# Sabancı University
## MATH 203, Recitation 2, Fall 2019-2020

1. Exercise 2.43: An experiment consists of rolling a die until a 3 appears. Describe the sample space and determine

   (a) how many elements of the sample space correspond to the event that the 3 appears on the $k$th roll of the die.

   (b) how many elements of the sample space correspond to the event that the 3 appears not later than the $k$th roll of the die.

   **Solution:** Left to TA.

2. In a lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs a random experiment that selects 8 of those 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the $\binom{40}{8}$ combinations, what is the probability that a player has

   (a) all 8 of the numbers selected?

   (b) 7 of the numbers selected?

   **Solution:**

   (a)
$$\frac{1}{\binom{40}{8}} \approx 1.3 \cdot 10^{-8}.$$

   (b)
$$\frac{\binom{8}{7}\binom{32}{1}}{\binom{40}{8}} \approx 3.3 \cdot 10^{-6}.$$

3. In a random experiment, it is given that
$$\mathbb{P}(A \cap B') = 0.47,\ \mathbb{P}(A' \cap B') = 0.1,\ \text{and } \mathbb{P}(A' \cup B') = 0.84.$$
   Find
$$\mathbb{P}(A),\ \mathbb{P}(A \cup B),\ \text{and } \mathbb{P}(B).$$

   **Solution:** The solution can be given using the Venn diagram.

   - $A \cap B$ and $A \cap B'$ are disjoint, and their union is $A$. Therefore
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B').$$
     In the above equation, it is known that $\mathbb{P}(A \cap B') = 0.47$. Further we have
$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}((A \cap B)') = 1 - \mathbb{P}(A' \cup B') = 1 - 0.84 = 0.16,$$
     where we use de Morgan's law in the second equality. Combination of the equalities then gives
$$\mathbb{P}(A) = 0.16 + 0.47 = 0.63.$$
   - Since $A \cup B = (A' \cap B')'$, we have
$$\mathbb{P}(A \cup B) = 1 - \mathbb{P}(A' \cap B') = 0.9.$$

- Since
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$$
we have
$$0.9 = 0.63 + \mathbb{P}(B) - 0.16.$$
Therefore $\mathbb{P}(B) = 0.43$.

4. A box of 20 light bulbs contain 3 defective and 17 non-defective ones. If 2 light bulbs are selected at random from this box what is the probability that the light bulbs will be

   (a) of the same type? (either both defective or both non-defective)

   (b) of different type? (one defective and one non-defective)

   Answer both items when sampling is with replacement or without replacement.

   **Solution:**

   (a) Under sampling without replacement, considering the sequence of the draws, you can compute
$$\frac{3 \cdot 2 + 17 \cdot 16}{20 \cdot 19} = \frac{3 + 17 \cdot 8}{10 \cdot 19} = \frac{3 + 136}{190} = \frac{139}{190} \approx 0.7315$$
.

   Alternatively, you can consider the subsets of size 2 of a set with 20 items. Then, the same probability can be computed as
$$\frac{\binom{3}{2} + \binom{17}{2}}{\binom{20}{2}} = \frac{3 + 136}{190} = \frac{139}{190}$$
giving the same result.

   Under sampling with replacement, each time the bulb is drawn, it is replaced back and can be drawn again. Then the probability of drawing 2 defective or 2 non-defective is,
$$\frac{3 \cdot 3 + 17 \cdot 17}{20 \cdot 20} = \frac{9 + 289}{400} = \frac{298}{400} = 0.745$$

   (b) Under sampling without replacement, considering the sequence of draws, the probability of drawing one defective and one non-defective is
$$\frac{3 \cdot 17 + 17 \cdot 3}{20 \cdot 19} = \frac{17 \cdot 3}{10 \cdot 19} = \frac{51}{190} \approx 0.2684$$
Considering the subsets of size 2, the same probability can be computed as
$$\frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{3 \cdot 17}{19 \cdot 10}.$$

   Under sampling with replacement, this probability is
$$\frac{3 \cdot 17 + 17 \cdot 3}{20 \cdot 20} = 0.255$$

5. Suppose you pick a card at random from a well shuffled deck. What is the probability that it is an Ace given that it is a Diamond?

   **Solution:** Here, the sample space $\Omega$ has 52 equally likely outcomes. Let $A$ be the event that the card is an Ace, and $D$ be the event that it is a Diamond. Then we compute
$$\mathbb{P}(A|D) = \frac{\mathbb{P}(A \cap D)}{\mathbb{P}(D)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}.$$

6. In answering a question with four choices in a multiple choice exam, a student either knows the answer of guesses it. Let $p \in (0,1)$ be the probability that the student knows the answer. A student guessing the answer will select the correct answer with probability $1/4$.

   What is the probability that the student knew the answer given that s/he answered correctly?

   **Solution:** Let $C$ be the event that the answer is correct and let $K$ be the event that the students knows the answer/subject.

   The question states that

$$\mathbb{P}(K) = p, \qquad \mathbb{P}(C|K') = 1/4, \qquad \text{(and obviously } \mathbb{P}(C|K) = 1).$$

   Then, we compute

$$\mathbb{P}(K|C) = \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C|K) \cdot \mathbb{P}(K)}{\mathbb{P}(C|K) \cdot \mathbb{P}(K) + \mathbb{P}(C|K') \cdot \mathbb{P}(K')} = \frac{1 \cdot p}{1 \cdot p + \frac{1}{4} \cdot (1-p)} = \frac{4p}{1+3p}$$

7. Exercise 2.95: A shipment of 1000 parts contains 1 percent defective parts. Find the probability that

   (a) the first four parts chosen arbitrarily for inspection are nondefective;

   (b) the first defective part found will be on the fourth inspection.

   **Solution:**

   (a) In set notation, let $A_i$ be the event that the $i$th item is non-defective. Then we are asked to compute $\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$. Note that we can write

$$\begin{aligned}\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) &= \frac{\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)}{\mathbb{P}(A_1 \cap A_2 \cap A_3)} \cdot \frac{\mathbb{P}(A_1 \cap A_2 \cap A_3)}{\mathbb{P}(A_1 \cap A_2)} \cdot \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_1)} \cdot \mathbb{P}(A_1) \\ &= \mathbb{P}(A_4|A_1 \cap A_2 \cap A_3) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_1).\end{aligned}$$

   In plain words, this is equal to (going backwards)

   the probability that first item is non-defective $\times$
   the probability that second item is non-defective given that so is the first $\times$
   the probability that third item is non-defective given that so are the first two $\times$
   the probability that fourth item is non-defective given that so are the first three .

   Clearly, the probability that first item is non-defective equals 990/1000, the probability that second item is non-defective given that so is the first is equal to 989/999, the probability that third item is non-defective given that so are the first two is 988/998, and finally the probability that fourth item is non-defective given that so are the first three is 987/997. Substituting these values we get

$$\frac{990}{1000} \cdot \frac{989}{999} \cdot \frac{988}{998} \cdot \frac{987}{997} \approx 0.9605.$$

   **<u>Note for TA's:</u>** The question is of course easy to solve, but it is also helpful to make an explanation as above to help students understand the math behind it in terms of conditional probabilities.

   (b) This time, the first three item will be nondefective and fourth will be defective. That is, we compute

$$\frac{990}{1000} \cdot \frac{989}{999} \cdot \frac{988}{998} \cdot \frac{10}{997} \approx 0.00973$$

8. It is known from experience that in a certain industry 60 percent of all labor-management disputed are over wages, 15 percent are over working conditions, and 25 percent are over fringe issues. Also, 45 percent of the disputes over wages are resolved without strikes, 70 percent of the disputed over working conditions are resolved without strikes, and 40 percent of the disputes over fringes are resolved without strikes.

   (a) What is the probability that a labor-management dispute in this industry will be resolved without a strike?

   (b) What is the probability that if there was a strike, it was about fringe issues?

   **Solution:**

   Let $W$ be the event that the dispute is over wages, $C$ be the event that it is over working conditions, and $F$ be the event that it is over fringe issues. Also, $S$ be the event that there is a strike.

   (a) We compute

$$\begin{aligned}\mathbb{P}(S') &= \mathbb{P}(S' \cap W) + \mathbb{P}(S' \cap C) + \mathbb{P}(S' \cap F) \\ &= \mathbb{P}(S'|W) \cdot \mathbb{P}(W) + \mathbb{P}(S'|C) \cdot \mathbb{P}(C) + \mathbb{P}(S'|F) \cdot \mathbb{P}(F) \\ &= (0.45)(0.6) + (0.7)(0.15) + (0.4)(0.25) = 0.475.\end{aligned}$$

   (b) We compute

$$\mathbb{P}(F|S) = \frac{\mathbb{P}(F \cap S)}{\mathbb{P}(S)} = \frac{\mathbb{P}(F) \cdot \mathbb{P}(S|F)}{1 - \mathbb{P}(S')} = \frac{(0.25)(0.6)}{0.525} = 0.286.$$

9. An urn contains two white balls and one black ball, whereas a second urn has one white and five black balls. A ball is drawn at random from the first urn and without looking at its color it is placed in the urn 2. Then a ball is drawn at random from urn 2.

   (a) What is the probability that the final ball is white?

   (b) If the ball drawn from urn 2 is white, what is the probability that the transferred ball was black? / was white?

   **Solution:** For $i = 1, 2$, let $W_i$ be the event that the ball picked in the $i$th draw is white (note that $W_i'$ be the event that the ball in the $i$th draw is black).

   (a) We have

$$\begin{aligned}\mathbb{P}(W_2) &= \mathbb{P}(W_2 \cap W_1) + \mathbb{P}(W_2 \cap W_1') \\ &= \mathbb{P}(W_2|W_1) \cdot \mathbb{P}(W_1) + \mathbb{P}(W_2|W_1') \cdot \mathbb{P}(W_1') \\ &= \frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3} = \frac{5}{21}.\end{aligned}$$

   (b) The probability that the transferred ball was black given that the ball drawn from urn 2 is white can be computed as

$$\begin{aligned}\mathbb{P}(W_1'|W_2) &= \frac{\mathbb{P}(W_1' \cap W_2)}{\mathbb{P}(W_2)} \\ &= \frac{\mathbb{P}(W_2|W_1') \cdot \mathbb{P}(W_1')}{\mathbb{P}(W_2)} \\ &= \frac{\frac{1}{7} \cdot \frac{1}{3}}{\frac{5}{21}} = \frac{1}{5}.\end{aligned}$$

Using this result, we cam compute the conditional probability that the transferred ball was white given that the ball drawn from urn 2 is white as

$$\mathbb{P}(W_1|W_2) = 1 - \mathbb{P}(W_1'|W_2) = \frac{4}{5}.$$