Exam I, Fall 2017
November 4, 2017, 10:45 - 13:15

Name:____ Student Number:________________

READ ALL THE ITEMS BELOW CAREFULLY BEFORE YOUR START THE EXAM

- **All unauthorized materials (textbooks, notes, electronic devices, bags, etc.), and in particular mobile phones, tablets, smart watches, computers, calculators etc. must be switched off and all must be left together with your bags, purses etc. in the area designated by the proctor(s), away from the seating area in the room.**
- **If one of the proctors suspects you of communicating with another student or cheating in other ways, you will be immediately removed from the exam room and your grade will be zero.**
- **Write your name and your student number on each page.**
- **Solve each question on the given page (not on another page).**
- **Read the questions carefully.**
- **Write your answers clearly, legibly, and bold enough.**
- **You must show all your work, steps, and computations in your solutions.**
- **If the final answer is a number, do not leave it in terms of factorials or product of other numbers. Do all the computations fully. Also, whenever the answer is a ratio, simplify it as much as possible.**
- **At any time during the exam, you are not allowed to leave the exam room and come back (for WC and other needs.)**
- **You are not allowed to hand in your exam paper within the first 30 minutes. You have to stay in the exam room during this period.**
- **You are not allowed to ask questions during the last 30 minutes.**
- **There are 10 questions. Check whether your exam sheet has all of them. If any is missing report this to the proctor.**

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READ ALL THE ITEMS ON THE COVER PAGE BEFORE YOU START THE EXAM

1. (a) (2.5 pts) Suppose you have two identical mathematics books and three other books of different subjects (for example, one psychology book, one history book, and one chemistry book).

   In how many different ways can you arrange them on a shelf if mathematics books will not be next to each other?

   Hint: Find the number of ways the books can be arranged in which the math books are together.

   Solution: Without any restriction, there are $5!\ /\ 2! = 60$ ways of arranging them. If we treat these two math books as one, there are $4! = 24$ ways of arranging them in which tow math books are next to each other. Hence, the answer is 60-24 = 36.

   (b) (2.5 pts) Suppose you have four different sweaters, and five different t-shirts. Your carry-on luggage has limited capacity, and it can take three items only.

   In how many different ways can you fill in your carry-on with three items if the number of sweaters you take with you in the luggage is at least one.

   Note 1: The luggage will be full. You will leave with exactly three items.

   Note 2: The order in which the items are placed in the luggage is not important.

   Hint: Find the number of ways you can fill in the luggage with no sweater in it.

   Solution: There are $\binom{9}{3} = 84$ different ways of arranging the luggage. Among them $\binom{5}{3} = 10$ consists of all t-shirts. Hence, the answer is 74.

   Alternatively, without using the hint:

   There are $\binom{4}{1}\binom{5}{2} = 40$ different ways of having one sweater and two t-shirts.

   There are $\binom{4}{2}\binom{5}{1} = 60$ different ways of having two sweaters and one t-shirt.

   There are $\binom{4}{3}\binom{5}{0} = 4$ different ways of having three sweaters.

   They all add up to 74.

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2. Let A be B be two sets with $P(A \cap B^c) = 0.2$, $P(A^c \cap B) = 0.1$ and $P(A^c \cap B^c) = 0.5$. Here, A^c and B^c denote the complements of the sets A and B respectively.

   Compute the following conditional probabilities.

   (a) (1 pt) $P(A|B)$;

   (b) (1 pt) $P(B|A)$;

   (c) (1 pt) $P(A|B^c)$;

   (d) (1 pt) $P(A|A \cup B)$;

   (e) (1 pt) $P(A|A \cap B)$.

   Hint: Draw a Venn diagram.

   Solution: From the Venn diagram we obtain $P(A \cap B) = 0.2$, $P(A) = 0.4$, $P(B) = 0.3$, $P(A \cup B) = 0.5$. Then we compute

   (a) (1 pt) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2}{3}$;

   (b) (1 pt) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1}{2}$;

   (c) (1 pt) $P(A|B^c) = \frac{P(A\cap B^c)}{1-P(B)} = \frac{2}{7}$;

   (d) (1 pt) $P(A|A \cup B) = \frac{P\big(A\cap(A\cup B)\big)}{P(A\cup B)} = \frac{P(A)}{P(A\cup B)} = \frac{4}{5}$;

   (e) (1 pt) $P(A|A \cap B) = \frac{P\big(A\cap(A\cap B)\big)}{P(A\cap B)} = \frac{P(A\cap B)}{P(A\cap B)} = 1$.

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3. A four-sided die is tossed twice and all 16 outcomes are equally likely (each outcome is pair of the form (i, j) where i is the number on the first toss and j on the second).

   Let A be the event that an even number comes up on the first toss, B be the event that an even number comes up on the second toss, and C be the event that both tosses result in the same number

   (a) (2.5 pts) Determine whether A, B, C are pairwise independent.

   (b) (2.5 pts) Determine whether A, B, C are independent (together as a collection).

   Hint: Make a 4×4 table for the sample space.

   Solution: Represent the sample space as a 4×4 table in which (i, j)th entry corresponds to the outcome for which i is the number on the first toss and j on the second. Then events are

$$\begin{aligned} A &= \big\{(2,1),(2,2),(2,3),(2,4),(4,1),(4,2),(4,3),(4,4)\big\} \\ B &= \big\{(1,2),(2,2),(3,2),(4,2),(1,4),(2,4),(3,4),(4,4)\big\} \\ C &= \big\{(1,1),(2,2),(3,3),(4,4)\big\} \\ A\cap B &= \big\{(2,2),(2,4),(4,2),(4,4)\big\} \\ A\cap C &= \big\{(2,2),(4,4)\big\} = B\cap C = A\cap B\cap C \end{aligned}$$

   Because all outcomes are equally likely, the number of elements in a set divided by 16 gives the probability of the set. Hence we have

$$\begin{aligned} P(A) &= \frac{1}{2} \\ P(B) &= \frac{1}{2} \\ P(C) &= \frac{1}{4} \\ p(A\cap B) &= \frac{1}{4} \\ P(A\cap C) = P(B\cap C) = P(A\cap B\cap C) &= \frac{1}{8} \end{aligned}$$

   (a) From the probabilities above we observe that

$$\begin{aligned} P(A)P(B) &= \frac{1}{2}\frac{1}{2} = \frac{1}{4} = P(A\cap B) \\ P(A)P(C) &= \frac{1}{2}\frac{1}{4} = \frac{1}{8} = P(A\cap C) \\ P(B)P(C) &= \frac{1}{2}\frac{1}{4} = \frac{1}{8} = P(B\cap C) \end{aligned}$$

   Hence they are pairwise independent.

   (b) We have $P(A)P(B)P(C) = \frac{1}{16} \neq \frac{1}{8} = P(A\cap B\cap C)$. Therefore they are not independent.

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4. A computer terminal is trying to send a file to another terminal over an unreliable connection. Suppose that at every trial, the file is delivered successfully with probability $\theta = \frac{1}{3}$. Trials are repeated independently until the file is delivered successfully.

   (a) (2.5 pts) Find the probability that the file is delivered successfully at the third trial.

   (b) (2.5 pts) Find the probability that the file is delivered successfully latest by the third trial.

Solution:

(a) We compute

$$P(\text{failure at trial \#1, failure at trial \#2, success at trial \#3})$$

By the independence of trials, this is equal to

$$P(\text{failure at trial \#1})\, P(\text{failure at trial \#2})\, P(\text{success at trial \#3}) = (1-\theta)^2\theta = \frac{4}{27}$$

(b) We compute

$$1 - P(\text{all failures in the first 3 trials}) = 1 - P(\text{failure at trial \#1, failure at trial \#2, failure at trial \#3})$$

By the independence of trials again, this is equal to

$$1 - P(\text{failure at trial \#1})\, P(\text{failure at trial \#2})\, P(\text{failure at trial \#3}) = 1 - (1-\theta)^3 = \frac{19}{27}$$

Alternatively, you can compute

$$P(\text{successful delivery at trial \#1}) = 1 - \theta = \frac{1}{3}$$
$$P(\text{successful delivery at trial \#2}) = (1-\theta)\theta = \frac{2}{9}$$

which gives

$$P(\text{successful delivery by the third trial}) = P(\text{successful delivery at trial \#1}) +$$
$$P(\text{successful delivery at trial \#2}) + P(\text{successful delivery at trial \#3}) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} = \frac{19}{27}$$

again.

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5. In a factory, it is known from past experience that a new worker who attends the company's training program later becomes a successful/productive worker with probability 0.8. A new worker who does not attend the training program later becomes a successful worker with probability 0.4. It is also known that 60% of the new workers attend the training program.

   (a) (2.5 pts) If a randomly selected worker is successful, what is the probability that s/he has attended the training program?

   (b) (2.5 pts) If a randomly selected worker is unsuccessful, what is the probability that s/he has attended the training program?

   Solution:

   Let T be the event that the worker attends the training program and S be the event that s/he is successful. Then the question states that

$$P(S|T) = 0.8 \quad P(S|T^c) = 0.4 \quad P(T) = 0.6$$

   From this we obtain

$$\begin{aligned} P(S \cap T) &= P(S|T)\,P(T) = 0.8 \cdot 0.6 = 0.48 \\ P(S \cap T^c) &= P(S|T^c)\,P(T^c) = 0.4 \cdot 0.4 = 0.16 \\ P(S) &= P(S \cap T) + P(S \cap T^c) = 0.64 \end{aligned}$$

   (a) We compute

$$P(T|S) = \frac{P(T \cap S)}{P(S)} = \frac{3}{4}$$

   (b) We have

$$P(T|S^c) = \frac{P(T \cap S^c)}{P(S^c)} = \frac{P(S^c|T)P(T)}{1 - P(S)} = \frac{\big(1 - P(S|T)\big)P(T)}{1 - P(S)} = \frac{(1 - 0.8)0.6}{0.36} = \frac{1}{3}$$

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6. There are two fair coins and one biased coin in an urn. The biased coin has a probability of 3/4 of coming up Heads and 1/4 of coming up Tails at every independent toss. These probabilities are 1/2 and 1/2 for the fair coins.

   Your friend selects one of these coins at random and tosses it independently three times.

   (a) (2.5 pts) What is the probability that the selected coin is a fair one if your friend observes the sequence H, T, H? (H for Heads, T for Tails)

   (b) (2.5 pts) Suppose that your friend tells you that 2 Heads and 1 Tail are observed but s/he does not tell you in which order they appear. What is the probability that the selected coin is a fair one?

   Solution: Let F be the event that the selected coin is a fair one, S be the event that we get the sequence H,T,H, and D be the event that we get 2H's and 1 T (without knowing the order). Then we have
   $\mathrm{P}(S|F) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$
   $\mathrm{P}(S|F^c) = \frac{3}{4}\frac{1}{4}\frac{3}{4} = \frac{9}{64}$
   There are $\binom{3}{2}$ many different sequences with 2 H and 1 T, and each has a probability of 1/8 with the fair coin and 9/64 with the biased one. Hence we have
   $\mathrm{P}(D|F) = \binom{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{3}{8}$
   $\mathrm{P}(D|F^c) = \binom{3}{2}\left(\frac{3}{4}\right)^2\frac{1}{4} = \frac{27}{64}$

   (a) We compute
   $$P(F|S) = \frac{\frac{1}{8}\frac{2}{3}}{\frac{1}{8}\frac{2}{3} + \frac{9}{64}\frac{1}{3}} = \frac{16}{25}$$

   (b) Similarly
   $$P(F|D) = \frac{\frac{3}{8}\frac{2}{3}}{\frac{3}{8}\frac{2}{3} + \frac{27}{64}\frac{1}{3}} = \frac{16}{25}$$

   and it gives the same answer.

   Note that because all coins behave independently over tosses with the same head and tail probabilities, we have
   $$P(D|F) = \binom{3}{2}P(S|F) \qquad \text{and} \qquad P(D|F^c) = \binom{3}{2}P(S|F^c)$$

   Hence,
   $$P(D \cap F) = P(D|F)P(F) = \binom{3}{2}P(S|F)P(F) = \binom{3}{2}P(S \cap F)$$
   $$P(D \cap F^c) = P(D|F^c)P(F^c) = \binom{3}{2}P(S|F^c)P(F^c) = \binom{3}{2}P(S \cap F^c)$$
   $$P(D) = P(D \cap F) + P(D \cap F^c) = \binom{3}{2}P(S \cap F) + \binom{3}{2}P(S \cap F^c) = \binom{3}{2}P(S)$$

   That is, all probabilities are multiplied by the same constant $\binom{3}{2}$ and we obtain the same result at the end.

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7. Let X be a discrete random variable with the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{1}{4} & \text{for } -1 \leq x < 1 \\ \dfrac{1}{2} & \text{for } 1 \leq x < 3 \\ \dfrac{3}{4} & \text{for } 3 \leq x < 5 \\ 1 & \text{for } x \geq 5 \end{cases}$$

Compute the following probabilities

(a) $\mathbb{P}(X \leq 3)$;

(b) $\mathbb{P}(X = 3)$;

(c) $\mathbb{P}(-1 < X < 1)$;

(d) $\mathbb{P}(X = 2)$;

(e) $\mathbb{P}(X \geq 5)$.

Solution:

(a) $\mathbb{P}(X \leq 3) = F(3) = \frac{3}{4}$;

(b) $\mathbb{P}(X = 3) = F(3) - \lim_{x\to 3^-} F(x) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$;

(c) $\mathbb{P}(-1 < X < 1) = \lim_{x\to 1^-} F(x) - F(-1) = \frac{1}{4} - \frac{1}{4} = 0$;

(d) $\mathbb{P}(X = 2) = F(2) - \lim_{x\to 2^-} F(x) = \frac{1}{2} - \frac{1}{2} = 0$;

(e) $\mathbb{P}(X \geq 5) = 1 - P(X < 5) = 1 - \lim_{x\to 5^-} F(x) = 1 - \frac{3}{4} = \frac{1}{4}$.

We can also solve this question by obtaining the probability distribution $f(x)$ of the random variable. The function $F(x)$ has discontinuities at the points $\{-1, 1, 3, 5\}$, hence X takes values in this set and we have

$$\begin{aligned} f(-1) = P(X = -1) = F(-1) - \lim_{x\to -1^-} F(x) &= \frac{1}{4} - 0 = \frac{1}{4} \\ f(1) = P(X = 1) = F(1) - \lim_{x\to 1^-} F(x) &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ f(3) = P(X = 3) = F(3) - \lim_{x\to 3^-} F(x) &= \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \\ f(5) = P(X = 5) = F(5) - \lim_{x\to 5^-} F(x) &= 1 - \frac{3}{4} = \frac{1}{4} \end{aligned}$$

Then we have

(a) $\mathbb{P}(X \leq 3) = f(-1) + f(3) = \frac{1}{2}$;

(b) $\mathbb{P}(X = 3) = f(3) = \frac{1}{4}$;

(c) $\mathbb{P}(-1 < X < 1) = 0$ because none of the values in the set $\{-1, 1, 3, 5\}$ is in the open interval $(-1, 1)$;

(d) $\mathbb{P}(X = 2) = 0$ because 2 is not an element of the set $\{-1, 1, 3, 5\}$;

(e) $\mathbb{P}(X \geq 5) = f(5) = \frac{1}{4}$.

8. Let X be a continuous random variable with the density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) (1 pt) Compute $P(X > 0.5)$;

(b) (1 pt) Compute $P(X = 0.5)$;

(c) (1 pt) Find a for which $P(X > a) = 0.5$.

(d) (2 pts) Find the distribution function $F(x)$ of X.

Solution:

(a)

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)dx = \int_{0.5}^{1} 2x\,dx = 1^2 - (1/2)^2 = \frac{3}{4}$$

(b)

$$P(X = 0.5) = \int_{0.5}^{0.5} f(x)dx = 0$$

(c) Since the density is non-zero only on (0,1), a will be on this interval. Then, we compute

$$P(X > a) = \int_{a}^{1} 2x\,dx = 1^2 - (a)^2 = \frac{1}{2}$$

This gives $a = \sqrt{\frac{1}{2}}$.

(d) Recall that the distribution function is given by $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$ for every $x \in \mathbb{R}$.
For $x \le 0$:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

For $x \in (0,1)$:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 2t\,dt = x^2$$

For $x \ge 1$:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} 2t\,dt = 1$$

Hence

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ x^2 & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 \le x \end{cases}$$

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9. In an urn there are four balls, numbered from 1 to 4. Suppose you pick two balls at random without replacement. Let X be the sum of the numbers on these two balls.

   (a) (2.5 pts) Find the probability distribution $f(x)$ of X;

   (b) (2.5 pts) Find the distribution function $F(x)$ of X.

   Note: Although it may help to draw them, we don't ask for the plots of the functions $f(x)$ and $F(x)$. Plots will not be considered as answers.

   Hint: List the elements of the sample space as pairs (i, j)'s where i is the number on the first ball and j is the number on the second.

   Solution: The sample space is as follows:

$$\Omega = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$$

Since we select the balls at random, each outcome in Ω has the same probability 1/12.

The random variable X takes values in the set $\{3,4,5,6,7\}$ and we have

$$\begin{aligned}
\{X=3\} &= \{(1,2),(2,1)\} \\
\{X=4\} &= \{(1,3),(3,1)\} \\
\{X=5\} &= \{(1,4),(2,3),(3,2),(4,1)\} \\
\{X=6\} &= \{(2,4),(4,2)\} \\
\{X=7\} &= \{(3,4),(4,3)\}
\end{aligned}$$

(a) By counting the number of outcomes in each event above and multiplying with 1/12, we obtain

$$\begin{aligned}
f(3) &= P(X=3) = \frac{2}{12} = \frac{1}{6} \\
f(4) &= P(X=4) = \frac{2}{12} = \frac{1}{6} \\
f(5) &= P(X=5) = \frac{4}{12} = \frac{1}{3} \\
f(6) &= P(X=6) = \frac{2}{12} = \frac{1}{6} \\
f(7) &= P(X=7) = \frac{2}{12} = \frac{1}{6}
\end{aligned}$$

(b) The distribution function is given by $F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$ for every $x \in \mathbb{R}$.
For $x < 3$, $F(x) = P(X \le x) = 0$.
For $3 \le x < 4$, $F(x) = P(X \le x) = f(3) = \frac{1}{6}$.
For $4 \le x < 5$, $F(x) = P(X \le x) = f(3) + f(4) = \frac{2}{6} = \frac{1}{3}$.
For $5 \le x < 6$, $F(x) = P(X \le x) = f(3) + f(4) + f(5) = \frac{4}{6} = \frac{2}{3}$.
For $6 \le x < 7$, $F(x) = P(X \le x) = f(3) + f(4) + f(5) + f(6) = \frac{5}{6}$.
For $7 \le x$, $F(x) = P(X \le x) = f(3) + f(4) + f(5) + f(7) = 1$.

Hence we have

$$F(x) = \begin{cases} 0, & \text{for } x < 3 \\ \dfrac{1}{6} & \text{for } 3 \leq x < 4 \\ \dfrac{1}{3} & \text{for } 4 \leq x < 5 \\ \dfrac{2}{3} & \text{for } 5 \leq x < 6 \\ \dfrac{5}{6} & \text{for } 6 \leq x < 7 \\ 1 & \text{for } 1 \leq x \end{cases}$$

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10. Let X be a continuous random variable having the density

$$f_X(x) = \begin{cases} 1+x & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Define a new random variable

$$Y = \begin{cases} -1 & \text{if } X < 0 \\ 1 & \text{if } X \geq 0 \end{cases}$$

(a) (2.5 pts) Find the probability distribution $f_Y(y)$ of Y;

(b) (2.5 pts) Find the distribution function $F_Y(y)$ of Y.

Note: Although it may help to draw them, we don't ask for the plots of the functions $f_Y(y)$ and $F_Y(y)$. Plots will not be considered as answers.

Solution:

(a) The random variable Y can take two values: -1 and 1. It is -1 if $X < 0$, and 1 if $X \geq 0$. Hence we have

$$f_Y(-1) = P(Y = -1) = P(X < 0) = \int_{-\infty}^{0} f_X(x)\,dx = \int_{-1}^{0} (1+x)\,dx = \frac{1}{2}$$

and

$$f_Y(1) = P(Y = 1) = P(X \geq 0) = \int_{0}^{\infty} f_X(x)\,dx = \int_{0}^{1} (1-x)\,dx = \frac{1}{2}$$

Or simply $f_Y(1) = 1 - f_Y(-1) = \frac{1}{2}$.

(b) The distribution function is given by $F_Y(y) = P(Y \leq y) = \sum_{y_i \leq y} f(y_i)$ for every $y \in \mathbb{R}$.
For $y < -1$, $F_Y(y) = P(Y \leq y) = 0$.
For $-1 \leq y < 1$, $F_Y(y) = P(Y \leq y) = f_Y(-1) = \frac{1}{2}$.
For $1 \leq y$, $F_Y(y) = P(Y \leq y) = f_Y(-1) + f_Y(1) = 1$.
Hence we have

$$F_Y(y) = \begin{cases} 0 & \text{for } y < -1 \\ \dfrac{1}{2} & \text{for } -1 \leq y < 1 \\ 1 & \text{for } 1 \leq y \end{cases}$$