Some More (mostly) Continuous Distrubutions

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StatR 101 - Lecture 7b
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The Volcano: Part I

UW vulcanologist says: “Mt. Rainier will definitely erupt in the next 100 years, but it could happen any time.”

Continuous random variable

- Eruption time X is a **continuous random variable**.
- If eruption is equally likely to occur at any moment between now and the 100 year interval, then:
  - $P(X < 50 \text{ years}) = 0.5$
  - $P(X < 10 \text{ years}) = 0.1$
  - $P(X > 80 \text{ years}) = 0.2$
  - $P(50 < X < 80 \text{ years}) = 0.3$
- BUT
  - $P(X = 50 \text{ years}) = P(X = 10 \text{ years}) = P(X = 86.593832715 \text{ years}) = 0$
- How do you calculate these numbers?

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Probability density function $f(x)$ (pdf)

- $f(x) \geq 0$ for all values of x.
- Total area under the curve of $f(x)$ is 1.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

- $P(a < X < b) = \int_a^b f(x)\, dx$

Uniform random variable

Models an event that can happen with equal probability at *any* moment between time α and time β.

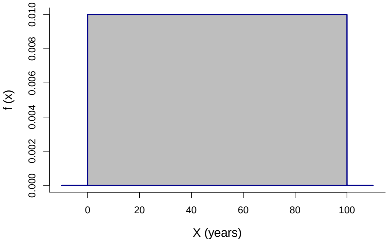
- $X \sim \text{Unif}(\alpha, \beta)$
- $f(x) = \begin{cases} \frac{1}{\beta-\alpha} \text{ where } \alpha \leq x \leq \beta \\ 0 \text{ otherwise.} \end{cases}$
-

$$\begin{aligned} P(a < X < b) &= \int_a^b \frac{1}{\beta - \alpha}\, dx \\ &= \frac{b-a}{\beta-\alpha} \end{aligned}$$

- Known as: **continuous uniform distribution**.

Uniform distribution

- $X \sim \text{Unif}(0, 100)$
- $f(x) = \begin{cases} \frac{1}{100} \text{ where } 0 \leq x \leq 100 \\ 0 \text{ else} \end{cases}$



Uniform distribution

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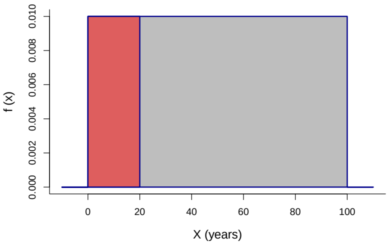
$$
\begin{aligned}
P(X < 20) &= \int_{-\infty}^{20} \frac{1}{100} dx \\
&= \frac{20}{100} - \frac{0}{100} \\
&= 20
\end{aligned}
$$

`punif(20,0,100)`



Uniform distribution

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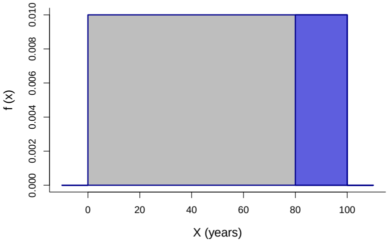
- $f(x) = \begin{cases} \frac{1}{100} \text{ where } 0 \leq x \leq 100 \\ 0 \text{ else} \end{cases}$

$$\begin{aligned} P(X > 80) &= \int_{80}^{\infty} \frac{1}{100} dx \\ &= \frac{100}{100} - \frac{80}{100} \\ &= 20 \end{aligned}$$

```
1 - punif(80,0,100)
```



Uniform distribution

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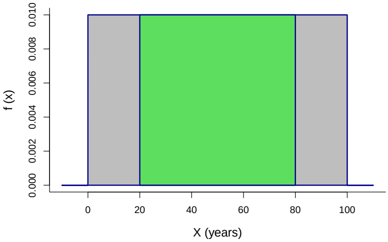
- $f(x) = \begin{cases} \frac{1}{100} \text{ where } 0 \leq x \leq 100 \\ 0 \text{ else} \end{cases}$

$$
\begin{aligned}
P(20 < X < 80) &= \int_{20}^{80} \frac{1}{100} dx \\
&= \frac{80}{100} - \frac{20}{100} \\
&= 60
\end{aligned}
$$



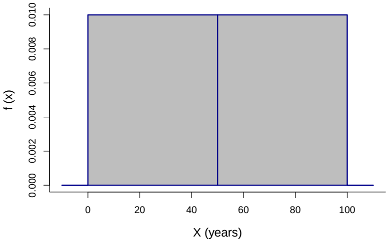

```
punif(80,0,100) - punif(20,0,100)
```

Uniform distribution

- $X \sim \text{Unif}(0, 100)$
- $f(x) = \begin{cases} \frac{1}{100} \text{ where } 0 \le x \le 100 \\ 0 \text{ else} \end{cases}$

$$\begin{aligned} P(X = 50) &= \int_{50}^{50} \frac{1}{100} dx \\ &= \frac{50}{100} - \frac{50}{100} \\ &= 0 \end{aligned}$$

`punif(20,0,100)`



Expected value and variance

- Refresher: Let X be a discrete random variable with possible values $x_1, x_2, x_3, ..., x_n$ and probability mass function $f(x)$
- $\mathrm{E}(X) = \sum_{i=1}^{n} x_i f(x_i)$
- $\mathrm{Var}(X) = \mathrm{E}(([X - \mathrm{E}(X)]^2)) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \sum_{i=1}^{n} (x_i - \mathrm{E}(X))^2 f(x_i)$

Expectation and variance for continuous random variables

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\
\mathrm{Var}(X) &= \mathrm{E}(([X - \mathrm{E}(X)]^2)) \\
&= \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2\, f(x)\, dx \\
\mathrm{SD}(X) &= \sqrt{\mathrm{Var}(X)}
\end{aligned}
$$

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Uniform random variable

Let $X \sim \text{Unif}(\alpha, \beta)$, $f(x) = \frac{1}{\beta-\alpha}$

$$
\begin{aligned}
\text{E}(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{\alpha}^{\beta} x \frac{1}{\beta-\alpha}\, dx \\
&= \frac{1}{\beta-\alpha} \frac{x^2}{2} \bigg|_{\alpha}^{\beta} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} \\
&= \frac{(\beta-\alpha)(\beta+\alpha)}{2(\beta-\alpha)} = \frac{\beta+\alpha}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} (x - \text{E}(X))^2\, f(x)\, dx \\
&= \int_{\alpha}^{\beta} \left(x - \frac{\beta+\alpha}{2}\right)^2 \left(\frac{1}{\beta-\alpha}\right) dx \\
&= \dots \\
\text{Var}(X) &= \frac{1}{12}(\beta-\alpha)^2
\end{aligned}
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Uniform random variable

Let $X \sim \text{Unif}(\alpha, \beta)$, $f(x) = \frac{1}{\beta - \alpha}$

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E(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{\alpha}^{\beta} x \frac{1}{\beta - \alpha}\, dx \\
&= \frac{1}{\beta - \alpha} \frac{x^2}{2} \Big|_{\alpha}^{\beta} = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} \\
&= \frac{(\beta - \alpha)(\beta + \alpha)}{2(\beta - \alpha)} = \frac{\beta + \alpha}{2}
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} (x - E(X))^2\, f(x)\, dx \\
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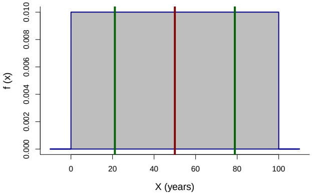
&= \ldots \\
\mathrm{Var}(X) &= \frac{1}{12}(\beta-\alpha)^2
\end{aligned}
$$

Uniform random variable

- Eruption time $X \sim \text{Unif}(0, 100)$
- $\text{E}(X) = 100/2 = 50$
- $\text{Var}(X) = 100^2/12 = 833.33$
- $\text{SD}(X) = 100/\sqrt{12} = 28.87$



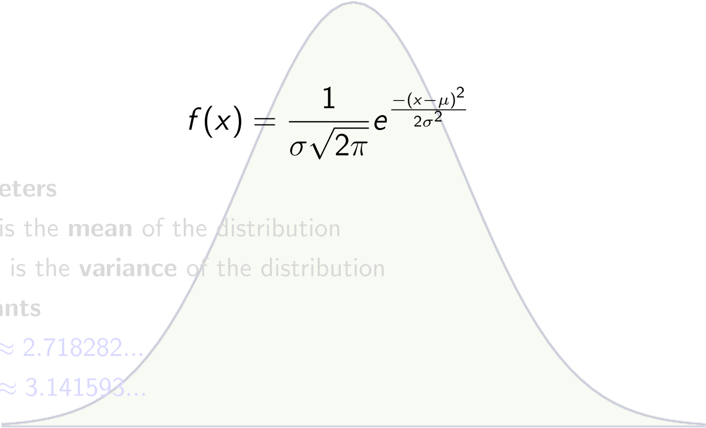
$X \sim \text{Unif}(\alpha, \beta)$

Name:	Uniform distribution	Models equal probability events within a continuous range of values
Support:	x	$(-\infty, \infty)$
Parameters:	$\alpha \in (-\infty, \beta)$ $\beta \in (\alpha, \infty)$	minimum maximum
pdf	$f(x \mid \alpha, \beta) =$	$\frac{1}{\beta-\alpha}$
mean	$\mathrm{E}(X)$	$\frac{\beta+\alpha}{2}$
variance	$\mathrm{Var}(X)$	$\frac{1}{12}(\beta-\alpha)^2$

The Volcano Part II

Japanese vulcanologist says: “Mt. Fuji is such a beautiful volcano, it will have the most beautiful distribution, and erupt with a normal distribution with mean 50 and standard deviation 20 years.”

Normal distribution: The formula



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Parameters

- μ is the **mean** of the distribution
- σ^2 is the **variance** of the distribution

Constants

- $e \approx 2.718282...$
- $\pi \approx 3.141593...$

Normal distribution: The formula

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- μ is the **mean** of the distribution
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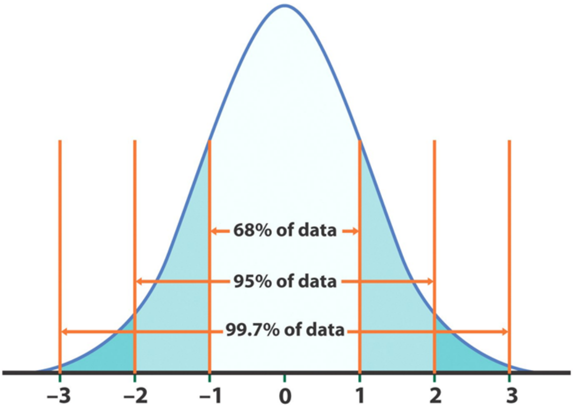
No matter what the parameters are...

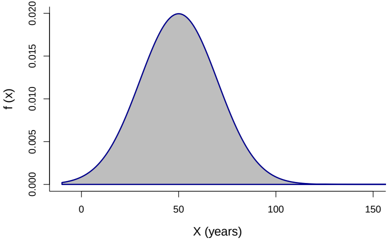


Approximately:

- 68% of the probability lies "within one standard deviations" $(\mu \pm \sigma)$.
- 95% of the probability lies "within 2 standard deviations" $(\mu \pm 2\sigma)$.
- 99.7% of the probability lies "within 3 standard deviations" $(\mu \pm 3\sigma)$.

Normal distribution

- $X \sim \text{N}(\mu = 50, \sigma = 20)$
- $f(x) = \frac{1}{20\sqrt{2\pi}} e^{\frac{-(x-50)^2}{220^2}}$



Normal distribution

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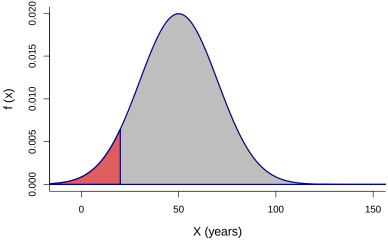
- $f(x) = \frac{1}{20\sqrt{2\pi}} e^{\frac{-(x-50)^2}{220^2}}$

- $P(X < 20) = \int_{-\infty}^{20} f(x)dx = 0.067$



Normal distribution

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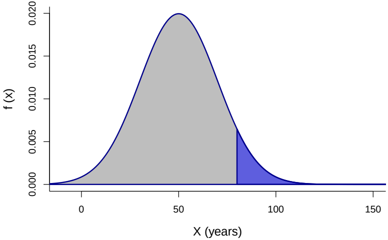
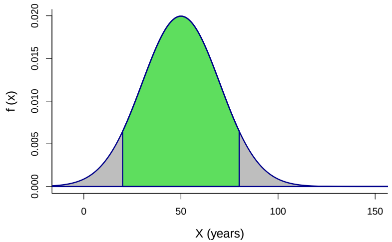
- $P(X > 80) = \int_{80}^{\infty} f(x)dx = 0.067$



Normal distribution

- $X \sim \mathrm{N}(\mu = 50, \sigma = 20)$
- $f(x) = \frac{1}{20\sqrt{2\pi}} e^{\frac{-(x-50)^2}{220^2}}$

- $P(20 < X < 80) = 0.866$



$X \sim \text{Normal}(\mu, \sigma)$

Name:	Normal distribution	Models bell-shaped continuous variables.
Support:	x	$(-\infty, \infty)$
Parameters:	$\mu \in (-\infty, \infty)$ $\sigma \in (0, \infty)$	measure of center measure of spread
pdf	$f(x\|\mu, \sigma) =$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mean	$E(X)$	μ
variance	$Var(X)$	σ^2

Note on the normal distribution

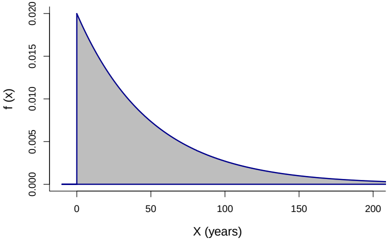
The normal distribution is a fantastic distribution for many things, but it is a lousy model for volcano eruptions!

The Volcano Part III

Russian vulcanologist says: “Mt. Avacha has been erupting at totally random times, but on average every 50 years.”

Exponential distribution

- $X \sim \text{Exp}(\gamma = 50)$
- $f(x) = \frac{1}{50} e^{-x/50}$, for $x > 0$



Exponential distribution

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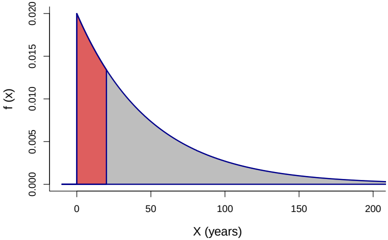
- $f(x) = \frac{1}{50} e^{-x/50}$, for $x > 0$

$P(X < 20) = \int_{-\infty}^{20} f(x)dx = 0.33$

```
pexp(20, rate=1/50)
```

Note, the use of "rate $= 1/\gamma$" instead of scale.

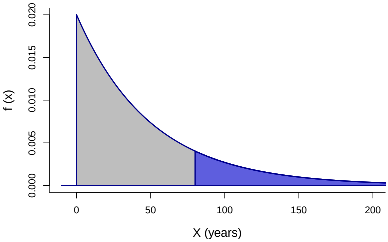


Exponential distribution

- $X \sim \text{Exp}(\gamma = 50)$
- $f(x) = \frac{1}{50} e^{-x/50}$, for $x > 0$

$P(X > 80) = \int_{80}^{\infty} f(x)dx = 0.20$

```
1-pexp(80, rate=1/50)
```

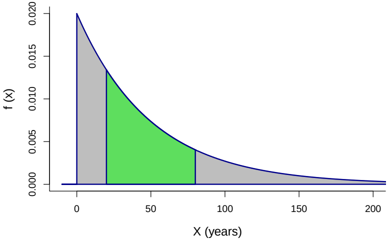


Exponential distribution

- $X \sim \text{Exp}(\gamma = 50)$
- $f(x) = \frac{1}{50} e^{-x/50}$, for $x > 0$

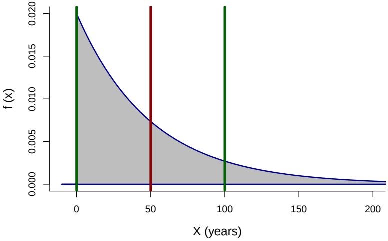
$P(20 < X < 80) = 0.47$

```
pexp(80, 1/50) - pexp(20,
1/50)
```



Exponential Distribution

- Eruption time
  $X \sim \text{Exp}(\gamma = 50)$
- $\text{E}(X) = 50$
- $\text{Var}(X) = 50^2$
- $\text{SD}(X) = 50$



Exponential random variable

Let $X \sim \text{Exp}(\gamma)$, $f(x) = \frac{1}{\gamma} e^{\frac{-x}{\gamma}}$

$$\begin{aligned} \text{E}(X) &= \int_0^\infty x\, f(x)\, dx = \int_0^\infty \frac{1}{\gamma} e^{\frac{-x}{\gamma}}\, dx \\ &= e^{\frac{-x}{\gamma}}(\gamma + x)\Big|_0^\infty \\ &= \gamma \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_0^\infty (x-\gamma)^2\, f(x)\, dx \\ &= \ldots \\ &= \gamma^2 \end{aligned}$$

$$\text{SD}(X) = \gamma$$

Exponential random variable

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&= e^{\frac{-x}{\gamma}}(\gamma + x)\Big|_0^\infty \\
&= \gamma
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_0^\infty (x-\gamma)^2\, f(x)\, dx \\
&= \dots \\
&= \gamma^2
\end{aligned}$$

$$\text{SD}(X) = \gamma$$

Exponential random variable

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$$\begin{aligned} \text{Var}(X) &= \int_0^\infty (x - \gamma)^2\, f(x)\, dx \\ &= \ldots \\ &= \gamma^2 \end{aligned}$$

$$\text{SD}(X) = \gamma$$

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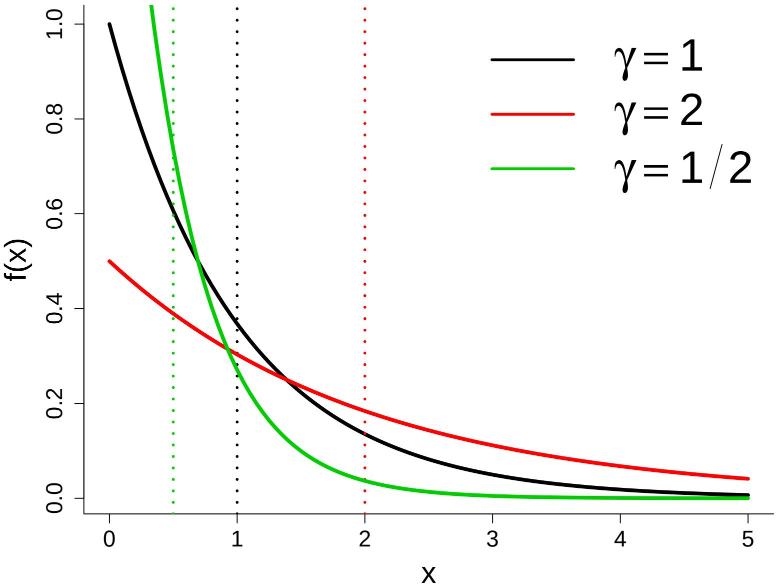
\text{Var}(X) &= \int_0^\infty (x-\gamma)^2\, f(x)\, dx \\
&= \ldots \\
&= \gamma^2
\end{aligned}$$

$$\text{SD}(X) = \gamma$$

$X \sim \text{Exp}(\gamma)$

Name:	Exponential distribution	Models time-independent random events
Support:	x	$(0, \infty)$
Parameters:	$\gamma \in (0, \infty)$	scale parameter
pdf	$f(x\|\mu, \sigma) =$	$\frac{1}{\gamma} e^{-\frac{x}{\gamma}}$
mean	$\text{E}(X)$	γ
variance	$\text{Var}(X)$	γ^2

Exponential Distribution: $X \sim \mathrm{Exp}(\gamma)$



Important comments on the exponential distribution

- The exponential distribution models the waiting time to any event which can occur with equal probability in time.
- It is the continuous analogue of the geometric distribution, and is also memoryless:
  - If Avacha erupts today, the expected time to next eruption is 50 years.
  - If Avacha hasn't erupted in 50 years, the expected time to next eruption is 50 years.
  - If Avacha hasn't erupted in 500 years, the expected time to next eruption is 50 years (though you might consider updating your Exp(50) model).
- It is readily identified by the standard deviation being similar to the mean.

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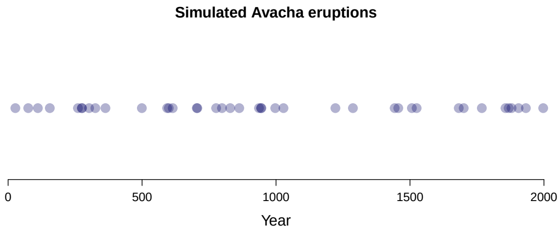
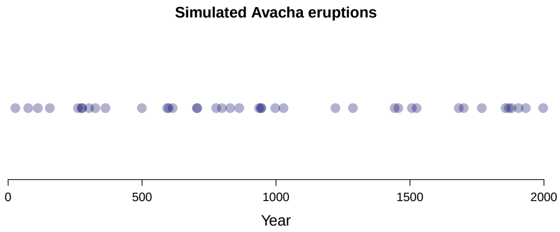
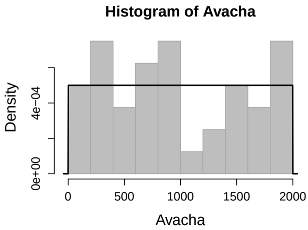
Relationship between exponential and uniform distributions




```
> Avacha <- runif(40,0,1000)
```

Relationship between exponential and uniform distributions



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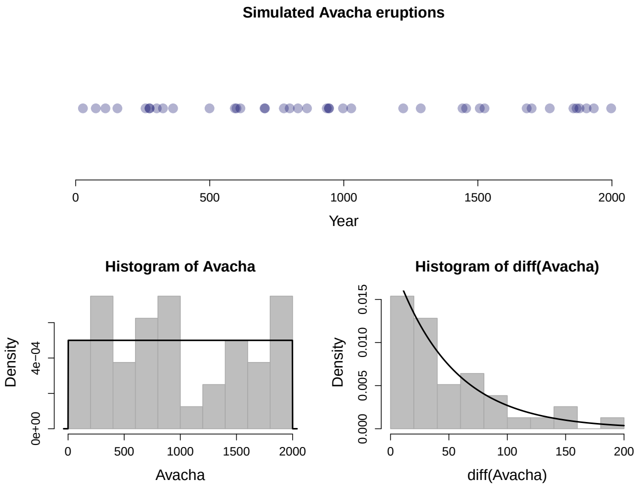
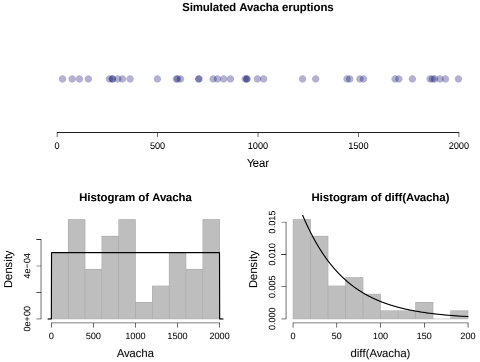
```

Relationship between exponential and uniform distributions




```
> Avacha <- runif(40,0,1000)
```

Relationship between exponential and uniform distributions

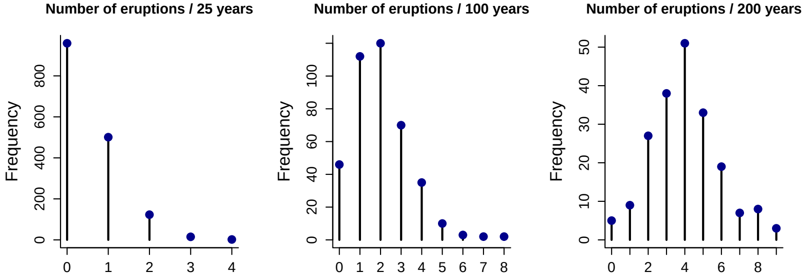


Important fact: If $X_{(1)}, X_{(2)}, X_{(3)} ... X_{(n)}$ are ordered uniform r.v.'s: $X \sim \text{Unif}(0, l)$, then:

$$W_i = X_{(i+1)} - X_{(i)} \sim \text{Exp}(\gamma = \frac{n}{l}).$$

More questions about exponential distribution

Ok, if Avacha erupts randomly with mean 50 years. How many eruptions can we *expect* in a century? 200 years? 25 years?



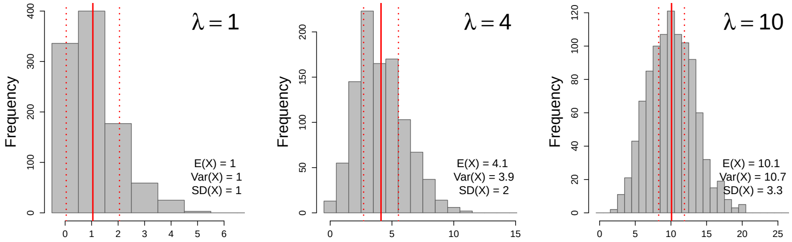
Poisson distribution

- Describes the number of times a random, independent event occurring with constant intensity in a given interval of time or space

$$\mathrm{P}(c = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

where λ = rate or intensity of occurrence

Poisson process



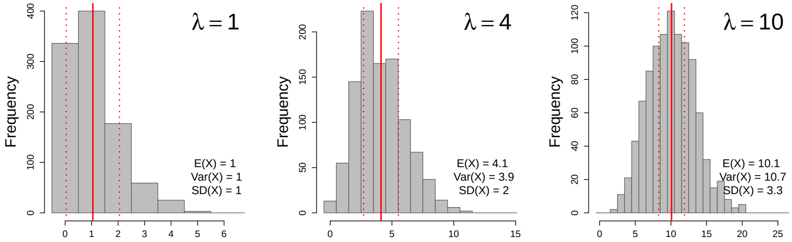
$X \sim \text{Poisson}(\lambda)$

$$\begin{aligned} \mathrm{E}[X] &= \lambda \\ \mathrm{Var}[X] &= \lambda \end{aligned}$$

i.e.: **Variance = Mean = Intensity!**

This is a very useful property for determining whether the Poisson distribution is an appopriate model.

Poisson process



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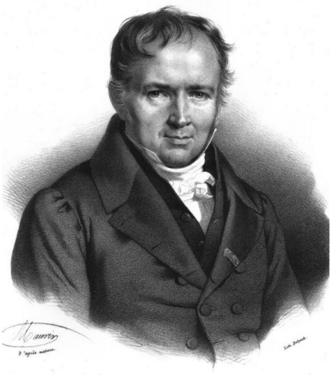
$X \sim \text{Poisson}(\lambda)$

Name:	Poisson distribution	Models frequency of discrete random events
Support:	x	$\{0, 1, 2, 3, ...\}$
Parameters:	$\lambda \in (0, \infty)$	intensity parameter
pdf	$P(X = k \vert \lambda) =$	$\frac{\lambda^k}{k!} e^{-\lambda}$
mean	$\text{E}(X)$	λ
variance	$\text{Var}(X)$	λ

Historical Aside



RECHERCHES

SUR LA

PROBABILITÉ DES JUGEMENTS

EN MATIÈRE CRIMINELLE

ET EN MATIÈRE CIVILE,

DES RÈGLES GÉNÉRALES DU CALCUL DES PROBABILITÉS;

PAR S.-D. POISSON,

Le calcul des probabilités s'applique également aux choses de toute espèce, morales ou physiques, et ne dépend aucunement de leur nature, pourvu que dans chaque cas, l'observation fournisse les données numériques, nécessaires à ses applications.

Siméon Denis Poisson (1781-1840) - French physicist and mathematician, developed the Poisson distribution to model the number of convictions in the civil courts in France, noting in 1837 that:

"The science of probability can be applied to any subject - be they moral or physical - regardless of their nature, as long as the observations provide the numerical data required for its application."

Volcano Model

Which of these distributions best describes actual eruption times?
R lab on data from Mt. Vesuvius.