CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 3

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April 30, 2024

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1. w_0 is $\frac{2\pi}{T} = \frac{\pi}{2}$. Using synthesis equation, we can write the $x(t)$ as the following:

$$x(t) = \sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{2}t} - 2\sum_{-\infty}^{\infty} e^{j2k\frac{\pi}{2}t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{2}t} - 2\sum_{-\infty}^{\infty} e^{jk\pi t}$$

For the first term, which is $\sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{2}t}$, all a_k's are 1. This means this part represents an impulse train with all constants are 1. From the look-up table, we already know the fact that the coefficients of an impulse train is $\frac{1}{T}$ where T represents period. In our case $a_k = \frac{1}{T} = \frac{1}{4}$. To make it equal to one, we must multiple it by 4. Therefore, the first part becomes $\sum_{k=-\infty}^{\infty} 4\delta(t-4k)$.

As of the second term, which is $\sum_{-\infty}^{\infty} e^{jk\pi t}$. Again for this term all a_K's are 1. However, this time period becomes 2 since $w_0 = \frac{2\pi}{T} = \pi$. Therefore, this time we have $a_k = \frac{1}{T} = \frac{1}{2}$. To make it equal to 1, we must multiply the term by 2. Therefore, the second term becomes (without the -2 coefficients in the above equation), $\sum_{k=-\infty}^{\infty} 2\delta(t-2k)$

Hence, the equation becomes

$$x(t) = \sum_{k=-\infty}^{\infty} 4\delta(t-4K) - 2\sum_{k=-\infty}^{\infty} 2\delta(t-2k)$$

$$x(t) = 4\sum_{k=-\infty}^{\infty} \delta(t-4K) - 4\sum_{k=-\infty}^{\infty} \delta(t-2k)$$

$$x(t) = 4\sum_{k=-\infty}^{\infty} \delta(t-4K) - \delta(t-2k)$$

$$x(t) = -4\sum_{k=-\infty}^{\infty} \delta(t-4K+2)$$

Finally, the result is shifted impulse train $x(t) = -4\sum_{k=-\infty}^{\infty} \delta(t-4K+2)$

2. (a) The formula for a_k is $\frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$.
We basically put $x(t)$ into the equation. We get the following. Period is $T = 4$.

$$a_k = \frac{1}{4}\int_0^2 2te^{-jkw_0t} + \frac{1}{4}\int_2^4 (4-t)e^{-jkw_0t}dt$$

Lets say $jkw_0 = m$ for the sake of easiness.

$$a_k = \frac{1}{4}\int_0^2 2te^{-mt} + \frac{1}{4}\int_2^4 (4-t)e^{-mt}dt$$

This integral requires integration by part. We apply it.

$$a_k = \frac{1}{4}[\int_0^2 2te^{-mt}dt + \int_2^4 (4-t)e^{-mt}dt]dt$$

$$a_k = \frac{1}{4}\left[\frac{2t}{-m}e^{-mt}\Big|_0^2 - \frac{1}{-m}\int_0^2 2e^{-mt}dt + \int_2^4 4e^{-mt}dt - \frac{t}{-m}e^{-mt}\Big|_2^4 + \frac{1}{-m}\int_2^4 e^{-mt}dt\right]$$

$$a_k = \frac{1}{4}\left[\frac{4}{-m}e^{-2m} - \frac{2}{m}\frac{e^{-mt}}{m}\Big|_0^2 + 4\frac{e^{-mt}}{-m}\Big|_2^4 + \frac{2e^{-2m}}{-m} - \frac{4e^{-4m}}{-m} + \frac{1}{m}\frac{e^{-mt}}{m}\Big|_2^4\right]$$

$$a_k = \frac{1}{4}\left[\frac{-4}{m}e^{-2m} - 2\frac{e^{-2m}}{m^2} + \frac{2}{m^2} - 4\frac{e^{-4m}}{m} + 4\frac{e^{-2m}}{m} + 4\frac{e^{-4m}}{m} - 2\frac{e^{-2m}}{m} + \frac{e^{-4m}}{m^2} - \frac{e^{-2m}}{m^2}\right]$$

$$a_k = \frac{1}{4}\left[-2\frac{e^{-2m}}{m^2} + \frac{2}{m^2} - 2\frac{e^{-2m}}{m} + \frac{e^{-4m}}{m^2} - \frac{e^{-2m}}{m^2}\right]$$

We replace m by jw_0k. We can also replace w_0 by $\frac{2\pi}{4} = \frac{\pi}{2}$. So, we can replace it by $j\frac{\pi}{2}k$

$$a_k = \frac{1}{4}\left[-2\frac{e^{-j\pi k}}{m^2} + \frac{2}{m^2} - 2\frac{e^{-j\pi k}}{m} + \frac{e^{-2-j\pi k}}{m^2} - \frac{e^{-j\pi k}}{m^2}\right]$$

$$a_k = \frac{1}{4}\left[-2\frac{e^{-j\pi k}}{m^2} + \frac{2}{m^2} - 2\frac{e^{-j\pi k}}{m} + \frac{1}{m^2} - \frac{e^{-j\pi k}}{m^2}\right]$$

$$a_k = \frac{1}{4}\left[-3\frac{e^{-j\pi k}}{m^2} + \frac{3}{m^2} - 2\frac{e^{-j\pi k}}{m}\right]$$

Again using $m = j\frac{\pi}{2}k$.

$$a_k = \frac{1}{2j\pi k}\left[-6\frac{e^{-j\pi k}}{j\pi k} + \frac{6}{j\pi k} - 2e^{-j\pi k}\right]$$

(b) Derivative property says

$$y(t) = x'(t)$$

$$x(t) \Leftrightarrow a_k$$

$$y(t) \Leftrightarrow b_k$$

$$b_k = jkw_0a_k$$

$$b_k = jk\frac{\pi}{2}\frac{1}{2j\pi k}\left[-6\frac{e^{-j\pi k}}{j\pi k} + \frac{6}{j\pi k} - 2e^{-j\pi k}\right]$$

$$b_k = \frac{1}{4}\left[-6\frac{e^{-j\pi k}}{j\pi k} + \frac{6}{j\pi k} - 2e^{-j\pi k}\right]$$

3. (a) First, note that both x_1 and x_2 has period 4, means $w_0 = \frac{\pi}{2}$.

Let's find the spectral coefficients of x_1
$a_k = \frac{1}{4}\Sigma_{<4>}cos(\frac{\pi}{2}n)e^{-jk\frac{\pi}{2}n}$
Let's choose n from 0 to 3. $cos(\frac{\pi}{2})$ and $cos(\frac{3\pi}{2})$ are equal to 0. Then;
$a_k = \frac{1}{4}[cos(0)e^0 + cos(\pi)e^{-jk\pi}] = \frac{1}{4}[1-(-1)^{-k}]$
We find that, spectral coefficients of x_0 are: $a_0 = 0$, $a_1 = \frac{1}{2}$, $a_2 = 0$, $a_3 = \frac{1}{2}$

Using the same method we can find that spectral coefficients of x_1 are:
$b_k = \frac{1}{4}[sin(\frac{\pi}{2})e^{-jk\frac{\pi}{2}} + sin(3\frac{\pi}{2})e^{-jk3\frac{\pi}{2}}] = \frac{1}{4}[(-1)^{-\frac{k}{2}} - (-1)^{-\frac{3k}{2}}]$
We find that, spectral coefficients of x_1 are: $b_0 = 0$, $b_1 = -\frac{j}{2}$, $b_2 = 0$, $b_3 = \frac{j}{2}$

Using the same method we can find that spectral coefficients of x_3 are:
$c_k = \frac{1}{4}\Sigma_{<4>}cos(\frac{\pi}{2}n)sin(\frac{\pi}{2}n)e^{-jk\frac{\pi}{2}n}$
Clearly the right hand side is equal to 0 for all integer n values so all spectral coefficients of x_2 are equal to 0.
$c_0 = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

(b) Using the multiplication property we know that:
$c_k = \Sigma_{l=<4>}a_l b_{k-l}$
$c_k = a_0 * b_k + a_1 * b_{k-1} + a_2 * b_{k-2} + a_3 * b_{k-3} = \frac{1}{2}[b_{k-1} + b_{k-3}]$
Clearly this is equal to 0 for all k values since $\frac{j}{2} + (-\frac{j}{2}) = 0 + 0 = 0$

Using multiplication property, we easily found the same result we found in part(a).

4. Note that x[n] has period 24 and, means $w_0 = \frac{\pi}{12}$.

$$a_k = cos(k\frac{\pi}{3}) + cos(k\frac{\pi}{4}) = \frac{1}{2}[e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}] = \frac{1}{24}\Sigma_{<24>}x[n]e^{-jk\frac{\pi}{12}n}$$

$$12[e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}] = \Sigma_{<24>}x[n]e^{-jk\frac{\pi}{12}n}$$

.

For the first term, $12e^{jk\frac{\pi}{3}}$, $n = -4$ in the summation.Therefore, $x[-4] = 12$.

For the second term, $12e^{-jk\frac{\pi}{3}}$, $n = 4$ in the summation.Therefore, $x[4] = 12$.

For the second term, $12e^{jk\frac{\pi}{4}}$, $n = -3$ in the summation.Therefore, $x[-3] = 12$.

For the second term, $12e^{-jk\frac{\pi}{4}}$, $n = 3$ in the summation.Therefore, $x[3] = 12$.

$x[-4] = x[4] = x[-3] = x[3] = 12$ and all other $x[n] = 0$ within the range $[-4, 19]$ with respect to Period $N = 24$. This means $x[n] = x[n \pm 24]$

5. (a) $2\pi m = \frac{6\pi N}{13}$ Where m are N integers.Giving $m = 3$, we get $N = 13$. So, the fundamental period of the formula is 13.

(b)

$$x[n] = \Sigma_{<13>}a_k e^{-jk\frac{2\pi}{13}n}$$

$$sin(\frac{6\pi}{13}n + \frac{\pi}{2}) = \frac{1}{2j}[e^{j(\frac{6\pi}{13}n+\frac{\pi}{2})} - e^{-j(\frac{6\pi}{13}n+\frac{\pi}{2})}] = \Sigma_{<13>}a_k e^{-jk\frac{2\pi}{13}n}$$

$$e^{j(\frac{6\pi}{13}n)} * (\frac{e^{\frac{j\pi}{2}}}{2j}) + e^{-j(\frac{6\pi}{13}n)} * (-\frac{e^{-\frac{j\pi}{2}}}{2j}) = \Sigma_{<13>}a_k e^{-jk\frac{2\pi}{13}n}$$

There are two terms that are non-zero. The first term is $\frac{1}{2j}e^{j\frac{\pi}{2}}$. This corresponds to the term where $k = -3$. The other term is $\frac{-1}{2j}e^{-j\frac{\pi}{2}}$. This corresponds to the term where $k = 3$.

These values contains both real and imaginary part. The first value is $\frac{1}{2j}e^{j\frac{\pi}{2}}$. We know $e^{j\frac{\pi}{2}} = j$. So, the overall term is $\frac{1}{2}$.

These values contains both real and imaginary part. The first value is $\frac{-1}{2j}e^{-j\frac{\pi}{2}}$. We know $e^{-j\frac{\pi}{2}} = -j$. So, the overall term is $\frac{1}{2}$. Below, there are plots for both magnitude and phase.

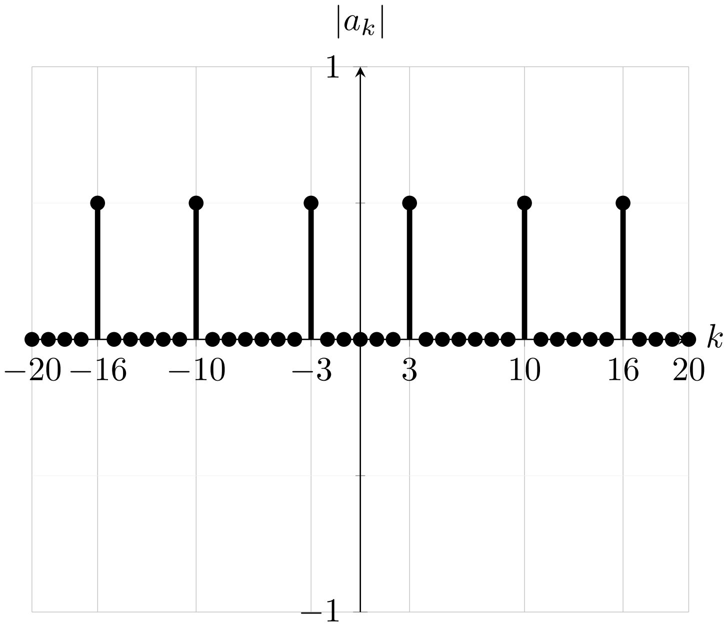

Figure 1: Magnitude plot of spectral coefficients.

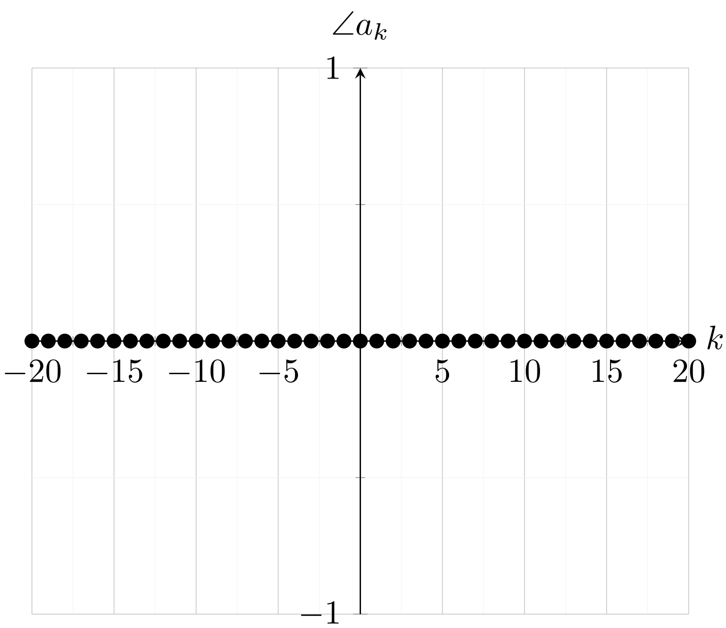

Figure 2: Angle plot of the coefficients

6. (a) First, we know the Fourier transform of $h(t)$ (impulse response) is equal to $H(jw)$. Since we know the formula of $H(jw)$, we can find $h(t)$ from the look up table.

The look up table says

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+jw}$$

In our case, $H(jw) = \frac{1}{4jw+3} = \frac{1}{4(jw+3/4)} = \frac{1}{4}[\frac{1}{jw+3/4}]$.

From the look up table equation and linearity property of Fourier Transformation, we can infer $h(t) = \frac{1}{4}e^{-3/4t}u(t)$

$h(t) = \frac{1}{4}e^{-3/4t}u(t)$ is the result.

(b) Let's find the $Y(jw)$ first. Using the look up table;

$Y(jw) = \frac{1}{jw+5} - \frac{1}{jw+10}$

We also know that $H(jw) = \frac{Y(jw)}{X(jw)}$, means $X(jw) = \frac{Y(jw)}{H(jw)}$

$X(jw) = \frac{4jw+3}{jw+5} - \frac{4jw+3}{jw+10} = (4 - \frac{17}{jw+5}) - (4 - \frac{37}{jw+10}) = \frac{37}{jw+10} - \frac{17}{jw+5}$

Again, using look up table we can simply find that $x(t) = (37e^{-10t} - 17e^{-5t})u(t)$

Listing 1: Python

7.
```python
import numpy as np
import matplotlib.pyplot as plt

def x(t):
    return (np.cos(np.pi / 3 * t) + 2*np.cos(np.pi * t + np.pi/2))

t = np.linspace(0, 6, 6000)

def a(k):
    return 1/6 * np.trapz(x(t)* np.exp( -1j * k * np.pi / 3 * t) ,t)

x_axis = [i for i in range(-20,21)]
y_axis = []



for i in x_axis:
    y_axis.append(a(i))

y_axis_mag = []
y_axis_phase = []

for i in y_axis:
    mag = np.sqrt(i.real**2 + i.imag**2)
    if round(mag,3) == 0:
        y_axis_phase.append(0)
    else:
        phas = np.arctan( i.imag / i.real  )
        y_axis_phase.append(phas / np.pi)
    y_axis_mag.append(mag)


print("The period is 6, w_0 is pi/3")
print("The Fourier series representation is -j * e^{-j*pi*t} + 0.5 * e^{-j*pi*t / 3} + 0.

# -3 -> -j, -1 -> 0.5 , 1 -> 0.5, 3 -> j

x1 =  x_axis
y1 = y_axis_mag


x2 = x_axis
y2 = y_axis_phase

fig = plt.figure(figsize=(12, 5))


plt.subplot(1, 2, 1)
plt.plot(x1, y1, marker='o', color='r', linestyle='',  label='Plot 1')  # Customizing line
plt.xlabel('X-axis')
plt.ylabel('Y-axis')
plt.title('Magnitude ')
plt.grid(True)
plt.legend()
```

```
plt.subplot(1, 2, 2)
plt.plot(x2, y2, marker='s', color='b', linestyle='', label='Plot 2')  # Customizing line
plt.xlabel('X-axis')
plt.ylabel('Y-axis')
plt.title('Phase')
plt.grid(True)
plt.legend()

plt.tight_layout()
plt.show()
```

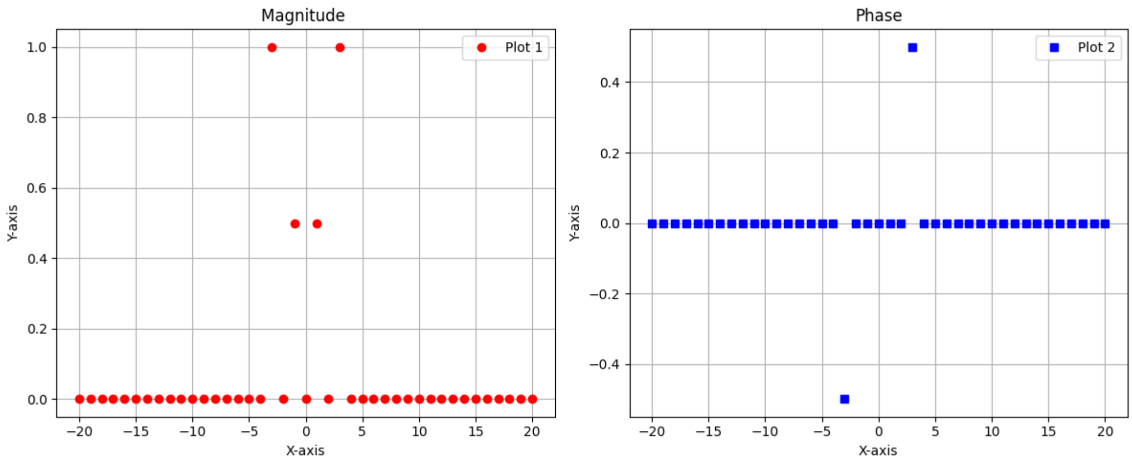

Figure 3: The plotting of the magnitude and phase of the Fourier series coefficients

Below is the standard output of the code above.

```
The period is 6, w_0 is pi/3
The Fourier series representation is -j * e ^{-j*pi*t} + 0.5 * e ^{-j*pi*t / 3} +
0.5 *e ^{j*pi*t/3} + j * e^{j*pi*t}
```