Lecture Summary: Vectors and Geometry

Generated by LectureMate

February 22, 2025

1 Introduction

This lecture marks the halfway point of the term, focusing on the equations of straight lines and planes, as well as the concepts of projections and the cross product.

2 Equations of Straight Lines

We began by discussing various forms of the equation of a straight line. The vector equation of a line passing through the origin can be expressed as:

$$\mathbf{x} = t\mathbf{d} + \mathbf{a}$$

where $\mathbf{d}$ is a direction vector and $\mathbf{a}$ is a point on the line.

2.1 Key Points

- If the line passes through the origin, then $\mathbf{a}$ must equal $\mathbf{0}$.
- The direction vector $\mathbf{d}$ must not be the zero vector, as this would not define a line.
- The scalar multiple relationship between $\mathbf{a}$ and $\mathbf{d}$ is crucial for understanding the line's direction.

2.2 Discussion

We explored the implications of setting $\mathbf{a}$ to zero and the necessity of $\mathbf{d}$ being non-zero. The equation $\mathbf{x} = t\mathbf{d}$ represents a line through the origin, where t is a scalar parameter.

3 Equations of Planes

Next, we transitioned to the equations of planes. The parametric form of a plane can be expressed as:

$$\mathbf{x} = \mathbf{p} + s\mathbf{a} + t\mathbf{b}$$

where $\mathbf{p}$ is a point on the plane, and $\mathbf{a}$ and $\mathbf{b}$ are direction vectors lying in the plane.

3.1 Normal Form of a Plane

The normal form of a plane is given by:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

where $\mathbf{n}$ is the normal vector to the plane.

3.2 Key Points

- The normal vector $\mathbf{n}$ is perpendicular to any vector lying in the plane.
- Understanding the relationship between points and vectors in the plane is essential for solving problems related to planes.

4 Projections

We then discussed projections, starting with projecting a vector onto a line. The projection of a vector $\mathbf{u}$ onto a line defined by direction vector $\mathbf{d}$ is given by:

$$\operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$$

4.1 Projection onto a Plane

To project a vector onto a plane, we can project onto two direction vectors of the plane. The projection can be calculated iteratively.

4.2 Example

Given a vector $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and a plane defined by its normal vector, we can find the projection of $\mathbf{u}$ onto the plane.

5 Cross Product

The cross product is defined only in three dimensions and is used to find a vector perpendicular to two given vectors $\mathbf{u}$ and $\mathbf{v}$. The magnitude of the cross product is equal to the area of the parallelogram formed by the two vectors:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)$$

where θ is the angle between the two vectors.

5.1 Geometric Interpretation

The right-hand rule can be used to determine the direction of the cross product. The cross product is zero if the vectors are parallel.

6 Conclusion

The lecture concluded with a discussion on the relationships between vectors in the context of planes and the implications of the cross product. Homework was assigned, and the next session will continue with these concepts.