Vector Calculus and Geometry

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1 Introduction

This lecture marks the halfway point of the first term. We will continue our discussion on the equations of straight lines and planes, as well as delve into projections and the cross product.

2 Equations of Straight Lines

We explored various forms of the equation of a straight line. The vector equation of a line passing through the origin can be expressed as:

$$\mathbf{x} = t\mathbf{d} + \mathbf{a}$$

where $\mathbf{d}$ is a direction vector and $\mathbf{a}$ is a point on the line.

2.1 Key Points

- If the line passes through the origin, then $\mathbf{a}$ must equal the zero vector.
- The direction vector $\mathbf{d}$ cannot be the zero vector, as this would not define a line.
- The equation can be manipulated to express different points along the line.

3 Equations of Planes

We discussed the parametric form of a plane, which can be expressed as:

$$\mathbf{x} = \mathbf{p} + s\mathbf{a} + t\mathbf{b}$$

where $\mathbf{p}$ is a point on the plane, and $\mathbf{a}$ and $\mathbf{b}$ are direction vectors lying in the plane.

3.1 Normal Form of a Plane

The normal form of a plane is given by:

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the normal vector to the plane.

3.2 Key Points

- The normal vector is perpendicular to any vector lying in the plane.
- The relationship between points and vectors in the plane is crucial for understanding geometric properties.

4 Projections

We introduced the concept of projecting a vector onto a line and a plane. The projection of a vector $\mathbf{u}$ onto a line defined by direction vector $\mathbf{d}$ is given by:

$$\operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$$

4.1 Projection onto a Plane

To project a vector onto a plane, we can project onto two direction vectors that lie in the plane. The projection can be calculated iteratively.

4.2 Key Points

- The projection provides the shortest distance from a point to a line or plane.
- Understanding projections is essential for applications in optimization and geometry.

5 Cross Product

The cross product is defined for vectors in three-dimensional space and is given by:

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)\mathbf{n}$$

where θ is the angle between the vectors and $\mathbf{n}$ is the unit normal vector.

5.1 Key Points

- The magnitude of the cross product represents the area of the parallelogram spanned by the two vectors.
- The direction of the cross product follows the right-hand rule.
- The cross product is zero if the vectors are parallel.

6 Conclusion

In this lecture, we covered the equations of lines and planes, projections, and the cross product. Understanding these concepts is fundamental for further studies in vector calculus and geometry.