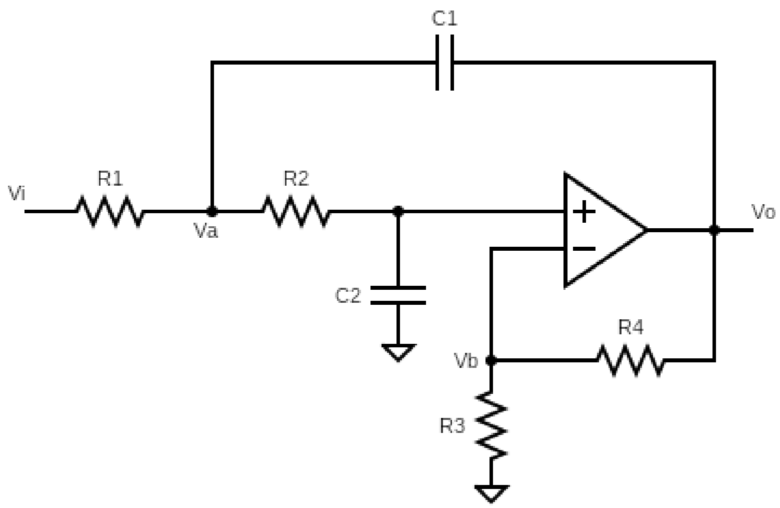


Figure 1: Sallen-Key Low-Pass Filter

*Proof.* Sallen-Key Low-Pass Filter Transfer Function

Writing KCL equation at node $V_a$ yields:

$$\frac{V_i - V_a}{R_1} + \frac{V_b - V_a}{R_2} + \frac{V_o - V_a}{Z_{C1}} = 0 \tag{1}$$

Writing voltage divison equation at node $V_b$ at non-inverting terminal yields:

$$V_b = V_a \cdot \frac{Z_{C2}}{R_2 + Z_{C2}} \tag{2}$$

Writing voltage division equation at node $V_b$ at inverting terminal yields:

$$V_b = V_o \cdot \frac{R_3}{R_3 + R_4} \tag{3}$$

Using equations 2, 3 and solving for $V_a$ gives:

$$V_a = V_o \cdot \frac{R_3}{R_3 + R_4} \cdot \frac{R_2 + Z_{C2}}{Z_{C2}} \tag{4}$$

Inserting $V_a$'s equivalent in Eq. 4 and $V_b$'s equivalent in Eq. 3 to Eq. 1

results:

$$\frac{V_i}{R_1} - \frac{V_o}{R_1} \cdot \frac{R_3}{R_3 + R_4} \cdot \frac{R_2 + Z_{C2}}{Z_{C2}} + \frac{V_o}{R_2} \cdot \frac{R_3}{R_3 + R_4} - \frac{V_o}{R_2} \cdot \frac{R_3}{R_3 + R_4} \cdot \frac{R_2 + Z_{C2}}{Z_{C2}} + \frac{V_o}{Z_{C1}} - \frac{V_o}{Z_{C1}} \cdot \frac{R_3}{R_3 + R_4} \cdot \frac{R_2 + Z_{C2}}{Z_{C2}} = 0 \tag{5}$$

Solving Eq. 5 for $\frac{V_o}{V_i}$ yields:

$$\frac{V_o}{V_i} = 1/[\frac{R_3}{R_3 + R4} \cdot (\frac{R_2 + Z_{C2}}{Z_{C2}} - \frac{R_1}{R_2} + \frac{R_1 \cdot (R_2 + Z_{C2})}{R_2 \cdot Z_{C2}} - \frac{R_1 \cdot (R_3 + R_4)}{R_3 \cdot Z_{C1}} + \frac{R_1 \cdot (R_2 + Z_{C2})}{Z_{C1} \cdot Z_{C2}})] \tag{6}$$

Then substituting $Z = \frac{1}{jwC}$ in to the Eq. 6, simpyfing it, and rewriting it in the $\frac{V_o}{V_i} = 1/[\frac{R_3}{R_3+R_4} \cdot (w^2 \cdot (...) + jw \cdot (...) + 1)]$ form results:

$$\frac{V_o}{V_i} = 1/[\frac{R_3}{R_3 + R4} \cdot (w^2 \cdot (-R_1 R_2 C_1 C_2) + jw \cdot (R_1 C_1 \frac{-R_4}{R_3} + C_2(R_1 + R_2)) + 1)] \tag{7}$$

Let $K = \frac{R_3 + R4}{R_3}$, using the equality $j = \sqrt{-1}$, and $w = 2\pi f$ Eq. 7 can be written as follows:

$$\frac{V_o}{V_i} = K \cdot 1/[(j2\pi f \sqrt{R_1 R_2 C_1 C_2})^2 + j2\pi f (R_1 C_1 \frac{-R_4}{R_3} + C_2(R_1 + R_2)) + 1] \tag{8}$$

And define $f_0$ as $\frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}}$. And rewriting Eq. 8 as follows:

$$\frac{V_o}{V_i} = K \cdot 1/[(\frac{jf}{1/(2\pi\sqrt{R_1 R_2 C_1 C_2})})^2 + j2\pi f (R_1 C_1 \frac{-R_4}{R_3} + C_2(R_1 + R_2)) + 1] \tag{9}$$

In Eq. 9 inserting $f_0$ to the squared term and multipying the $j2\pi f$ term in the denominater with $\frac{f_0}{f_0}$ ($2\pi$ are cancelled) gives:

$$\frac{V_o}{V_i} = K \cdot 1/[(\frac{jf}{f_0})^2 + \frac{jf}{f_0}(\frac{1}{\sqrt{R_1 R_2 C_1 C_2}}(R_1 C_1 \frac{-R_4}{R_3} + C_2(R_1 + R_2))) + 1] \tag{10}$$

Observing that $1 - K = \frac{-R_4}{R_3}$ then Eq. 10 can be written as:

$$\frac{V_o}{V_i} = K \cdot 1/[(\frac{jf}{f_0})^2 + \frac{jf}{f_0}(\frac{1}{\frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K) R_1 C_1 + C_2(R_1 + R_2)}}) + 1] \tag{11}$$

And let $Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K)R_1 C_1 + C_2(R_1+R_2)}$. Which is the quality factor of the filter. Then Eq. 11 can be written as follows:

$$\frac{V_o}{V_i} = K \cdot \frac{1}{(\frac{jf}{f_0})^2 + \frac{jf}{f_0}\frac{1}{Q} + 1} \tag{12}$$

□