Calculus and Its Applications - Week 4

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1 Introduction

Welcome to week four of Calculus and its Applications. In this lecture, we will discuss the applications of derivatives, building on the foundational concepts introduced in the previous week.

2 Linear Approximation

The derivative can be viewed as a linear approximation of a function at a specific point. For example, we consider the function $f(x) = \sqrt[3]{1+x}$ and find its linear approximation at the point $A = 0$.

2.1 Finding the Derivative

To find the linear approximation, we calculate the derivative at the point of interest:

$$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$$

Evaluating this at $x = 0$ gives:

$$f'(0) = \frac{1}{3}$$

Thus, the linear approximation is:

$$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{3}x$$

3 Maximizing a Product

Next, we explore the problem of finding two positive numbers whose sum is 300 and whose product is maximized. Let the two numbers be x and $300 - x$.

3.1 Setting Up the Function

The product P can be expressed as:

$$P(x) = x(300-x) = 300x - x^2$$

To find the maximum, we take the derivative:

$$P'(x) = 300 - 2x$$

Setting the derivative equal to zero:

$$300 - 2x = 0 \implies x = 150$$

Thus, the two numbers are 150 and 150.

3.2 Using the Extreme Value Theorem

To confirm that this is a maximum, we apply the Extreme Value Theorem, which states that a continuous function on a closed interval will have a maximum at either a critical point or an endpoint. Evaluating P at the endpoints (0 and 300) gives $P(0) = 0$ and $P(300) = 0$, confirming that the maximum occurs at $x = 150$ with a product of 22500.

4 Mean Value Theorem Application

We consider a scenario where Alex drives from Edinburgh to Glasgow. The first speed camera measures his speed at 59 mph after 14 miles, and the second measures 58 mph after 31 miles.

4.1 Calculating Average Speed

The average speed over the distance of 17 miles in 15 minutes is:

$$\text{Average Speed} = \frac{17 \text{ miles}}{\frac{15}{60} \text{ hours}} = 68 \text{ mph}$$

Since this average speed exceeds the speed limit of 60 mph, by the Mean Value Theorem, Alex must have exceeded the speed limit at some point.

5 Inflection Points

We examine whether the origin is an inflection point for the function $f(x) = x^4$.

5.1 Determining Concavity

An inflection point occurs where the function changes concavity, determined by the second derivative:

$$f''(x) = 12x^2$$

Since $f''(x)$ is positive for all $x \neq 0$ and does not change sign at $x = 0$, the origin is not an inflection point.

6 Limits and L'Hôpital's Rule

Finally, we discuss evaluating limits using derivatives, specifically applying L'Hôpital's Rule.

6.1 Example Limit

To find the limit:

$$\lim_{x \to 1} \frac{\ln x}{e^x - e}$$

we apply L'Hôpital's Rule since both the numerator and denominator approach 0. The derivatives are:

$$\lim_{x \to 1} \frac{\frac{1}{x}}{e^x} = \frac{1}{e}$$

7 Conclusion

In this lecture, we covered various applications of derivatives, including linear approximation, maximizing products, the Mean Value Theorem, inflection points, and evaluating limits. We will continue exploring more applications in the next session.