2024 Formal Languages and Automata Theory HW6 r12631055 林東甫

Problem 1.

(a) pf. Known $f, g : \mathbb{N} \to \mathbb{R}^+$, If $\exists c > 0, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0, n \in \mathbb{N}$

$f(n) \leq cg(n)$ then $f(n) = O(g(n))$

opposite: $\forall c > 0, n_0 \in \mathbb{N}, \exists n \geq n_0, n \in \mathbb{N}$ s.t. $f(n) > cg(n)$

For any $0 < c < e$, $n_0 \in \mathbb{N}$, $\exists n \geq \max\{n_0, 1\}$ ~~We simply choose $n_0 = 1$~~, ~~$\exists n \geq n_0$~~ s.t.

$$\frac{e^{2n}}{e^n} = e^n > c$$

For any $c \geq e$, $n_0 \in \mathbb{N}$, $\exists n \geq \max\{\lceil \ln c \rceil, n_0\}$ s.t.

$$\frac{e^{2n}}{e^n} = \frac{e \cdot e \cdots e}{1 \cdot 1 \cdots 1} > c$$

Combining both cases, opposite has been proved, thus (a) is disproved

(b) pf. For any $c \geq 1$, choose $n_0 = 1$ so that for $n \geq n_0$.

$$\frac{n!}{n^n} \leq 1$$

For any $0 \leq c < 1$, choose $n_0 = \lceil \frac{1}{c} \rceil$ so that for $n \geq n_0$

$$\frac{n!}{n^n} = \frac{n \cdot (n-1) \cdots 2}{n \cdot n \cdots n} \cdot \frac{1}{n} \leq 1 \cdot c$$

Therefore, $n! \leq c \cdot n^n$. Combining both cases, $f(n) = o(g(n))$ #