CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 4

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1. (a) The inverse of the integrator operator is the derivative operator. By using this fact, we can get the differential equation as the following.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{d(t)} + 6y(t) = 4\frac{dx(t)}{d(t)} + x(t)$$

(b) We know the following formula.

$$H(jw) = \frac{\sum^M b_k(jw)^k}{\sum^N a_k(jw)^k}$$

By applying this formula, we find $H(jw) = \frac{1+4jw}{(jw)^2+5jw+6}$. We can write this as the following.

$$\frac{1+4jw}{(jw)^2+5jw+6} = \frac{A}{jw+2} + \frac{B}{jw+3}$$

$$A + B = 4$$

$$3A + 2B = 1$$

We get $A = -7$ and $B = 1$.
So,

$$H(jw) = \frac{1+4jw}{(jw)^2+5jw+6} = \frac{-7}{jw+2} + \frac{11}{jw+3}$$

is the answer.

(c) Fourier transform of Impulse response is equal to frequence response. We have already found the frequence response. We can find impulse response from the look up table. From the look up table, we know that

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+jw}$$

We know $H(jw) = \frac{-7}{jw+2} + \frac{11}{jw+3}$. So, by using linearity property and the above equation, we find $h(t)$.

$$h(t) = \Bigg[-7e^{-2t} + 11e{-3t}\Bigg]u(t)$$

(d) For this part, we should make use of the formula

$$Y(jw) = H(jw)X(jw)$$

We already know the $H(jw)$. Lets find $X(jw)$. From the same formula in the look up table, we find

$$X(jw) = \frac{1}{4}\frac{1}{\frac{1}{4}+jw}$$

By applying $Y(jw) = H(jw)X(jw)$.

$$Y(jw) = \frac{1}{4}\left[\frac{1}{\frac{1}{4}+jw}\right]\left[\frac{-7}{jw+2}+\frac{11}{jw+3}\right]$$

$$Y(jw) = \frac{1}{4}\left[\frac{-7}{(\frac{1}{4}+jw)(jw+2)}+\frac{11}{(\frac{1}{4}+jw)(jw+3)}\right]$$

$$Y(jw) = \frac{1}{4}\left[\frac{A}{\frac{1}{4}+jw}+\frac{B}{jw+2}+\frac{C}{\frac{1}{4}+jw}+\frac{D}{jw+3}\right]$$

We find $A = -4, B = 4, C = 4, D = -4$. Thus, we get.

$$Y(jw) = \frac{1}{jw+2} - \frac{1}{jw+3}$$

2. (a)

$$H(jw) = \frac{jw+4}{-w^2+5jw+6} = \frac{\sum b_k(jw)^k}{\sum a_k(jw)^k}$$

$$b_0 = 4$$

$$b_1 = 1$$

$$a_0 = 6$$

$$a_1 = 5$$

$$a_2 = 1$$

So, the difference equation can be written as the following.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(b)

$$H(jw) = \frac{jw+4}{-w^2+5jw+6} = \frac{jw+4}{(jw)^2+5jw+6} = \frac{A}{jw+3} + \frac{B}{jw+2}$$

$$A + B = 1$$

$$2A + 3B = 4$$

$$A = -1$$

$$B = 2$$

We can use the look up table to convert $H(jw)$ into the time domain.
We basically use the formula $e^{-at}u(t) \Leftrightarrow \frac{1}{a+jw}$

$$H(jw) = \frac{-1}{jw+3} + \frac{2}{jw+2} \Leftrightarrow (-e^{-3t} + 2e^{-2t})u(t) = h(t)$$

So, the result is

$$h(t) = (-e^{-3t} + 2e^{-2t})u(t)$$

(c)

$$x(t) \Leftrightarrow \frac{1}{jw+4} - \frac{1}{(jw+4)^2}$$

$Y(jw) = X(jw)H(jw)$ and we already know the formula for $H(jw)$.

$$Y(jw) = \left[\frac{1}{jw+4} - \frac{1}{(jw+4)^2}\right]\left[\frac{jw+4}{(jw+3)(jw+2)}\right]$$

$$Y(jw) = \frac{jw+3}{(jw+4)^2} + \frac{jw+4}{(jw+3)(jw+2)}$$

$$Y(jw) = \frac{1}{(jw+4)(jw+2)} = \frac{A}{jw+4} + \frac{B}{jw+2}$$

$$A + B = 0$$
$$2A + 4B = 1$$
$$A = -\frac{1}{2}$$
$$B = \frac{1}{2}$$

So, we get

$$Y(jw) = \frac{1}{2}\left[\frac{-1}{jw+4} + \frac{1}{jw+2}\right]$$

(d) Once again, we make use of the formula from the look up table. $e^{-at}u(t) \Leftrightarrow \frac{1}{a+jw}$. We find the answer as

$$y(t) = \frac{1}{2}\left[-e^{-4t} + e^{-2t}\right]u(t)$$

3. (a)

From lookup table;

$$1)\quad a^n u[n] \Leftrightarrow \frac{1}{1-ae^{-jw}}$$

$$2)\quad (n+1)a^n u[n] \Leftrightarrow \frac{1}{(1-ae^{-jw})^2}$$

$$\text{Using property 1,}\quad x[n] = (\frac{2}{3})^n u[n] \Leftrightarrow \frac{1}{1-\frac{2}{3}e^{-jw}} = X(e^{jw})$$

$$\text{Using property 2,}\quad x[n] + \frac{3}{2}y[n] = (n+1)(\frac{2}{3})^n u[n] \Leftrightarrow \frac{1}{(1-\frac{2}{3}e^{-jw})^2} = X(e^{jw}) + \frac{3}{2}Y(e^{jw})$$

$$\text{Hence}\quad Y(e^{jw}) = \frac{2}{3}\left[\frac{1}{(1-\frac{2}{3}e^{-jw})^2} - \frac{1}{1-\frac{2}{3}e^{-jw}}\right]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{2}{3}\frac{\left[\frac{1}{(1-\frac{2}{3}e^{-jw})^2} - \frac{1}{1-\frac{2}{3}e^{-jw}}\right]}{\frac{1}{1-\frac{2}{3}e^{-jw}}}$$

$$\text{After simplification,}\quad H(e^{jw}) = \frac{2}{3}\left[\frac{1}{1-\frac{2}{3}e^{-jw}} - 1\right]$$

(b)

From lookup table;

$$h[n] = \frac{2}{3}\left[\left(\frac{2}{3}\right)^n u[n] - \delta[n]\right]$$

(c)

$$\text{From part (a),}\quad H(e^{jw}) = \frac{2}{3}\left[\frac{1}{1-\frac{2}{3}e^{-jw}} - 1\right] = \frac{2}{3}\left[\frac{1-1+\frac{2}{3}e^{-jw}}{1-\frac{2}{3}e^{-jw}}\right] = \frac{2}{3}\left[\frac{\frac{2}{3}e^{-jw}}{1-\frac{2}{3}e^{-jw}}\right]$$

$$H(e^{jw}) = \frac{\frac{4}{3}e^{-jw}}{3-2e^{-jw}}$$

$$\text{We also know,}\quad H(e^{jw}) = \frac{\Sigma\ b_k\ e^{-jwk}}{\Sigma\ a_k\ e^{-jwk}}$$

$$b_1 = \frac{4}{3},\quad a_0 = 3,\quad a_1 = -2,\quad \text{and all other coefficients are equal to 0.}$$

$$\text{The final result is,}\quad 3y[n] - 2y[n-1] = \frac{4}{3}x[n-1]$$

(d)

$$\text{Block diagram of the equation} \quad 3y[n] - 2y[n-1] = \frac{4}{3}x[n-1]$$

$$y[n] = \frac{4}{9}x[n-1] + \frac{2}{3}y[n-1]$$

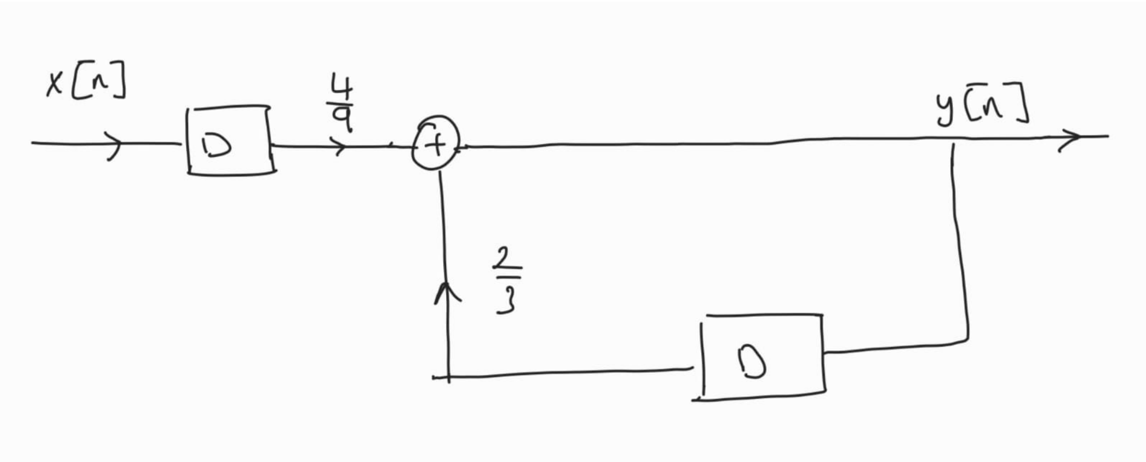

Figure 1: 3 - d Block Diagram

4. (a)

The difference equation that represents this system is:

$$2x[n] - \frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] = y[n]$$

or

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(b)

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$H(e^{jw})X(e^{jw}) - \frac{3}{4}e^{-jw}H(e^{jw})X(e^{jw}) + \frac{1}{8}e^{-2jw}H(e^{jw})X(e^{jw}) = 2X(e^{jw})$$

$$H(e^{jw})\left[1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}\right] = 2 \quad \longrightarrow \quad H(e^{jw}) = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}}$$

$$H(e^{jw}) = \frac{16}{8 - 6e^{-jw} + e^{-2jw}}$$

(c)

$$H(e^{jw}) = \frac{16}{8 - 6e^{-jw} + e^{-2jw}} = \frac{A}{e^{-jw} - 4} + \frac{B}{e^{-jw} - 2}$$

$$A + B = 0 \quad \text{and,} \quad -2A - 4B = 16 \quad \longrightarrow \quad A = 8 \quad \text{and,} \quad B = -8$$

$$H(e^{jw}) = \frac{8}{-4(1 - \frac{1}{4}e^{-jw})} + \frac{-8}{-2(1 - \frac{1}{2}e^{-jw})} = \frac{-2}{1 - \frac{1}{4}e^{-jw}} + \frac{4}{1 - \frac{1}{2}e^{-jw}}$$

From lookup table;

$$h[n] = \left(-2\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n\right)u[n]$$

(d)

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \Leftrightarrow \frac{1}{1-\frac{1}{4}e^{-jw}} = X(e^{jw})$$

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$Y(e^{jw}) = \frac{-2}{(1-\frac{1}{4}e^{-jw})^2} + \frac{4}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}$$

$$Y(e^{jw}) = \frac{-2}{(1-\frac{1}{4}e^{-jw})^2} + \frac{A}{(1-\frac{1}{2}e^{-jw})} + \frac{B}{(1-\frac{1}{4}e^{-jw})}$$

After the calculations, we find that: $A = 8, B = -4$

$$Y(e^{jw}) = \frac{-2}{(1-\frac{1}{4}e^{-jw})^2} + \frac{8}{(1-\frac{1}{2}e^{-jw})} + \frac{-4}{(1-\frac{1}{4}e^{-jw})}$$

From lookup table;

$$y(t) = \left[-2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n\right] u[n]$$

5.

$$Y(e^{jw}) = H_1(e^{jw})X(e^{jw}) + H_2(e^{jw})X(e^{jw})$$

$$Y(e^{jw}) = X(e^{jw})\left[H_1(e^{jw}) + H_2(e^{jw})\right]$$

Using Lookup Table; $H_1(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}}$

Also; $\frac{5e^{-jw}-12}{e^{-2jw}-7e^{-jw}+12} = \frac{A}{e^{-jw}-4} + \frac{B}{e^{-jw}-3}$

After the calculations, we find that: $A = 8, B = -3$

We also know: $H_1(e^{jw}) + H_2(e^{jw}) = H(e^{jw})$

$$\frac{1}{1-\frac{1}{3}e^{-jw}} + H_2(e^{jw}) = \frac{8}{e^{-jw}-4} + \frac{-3}{e^{-jw}-3} == \frac{-2}{1-\frac{1}{4}e^{-jw}} + \frac{1}{1-\frac{1}{3}e^{-jw}}$$

$$H_2(e^{jw}) = \frac{-2}{1-\frac{1}{4}e^{-jw}}$$

From lookup table;

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

6.

```python
import numpy as np
import matplotlib.pyplot as plt

N = 10**3
def x(n): # defining x[n]
    if n < 0:
        return (1/2)**(-n)
    return 1/2**(n)

w_axis = np.linspace(-20, +20, 600)
def X(w): # defining x[e^(jw)]
    return np.sum([x(n) * np.exp(1j * w * -n) for n in range(-N,N+1)])

result = [X(w) for w in w_axis]

magnitudes = []
phases = []

for i in result:
    re = i.real
    im = i.imag
    magnitude = np.sqrt(re**2 + im**2)
    magnitudes.append(magnitude)
    # print(f"i:{i} - real:{re} - imaginary:{im} - magnitude is:{magnitude}") // debugging
    if re == 0:
        if im == 0:
            phases.append(0)
        elif im > 0:
            phases.append(np.pi / 2)
        else:
            phases.append( -np.pi / 2)
    else:
        phase = np.arctan(im/re)
        phases.append(round(phase,3))

fig = plt.figure(figsize=(12, 5))

w1 = w_axis
w2 = w_axis

plt.subplot(1, 2, 1)
plt.plot(w1, magnitudes, color='r',  label='Magnitude')
plt.xlabel('w-axis')
plt.ylabel('X(e^(jw))-axis')
plt.title('Magnitude')
plt.grid(True)
plt.legend()

plt.subplot(1, 2, 2)
plt.plot(w2, phases, color='b', linestyle='-',  label='Phase')
plt.xlabel('w-axis')
plt.ylabel('X(e^(jw))-axis')
plt.title('Phase')
plt.grid(True)
plt.legend()

plt.tight_layout()
plt.show()
```

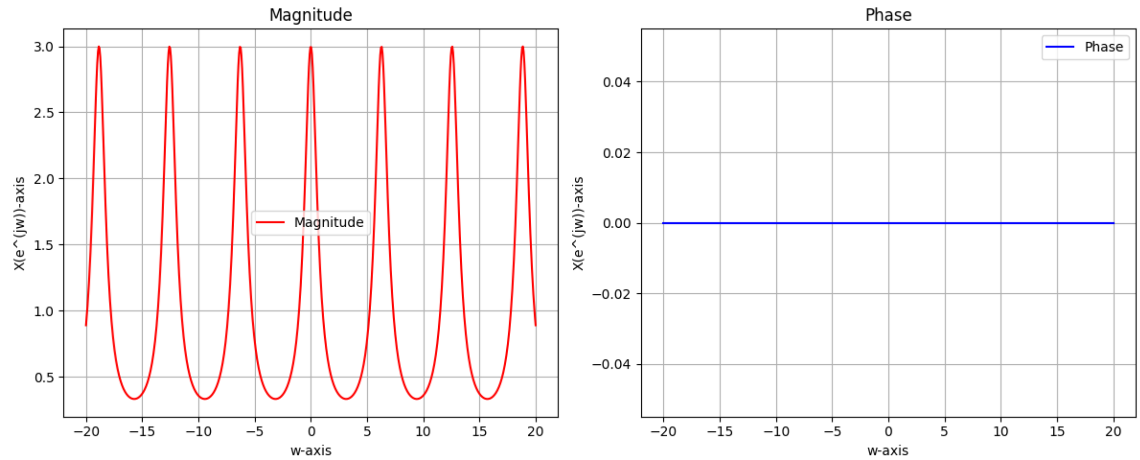

Figure 2: Plotting of the given signaş

From the figure, we can observe that the phase of the Fourier transform of the given signal is always zero. This means the corresponding Fourier Transform is always a real number.