Lecture Summary: Eigenvalues and Eigenvectors

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1 Introduction

This lecture marks the fifth week of the course, focusing on eigenvalues and eigenvectors. These concepts are fundamental in various areas of mathematics and have numerous applications in computational mathematics.

2 Linear Transformations

2.1 Review of Linear Transformations

We began with a review of linear transformations, discussing which transformations are linear. The key properties of linear transformations include:

- The origin must be mapped to itself.
- The transformation must preserve vector addition and scalar multiplication.

We analyzed several transformations:

- Reflection and rotation around the origin are linear transformations.
- Translation is not a linear transformation as it does not fix the origin.
- Stretching the y-component by a factor of four is a linear transformation.
- Mapping every point to the origin is also a linear transformation.

2.2 Geometric Interpretation

We illustrated the geometric implications of these transformations, emphasizing the importance of fixed points and lines. For example, reflecting in the x-axis keeps the x-axis fixed while mapping other points accordingly.

3 Eigenvalues and Eigenvectors

3.1 Definitions

An eigenvector $\mathbf{x}$ of a matrix A is a non-trivial vector such that:

$$A\mathbf{x} = \lambda\mathbf{x}$$

where λ is the corresponding eigenvalue. The eigenvalue represents the factor by which the eigenvector is scaled during the transformation.

3.2 Historical Context

The term "eigen" was coined by David Hilbert, meaning "own" in German, indicating that the eigenvector is intrinsic to the matrix transformation.

3.3 Examples and Applications

We discussed practical examples, such as reflecting across a line. The eigenvalues for a reflection are typically 1 and -1, indicating that points on the line remain unchanged while points off the line are inverted.

4 Calculating Eigenvalues

4.1 Characteristic Polynomial

To find the eigenvalues of a matrix A, we solve the characteristic polynomial:

$$\det(\lambda I_n - A) = 0$$

where I_n is the identity matrix. We demonstrated this with a sample matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 9 \end{pmatrix}$$

Calculating the determinant leads to the eigenvalues.

4.2 Proof of Eigenvalue Condition

We proved that λ is an eigenvalue of A if and only if the matrix $\lambda I_n - A$ is not invertible, which is equivalent to its determinant being zero.

5 Conclusion

The lecture concluded with a discussion on the importance of eigenvalues and eigenvectors in understanding linear transformations. We also previewed the next lecture, which will cover finding eigenvectors and diagonalizing matrices.