CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 2

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1. The formula of the convolution integral is $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$. We can directly find the result from the overlapping area of the $x(t)$ and $h(-t)$ as we shift $h(-t)$ along the x axis. The calculation is similar to dot product.

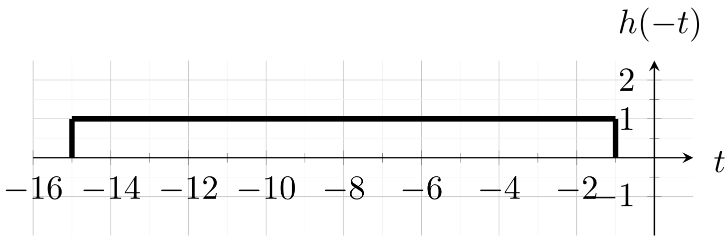


Figure 1: A piecewise function illustrating $h(-t)$.

If we shift $h(-t)$ to left 2 units, then there will not be overlapping area. This means $y(-2) = 0$. The overlapping area will increase linearly as we shift to right starting from $x = -2$. The area will be maximized when $x = 8$. This is the case where $y(t)$ is completely inside of the $h(-t)$. This case will continue until $x = 12$. Afterwards, the overlapping are will decrease linearly until there is no overlapping area, which occurs if $x = 22$.

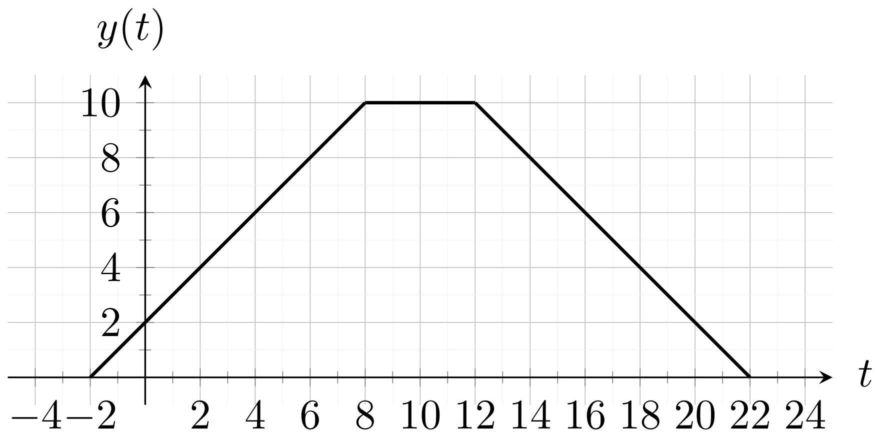


Figure 2: A piecewise function illustrating y(t).

We get the formula of the $y(t)$ as

$$y(t) = \begin{cases} t+2 & \text{if } -2 \leq t \leq 8 \\ 10 & \text{if } 8 < t \leq 12 \\ 22-t & \text{if } 12 < t \leq 22 \\ 0 & \text{Elsewhere} \end{cases}$$

2. We make use of the distributive and commutative rules of convolution for this problem.

(a) $x[n]*h[n] = \delta[n]*2\delta[n+2]+\delta[n]*\delta[n-2]+2\delta[n-2]*2\delta[n+2]+2\delta[n-2]*\delta[n-2]-3\delta[n-4]*2\delta[n+2]-3\delta[n-4]*\delta[n-2]$
$= 2\delta[n+2]+\delta[n-2]+4\delta[n]+2\delta[n-4]-6\delta[n-2]-3\delta[n-6] = 2\delta[n+2]-5\delta[n-2]+4\delta[n]+2\delta[n-4]-3\delta[n-6]$

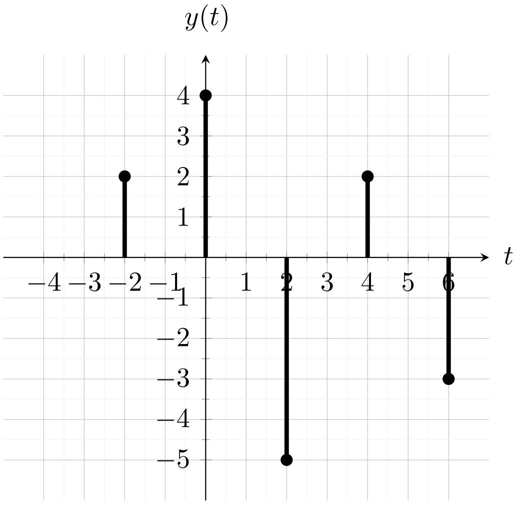


Figure 3: A piecewise function illustrating $y_1(t)$.

(b) Again, we make use of distributive and commutative rules of the convolution.
$y_2[n] = x[n+2]*h[n] = x[n]*\delta[n+2]*h[n] = x[n]*h[n]*\delta[n+2] = y_1[n]*\delta[n+2]$. We know the formula and the plot of y_1 from the above part. We should just shift to left by 2 units.
$2\delta[n+4]-5\delta[n]+4\delta[n+2]+2\delta[n-2]-3\delta[n-4]$ is the formula.

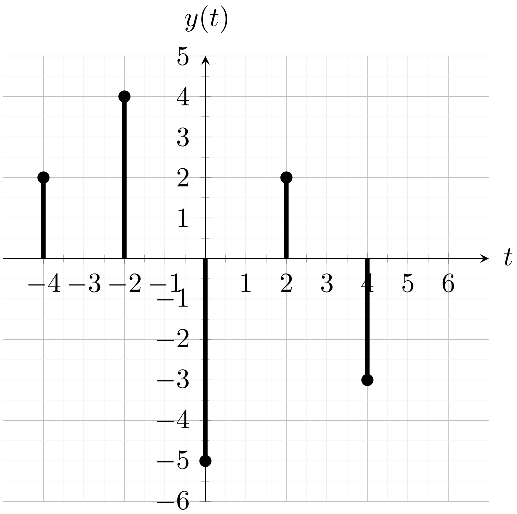


Figure 4: A piecewise function illustrating $y_2(t)$.

(c) Again, we make use of distributive and commutative rules of the convolution.
$y_3[n] = x[n+2]*h[n-2] = x[n]*\delta[n+2]*h[n]*\delta[n-2] = x[n]*h[n]*\delta[n+2]*\delta[n-2] = y_1[n]*\delta[n] = y_1[n]$.
since $\delta[n-2]*\delta[n+2] = \delta[n]$ and this is the identity for convolution.
We know the formula and the plot of y_1 from the above part. This is exactly the same as y_1.

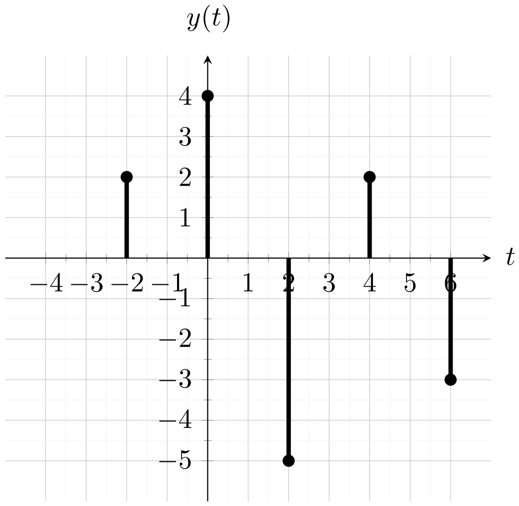


Figure 5: A piecewise function illustrating $y_3(t)$.

3. (a) To find the impulse response, we feed the system with $\delta[n]$. The result will be $h[n] = \frac{1}{5}\delta[n-1] + \delta[n]$.

$$h[n] = \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{5} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) We just replace $x[n]$ with $\delta[n-2]$. Then we get $\frac{1}{5}\delta[n-3] + \delta[n-2]$

(c) From Jensen's equation, the system is stable if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$. We replace $h[n]$ and get the following.

$$\sum_{k=-\infty}^{\infty} \frac{1}{5}\delta[k-1] + \sum_{k=-\infty}^{\infty} \delta[k] = \frac{1}{5} + 1 = \frac{6}{5}$$

The result is a bounded value. The system is BIBO stable.

(d) For the system to be memoryless, it must only depend on $x[n]$. However, our system also depends on $x[n-1]$. So, the system is not memoryless.So, YES, the system has a memory.

(e) For the system to be invertible, distinct inputs must lead to distinct outputs. However, this not the case since there is an addittion operation which is not invertible. For example, $5 + 5 = 4 + 6$. Hence, the system is NOT invertible.

4. (a) The formula for Transfer function is the following.

$$H(\lambda) = \frac{\sum b_k \lambda^k}{\sum a_k \lambda^k}$$

From this equation we can find the corresponding constants as below.

$b_0 = 0$, $b_1 = 2$, $a_0 = 1$, $a_1 = -2$, $a_2 = 1$, all else is 0.

Also, $\sum a_k \frac{d^k y(t)}{dt^k} = \sum b_k \frac{d^k x(t)}{dt^k}$

$y''(t) - 2y'(t) + y(t) = 2x'(t)$

(b) For homogeneous part, we should first find the root of the characteristic equation, which is $\lambda^2 - 2\lambda + 1 = 0$. The solution is $(\lambda - 1)^2 = 0$, and this gives $\lambda = 1$. However, this is repeated roots. So, the equation $y_h(t) = c_1 te^t + c_2 e^t$ must hold where c_1 and c_2 are constants.

For particular solution, we assume $y_p(t) = at + b$
$0 - 2a + at + b = 0$, means a = b = 0 and $y_p(t) = 0$.

We have found homogeneous and particular solutions $y_p(t)$ and $y_h(t)$. The final solution is $y(t) = y_h(t) + y_p(t)$, which is equal to $y(t) = c_1 te^t + c_2 e^t$.

Now, we can find the constants c_1 and c_2 from the fact that the system is initial at rest. This gives $y(0) = y'(0) = 0$.

$$y(0) = c_2 = 0$$

We got c_2, now we can find c_1.

$$y'(0) = c_1 = 0$$

We found $c_1 = c_2 = 0$. If we replace these constants into the equation, we find $y(t) = 0$ for $x(t) = 0$.

(c) The homogeneous solution remains the same. However, particular changes since the input $x(t)$ also changes.

For particular solution, we assume $y_p(t) = at + b$
$0 - 2a + at + b = 4$, means a = 0, b = 4 and $y_p(t) = 4$
$y(t) = y_h(t) + y_p(t) = c_1 te^t + c_2 e^t + 4$

using initial rest conditions:
$y(0) = c_1 * 0 * e^0 + c_2 e^0 + 4 = 0$ means $c_2 = -4$
$y'(0) = c_1 e^0 + c_1 * 0 * e^0 + c_2 e^0 = 0$ means $c_1 = -c_2$, so $c_1 = 4$
$y(t) = (4te^t - 4e^t + 4)u(t)$

5. (a) We replace $x[n]$ with $\delta[n]$. The equation becomes

$$h[n] = \frac{1}{5}h[n-1] + 2\delta[n-2]$$

We know system is initially at rest. This means $h[n] = 0$ for $n < 0$. We will solve it with recursive method.

For n = 0, $h[0] = \frac{1}{5} * 0 + 0$ So, h[0] = 0.

For n = 1, $h[1] = \frac{1}{5} * 0 + 0$ So, h[1] = 0.

For n = 2, $h[2] = \frac{1}{5} * 0 + 2$ So, h[2] = 2.

For n = 3, $h[3] = \frac{1}{5} * 2 + 0$ So, h[3] = $\frac{2}{5}$

For n = 4, $h[4] = \frac{1}{5} * (2 * (\frac{1}{5}) + 0)$ So, h[4] = $\frac{2}{25}$

For n = 5, $h[5] = \frac{1}{5} * (\frac{2}{25}) + 0)$ So, h[5] = $\frac{2}{125}$

It goes like that. The pattern can be seen as $h[n] = \frac{2}{5^{n-2}} u[n-2]$

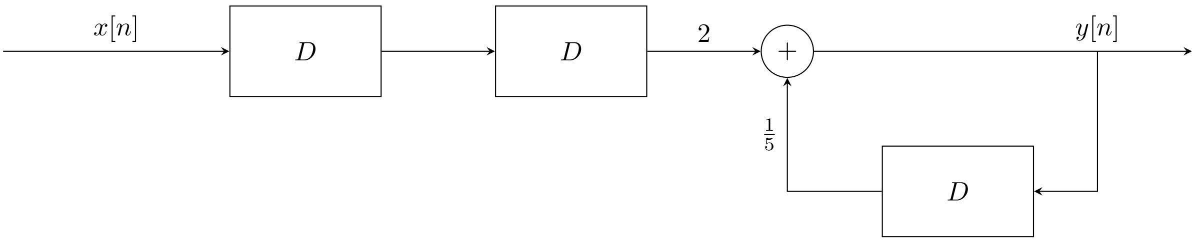
(b)

$$y[n] - \frac{1}{5}y[n-1] = 2x[n-2]$$

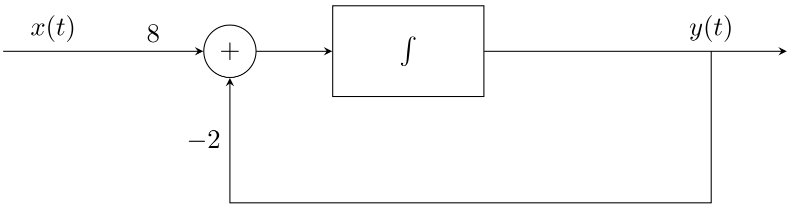
From the equation above, we can see that $a_0 = 1, a_1 = -\frac{1}{5}$ and $b_2 = 2$. All else is 0.

The formula for the transfer function is $H(\lambda) = \frac{\sum b_k \lambda^k}{\sum a_k \lambda^k}$. The result becomes $\frac{2\lambda^2}{1-\frac{1}{5}\lambda}$

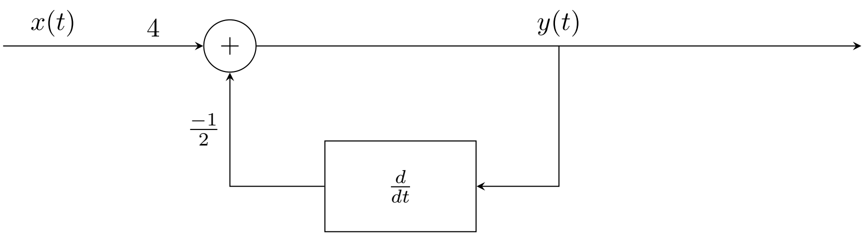
(c) The below figure is the **Part C**.



6. (a) The solution for **6-(a)**



(b) The solution for **6-(b)**



Listing 1: Python

7.
```
import matplotlib.pyplot


def x(n):
    return 1 if n == 1 else 0

def y(n):
    if n < 0:
        return 0
    return 1/4*y(n-1) + x(n)


x_axis = [0,1,2,3,4]

y_axis = []

for i in x_axis:
    y_axis.append(y(i))

matplotlib.pyplot.stem(x_axis, y_axis)
matplotlib.pyplot.grid(True)
matplotlib.pyplot.xticks(range(0,6))
matplotlib.pyplot.yticks([0,0.25,0.5,0.75,1,1.25])
matplotlib.pyplot.show()
```

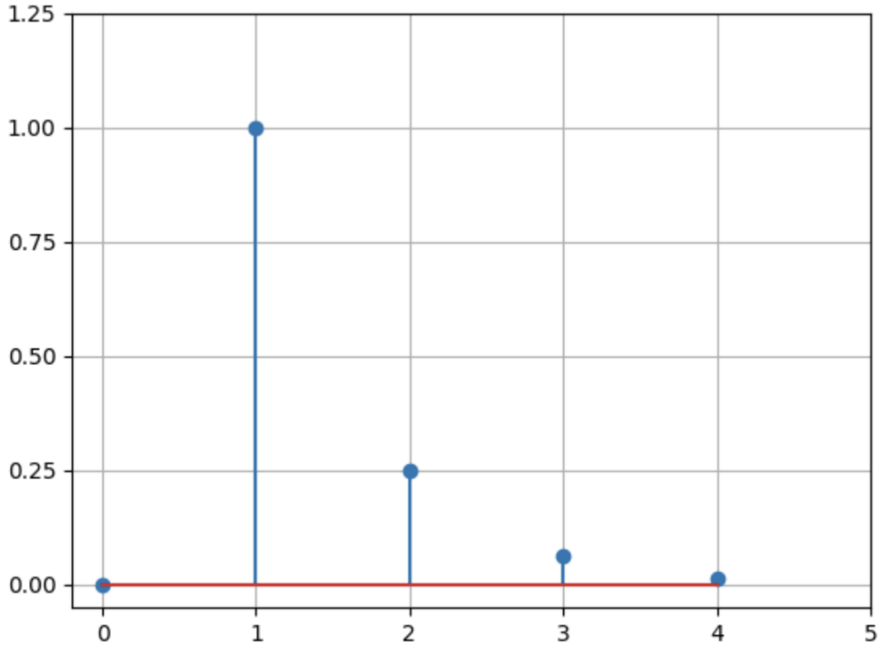


Figure 6: Plot for the first 5 samples of the LTI system