Calculus and Its Applications

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1 Introduction

Welcome back to our calculus course. This is our second lecture of the week, where we continue to explore limits and continuity of functions.

2 Squeeze Theorem

We began with a club question that involved determining the limit of a function. The key takeaway was the application of the Squeeze Theorem. Whenever you encounter an inequality involving three functions and are asked about the limit, your intuition should immediately suggest the Squeeze Theorem.

2.1 Statement of the Squeeze Theorem

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ for all x in some interval around a (except possibly at a), and if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

2.2 Example Problem

Consider the functions:

$$f(x) = -\frac{1}{3}x^3 + x^2 - \frac{7}{3}, \quad g(x) = h(x) = \cos\left(\frac{\pi}{2}x\right).$$

We want to find the limit as $x \to 2$.

2.2.1 Checking Assumptions

We verified that both $f(x)$ and $h(x)$ are defined for all x not equal to 2 in an open interval containing 2. Since polynomials and cosine functions are continuous everywhere, the first assumption of the Squeeze Theorem is satisfied.

Next, we checked the behavior of the functions around $x = 2$ using graphs. We confirmed that the inequalities hold in the interval around 2.

2.2.2 Applying the Squeeze Theorem

Since both assumptions of the Squeeze Theorem are satisfied, we can conclude:

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} f(x) = \lim_{x \to 2} h(x) = -1.$$

3 Continuity of Functions

Next, we discussed the continuity of the function

$$\frac{x^2 - 4x + 3}{x^2 - 1}$$

over the closed interval $[0, 3]$.

3.1 Closed vs Open Intervals

We defined closed intervals as those that include their endpoints, while open intervals do not. The function is continuous over a closed interval if it is continuous at every point in the open interval and continuous from the right at the left endpoint and from the left at the right endpoint.

3.2 Identifying Points of Discontinuity

We found that the function is not defined at $x = 1$, which is within the interval $[0, 3]$. Therefore, the function is not continuous over this interval.

4 Piecewise Functions

We also examined piecewise functions, specifically:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

4.1 Checking Continuity at the Junction

To check continuity at $x = 1$, we evaluated:

$$\lim_{x \to 1^-} f(x) = 1, \quad \lim_{x \to 1^+} f(x) = 1, \quad f(1) = 1.$$

Since the left limit, right limit, and function value at $x = 1$ are all equal, the function is continuous at that point.

5 Conclusion

In summary, we explored the Squeeze Theorem, continuity of functions, and the behavior of piecewise functions. Understanding these concepts is crucial for further studies in calculus and analysis.

Thank you for your participation, and I look forward to seeing you in the next lecture.