Linear Algebra Lecture Summary: Elementary Matrices and Linear Transformations

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1 Introduction

This lecture focuses on two main topics: elementary matrices and linear transformations. The content builds on previous weeks' discussions about Gaussian elimination and matrix algebra, highlighting the connections between these concepts.

2 Elementary Matrices

2.1 Definition

An elementary matrix is defined as a square matrix that can be obtained from the identity matrix by a single elementary row operation. The three types of elementary row operations are:

- Type 1: Swapping two rows.
- Type 2: Multiplying a row by a non-zero scalar.
- Type 3: Adding a multiple of one row to another row.

2.2 Examples of Elementary Matrices

To illustrate, consider the following matrices:

- Matrix A: Swapping rows 1 and 2.
- Matrix B: Multiplying row 2 by 2.
- Matrix C: Adding 5 times row 1 to row 2.

Only matrices A and C are elementary matrices, as they result from a single row operation.

2.3 Relation to Gaussian Elimination

The connection between elementary matrices and Gaussian elimination is significant. If an elementary row operation is performed on an $m \times n$ matrix, the result can be expressed as:

$$\mathbf{A}' = \mathbf{E} \cdot \mathbf{A}$$

where $\mathbf{E}$ is the elementary matrix corresponding to the row operation.

2.4 Properties of Elementary Matrices

All elementary matrices are invertible. The inverse of an elementary matrix corresponds to the inverse of the row operation it represents. For example, if an elementary matrix multiplies a row by a scalar, its inverse will multiply that row by the reciprocal of that scalar.

3 Linear Transformations

3.1 Definition and Properties

A linear transformation can be represented by a matrix multiplication. If $\mathbf{T}$ is a linear transformation represented by matrix $\mathbf{A}$, then for any vector $\mathbf{x}$, we have:

$$\mathbf{T}(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$$

Linear transformations preserve vector addition and scalar multiplication.

3.2 Composition of Linear Transformations

If $\mathbf{A}$ and $\mathbf{B}$ are matrices representing linear transformations, the composition of these transformations can be expressed as:

$$\mathbf{T}(\mathbf{x}) = \mathbf{B}(\mathbf{A}(\mathbf{x})) = (\mathbf{B} \cdot \mathbf{A}) \cdot \mathbf{x}$$

This shows that the order of transformations matters.

4 Conclusion

In summary, elementary matrices serve as the building blocks for understanding matrix operations and transformations. They provide a framework for performing Gaussian elimination and understanding the nature of linear transformations. The lecture emphasized the importance of these concepts in both theoretical and practical applications in linear algebra.