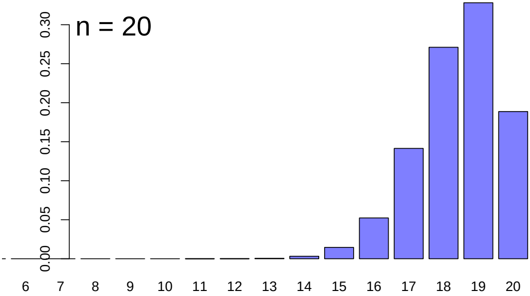

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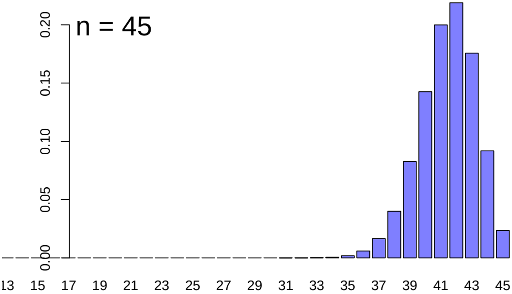

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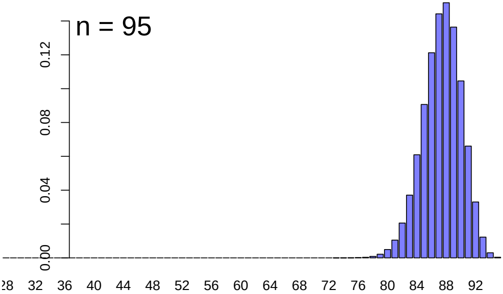

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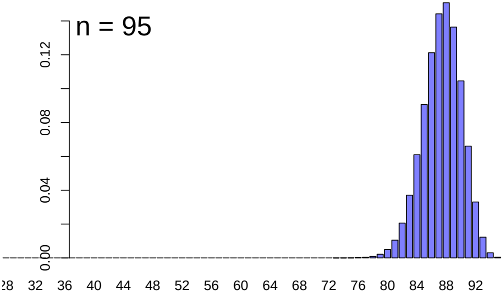

We have to dig a bit deeper, but in the long run ... it ALSO looks symmetric, and bell-shaped!

A hypothesis:

- If we repeat an Bernoulli trial many, many times, we KNOW it is a binomial...
- But Binomial(n, p) at large n looks like a Normal(μ, σ^2) .

But what are the mean and variance?

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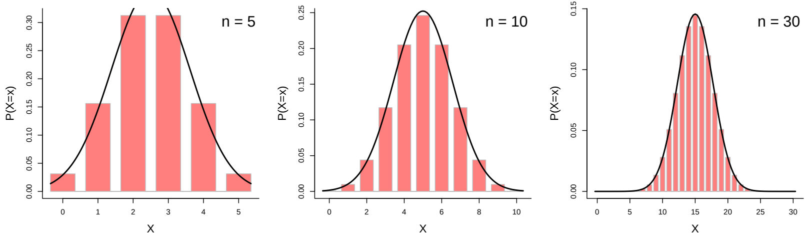

$$p = 0.5,\ n = (5, 10, 30),$$
$$\mu = 2.5, 5, 15,$$
$$\sigma^2 = 1.25, 2.54, 7.5$$

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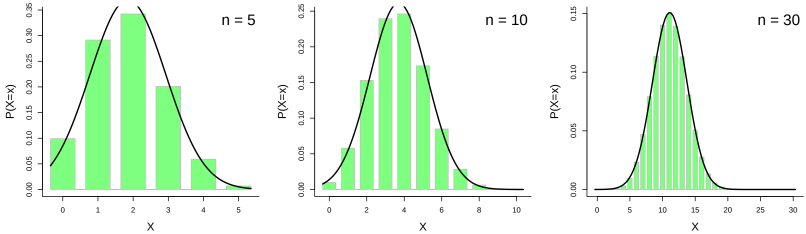
- Match the binomial distribution's mean (np) and variance ($np(1-p)$)



$$p = 0.5,\ n = (5, 10, 30),$$
$$\mu = 1.85, 3.7, 11.1,$$
$$\sigma^2 = 1.16, 2.33, 7.00$$

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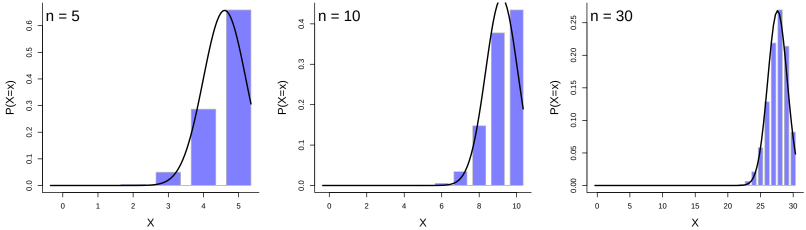

$$p = 0.5,\ n = (5, 10, 30),$$
$$\mu = 4.6, 9.2, 27.6,$$
$$\sigma^2 = 0.368, 0.736, 2.208$$

Normal approximation to the Binomial

- A variable $X \sim \text{Binomial}(n, p)$at large n is approximated by a continuous normal distribution

$$\mathcal{N}(\mu = np, \sigma^2 = np(1-p))$$

- This is useful because: $n!$ can be difficult to compute.

Caution:

The normal distribution is *continuous* - so it can not (easily) tell you the probability of a single discrete value ($P(X = x)$) ... but it is quite good for calculating ranges ($P(a < X < b)$).

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Example of normal approximation of binomial

About 4% of students have tattoos. Let X be the number of students that have tattoos in a review section of $n_1 = 30$, and Y be the number of students that have tattoos in a lecture of $n_2 = 200$ students. **What is the probability that no more than 3 students have a tattoo? ($Pr(X \leq 3)$ and $Pr(Y \leq 3)$)**

- $X \sim \text{Binomial}(n_1 = 30, p = 0.04)$, is approximated as: $X \sim \mathcal{N}(\mu = 1.2, \sigma^2 = 1.152)$
- $Y \sim \text{Binomial}(n_2 = 200, p = 0.04)$, is approximated as: $Y \sim \mathcal{N}(\mu = 8, \sigma = 2.77)$

True values:

- $P(X \leq 3) = \sum_{i=0}^{3} f(x|30, .04) =$ `pbinom(3,n1,sqrt(n1*p*(1-p)))`$= 0.9694$
- $P(Y \leq 3) = \sum_{i=0}^{3} f(y|200, .04) =$ `pbinom(3,n2,sqrt(n2*p*(1-p)))`$= 0.0395$

Approximate values:

- $P(X \leq 3) = \int_{-\infty}^{3} f(x|1.2, 1.152) =$ `pnorm(3,mean=1.2, sd=1.07)`$= 0.9532$
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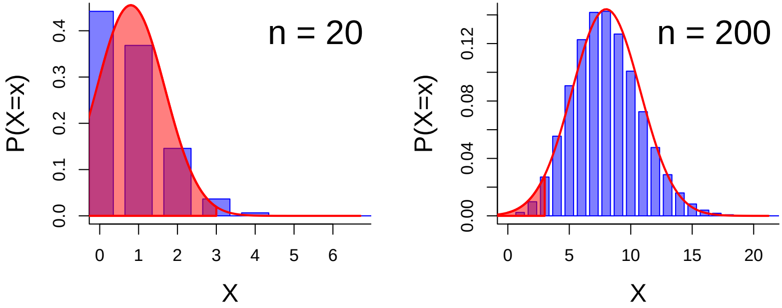
Note that the approximation is best for higher n, but the estimates are worse away from the mean of the distribution.



- True: $P(X \leq 3) = 0.9694$
- Approx: $P(X \leq 3) = 0.9532$

- True: $P(Y \leq 3) = 0.0355$
- Approx: $P(Y \leq 3) = 0.0395$

A hypothesis:

- If we repeat an Bernoulli trial many, many times, it looks like a Normal(μ, σ) distribution.

But what are the mean and variance?

- Recall the summation rules of Expectation and Variance
  - $\mathrm{E}(X_1 + X_2 + X_3 + \ldots) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3) + \ldots$
  - More generally
$$\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{E}(X_i)$$
  - $\mathrm{Var}(X_1 + X_2 + X_3 + \ldots) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) + \ldots$
  - More generally
$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

(**Note:** the variance rule is only for independent X.)

Normal approximation to many Bernoulli trials

- If a variable $Y = \sum_{i=1}^{n} X_i$ where X_i is a Bernoulli random variable with probability p $(X \sim \text{Bernoulli}(p))$, then (when n is large), Y is distributed approximately:

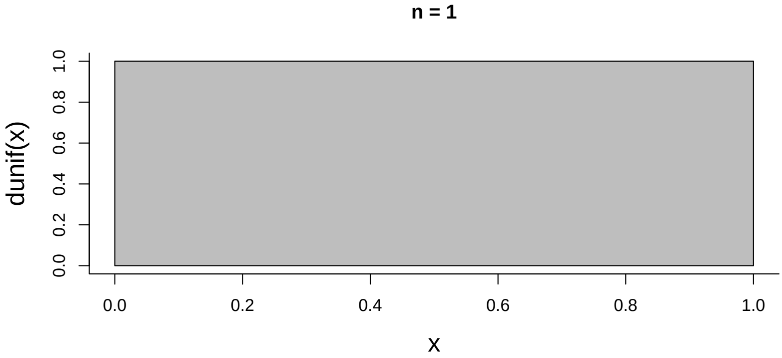
$$\mathcal{N}(\mu = np, \sigma = \sqrt{np(1-p)})$$

- Recall that $\mathrm{E}\left(X\right) = p$ and $\mathrm{Var}\left(X\right) = p(1-p)$ ... so we can say (in this case) that:

$$\mathcal{N}(\mu = n\,\mathrm{E}\left(X\right), \sigma = \sqrt{n\,\mathrm{Var}\left(X\right)})$$

What about other distributions?

$X \sim \text{Unif}(0, 1)$



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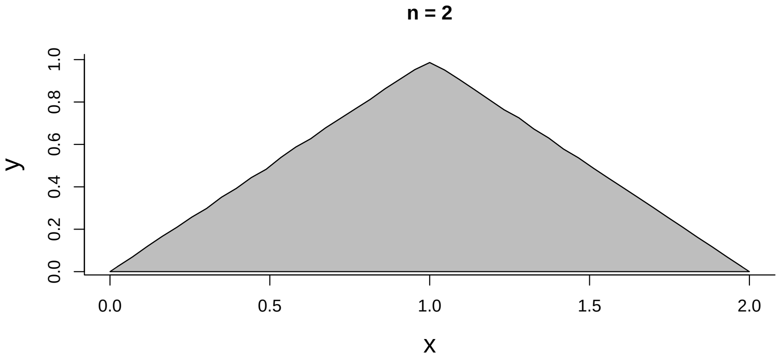

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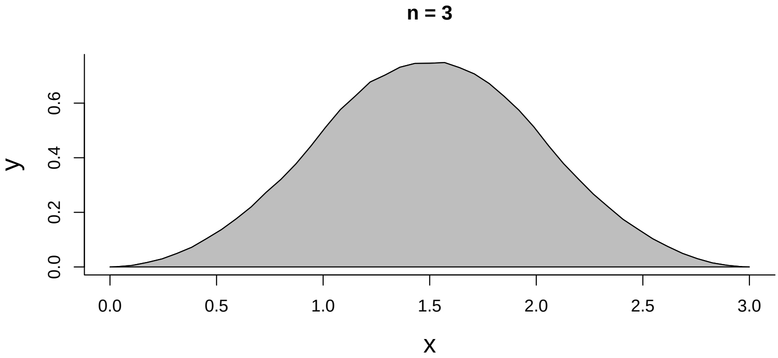
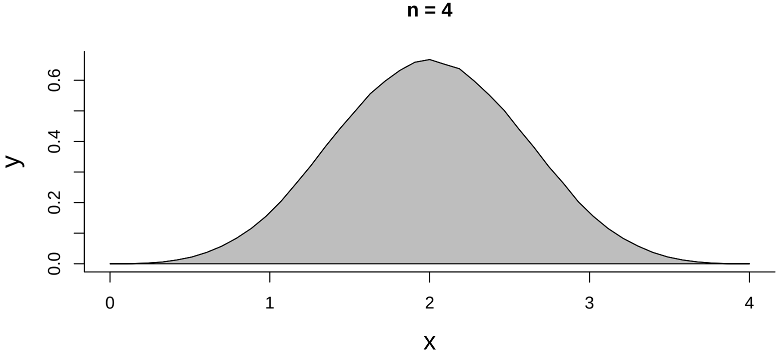


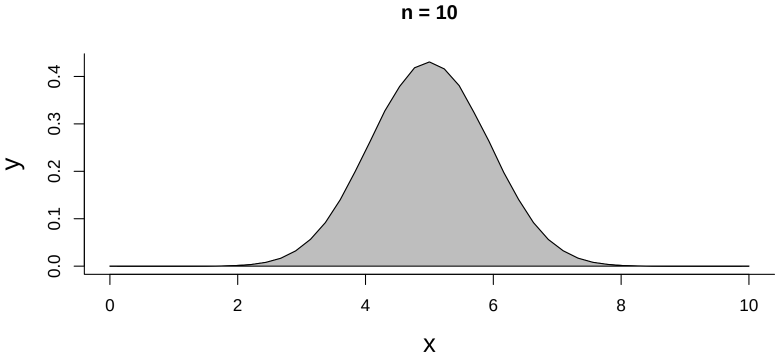
What about other distributions?

$Y = X_1 + X_2 + X_3 + X_4$



What about other distributions?

$Y = \sum_{i=1}^{10} X_i$



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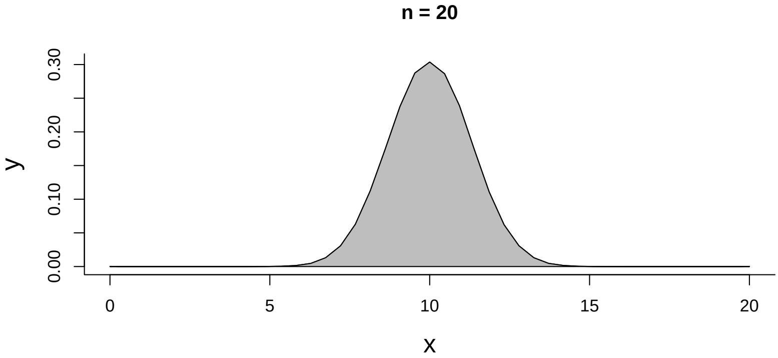

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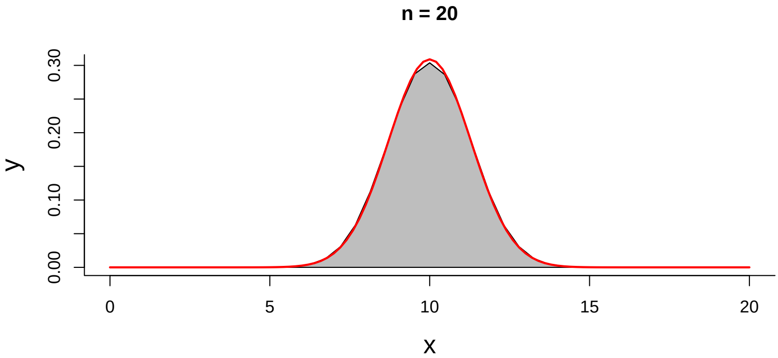
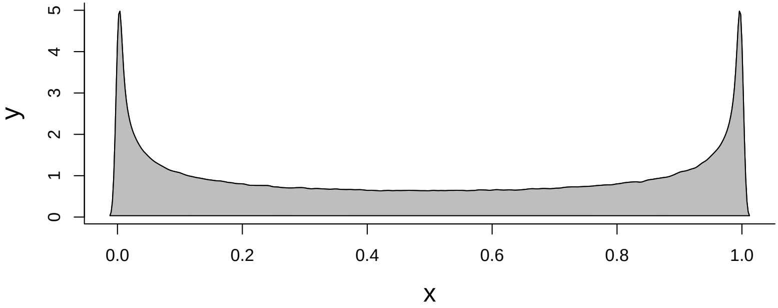


$\mathrm{E}(X) = \frac{\alpha+\beta}{2}$, $\mathrm{Var}(X) = \frac{(\beta-\alpha)^2}{12}$
$\mathrm{E}(Y) = 20 \times \frac{1}{2} = 10$, $\mathrm{Var}(Y) = 20 \times \frac{1}{12} = 5/3$

What about a crazy distributions?

$X \sim \text{Beta}(.5, .5)$ .... $\text{E}(X) = 1/2$, $\text{Var}(X) = 1/8$



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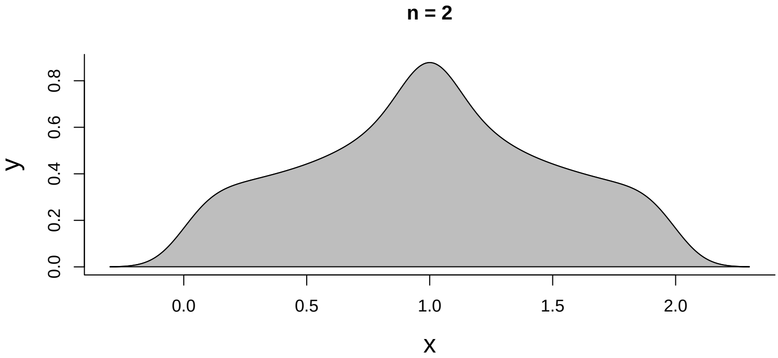
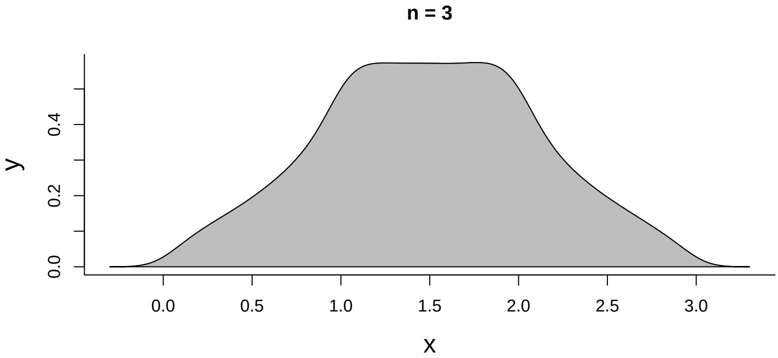


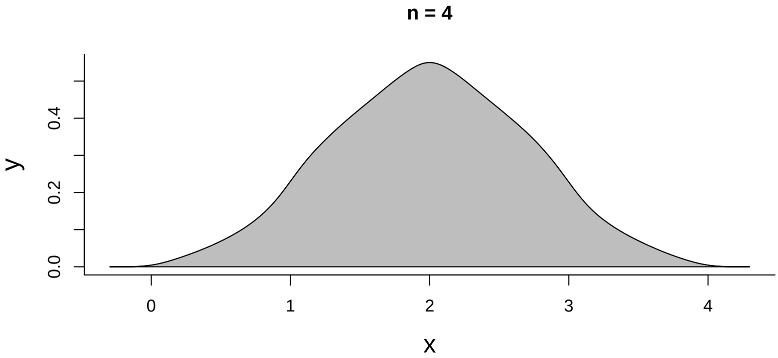
What about a crazy distributions?

$Y = X_1 + X_2 + X_3$



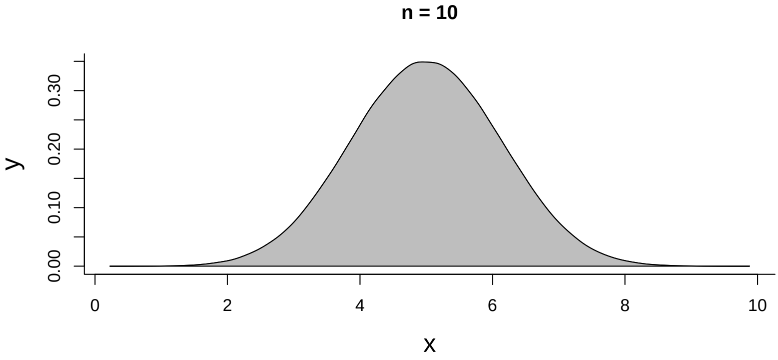
What about a crazy distributions?

$Y = X_1 + X_2 + X_3 + X_4$



What about a crazy distributions?

$Y = \sum_{i=1}^{10} X_i$



What about a crazy distributions?

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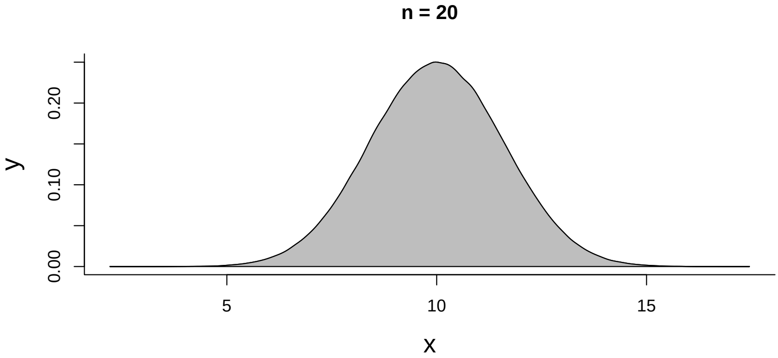
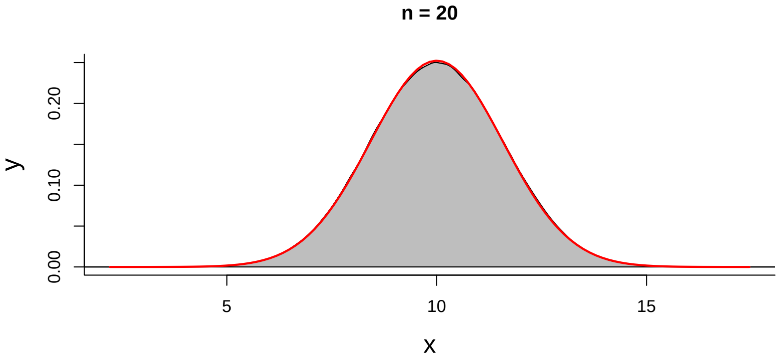


What about a crazy distributions?

$Y = \sum_{i=1}^{20} X_i$



$\mathrm{E}\,(X) = \frac{\alpha}{\alpha+\beta}$, $\mathrm{Var}\,(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$\mathrm{E}\,(Y) = 20 \times \frac{1}{2} = 10$, $\mathrm{Var}\,(Y) = 20 \times \frac{1}{8} = 5/2$

Central Limit Theorem (CLT)

If X_1, X_2, X_3 ... X_n are **any, independent, identically distributed (iid)** random variables with mean μ_x and variance σ_x^2, and

$$Y = \sum_{i=1}^{n} X_i$$

then, as n becomes large

$$Y \sim \mathcal{N}(n\mu_x, n\sigma_x^2)$$

In words: If you add up a BUNCH OF IID RANDOM VARIABLES, the result will be distributed approximately as a NORMAL distribution!

Central Limit Theorem (CLT)

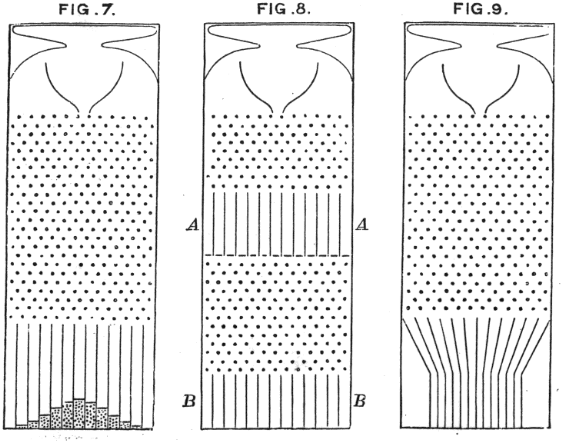
- The single most important theorem in Statistics!
- Because:
  - Many processes in nature are "additive" - (e.g. growth)
  - Many statistical objects (including many we have seen) are based on sums!
    - Remember: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
    - According to the CLT, if X has a distribution with mean μ_x and σ_x,

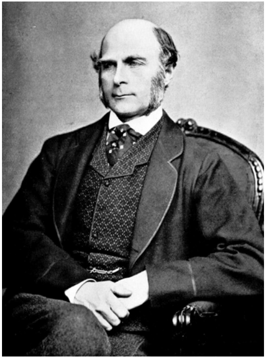
      then

$$\overline{X} \sim \mathcal{N}\left(\mu_x, \frac{\sigma_x^2}{n}\right)$$

  - This fact is the basis of A LOT of **classical inference** techniques!

Central Limit Theorem: Galton's Box (the Quincunx)





Francis Galton (1822-1911)
Founder of regression, correlation, weather maps, fingerprinting, questionnaires...

Video: http://www.youtube.com/watch?v=9xUBhhM4vbM