Linear Algebra: Linear Independence and Dimension

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1 Introduction

This lecture builds upon the previous discussion about subspaces and spanning sets. We will explore the concepts of linear independence and dimension, which are crucial in understanding vector spaces.

2 Recap of Subspaces and Spanning Sets

A **subspace** is defined as a subset of R^n that is closed under vector addition and scalar multiplication. This means that any linear combination of vectors in the subspace remains within the subspace.

The **span** of a set of vectors is the smallest subspace that contains those vectors. It consists of all possible linear combinations of the vectors.

3 Linear Independence

3.1 Definition

Vectors are said to be **linearly independent** if no vector in the set can be expressed as a linear combination of the others. Conversely, if at least one vector can be expressed as a linear combination of the others, the vectors are **linearly dependent**.

3.2 Criterion for Linear Independence

A set of vectors $\{x_1, x_2, \ldots, x_k\}$ is linearly independent if the only solution to the equation

$$t_1x_1 + t_2x_2 + \ldots + t_kx_k = 0$$

is $t_1 = t_2 = \ldots = t_k = 0$. If there exists a non-trivial solution (where not all t_i are zero), then the vectors are linearly dependent.

3.3 Example

Consider the vectors $x_1 = (1, 0)$, $x_2 = (0, 1)$, and $x_3 = (1, 1)$ in R^2. The set $\{x_1, x_2\}$ is linearly independent, while $\{x_1, x_2, x_3\}$ is linearly dependent since x_3 can be expressed as $x_1 + x_2$.

4 Dimension of a Subspace

The **dimension** of a subspace is defined as the maximum number of linearly independent vectors that can span the subspace.

4.1 Properties of Dimension

1. If a subspace is spanned by m vectors, then the dimension is at most m. 2. All bases of a subspace have the same number of elements, which is the dimension of that subspace.

4.2 Example of Dimension

In R^3, the only possible dimensions for subspaces are 0, 1, 2, or 3. A dimension of 0 corresponds to the zero vector, dimension 1 corresponds to a line through the origin, dimension 2 corresponds to a plane through the origin, and dimension 3 corresponds to the entire space R^3.

5 Basis of a Subspace

A **basis** of a subspace is a set of vectors that is both linearly independent and spans the subspace.

5.1 Existence of Basis

If a subspace has dimension d, then it has a basis consisting of d vectors.

5.2 Example of Basis

For the subspace spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$ in R^3, the basis is $\{(1, 0, 0), (0, 1, 0)\}$, which is linearly independent and spans the plane defined by these vectors.

6 Conclusion

In summary, we have discussed the concepts of linear independence and dimension, establishing their importance in the study of vector spaces. We have also defined what constitutes a basis and explored its properties. Understanding these concepts is essential for further studies in linear algebra.