Calculus and Its Applications - Week 4

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1 Introduction

Welcome to week four of Calculus and its Applications. In this lecture, we will discuss the applications of derivatives, building on the definitions and properties studied in the previous week.

2 Linear Approximation

The derivative can be viewed as a linear approximation of a function at a point. For example, we consider the function $f(x) = \sqrt[3]{1+x}$ and find its linear approximation at the point $A = 0$.

2.1 Finding the Derivative

To find the linear approximation, we first compute the derivative:

$$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$$

Evaluating this at $x = 0$:

$$f'(0) = \frac{1}{3}$$

Thus, the linear approximation at $A = 0$ is given by:

$$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{1}{3}x$$

3 Maximizing a Product

Next, we explore the problem of finding two positive numbers whose sum is 300 and whose product is maximized.

3.1 Setting Up the Problem

Let the two numbers be x and $300 - x$. We want to maximize the product:

$$P(x) = x(300 - x) = 300x - x^2$$

3.2 Finding Critical Points

To find the maximum, we take the derivative and set it to zero:

$$P'(x) = 300 - 2x = 0 \implies x = 150$$

We check the endpoints and find that the product at $x = 150$ is:

$$P(150) = 150 \times 150 = 22500$$

Thus, the maximum product occurs at $x = 150$.

4 Mean Value Theorem Application

We consider a scenario where Alex drives from Edinburgh to Glasgow, and we need to determine if he receives a speeding ticket based on his average speed.

4.1 Calculating Average Speed

Alex's speed is measured at two points: - At 14 miles, speed = 59 mph at 10:15 a.m. - At 31 miles, speed = 58 mph at 10:30 a.m.

The average speed over the 17 miles is:

$$\text{Average Speed} = \frac{17 \text{ miles}}{15 \text{ minutes}} = 68 \text{ mph}$$

Since the speed limit is 60 mph, by the Mean Value Theorem, Alex must have exceeded the speed limit at some point.

5 Inflection Points

We examine whether the origin is an inflection point for the function $f(x) = x^4$.

5.1 Determining Inflection Points

An inflection point occurs where the second derivative changes sign. The second derivative is:

$$f''(x) = 12x^2$$

Since $f''(x)$ is positive for all $x \neq 0$ and does not change sign, the origin is not an inflection point.

6 Evaluating Limits Using Derivatives

We conclude with an example of using derivatives to evaluate limits. We need to find:

$$\lim_{x \to 1} \frac{\ln x}{e^x - e}$$

6.1 Applying L'Hôpital's Rule

Since both the numerator and denominator approach 0 as $x \to 1$, we can apply L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{e^x - e} = \lim_{x \to 1} \frac{\frac{1}{x}}{e^x} = \frac{1}{e}$$

7 Conclusion

In this lecture, we covered several applications of derivatives, including linear approximation, maximizing products, the Mean Value Theorem, inflection points, and evaluating limits. We will continue exploring more applications in the next session.