Linear Transformations and Matrix Algebra

Generated by LectureMate

February 22, 2025

1 Introduction

This lecture focuses on the concept of transformations, particularly linear transformations, and their relationship with matrices. We will explore definitions, properties, examples, and practical applications of linear transformations.

2 Warm-Up Poll

The lecture began with a poll regarding composite functions. The majority of students correctly identified that for functions g and f, the expression $g(f(u))$ can be interpreted as applying f to u first, followed by applying g to the result.

3 Linear Transformations

3.1 Definition

A linear transformation is defined by two key properties:

- It preserves addition: $T(x + y) = T(x) + T(y)$ for all vectors x and y.
- It preserves scalar multiplication: $T(ax) = aT(x)$ for all vectors x and scalars a.

This definition is central to the course, as it allows us to identify linear transformations through their properties rather than explicit formulas.

3.2 Examples of Linear Transformations

We examined two functions:

- $f(x) = x + 1$
- $g(x) = -2x$

For $f(x)$, we found that it does not satisfy the linear transformation properties, as shown by the counterexample:

$$f(x + y) = (x + y) + 1 \neq f(x) + f(y) = (x + 1) + (y + 1)$$

Conversely, $g(x)$ satisfies both properties:

$$g(x + y) = -2(x + y) = -2x - 2y = g(x) + g(y)$$
$$g(ax) = -2(ax) = a(-2x) = ag(x)$$

4 Geometric Interpretations

We discussed various transformations geometrically:

- The identity transformation sends x to x.
- Stretching a vector by a factor of 3 is represented by $g(x) = 3x$.
- Sending every vector to $(1, 0)$ is not linear.
- Rotating vectors in the plane is a linear transformation.

4.1 Rotation Matrix

The rotation of the plane through an angle θ is induced by the matrix:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This matrix rotates vectors in the plane while preserving linearity.

5 Matrix Transformations

5.1 Induced Transformations

For any matrix A, the transformation induced is given by $T(x) = Ax$. We established that every matrix transformation is linear.

5.2 Constructing Matrices from Transformations

Given a linear transformation, we can construct its matrix by applying the transformation to the standard basis vectors. The images of these vectors form the columns of the matrix.

6 Conclusion

We concluded that every linear transformation can be represented by a matrix, and conversely, every matrix induces a linear transformation. This relationship is fundamental in linear algebra and will be explored further in future lectures.