Applications of Derivatives

Generated by LectureMate

February 22, 2025

1 Introduction

In today's lecture, we continued our discussion on the applications of derivatives. We focused on optimization problems, specifically exploring maxima and minima, and also covered some interesting limits.

2 Mid-Semester Feedback

Before diving into the main content, a reminder was given regarding the mid-semester feedback, which is open until February 14th. Students were encouraged to submit their feedback to improve the course experience.

3 Optimization Problems

We began with an optimization problem where we needed to find the maximum value of a revenue function under certain constraints.

3.1 Example 1: Revenue Function

The problem involved maximizing a revenue function given by:

$$R(x) = 900 - 60x^{\frac{1}{3}}$$

with constraints that x (the number of units sold) must be non-negative and the revenue must also be non-negative.

3.1.1 Finding Constraints

To find the constraints, we set:

$$900 - 60x^{\frac{1}{3}} \geq 0$$

Solving this inequality gives:

$$x \leq 3375$$

Thus, we have an optimization problem on the closed interval $[0, 3375]$.

3.1.2 Extreme Value Theorem

According to the Extreme Value Theorem, to find the absolute maximum of a continuous function on a closed bounded interval, we need to evaluate the function at critical points and endpoints.

3.1.3 Critical Points

We found the derivative of the revenue function:

$$R'(x) = -20x^{-\frac{2}{3}}$$

Setting the derivative equal to zero, we find the critical point:

$$x = 1000$$

3.1.4 Evaluating the Function

Next, we evaluate the revenue function at the critical point and endpoints:

$$R(0) = 900, \quad R(1000) = 3000, \quad R(3375) = 0$$

Thus, the maximum revenue occurs at $x = 1000$ with a maximum value of $R(1000) = 3000$.

3.2 Example 2: Two Positive Numbers

The next problem involved finding two positive numbers whose product is 750, and for which the sum of one and ten times the other is minimized.

3.2.1 Setting Up the Problem

Let x be one number and $y = \frac{750}{x}$ be the other. We want to minimize the function:

$$S(x) = x + 10y = x + 10\left(\frac{750}{x}\right)$$

This leads to:

$$S(x) = x + \frac{7500}{x}$$

3.2.2 Finding Critical Points

Taking the derivative and setting it to zero:

$$S'(x) = 1 - \frac{7500}{x^2} = 0$$

Solving gives:

$$x^2 = 7500 \quad \Rightarrow \quad x = 50\sqrt{3}$$

3.2.3 Second Derivative Test

To confirm that this is a minimum, we compute the second derivative:

$$S''(x) = \frac{15000}{x^3}$$

Since $S''(x) > 0$ for $x > 0$, we conclude that $x = 50\sqrt{3}$ is indeed a local minimum.

4 Limits

We also discussed limits, particularly focusing on indeterminate forms.

4.1 Example: Limit of a Function

We evaluated the limit:

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) \cos(x)$$

As x approaches $\frac{\pi}{2}$, $\tan(x)$ approaches infinity while $\cos(x)$ approaches zero, leading to an indeterminate form.

4.1.1 Using L'Hôpital's Rule

To resolve this, we applied L'Hôpital's Rule. We transformed the limit into a suitable form and calculated:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\tan(x))}{\frac{1}{\cos(x)}}$$

After applying L'Hôpital's Rule, we found that the limit evaluates to:

$$e^0 = 1$$

5 Conclusion

In conclusion, we explored various optimization problems and limits, reinforcing the importance of theoretical justifications for our computations. Students were encouraged to practice these concepts further.