HW5

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First, load the package.

```
library(MASS)
```

1. Conduct the eigenvalue decomposition for the covariance matrix. Comment on the relationship between the variance and eigenvalues.

```
s <- matrix(c(0.14, 0.03, 0.02,-0.10,-0.01, 0.08,
              0.03, 0.12, 0.10, 0.21, 0.10,-0.21,
              0.02, 0.10, 0.16, 0.28, 0.12,-0.24,
             -0.10, 0.21, 0.28, 2.07, 0.16,-1.03,
             -0.01, 0.10, 0.12, 0.16, 0.64,-0.54,
              0.08,-0.21,-0.24,-1.03,-0.54, 1.32), nrow=6, byrow=TRUE)

eigen_decomp <- eigen(s)


eigen_decomp$values
```

```
## [1] 2.97407033 0.92686534 0.23906271 0.18851360 0.08582683 0.03566120
```

```
eigen_decomp$vectors
```

```
##             [,1]        [,2]       [,3]       [,4]        [,5]        [,6]
## [1,] -0.04205408  0.01735469 -0.2755460 -0.6105384  0.72616984  0.14807064
## [2,]  0.11060064  0.07172518 -0.1550414 -0.4763540 -0.29059835 -0.80450273
## [3,]  0.13806701  0.06173944 -0.2400151 -0.4782233 -0.60321951  0.57179297
## [4,]  0.76888874 -0.56088394 -0.2537564  0.1474821  0.08864876 -0.01474458
## [5,]  0.19886996  0.66018146 -0.6285631  0.3540774  0.05980409 -0.02392170
## [6,] -0.57982524 -0.49021648 -0.6188258  0.1565582 -0.11367083 -0.05579957
```

By observe how eigenvalues relate to the variance, it seem larger eigenvalues correspond to directions (eigenvectors) with higher variance, indicating the principal components of the data in those directions.

2. Express the singular value decomposition of matrice A and A^T using eigenvalues and eigenvectors of matrices AA^T and A^TA.

Let A ~~\#~~ is a $m \times n$ matrix, $\operatorname{rank}(A^TA) = r$, the set of orthonormal eigenvectors of A^TA are $v_1, v_2, \cdots v_n$ in $\mathbb{R}^n$ and eigenvalue: $\lambda_1 \cdots \lambda_n$ where singular values of A are defined as the square root of the eigenvalues of A^TA: $\sigma_i = \sqrt{\lambda_i} \geq 0$, $\|Av_i\|^2 = v_i^T A^T A v_i = v_i^T \lambda_i v_i = \sigma_i^2$

$\therefore \|Av_i\| = \sigma_i$. the unit vectors u_i: $u_i = \frac{1}{\|Av_i\|} Av_i = \frac{1}{\sigma_i} Av_i$ where $\|u_i\| = 1$.

$\therefore Av_i = \sigma_i u_i$, when $i = 1, 2, \cdots, r$, $Av_i = 0$ when $i = r+1, \ldots, n$

so let $V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}_{n \times n}$, $U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix}_{m \times m}$

$$AV = A \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ Av_1 & Av_r & Av_n \\ | & | & | \end{bmatrix}_{m \times n} = \begin{bmatrix} | & & | & & \vdots \\ \sigma_1 u_1 & \cdots & \sigma_r u_r & \cdots & 0 \\ | & & | & & \vdots \end{bmatrix}_{m \times n}$$

$$= \begin{bmatrix} | & & | & & | \\ u_1 & \cdots & u_r & \cdots & u_m \\ | & & | & & | \end{bmatrix}_{m \times m} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_r & \\ 0 & \cdots & & 0 \end{bmatrix}_{m \times n} = U\Sigma$$

* where Σ is diagonal matrix of singular values of A^TA.

$\therefore A = U\Sigma V^{-1} = U\Sigma V^T$. A^T is the same way through AA^T \#

$\therefore A = V\Sigma W^T$: S are eigenvectors of A^TA, W are eigenvectors of AA^T \#