Linear Algebra - Week 8 Summary

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1 Introduction

Welcome to week eight of Linear Algebra. This week is significant as we revisit concepts previously covered and extend them to more general structures. We will explore linear transformations in R^2 and R^3, and extend our understanding to R^n.

2 Topics Covered

This week's lessons are divided into two main topics:

- Dot product, orthogonality, length, Cauchy-Schwarz inequality, and triangle inequality.
- Similarity, eigenvalues, eigenvectors, and eigenspaces.

3 Dot Product and Orthogonality

The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$$

where θ is the angle between the vectors. Two vectors are orthogonal if their dot product is zero.

3.1 Length of Vectors

The length (or norm) of a vector $\mathbf{a}$ is given by:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

The square of the length is non-negative, and thus $\|\mathbf{a}\|^2 \geq 0$.

3.2 Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality states that for any vectors $\mathbf{a}$ and $\mathbf{b}$:

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

Equality holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

3.3 Triangle Inequality

The triangle inequality states that for any vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are collinear.

4 Exercises and Examples

The lecture included several exercises to reinforce these concepts. For example, given the lengths of two vectors and their scalar product, students were asked to find the length of a linear combination of these vectors.

4.1 Proof of Orthogonality Implies Linear Independence

A key theorem discussed was that every orthogonal set in R^n is linearly independent. The proof involves showing that if a linear combination of orthogonal vectors equals zero, then all coefficients must be zero.

5 Eigenvalues and Eigenvectors

The second part of the lecture focused on eigenvalues and eigenvectors. An eigenvector $\mathbf{v}$ of a matrix $\mathbf{A}$ satisfies:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

where λ is the corresponding eigenvalue. The eigenspace associated with λ is the set of all eigenvectors corresponding to λ, along with the zero vector.

6 Conclusion

This week's lecture emphasized the importance of understanding linear transformations and their properties in higher dimensions. The concepts of orthogonality, the Cauchy-Schwarz inequality, and eigenvalues are foundational in linear algebra and have numerous applications in various fields.