Linear Algebra Lecture Summary - Week 10

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February 22, 2025

1 Introduction

This lecture marks the end of new material in the linear algebra course. The focus is on orthogonal concepts, particularly in the context of vector spaces and projections.

2 Vector Spaces

2.1 Generalization of Vectors

- Vectors have been discussed primarily in R^n, which are lists of real numbers. - Chapters 6 and 7 introduce generalized vector spaces, but the course will focus on real vectors.

3 Orthogonality

3.1 True or False Question

Consider a list of vectors that forms a basis for a subspace in R^n. Given a point x in this subspace, can x be expressed as:

$$x = \sum_{i=1}^{k} (x \cdot v_i)\, v_i$$

where v_i are the basis vectors?

- The answer is **false** unless the basis is orthonormal.
- Normalization is crucial; without it, the projection does not yield the correct vector.

3.2 Importance of Orthogonality

- If the basis vectors are not orthogonal, the projection of x onto the subspace may not align with the expected direction. - Orthogonal bases ensure that projections are consistent regardless of the basis used.

4 Gram-Schmidt Process

4.1 Overview

The Gram-Schmidt process is a method for converting a set of vectors into an orthogonal basis.

4.2 Steps of the Process

1. Set the first vector as the first basis vector. 2. For each subsequent vector, project it onto the existing basis vectors and subtract these projections to find the new orthogonal vector.

4.3 Example

Given three vectors u_1, u_2, u_3 in R^4, the steps are: - Set $v_1 = u_1$. - For v_2, project u_2 onto v_1 and subtract:

$$v_2 = u_2 - \mathrm{proj}_{v_1}(u_2)$$

- For v_3, project u_3 onto both v_1 and v_2 and subtract:

$$v_3 = u_3 - \mathrm{proj}_{v_1}(u_3) - \mathrm{proj}_{v_2}(u_3)$$

5 Orthogonal Complements

5.1 Definition

The orthogonal complement of a subspace U consists of all vectors in the space that are orthogonal to every vector in U.

5.2 Properties

- The dimension of the orthogonal complement is given by:

$$\dim(U^{\perp}) = n - \dim(U)$$

where n is the dimension of the entire space.

6 Linear Transformations and Projections

6.1 Projection as a Linear Transformation

- The projection onto a subspace U is a linear transformation. - The image of this transformation is the subspace U, while the kernel consists of vectors in the orthogonal complement $U^{\perp}$.

6.2 Rank-Nullity Theorem

The rank-nullity theorem states:

$$\dim(\text{Image}) + \dim(\text{Kernel}) = n$$

This theorem applies to projections, where the image is the dimension of U and the kernel is the dimension of $U^{\perp}$.

7 Conclusion

The lecture concludes with a discussion on the properties of projections and their implications in linear algebra. The importance of orthonormal bases and the Gram-Schmidt process is emphasized as essential tools for working with vector spaces.