Differential-Geometric Analysis of the Conical Helix

A Complete Treatment of Curvature, Torsion,
Frenet Frame, Arc Length, and Geometric Properties

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Abstract

This paper presents a complete differential-geometric analysis of the conical helix defined by the parametrization

$$\gamma(\theta) = \begin{pmatrix} \dfrac{6}{\pi^2}\theta\cos(\theta) \\ \dfrac{6}{\pi^2}\theta\sin(\theta) \\ -\dfrac{3}{4} + \dfrac{3}{\pi}\theta \end{pmatrix}, \qquad \theta \geq 0.$$

The curve combines the properties of an Archimedean spiral (in the xy-projection) with a linear increase in height, and lies entirely on the cone $z = -\frac{3}{4} + \frac{\pi}{2}r$. We compute in closed form: the tangent vector, principal normal, binormal (Frenet frame), curvature, torsion, arc length, pitch angle, and the asymptotic behaviour of all these quantities. A key result is that the spiral constant $a = 6/\pi^2 = 1/\zeta(2)$ connects number theory and differential geometry directly.

Contents

1 Introduction and Background

1.1 The Conical Helix

A *helix* is a space curve that winds spirally around an axis [4, 5]. In the classical case of the *cylindrical helix*, the radius remains constant and both curvature and torsion are constant. The curve studied here is a fundamental generalisation: the radius grows linearly with the angle (Archimedean spiral), while the height also increases linearly. The curve therefore winds on a *cone* and is called a **conical helix**.

Definition 1.1 (Conical Helix)**.** The conical helix is the space curve $\gamma : [0, \infty) \to \mathbb{R}^3$ defined by:

$$\boxed{\gamma(\theta) = \begin{pmatrix} \dfrac{6}{\pi^2}\,\theta\cos(\theta) \\ \dfrac{6}{\pi^2}\,\theta\sin(\theta) \\ -\dfrac{3}{4} + \dfrac{3}{\pi}\,\theta \end{pmatrix}} \tag{1}$$

with structural constants

$$a := \frac{6}{\pi^2} \approx 0.607927101854027, \qquad b := \frac{3}{\pi} \approx 0.954929658551372. \tag{2}$$

Remark 1.1 (Structural Constants and the Basel Problem). The spiral constant $a = 6/\pi^2$ is the reciprocal of the famous Euler sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, known as the Basel problem (L. Euler, 1734/1740) [16, 17]. The ratio of the two constants is remarkable:

$$\frac{b}{a} = \frac{3/\pi}{6/\pi^2} = \frac{\pi}{2}. \tag{3}$$

As shown in Section 2, this ratio equals exactly the slope of the cone.

1.2 Component Form

In component form, Eq. (1) reads:

$$x(\theta) = a\theta\cos(\theta), \tag{4}$$
$$y(\theta) = a\theta\sin(\theta), \tag{5}$$
$$z(\theta) = -\tfrac{3}{4} + b\theta. \tag{6}$$

The xy-projection of γ is the *Archimedean spiral* $r(\theta) = a\theta$ [15, 14, 1], while $z(\theta)$ depends linearly on θ.

1.3 Regularity

Proposition 1.2 (Regularity)**.** *The curve γ is regular for all $\theta > 0$, i.e. $\dot{\gamma}(\theta) \neq \mathbf{0}$.*

Proof. We have $\dot{z}(\theta) = b = 3/\pi > 0$ for all θ. Therefore the third component of the tangent vector is always non-zero, which implies $\dot{\gamma} \neq \mathbf{0}$. □

2 Position on the Cone

Theorem 2.1 (Cone Equation)**.** *The conical helix γ lies entirely on the cone*

$$\mathcal{K} := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = -\frac{3}{4} + \frac{\pi}{2}\sqrt{x^2 + y^2} \right\}. \tag{7}$$

Proof. The distance of a curve point from the z-axis is:

$$r(\theta) = \sqrt{x(\theta)^2 + y(\theta)^2} = \sqrt{a^2\theta^2 \cos^2\theta + a^2\theta^2 \sin^2\theta} = a\theta.$$

Substituting into the right-hand side of Eq. (7):

$$-\frac{3}{4} + \frac{\pi}{2} \cdot a\theta = -\frac{3}{4} + \frac{\pi}{2} \cdot \frac{6}{\pi^2}\theta = -\frac{3}{4} + \frac{3}{\pi}\theta = z(\theta).$$

□

Corollary 2.2 (Cone Slope)**.** *The slope of the cone generator is $\pi/2$; the corresponding cone angle from the horizontal (xy-plane) is:*

$$\psi_{\mathcal{K}} = \arctan\left(\frac{\pi}{2}\right) \approx 57.518°. \tag{8}$$

Remark 2.1 (Two Cone Angles)*.* Two different angles are used in the literature to characterise a cone, depending on convention:

- The *elevation angle from the horizontal* (from the xy-plane): $\psi_{\mathcal{K}} = \arctan(\pi/2) \approx 57.52°$.
- The *opening half-angle from the rotation axis* (from the z-axis): $\alpha = \arctan(2/\pi) \approx 32.48°$.

Both angles are complementary: $\psi_{\mathcal{K}} + \alpha = 90°$.

3 Tangent Vector and Arc Length

3.1 Tangent Vector

Proposition 3.1 (Tangent Vector)**.** *The tangent vector $\dot{\gamma}(\theta)$ is:*

$$\dot{\gamma}(\theta) = \begin{pmatrix} a(\cos\theta - \theta\sin\theta) \\ a(\sin\theta + \theta\cos\theta) \\ b \end{pmatrix}. \tag{9}$$

Proposition 3.2 (Speed)**.**

$$|\dot{\gamma}(\theta)| = \sqrt{a^2(1 + \theta^2) + b^2}. \tag{10}$$

Proof.

$$\begin{aligned}
|\dot\gamma|^2 &= a^2(\cos\theta - \theta\sin\theta)^2 + a^2(\sin\theta + \theta\cos\theta)^2 + b^2 \\
&= a^2\left[\cos^2\theta - 2\theta\sin\theta\cos\theta + \theta^2\sin^2\theta\right] \\
&\quad + a^2\left[\sin^2\theta + 2\theta\sin\theta\cos\theta + \theta^2\cos^2\theta\right] + b^2 \\
&= a^2\left[\underbrace{\cos^2\theta + \sin^2\theta}_{=1} + \theta^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}\right] + b^2 \\
&= a^2(1+\theta^2) + b^2.
\end{aligned}$$

□

Corollary 3.3 (Initial Speed)**.** *Bei $\theta = 0$ gilt:*

$$|\dot\gamma(0)| = \sqrt{a^2+b^2} = \sqrt{\frac{36}{\pi^4} + \frac{9}{\pi^2}} = \frac{3}{\pi}\sqrt{\frac{4}{\pi^2}+1} \approx 1.13202.$$

Corollary 3.4 (Asymptotic Speed)**.** *For $\theta \to \infty$:*

$$|\dot\gamma(\theta)| \sim a\theta = \frac{6}{\pi^2}\theta.$$

3.2 Second and Third Derivatives

Proposition 3.5 (Second Derivative)**.**

$$\ddot\gamma(\theta) = \begin{pmatrix} a(-2\sin\theta - \theta\cos\theta) \\ a(2\cos\theta - \theta\sin\theta) \\ 0 \end{pmatrix}. \tag{11}$$

Proposition 3.6 (Third Derivative)**.**

$$\dddot\gamma(\theta) = \begin{pmatrix} a(-3\cos\theta + \theta\sin\theta) \\ a(-3\sin\theta - \theta\cos\theta) \\ 0 \end{pmatrix}. \tag{12}$$

3.3 Arc Length

Theorem 3.7 (Arc Length)**.** *The arc length of the conical helix from $\theta = 0$ to $\theta = T$ is:*

$$L(T) = \frac{a}{2}\left[T\sqrt{T^2+c^2} + c^2\,\operatorname{arcsinh}\left(\frac{T}{c}\right)\right], \tag{13}$$

where

$$c^2 := \frac{a^2+b^2}{a^2} = 1 + \frac{\pi^2}{4} \approx 3.4674, \qquad c \approx 1.8621. \tag{14}$$

Proof. From Theorem 3.2:

$$L(T) = \int_0^T \sqrt{a^2(1+t^2)+b^2}\,dt = a\int_0^T \sqrt{t^2 + \frac{a^2+b^2}{a^2}}\,dt = a\int_0^T \sqrt{t^2+c^2}\,dt.$$

Das Integral $\int \sqrt{t^2+c^2}\,dt = \frac{1}{2}\left[t\sqrt{t^2+c^2} + c^2\operatorname{arcsinh}(t/c)\right]$ is a standard integral [12, 13]. □

Remark 3.1 (The Constant c). The constant $c = \sqrt{1 + \pi^2/4}$ follows directly from the ratio $b/a = \pi/2$ (cf. Remark 1.1). It governs the long-run growth rate of the arc length.

Proposition 3.8 (Asymptotic Arc Length)**.** *For $T \to \infty$:*

$$L(T) \sim \frac{a}{2} T^2 = \frac{3}{\pi^2} T^2. \tag{15}$$

Proof. For $T \gg c$: $\sqrt{T^2 + c^2} \approx T$ and $\operatorname{arcsinh}(T/c) \approx \ln(2T/c) = o(T^2)$, so the dominant term is $\frac{a}{2} T^2$. □

Example 3.1 (Arc Length per Revolution)**.** Arc length of the first five revolutions:

Table 1: Arc length of the conical helix per revolution

Revolution n	θ: from	θ: to	**Arc length** L
1	0	2π	14.5507
2	2π	4π	36.7221
3	4π	6π	60.4258
4	6π	8π	84.3867
5	8π	10π	108.3747

Cumulative arc length after n full revolutions:

Table 2: Cumulative arc length after n revolutions

Revolutions n	$\theta_{\max}$	$r(\theta_{\max})$	**Arc length** L
1	2π	3.820	14.5507
2	4π	7.639	51.2728
3	6π	11.459	111.6985
5	10π	19.099	304.2361
10	20π	38.197	1204.9663

4 The Frenet–Serret Frame

4.1 Overview

The *Frenet–Serret frame* (also called the moving trihedron, cf. [4, 8, 5]) is an orthonormal coordinate system that moves along the curve and completely describes its local geometry. Er besteht aus drei Vektoren:

- $\boldsymbol{T}(\theta)$ – unit tangent vector (direction of motion),

- $\boldsymbol{N}(\theta)$ – principal normal unit vector (direction of curvature),
- $\boldsymbol{B}(\theta)$ – binormal unit vector ($\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$).

4.2 Unit Tangent Vector

Definition 4.1 (Unit Tangent Vector)**.**

$$\boldsymbol{T}(\theta) = \frac{\dot{\gamma}(\theta)}{|\dot{\gamma}(\theta)|} = \frac{1}{\sqrt{a^2(1+\theta^2)+b^2}} \begin{pmatrix} a(\cos\theta - \theta\sin\theta) \\ a(\sin\theta + \theta\cos\theta) \\ b \end{pmatrix}. \tag{16}$$

Proposition 4.2 (Pitch Angle)**.** *The angle $\psi(\theta)$ between $\boldsymbol{T}$ and the xy-plane (pitch angle) is:*

$$\psi(\theta) = \arctan\left(\frac{b}{a\sqrt{1+\theta^2}}\right) = \arctan\left(\frac{\pi}{2\sqrt{1+\theta^2}}\right). \tag{17}$$

Remark 4.1. At $\theta \to 0$: $\psi(0) = \arctan(\pi/2) \approx 57.52°$, exactly the cone angle from Section 2. As $\theta \to \infty$, $\psi \to 0°$: the curve becomes increasingly flat.

Table 3: Pitch angle $\psi(\theta)$ of the conical helix

θ	$\psi(\theta)$ (rad)	$\psi(\theta)$ (degrees)
$\theta \to 0$	1.0039	57.517°
$\pi/2$	0.7007	40.150°
π	0.4446	25.475°
2π	0.2421	13.869°
4π	0.1240	7.103°
10π	0.0499	2.861°

4.3 The Cross Product $\dot{\gamma} \times \ddot{\gamma}$

Lemma 4.3 (Cross Product)**.** *The cross product $\dot{\gamma} \times \ddot{\gamma}$ has components:*

$$(\dot{\gamma} \times \ddot{\gamma})_x = -ab(2\cos\theta - \theta\sin\theta), \tag{18}$$
$$(\dot{\gamma} \times \ddot{\gamma})_y = -ab(2\sin\theta + \theta\cos\theta), \tag{19}$$
$$(\dot{\gamma} \times \ddot{\gamma})_z = a^2(2+\theta^2). \tag{20}$$

Proof. Using $\dot{\gamma}$ from Eq. (9) and $\ddot{\gamma}$ from Eq. (11):

$$\begin{aligned}(\dot{\gamma} \times \ddot{\gamma})_z &= \dot{x}\ddot{y} - \dot{y}\ddot{x} \\ &= a(\cos\theta - \theta\sin\theta) \cdot a(2\cos\theta - \theta\sin\theta) \\ &\quad - a(\sin\theta + \theta\cos\theta) \cdot a(-2\sin\theta - \theta\cos\theta) \\ &= a^2\left[\cos^2\theta(2-\theta^2) + \sin^2\theta(2-\theta^2) + 2\theta^2\right] \\ &= a^2(2 - \theta^2 + 2\theta^2) = a^2(2+\theta^2).\end{aligned}$$

□

Proposition 4.4 (Norm of the Cross Product)**.**

$$|\dot{\gamma} \times \ddot{\gamma}|^2 = a^2 b^2 (4 + \theta^2) + a^4 (2 + \theta^2)^2. \tag{21}$$

Proof. Inserting Eqs. (18) to (20):

$$\begin{aligned} |\dot{\gamma} \times \ddot{\gamma}|^2 &= a^2 b^2 (2\cos\theta - \theta\sin\theta)^2 + a^2 b^2 (2\sin\theta + \theta\cos\theta)^2 + a^4 (2 + \theta^2)^2 \\ &= a^2 b^2 \left[(2\cos\theta - \theta\sin\theta)^2 + (2\sin\theta + \theta\cos\theta)^2 \right] + a^4 (2 + \theta^2)^2 \\ &= a^2 b^2 (4 + \theta^2) + a^4 (2 + \theta^2)^2. \end{aligned}$$

□

5 Curvature

Theorem 5.1 (Curvature of the Conical Helix)**.** *The curvature $\kappa(\theta)$ of the conical helix is:*

$$\kappa(\theta) = \frac{\sqrt{a^2 b^2 (4 + \theta^2) + a^4 (2 + \theta^2)^2}}{\left[a^2 (1 + \theta^2) + b^2\right]^{3/2}}. \tag{22}$$

Using $b = \frac{\pi}{2} a$, this simplifies to:

$$\kappa(\theta) = \frac{a\sqrt{\frac{\pi^2}{4}(4 + \theta^2) + (2 + \theta^2)^2}}{\left[a^2 \left(1 + \theta^2 + \frac{\pi^2}{4}\right)\right]^{3/2}}. \tag{23}$$

Proof. Follows from the Frenet formula $\kappa = |\dot{\gamma} \times \ddot{\gamma}| \,/\, |\dot{\gamma}|^3$ together with Theorems 3.2 and 4.4. □

Proposition 5.2 (Initial Curvature)**.** *Bei $\theta = 0$ gilt:*

$$\kappa(0) = \frac{2a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{2a}{a^2 + b^2} = \frac{4\pi^2}{3(\pi^2 + 4)} \approx 0.948799. \tag{24}$$

Proof. Setting $\theta = 0$ in Eq. (22), the numerator becomes:

$$\sqrt{a^2 b^2 (4 + 0) + a^4 (2 + 0)^2} = \sqrt{4a^2 b^2 + 4a^4} = 2a\sqrt{b^2 + a^2}.$$

The denominator is $(a^2 \cdot 1 + b^2)^{3/2} = (a^2 + b^2)^{3/2}$. Hence:

$$\kappa(0) = \frac{2a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{2a}{a^2 + b^2}.$$

□

Proposition 5.3 (Asymptotic Curvature)**.** *For $\theta \to \infty$:*

$$\kappa(\theta) \sim \frac{a}{\theta} = \frac{6}{\pi^2 \theta}. \tag{25}$$

Proof. For large θ: numerator $\sim a^2\theta^2$, denominator $\sim a^3\theta^3$, so $\kappa \sim 1/(a\theta)$. □

Table 4: Curvature and radius of curvature of the conical helix

θ	$\kappa(\theta)$	$\varrho = 1/\kappa$	$\lvert\dot\gamma(\theta)\rvert$
0	0.948799	1.054	1.1320
$\pi/2$	0.681835	1.467	1.4810
π	0.446906	2.238	2.2201
2π	0.249876	4.002	3.9839
4π	0.129307	7.734	7.7229
10π	0.052256	19.137	19.132

Remark 5.1 (Comparison with the Cylindrical Helix). For a *cylindrical helix* with radius R and pitch h, both curvature $\kappa = R/(R^2+h^2)$ and torsion $\tau = h/(R^2+h^2)$ are *constant* [4, 5]. For the conical helix both decrease monotonically ($\kappa \sim 1/\theta$, $\tau \sim 1/\theta^2$) – a fundamental geometric generalisation [7, 6, 18].

6 Torsion

Theorem 6.1 (Torsion of the Conical Helix). *The torsion $\tau(\theta)$ of the conical helix is:*

$$\tau(\theta) = \frac{a^2 b(6+\theta^2)}{a^2b^2(4+\theta^2) + a^4(2+\theta^2)^2} = \frac{b(6+\theta^2)}{b^2(4+\theta^2)+a^2(2+\theta^2)^2}. \tag{26}$$

Proof. Using the formula $\tau = (\dot\gamma \times \ddot\gamma)\cdot \dddot\gamma / |\dot\gamma \times \ddot\gamma|^2$.

Numerator. Using $\dddot\gamma$ from Eq. (12) and the cross product from Theorem 4.3:

$$\begin{aligned}(\dot\gamma \times \ddot\gamma)\cdot\dddot\gamma &= -ab(2\cos\theta - \theta\sin\theta)\cdot a(-3\cos\theta + \theta\sin\theta)\\ &\quad + (-ab)(2\sin\theta + \theta\cos\theta)\cdot a(-3\sin\theta - \theta\cos\theta) + 0\\ &= a^2 b\big[(2\cos\theta - \theta\sin\theta)(3\cos\theta - \theta\sin\theta)\\ &\qquad + (2\sin\theta + \theta\cos\theta)(3\sin\theta + \theta\cos\theta)\big]\\ &= a^2b\big[6\cos^2\theta + 6\sin^2\theta + \theta^2\cos^2\theta + \theta^2\sin^2\theta\big]\\ &= a^2 b(6+\theta^2).\end{aligned}$$

The denominator is $|\dot\gamma \times \ddot\gamma|^2$ from Theorem 4.4. □

Proposition 6.2 (Asymptotic Torsion). *For $\theta \to \infty$:*

$$\tau(\theta) \sim \frac{b}{a^2\theta^2} = \frac{3/\pi}{(6/\pi^2)^2\theta^2} = \frac{\pi^3}{12\theta^2}. \tag{27}$$

Table 5: Torsion of the conical helix

θ	$\tau(\theta)$
$\theta \to 0$	1.11774
$\pi/2$	0.60917
π	0.23417
2π	0.06429
4π	0.01631
10π	0.00262

Proposition 6.3 (Ratio κ/τ)**.** *The ratio of curvature and torsion satisfies:*

$$\frac{\kappa(\theta)}{\tau(\theta)} = \frac{\sqrt{a^2b^2(4+\theta^2)+a^4(2+\theta^2)^2}}{a^2b(6+\theta^2)/(a^2b^2(4+\theta^2)+a^4(2+\theta^2)^2)^{1/2}} \cdot \frac{1}{a^2b^2(4+\theta^2)+a^4(2+\theta^2)^2}. \tag{28}$$

For $\theta \to \infty$:

$$\frac{\kappa(\theta)}{\tau(\theta)} \sim \frac{a\theta}{b} = \frac{2\theta}{\pi}. \tag{29}$$

7 Principal Normal and Binormal Vectors

7.1 Principal Normal Vector

Definition 7.1 (Principal Normal Vector)**.** The principal unit normal vector is:

$$\boldsymbol{N}(\theta) = \frac{1}{\kappa(\theta)\,|\dot\gamma(\theta)|^2}\left(\ddot\gamma(\theta) - \frac{\dot\gamma(\theta)\cdot\ddot\gamma(\theta)}{|\dot\gamma(\theta)|^2}\dot\gamma(\theta)\right). \tag{30}$$

Lemma 7.2 (Inner Product $\dot\gamma\cdot\ddot\gamma$)**.**

$$\dot\gamma\cdot\ddot\gamma = a^2\theta. \tag{31}$$

Proof.

$$\begin{aligned}\dot\gamma\cdot\ddot\gamma = {}& a(\cos\theta-\theta\sin\theta)\cdot a(-2\sin\theta-\theta\cos\theta)\\ &+ a(\sin\theta+\theta\cos\theta)\cdot a(2\cos\theta-\theta\sin\theta) + b\cdot 0.\end{aligned}$$

Expanding the two products individually:

$$\begin{aligned}&a^2(\cos\theta-\theta\sin\theta)(-2\sin\theta-\theta\cos\theta)\\ &\quad= a^2\big[-2\sin\theta\cos\theta-\theta\cos^2\theta+2\theta\sin^2\theta+\theta^2\sin\theta\cos\theta\big],\\ &a^2(\sin\theta+\theta\cos\theta)(2\cos\theta-\theta\sin\theta)\\ &\quad= a^2\big[2\sin\theta\cos\theta-\theta\sin^2\theta+2\theta\cos^2\theta-\theta^2\sin\theta\cos\theta\big].\end{aligned}$$

On addition the mixed terms $\pm 2\sin\theta\cos\theta$ and $\pm\theta^2\sin\theta\cos\theta$ cancel:

$$\dot\gamma\cdot\ddot\gamma = a^2\big[\theta(\cos^2\theta+\sin^2\theta)\big] = a^2\theta.$$

□

7.2 Binormal Vector

Definition 7.3 (Binormal Vector)**.** The unit binormal vector is:

$$\boldsymbol{B}(\theta) = \boldsymbol{T}(\theta) \times \boldsymbol{N}(\theta) = \frac{\dot{\gamma} \times \ddot{\gamma}}{|\dot{\gamma} \times \ddot{\gamma}|}. \tag{32}$$

Example 7.1 (Frenet Frame at $\theta = 2\pi$)**.** At $\theta = 2\pi$, numerically:

$$\boldsymbol{T}(2\pi) = \begin{pmatrix} 0{,}15259 \\ 0{,}95878 \\ 0{,}23970 \end{pmatrix}, \qquad \boldsymbol{N}(2\pi) = \begin{pmatrix} -0{,}98555 \\ 0{,}16566 \\ -0{,}03523 \end{pmatrix}, \qquad \boldsymbol{B}(2\pi) = \begin{pmatrix} -0{,}07348 \\ -0{,}23086 \\ 0{,}97021 \end{pmatrix}.$$

One verifies: $|\boldsymbol{T}| = |\boldsymbol{N}| = |\boldsymbol{B}| = 1$ and $\boldsymbol{T} \perp \boldsymbol{N}$, $\boldsymbol{T} \perp \boldsymbol{B}$, $\boldsymbol{N} \perp \boldsymbol{B}$.

8 The Frenet–Serret Equations

Theorem 8.1 (Frenet–Serret Equations [9, 10])**.** *With respect to the natural parametrization (arc length s):*

$$\frac{d\boldsymbol{T}}{ds} = \kappa\,\boldsymbol{N}, \qquad \frac{d\boldsymbol{N}}{ds} = -\kappa\,\boldsymbol{T} + \tau\,\boldsymbol{B}, \qquad \frac{d\boldsymbol{B}}{ds} = -\tau\,\boldsymbol{N}. \tag{33}$$

Remark 8.1. The Frenet–Serret equations describe how the trihedron $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$ rotates along the curve. Curvature κ governs rotation in the osculating plane; torsion τ governs the twisting about the tangent axis. By Lancret's theorem [11, 6], a space curve is a (generalised) helix if and only if $\kappa/\tau = \text{const}$. Since for the conical helix $\kappa/\tau \sim 2\theta/\pi \to \infty$, it is not a helix in this classical sense, but a geometrically richer generalisation.

9 Osculating, Normal, and Rectifying Planes

Definition 9.1 (Planes of the Frenet Frame)**.** At every regular point $\gamma(\theta)$, the Frenet frame defines three distinguished planes:

- **Osculating plane**: spanned by $\boldsymbol{T}$ and $\boldsymbol{N}$; contains the tangent and the principal normal.
- **Normal plane**: spanned by $\boldsymbol{N}$ and $\boldsymbol{B}$; perpendicular to the tangent.
- **Rectifying plane**: spanned by $\boldsymbol{T}$ and $\boldsymbol{B}$; contains the tangent and the binormal.

Proposition 9.2 (Normal Plane Equation)**.** *The normal plane at the point $\gamma(\theta_0)$ has the equation:*

$$\boldsymbol{T}(\theta_0) \cdot \big((x, y, z) - \gamma(\theta_0)\big) = 0. \tag{34}$$

10 Asymptotic Behaviour

Theorem 10.1 (Asymptotic Behaviour). *As $\theta \to \infty$, the following asymptotic approximations hold:*

$$|\dot{\gamma}(\theta)| \sim a\theta, \tag{35}$$

$$L(\theta) \sim \frac{a}{2}\theta^2, \tag{36}$$

$$\kappa(\theta) \sim \frac{1}{a\theta} = \frac{\pi^2}{6\theta}, \tag{37}$$

$$\tau(\theta) \sim \frac{b}{a^2\theta^2} = \frac{\pi^3}{12\theta^2}, \tag{38}$$

$$\psi(\theta) \sim \frac{b}{a\theta} = \frac{\pi}{2\theta}. \tag{39}$$

Corollary 10.2 (Asymptotic Ratios). *For $\theta \to \infty$:*

$$\frac{\kappa(\theta)}{\tau(\theta)} \sim \frac{2\theta}{\pi}, \qquad \frac{\kappa(\theta)}{\psi(\theta)} \sim \frac{2}{\pi^2} \cdot \frac{1}{\theta}. \tag{40}$$

Remark 10.1 (Geometric Interpretation). Asymptotically the curve approaches a flat spiral in the xy-plane since $\psi(\theta) \to 0$ and $\kappa(\theta) \to 0$. At the same time, the radius $r = a\theta \to \infty$: the curve moves unboundedly away from the origin.

11 Numerical Curve Points

Table 6: Selected points on the conical helix

θ	$x(\theta)$	$y(\theta)$	$z(\theta)$	$r(\theta)$	$\kappa(\theta)$	$\tau(\theta)$
0	0	0	−0.750	0	0.9488	1.1177
π	−1,905	0	2.250	1.905	0.4469	0.2342
2π	3.810	0	5.250	3.810	0,2499	0.0643
3π	−5,715	0	8.250	5.715	0.1680	0.0286
4π	7.620	0	11.250	7.620	0.1293	0.0163
5π	−9,524	0	14.250	9.524	0.1037	0.0104

12 Summary of Formulae

Table 7: Formula Summary Part 1: Parametrization, Derivatives, and Arc Length

Quantity	**Formula**
Parametrization	
Constants	$a = \dfrac{6}{\pi^2} = \dfrac{1}{\zeta(2)}, \quad b = \dfrac{3}{\pi}, \quad \dfrac{b}{a} = \dfrac{\pi}{2}$
Curve $\gamma(\theta)$	$\big(a\theta\cos\theta,\ a\theta\sin\theta,\ -\tfrac{3}{4} + b\theta\big)$
Cone equation	$z = -\dfrac{3}{4} + \dfrac{\pi}{2} r$
Derivatives	
$\dot\gamma(\theta)$	$(a(\cos\theta - \theta\sin\theta),\ a(\sin\theta + \theta\cos\theta),\ b)$
$\ddot\gamma(\theta)$	$(a(-2\sin\theta - \theta\cos\theta),\ a(2\cos\theta - \theta\sin\theta),\ 0)$
$\lvert\dot\gamma(\theta)\rvert$	$\sqrt{a^2(1+\theta^2) + b^2}$
Arc Length	
$L(T)$	$\dfrac{a}{2}\left[T\sqrt{T^2+c^2} + c^2 \operatorname{arcsinh}\left(\dfrac{T}{c}\right)\right]$
c^2	$1 + \dfrac{\pi^2}{4} \approx 3{,}467$
Asymptotic	$L(T) \sim \dfrac{a}{2} T^2$

Table 8: Formula Summary Part 2: Frenet Quantities and Identities

Quantity	**Formula**
Frenet Quantities	
Curvature $\kappa(\theta)$	$\dfrac{\sqrt{a^2b^2(4+\theta^2)+a^4(2+\theta^2)^2}}{[a^2(1+\theta^2)+b^2]^{3/2}}$
Asymptotic curvature	$\kappa \sim \dfrac{\pi^2}{6\theta}$
Torsion $\tau(\theta)$	$\dfrac{b(6+\theta^2)}{b^2(4+\theta^2)+a^2(2+\theta^2)^2}$
Asymptotic torsion	$\tau \sim \dfrac{\pi^3}{12\theta^2}$
Pitch angle	$\psi = \arctan\left(\dfrac{\pi}{2\sqrt{1+\theta^2}}\right)$
Identities	
	$\dot{\gamma}\cdot\ddot{\gamma} = a^2\theta$
	$(\dot{\gamma}\times\ddot{\gamma})\cdot\dddot{\gamma} = a^2b(6+\theta^2)$
	$\kappa/\tau \sim 2\theta/\pi$

13 Conclusion

This paper has presented a complete differential-geometric analysis of the conical helix $\gamma(\theta) = \left(a\theta\cos\theta,\, a\theta\sin\theta,\, -\frac{3}{4}+b\theta\right)^T$ präsentiert. The main results are:

1. The curve lies on the cone $z = -\frac{3}{4} + \frac{\pi}{2}r$ (cf. Theorem 2.1), because $b/a = \pi/2$ provides exactly the cone slope.

2. The tangent vector and arc length are given in closed form; the arc length grows asymptotically as θ^2 (cf. Theorems 3.7 and 3.8).

3. Curvature $\kappa \sim 1/\theta$ and torsion $\tau \sim 1/\theta^2$ both decrease monotonically, in contrast to the cylindrical helix with constant values (cf. Theorems 5.1 and 6.1). The non-constant ratio $\kappa/\tau \sim 2\theta/\pi$ confirms by Lancret's theorem that the conical helix is not a classical helix.

4. The triple product $(\dot{\gamma}\times\ddot{\gamma})\cdot\dddot{\gamma} = a^2b(6+\theta^2)$ yields an elegant closed form for the torsion (cf. Theorem 6.1).

5. The Frenet frame is completely determined; the Frenet–Serret equations (Theorem 8.1) describe its evolution along the curve.

6. The spiral constant $a = 6/\pi^2 = 1/\zeta(2)$, the reciprocal of the Euler sum, gives the curve a deep number-theoretic connection (cf. Remark 1.1).

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