Two-way ANOVA
Factor Level Means Study

Main effects and Interaction effects

- Cells defined by combinations of two or more **discrete factors**
- Allows effects to be decomposed into *main effects* and *interactions*
- Model assumptions remain unchanged

Cell means notation for two-way ANOVA

For $i = 1, \ldots, a$ levels in Factor A and $j = 1, \ldots, b$ levels in Factor B, there are $k = 1 \ldots, n_{i,j}$ individual observations in cell (i, j).

Cell means model: $Y_{i,j,k} = \mu_{i,j} + \varepsilon_{i,j,k}$

- $\mu_{i,j}$ is the expected value (true mean) of cell (i, j), estimated by $\overline{Y}_{i,j}$
- $\varepsilon \overset{\text{iid}}{\sim} N(0, \sigma^2)$

There are $ab + 1$ parameters in this model (including σ^2)

Main effects and Interaction effects

A *main effect* describes the difference between a baseline reference (μ) and the marginal mean for a factor level ($\mu_{i\cdot}$ or $\mu_{\cdot j}$).

The *marginal mean* is the average value of the response across all data points that belong to a particular level of a factor.

An *interaction effect* gives the difference between the mean for a particular cell ($\mu_{i,j}$) and the sum of the baseline and main effects for belonging to level i of factor A and level j of factor B.

Cell Means and Marginal Means in Two-way ANOVA

		Factor B			
		1	2	3	
Factor A	1	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$	$\mu_{1\cdot}$
	2	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$	$\mu_{2\cdot}$
	3	$\mu_{3,1}$	$\mu_{3,2}$	$\mu_{3,3}$	$\mu_{3\cdot}$
		$\mu_{\cdot 1}$	$\mu_{\cdot 2}$	$\mu_{\cdot 3}$	$\mu_{\cdot\cdot}$

$$\mu_{i\cdot} = \frac{1}{n_A}\sum_{j=1}^{3}\left(n_{i,j}\mu_{i,j}\right)$$

$$\mu_{\cdot\cdot} = \frac{1}{n_T}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(n_{i,j}\mu_{i,j}\right)$$

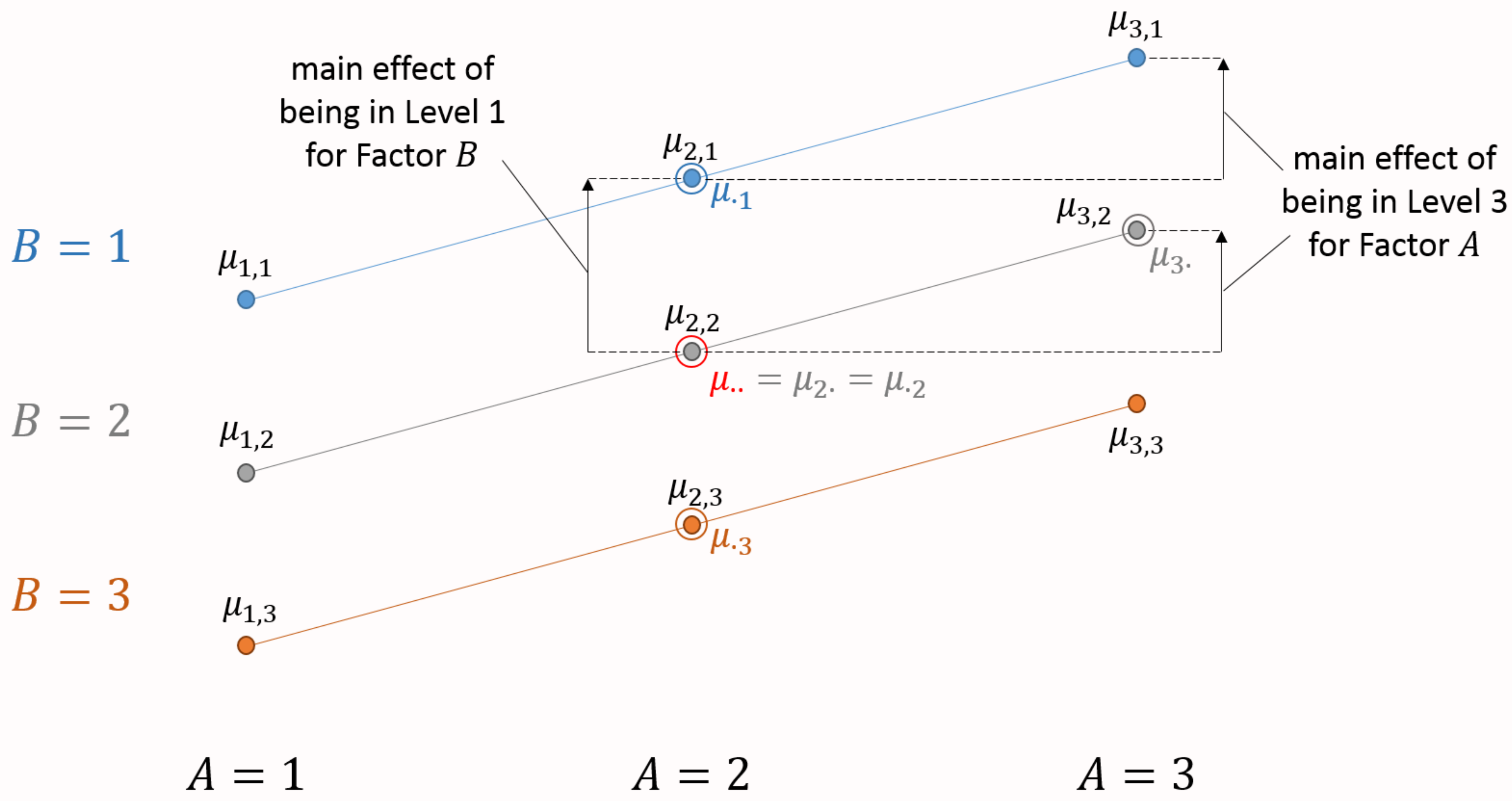
Factor effects notation for two-way ANOVA

Factor effects model: $Y_{i,j,k} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{i,j} + \varepsilon_{i,j,k}$

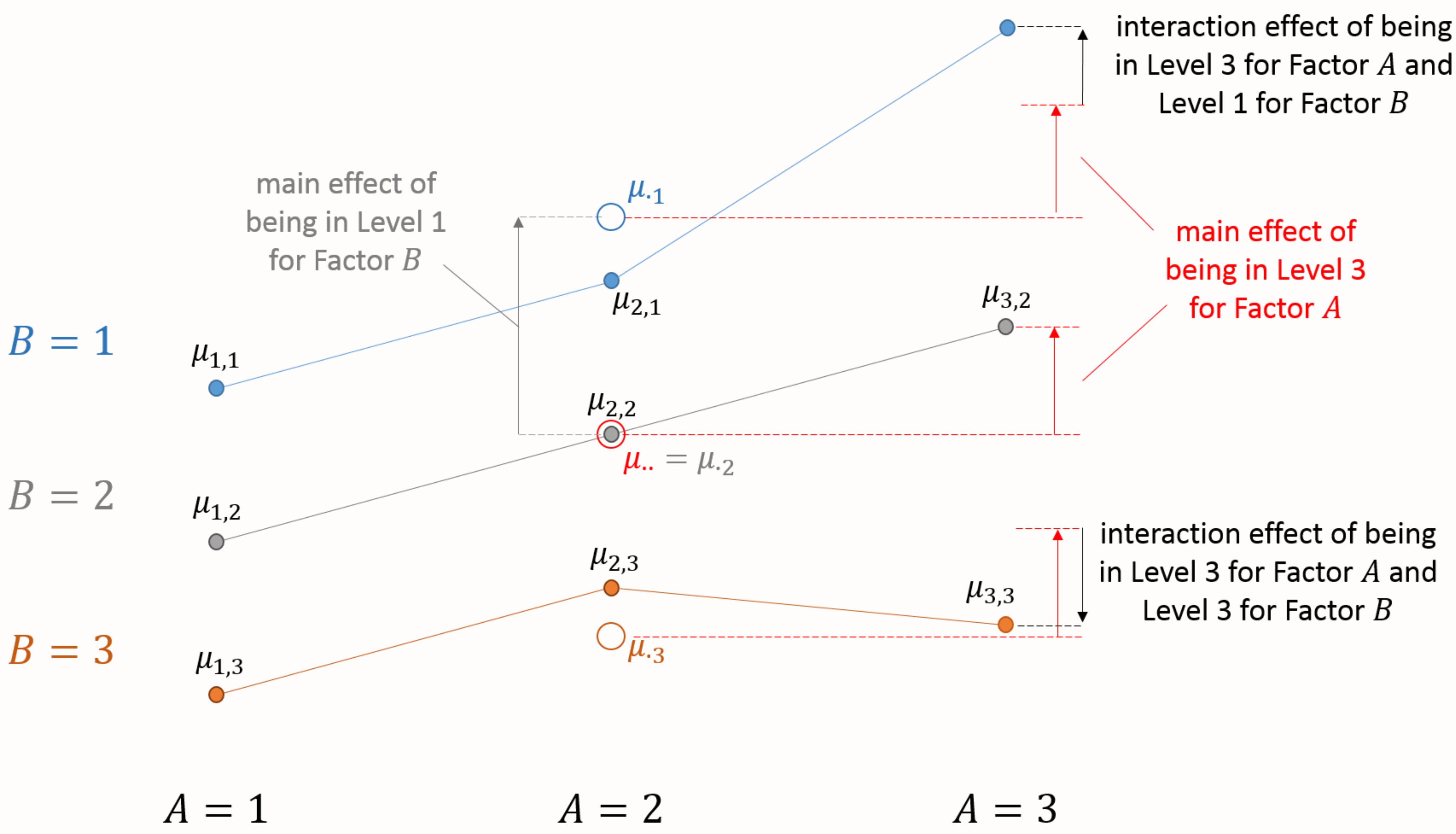
where,

- μ is grand mean, estimated by $\overline{Y}$ ...
- α_i is the main effect of belonging to level i of factor A, estimated by $\bar{Y}_{i..} - \bar{Y}_{...}$
- β_j is the main effect of belonging to level j of factor B, estimated by $\bar{Y}_{.j.} - \bar{Y}_{...}$
- $(\alpha\beta)_{i,j}$ is the interaction effect of belonging to both i and jestimated by $\bar{Y}_{\mathrm{i}j.} - \bar{Y}_{\mathrm{i}..} - \bar{Y}_{.j.} - \bar{Y}_{...}$

Note that “(αβ)i,j ” is ONE parameter, NOT a product!



Two-way ANOVA with no interactions: $\mu_{i,j} = \mu + \alpha_i + \beta_j$



Two-way ANOVA with interactions: $\mu_{i,j} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{i,j}$

Development of two-way ANOVA Model

$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$, where $i = 1\ to\ a, j = 1\ to\ b$

For two variables with $a\ and\ b\ levels\ respetively$, we need to define $(a-1)$ and $(b-1)$ dummy variables for main effect, and $(a-1)(b-1)$ dummy variables for interaction.

For example, $a = 3, b = 2$, the design matrix X is

Note: the reference baseline ($\mu_{..}$): unweighted mean

	X1	X2
level 1 in factor A	1	0
level 2 in factor A	0	1
level 3 in factor A	-1	-1

	X3
level 1 in factor B	1
level 2 in factor B	-1

Main effect

	X1X3	X2X3
level 1 in factor A and level 1 in factor B	1	0
level 1 in factor A and level 2 in factor B	-1	0
level 2 in factor A and level 1 in factor B	0	1
level 2 in factor A and level 2 in factor B	0	-1
level 3 in factor A and level 1 in factor B	-1	-1
level 3 in factor A and level 2 in factor B	1	1

Interaction effect

Constraints in two-way ANOVA

Equivalent of Constraint C ($\sum_{i=1}^{r} \tau_i = 0$) in one-way ANOVA:

$$\sum_{i=1}^{a} \alpha_i = 0 \qquad \sum_{j=1}^{b} \beta_i = 0 \qquad \sum_{i=1}^{a} (\alpha\beta)_{i,j} = 0 \; \forall j \qquad \sum_{j=1}^{b} (\alpha\beta)_{i,j} = 0 \; \forall i$$

- μ is the *grand mean* of the population ($\mu_{..}$, estimated by $\bar{Y}_{..}$)
- $\mu + \alpha_i$ is the *marginal mean* for level i of Factor A ($\mu_{i\cdot}$, estimated by $\bar{Y}_{i\cdot}$)
- $\mu + \beta_j$ is the marginal mean for level j of Factor B ($\mu_{\cdot j}$, estimated by $\bar{Y}_{\cdot j}$)
- $\mu + \alpha_i + \beta_j + (\alpha\beta)_{i,j}$ is the *cell mean* ($\mu_{i,j}$, estimated by $\bar{Y}_{i,j}$)

  (in a purely additive model, the cell mean would be $\mu + \alpha_i + \beta_j$)

Constraint $(\alpha\beta)_{a,b} = 0$ appears twice, so this is
a total of $1 + 1 + a + b - 1 = 1 + a + b$ constraints.

Development of the Regression Model

Example: Bread sales In this example, we use data from a designed experiment to determine how the height and width of a display shelf affects bread sales at a bakery. Twelve supermarkets, similar in sales volume and clientele were studied (bakery.txt).

$$a = 3, b = 2, n = 2, n_T = 12$$

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}\text{, where } i = 1 \text{ to } 3, j = 1 \text{ to } 2$$

Factor A (height)	Factor B (width) B1 (regular)	B2(wide)	row total	height average
A1 (bottom)	47	46		
	43	40		
Total	90	86	176	
average	45	43		44
A2 (middle)	62	67		
	68	71		
Total	130	138	268	
average	65	69		67
A3 (top)	41	42		
	39	46		
Total	80	88	168	
average	40	44		42
Column total	300	312	612	
width average	50	52		51

y	a (weight)	b(height)	Int.	x1	x2	x3	x1x3	x2x3
47	1	1	1	1	0	1	1	0
43	1	1	1	1	0	1	1	0
46	1	2	1	1	0	-1	0	-1
40	1	2	1	1	0	-1	0	-1
62	2	1	1	0	1	1	0	1
68	2	1	1	0	1	1	0	1
67	2	2	1	0	1	-1	0	-1
71	2	2	1	0	1	-1	0	-1
41	3	1	1	-1	-1	1	-1	1
39	3	1	1	-1	-1	1	-1	1
42	3	2	1	-1	-1	-1	-1	1
46	3	2	1	-1	-1	-1	-1	1

Design matrix

Development of the Regression Model

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width average	50	52		51

$\hat{Y}_{11} = \mu_{..} + \alpha_1 + \beta_1 + (\alpha\beta)_{11} = 55 - 7 - 1 + 2 = 44$

```
summary(lm(y~height*width, bakery))
```

```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       51.000      0.928  54.959 2.44e-09 ***
height1           -7.000      1.312  -5.334  0.00177 **
height2           16.000      1.312  12.192 1.85e-05 ***
width1            -1.000      0.928  -1.078  0.32261
height1:width1     2.000      1.312   1.524  0.17835
height2:width1    -1.000      1.312  -0.762  0.47494
```

$\mu_{..} = 51$

$\alpha_1 = \mu_{1.} - \mu_{..} = -7$

$\alpha_2 = \mu_{2.} - \mu_{..} = 16$

$\alpha_3 = -\alpha_1 - \alpha_2 =$

$\beta_1 = \mu_{.1} - \mu_{..} = -1$

$\beta_2 = -\beta_1 =$

$(\alpha\beta)_{11} = \mu_{11} - \mu_{1.} - \mu_{.1} + \mu_{..} = 2$

$(\alpha\beta)_{21} = \mu_{11} - \mu_{1.} - \mu_{.1} + \mu_{..} = -1$

$(\alpha\beta)_{31} = -(\alpha\beta)_{11} - (\alpha\beta)_{21} =$

$(\alpha\beta)_{12} = -(\alpha\beta)_{11} =$

$(\alpha\beta)_{22} = -(\alpha\beta)_{21} =$

$(\alpha\beta)_{32} = -(\alpha\beta)_{31} =$

Building the analysis of variance table in two-way ANOVA

In regression and one-way ANOVA, we broke the total sum of squares down into the model sum of squares (SSM) and error sum of squares (SSE).

In two-way ANOVA, SSM is further broken down into the main and interaction effects.

(This is really just an application of the extra sum of squares)

Rules for degrees of freedom

Degrees of freedom in the two-way ANOVA analysis are allocated as follows:

- Main effects for each factor take $r-1$ df, where r is the number of levels in the factor

  i.e., $df_A = (a-1)$ and $df_B = (b-1)$

- Interactions take df's equal to the product of the main effect df's:

  $(a-1)(b-1)$ for the interaction between factors A and B.

- Total sum of squares: $df_T = n_T - 1$ (as usual)

- Model df's are given by the sum of the df's for all main and interactions in the model:

  $$df_M = a + b - 2 + (a-1)(b-1)$$

- Error: $df_E = df_T - df_M$ (as usual)

F-tests

Two-way ANOVA adds several secondary F-tests to the standard global F-test.

- In *fixed effects* models, all of the F-tests use MSE in the denominator.
- The numerators for the secondary F-tests may use either the Type I (sequential) or Type II (last-variable-added) extra sums of squares.

ANOVA Table

Source	df	SS	MS	F
Factor A	$a-1$	SSA	$MSA = \frac{SSA}{a-1}$	$\frac{MSA}{MSE}$
Factor B	$b-1$	SSB	$MSB = \frac{SSA}{b-1}$	$\frac{MSB}{MSE}$
Interaction AB	$(a-1)(b-1)$	$SSAB$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$\frac{MSAB}{MSE}$
Error	$ab(n-1)$	SSE	$MSE = \frac{SSE}{ab(n-1)}$	
Total	n_T-1	SST	$MSA = \frac{SST}{n_T-1}$	

Expected mean squares

With the zero-sum constraints and a balanced design (so $n_{i,j} = n \; \forall i, j$):

$$\mathsf{E}(MSE) = \sigma^2$$

$$\mathsf{E}(MSA) = \sigma^2 + \frac{nb}{(a-1)} \sum_i \alpha_i^2$$

$$\mathsf{E}(MSB) = \sigma^2 + \frac{na}{(b-1)} \sum_j \beta_j^2$$

$$\mathsf{E}(MSAB) = \sigma^2 + \frac{n}{(a-1)(b-1)} \sum_{i,j} (\alpha\beta)_{i,j}^2$$

Analytical strategy

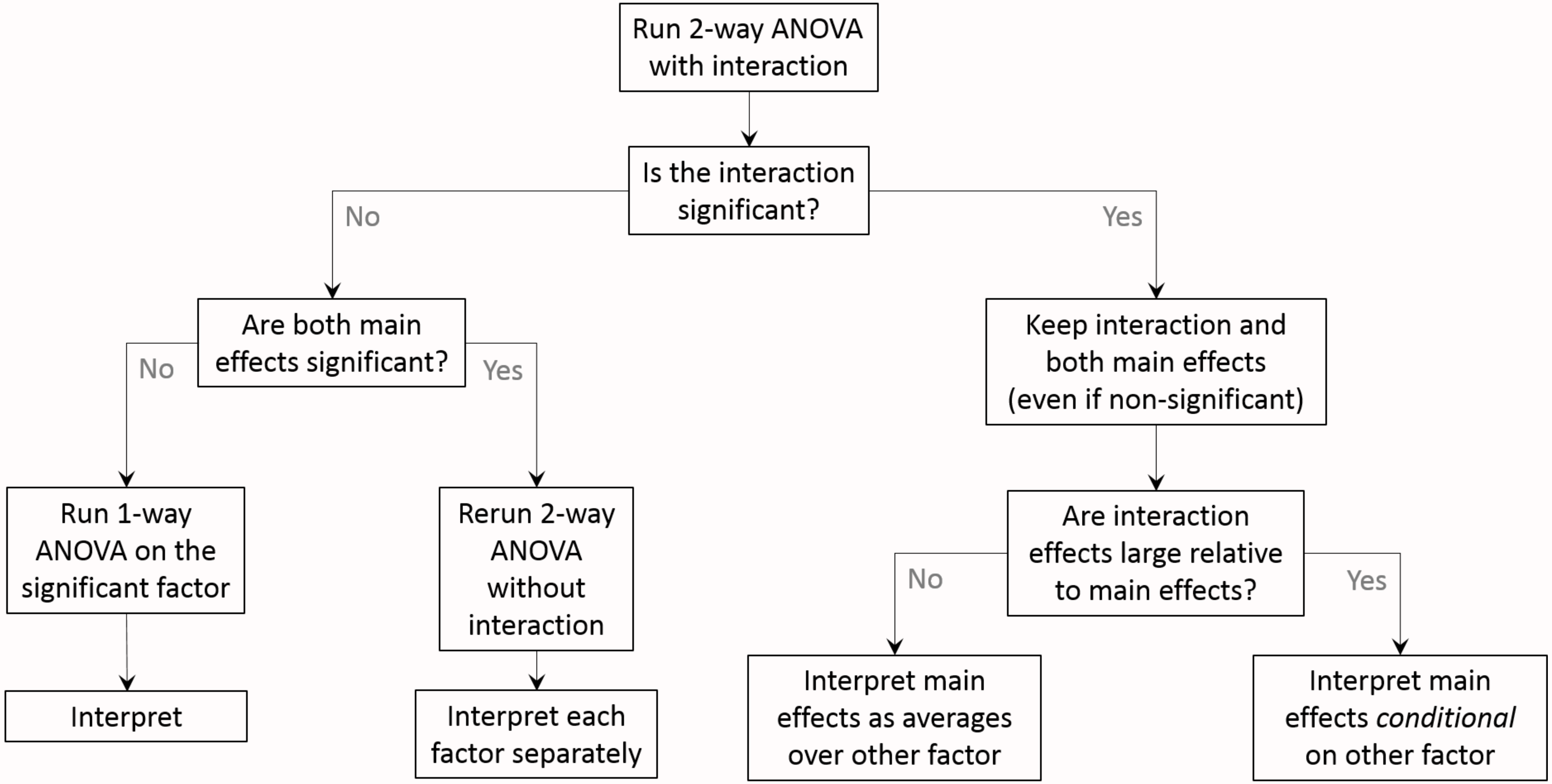
If the model contains interaction terms that are significantly different from zero, then the relationship between each factor and the response is not consistent.

It depends on the level of the other factor.

Always check for an interaction first.

- If the interaction is *not* significant, you can remove it.
- If the interaction term *is* significant, *leave the main effects in the model*, even if they are not significant.

Analytic strategy for two-way ANOVA



Compare main factor effects

Estimation of factor level mean

$$\hat{\mu}_{i.} = \bar{Y}_{i..} \qquad \sigma^2\{\bar{Y}_{i..}\} = \frac{\sigma^2}{bn} \qquad s^2\{\bar{Y}_{i..}\} = \frac{MSE}{bn}$$

$$\hat{\mu}_{.j} = \bar{Y}_{.j.} \qquad \sigma^2\{\bar{Y}_{.j.}\} = \frac{\sigma^2}{an} \qquad s^2\{\bar{Y}_{.j.}\} = \frac{MSE}{an}$$

Confidence interval for $\mu_{i..}$ and $\mu_{.j.}$

$$\bar{Y}_{i..} \pm t[1-\alpha/2; (n-1)ab]s\{\bar{Y}_{i..}\}$$

$$\bar{Y}_{.j.} \pm t[1-\alpha/2; (n-1)ab]s\{\bar{Y}_{.j.}\}$$

Estimation of contrast (or just general linear combination without the constriction) of factor level means

For factor A means $\mu_{i.}$

$$L = \sum c_i \mu_{i.} \qquad \text{where } \sum c_i = 0 \qquad \hat{L} = \sum c_i \bar{Y}_{i..}$$

$$\sigma^2\{\hat{L}\} = \sum c_i^2 \sigma^2\{\bar{Y}_{i..}\} = \frac{\sigma^2}{bn}\sum c_i^2 \qquad s^2\{\hat{L}\} = \frac{MSE}{bn}\sum c_i^2$$

For factor B means $\mu_{.j}$

$$L = \sum c_j \mu_{.j} \qquad \text{where } \sum c_j = 0 \qquad \hat{L} = \sum c_j \bar{Y}_{.j.}$$

$$s^2\{\hat{L}\} = \frac{MSE}{an}\sum c_j^2$$

Finally, the appropriate $1-\alpha$ confidence limits for L are:

$$\hat{L} \pm t[1-\alpha/2; (n-1)ab]s\{\hat{L}\}$$

The test statistic is $\frac{L}{s\{\hat{L}\}} \sim t((n-1)ab)$

Compare main factor effects

Bonferroni procedure comparison of factor level means

Compare g groups in factor A, each being D

$$D = \mu_{i\cdot} - \mu_{i'\cdot}$$

$$\hat{D} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}$$

$$s^2\{\hat{D}\} = \frac{2MSE}{bn}$$

Compare g groups in factor B, each being D

$$D = \mu_{\cdot j} - \mu_{\cdot j'}$$

$$\hat{D} = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'\cdot}$$

$$s^2\{\hat{D}\} = \frac{2MSE}{an}$$

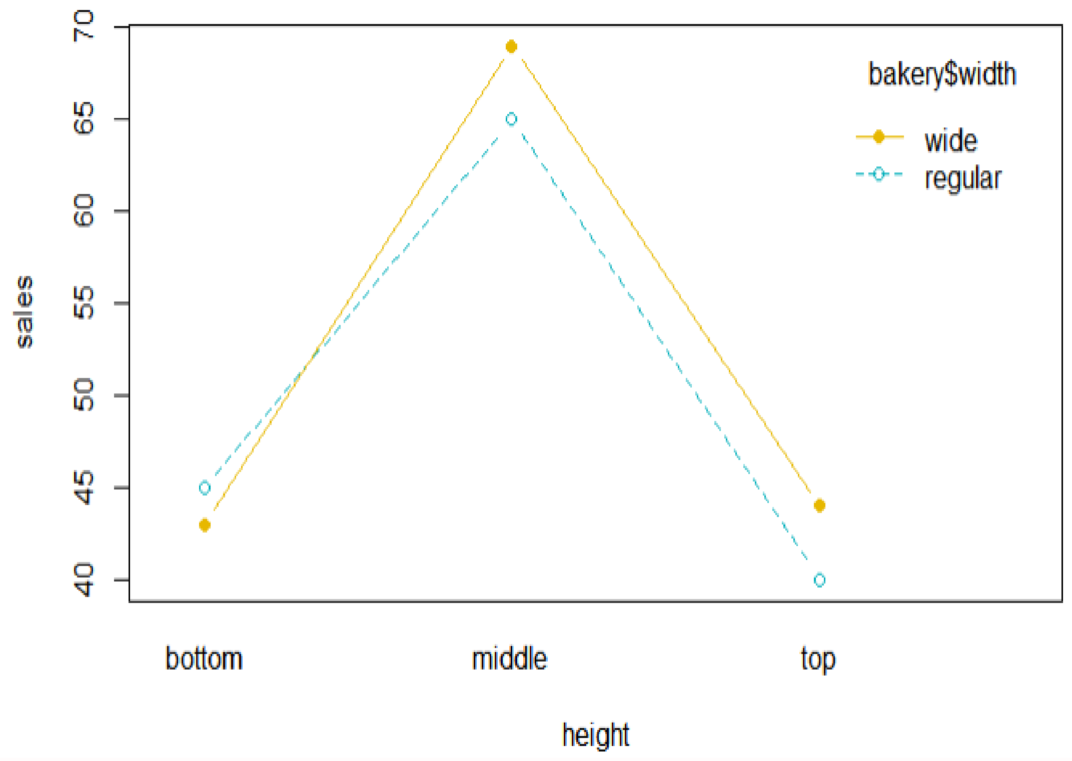
$$B = t[1 - \alpha/2g; (n-1)ab]$$

The $1-\alpha$ confidence interval for D are

$$\hat{D} \pm Bs\{\hat{D}\}$$

The test statistic are

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |t^*| > t[1-\alpha/2g; (n-1)ab], \text{ conclude } H_a$$



For example (main effect comparison)

Which of the following is the correct equation to compare average sale between regular and wide width.

A) $\mu_{1\cdot} - \mu_{2\cdot}$
B) $\mu_{\cdot 1} - \mu_{\cdot 2}$
C) $\mu_{11} - \mu_{21}$
D) $\mu_{11} - \mu_{12}$

There are also other procedures (Turkey, LSD, Sheaffe etc.) comparison of factor level means. Check out the text book for more details.

Compare interaction effects

Simultaneously compare multiple cell means

$$D = \mu_{ij} - \mu_{i'j'} \quad = \hat{Y}_{ij\cdot} - \hat{Y}_{i'j'\cdot} \qquad s^2\{\hat{D}\} = 2\left(\frac{MSE}{n}\right)$$

For example, $D_1 = \mu_{11} - \mu_{12}$
$D_2 = \mu_{22} - \mu_{22}$
$D_3 = \mu_{31} - \mu_{32}$

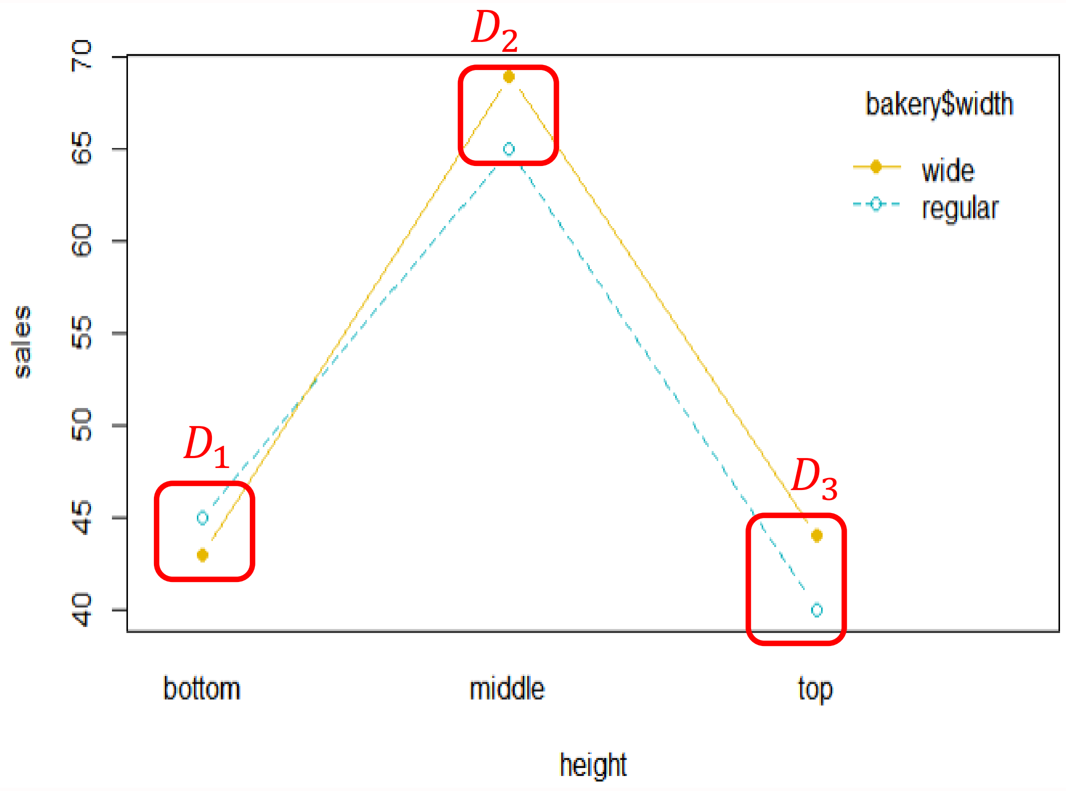
$$B = t[1 - \alpha/2g; (n-1)ab]$$

The $1 - \alpha$ confidence interval for D are

$$\hat{D} \pm Bs\{\hat{D}\}$$

The test statistic are

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |t^*| > t[1 - \alpha/2g; (n-1)ab], \text{ conclude } H_a$$



Comparison multiple cell means is necessary especially when the interaction effect is significant.

Example: Bread sales

In this example, we use data from a designed experiment to determine how the height and width of a display shelf affects bread sales at a bakery. Twelve supermarkets, similar in sales volume and clientele were studied (bakery.txt).

	Factor B (width)		row total	height average
Factor A (height)	B1 (regular)	B2(wide)		
	47	46		
A1 (bottom)	43	40		
Total	90	86	176	
average	45	43		44
	62	67		
A2 (middle)	68	71		
Total	130	138	268	
average	65	69		67
	41	42		
A3 (top)	39	46		
Total	80	88	168	
average	40	44		42
Column total	300	312	612	
width average	50	52		51

```
anova(lm(y~height*width, bakery))
```

```
Analysis of Variance Table

Response: y
             Df Sum Sq Mean Sq F value    Pr(>F)    
height        2   1544  772.00 74.7097 5.754e-05 ***
width         1     12   12.00  1.1613    0.3226    
height:width  2     24   12.00  1.1613    0.3747    
Residuals     6     62   10.33                      
```

$Ho: all\ (\alpha\beta_{ij}) = 0,\qquad Ha: not\ all\ (\alpha\beta_{ij}) = 0$

Not significant, p-value=0.3747

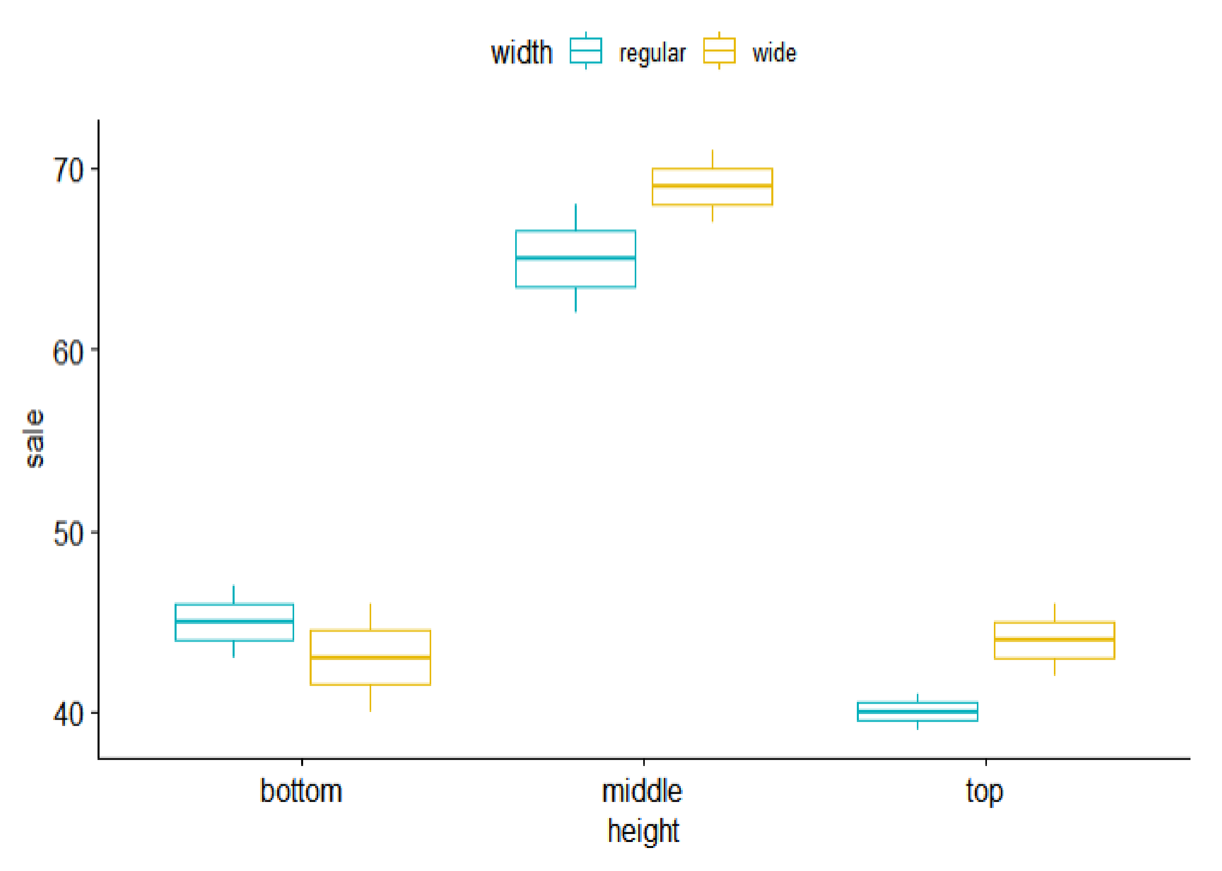
$Ho: all\ \alpha_i = 0,\qquad Ha: not\ all\ (\alpha_i) = 0$

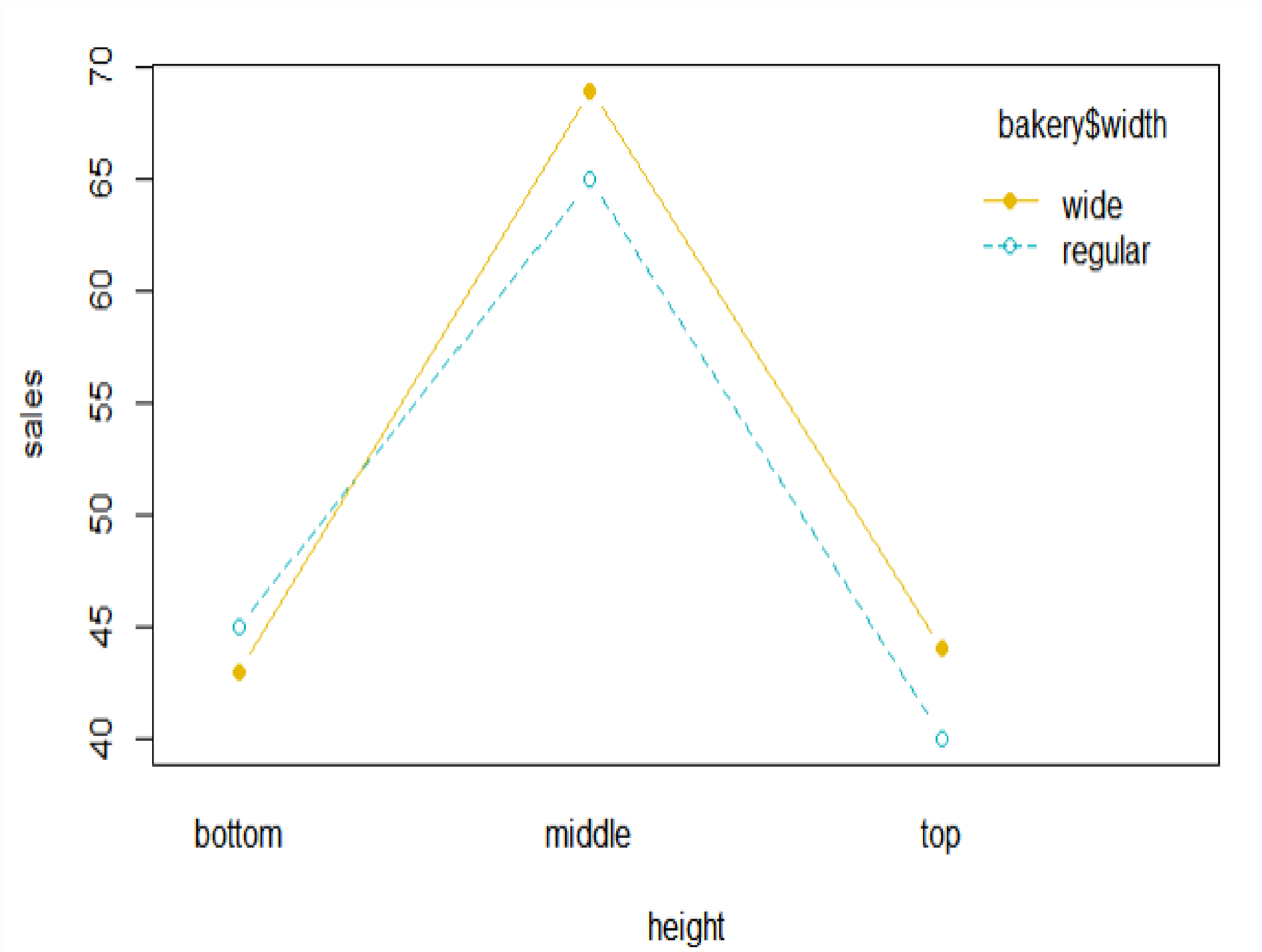
Significant, p-value<0.0001

$Ho: all\ \beta_i = 0,\qquad Ha: not\ all\ (\beta_i) = 0$

Not significant, p-value=0.3226

Example: Bread sales





1. Check the interaction term: not significant ($F_{2,6} = 1.16, p = 0.3747$)
2. Since the interaction is not significant, we can interpret the main effects independently of each other.
3. Main effect of height is significant ($F_{2,6} = 74.71, p < 0.0001$)
4. Main effect of width is not significant ($F_{1,6} = 1.16, p = 0.3226$)

Example: Bread sales

The height of the display affects sales, and has a similar effect at both widths.

Width has no effect on sales.

Further analyses are needed to tell which levels of height differ from the others:

1. Rerun the analysis as a one-way ANOVA on height

2. Compare the individual pairwise differences between levels

3. Look at a plot or at the cell means to see which height(s) maximize sales.

Example: Bread sales

Since the interaction effect is not significant. We can do comparison based on the marginal means (main effect)

Factor A (display height, i)	Factor B (display width, j)	Mean	n
i=1 bottom	j=1 (regular)	45	2
i=2 middle	j=1 (regular)	65	2
i=3 top	j=1 (regular)	40	2
i=1 bottom	j=2 (wide)	43	2
i=2 middle	j=2 (wide)	69	2
i=3 top	j=2 (wide)	44	2
MSE=10.3 a=3, b=2			

1. Compare the average sale between bottom and middle height

D1 $= \mu_{1.} - \mu_{2.}$

2. Compare the average sale between regular and wide width

$D2 = \mu_{.1} - \mu_{.2}$

3. Compare the average sale between the average of middle and top regular (21 and 31) and middle and top wide (22 and 32) width

$$D3 = \frac{(\mu_{21}+\mu_{31})}{2} - \frac{(\mu_{22}+\mu_{32})}{2}$$

4. Is the average sale the highest in the middle height? Consider a simultaneous comparison with Bonferroni procedure at 95% level.

Example: Bread sales

Since the interaction effect is not significant. We can do comparison based on the marginal means (main effect)

1. Compare the average sale between bottom and middle height

D1 $= \mu_{1.} - \mu_{2.}$

$$\hat{D} = \hat{Y}_{1.} - \hat{Y}_{2.} = (\hat{Y}_{11}+\hat{Y}_{12})/2 - (\hat{Y}_{21}+\hat{Y}_{22})/2 = -23$$

$$s^2\{\hat{D}\} = \frac{2MSE}{bn} = \frac{2(10.3)}{2(2)} = 5.15, \qquad so\ s\{\hat{D}\} = 2.27$$

$$t_s = \frac{\hat{D}}{s\{\hat{D}\}} = 10.1$$

Factor A (display height, i)	Factor B (display width, j)	Mean	n
i=1 bottom	j=1 (regular)	45	2
i=2 middle	j=1 (regular)	65	2
i=3 top	j=1 (regular)	40	2
i=1 bottom	j=2 (wide)	43	2
i=2 middle	j=2 (wide)	69	2
i=3 top	j=2 (wide)	44	2
MSE=10.3 a=3, b=2			

```
library(gmodels)
bm<-lm(y~height:width+0, data=bakery)
summary(bm)
anova(bm)
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
height1:width1   45.000      2.273   19.80 1.08e-06 ***
height2:width1   65.000      2.273   28.60 1.21e-07 ***
height3:width1   40.000      2.273   17.60 2.16e-06 ***
height1:width2   43.000      2.273   18.92 1.41e-06 ***
height2:width2   69.000      2.273   30.36 8.48e-08 ***
height3:width2   44.000      2.273   19.36 1.23e-06 ***
```

```
Response: y
             Df Sum Sq Mean Sq F value    Pr(>F)    
height:width  6  32792  5465.3   528.9 6.702e-08 ***
Residuals     6     62    10.3
```

```
cm<-c(1/2, -1/2 , 0 , 1/2, -1/2 , 0)
estimable(bm,cm)
```

Estimate <dbl>	Std. Error <dbl>	t value <dbl>	DF <dbl>	Pr(>|t|) <dbl>
-23	2.27303	-10.11865	6	5.415009e-05

To underhand the cm setting

$$D1 = \mu_{1.} - \mu_{2.} = (\mu_{11}+\mu_{12})/2 - (\mu_{21}+\mu_{22})/2$$
$$= \frac{\mu_{11}}{2} + \frac{\mu_{12}}{2} - \frac{\mu_{21}}{2} - \frac{\mu_{22}}{2}$$
$$= \frac{1}{2}(\mu_{11}) - \frac{1}{2}(\mu_{21}) + 0(\mu_{31}) + \frac{1}{2}(\mu_{12}) - \frac{1}{2}(\mu_{22}) + 0(\mu_{22})$$

Example: Bread sales

Since the interaction effect is not significant. We can do comparison based on the marginal means (main effect)

Factor A (display height, i)	Factor B (display width, j)	Mean	n
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MSE=10.3 a=3, b=2			

2. Compare the average sale between regular and wide width

$D2 = \mu_{.1} - \mu_{2.}$

$$\hat{D} = \hat{Y}_{.1} - \hat{Y}_{.2} = (\hat{Y}_{11} + \hat{Y}_{21} + \hat{Y}_{31})/3 - (\hat{Y}_{12} + \hat{Y}_{22} + \hat{Y}_{32})/3 = -2$$

$$s^2\{\hat{D}\} = \frac{2MSE}{an} = \frac{2(10.3)}{3(2)} = 3.43, \qquad so\ s\{\hat{D}\} = 1.86$$

$$t_s = \frac{\hat{D}}{s\{\hat{D}\}} = 1.08$$

```
library(gmodels)
bm<-lm(y~height:width+0, data=bakery)
summary(bm)
anova(bm)
```

```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
height1:width1    45.000      2.273   19.80 1.08e-06 ***
height2:width1    65.000      2.273   28.60 1.21e-07 ***
height3:width1    40.000      2.273   17.60 2.16e-06 ***
height1:width2    43.000      2.273   18.92 1.41e-06 ***
height2:width2    69.000      2.273   30.36 8.48e-08 ***
height3:width2    44.000      2.273   19.36 1.23e-06 ***
```

```
Response: y
             Df Sum Sq Mean Sq F value    Pr(>F)
height:width  6  32792  5465.3   528.9 6.702e-08 ***
Residuals     6     62    10.3
```

```
cm<-c(1/3, 1/3 , 1/3 , -1/3, -1/3 , -1/3)
estimable(bm,cm)
```

Estimate <dbl>	Std. Error <dbl>	t value <dbl>	DF <dbl>	Pr(>|t|) <dbl>
-2	1.855921	-1.077632	6	0.3226055

Example: Bread sales

3. Compare the average sale between
the average of middle and top regular (21 and 31) and
middle and top wide (22 and 32) width

$$D3 = \frac{(\mu_{21}+\mu_{31})}{2} - \frac{(\mu_{22}+\mu_{32})}{2}$$

$$\hat{D} = \frac{(\hat{Y}_{21}+\hat{Y}_{31})}{2} - \frac{\hat{Y}_{22}+\hat{Y}_{32}}{2} = -4$$

$$s^2\{\hat{D}\} = \frac{MSE}{n}\Sigma c_i^2 = \frac{(10.3)}{2}1 = 5.15, \qquad so\ s\{\hat{D}\} = 2.27$$

$$t_s = \frac{\hat{D}}{s\{\hat{D}\}} = -1.76$$

Factor A (display height, i)	Factor B (display width, j)	Mean	n
i=1 bottom	j=1 (regular)	45	2
i=2 middle	j=1 (regular)	65	2
i=3 top	j=1 (regular)	40	2
i=1 bottom	j=2 (wide)	43	2
i=2 middle	j=2 (wide)	69	2
i=3 top	j=2 (wide)	44	2
MSE=10.3 a=3, b=2			

```
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summary(bm)
anova(bm)
```

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Coefficients:
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height2:width2  69.000      2.273   30.36 8.48e-08 ***
height3:width2  44.000      2.273   19.36 1.23e-06 ***
```

```
Response: y
             Df Sum Sq Mean Sq F value    Pr(>F)
height:width  6  32792  5465.3   528.9 6.702e-08 ***
Residuals     6     62    10.3
```

```
cm<-c(0, 1/2 , 1/2 , 0, -1/2 , -1/2)
estimable(bm,cm)
```

Estimate <dbl>	Std. Error <dbl>	t value <dbl>	DF <dbl>	Pr(>|t|) <dbl>
-4	2.27303	-1.759765	6	0.1289371

Example: Bread sales

Factor A (display height, i)	Factor B (display width, j)	Mean	n
i=1 bottom	j=1 (regular)	45	2
i=2 middle	j=1 (regular)	65	2
i=3 top	j=1 (regular)	40	2
i=1 bottom	j=2 (wide)	43	2
i=2 middle	j=2 (wide)	69	2
i=3 top	j=2 (wide)	44	2
MSE=10.3 a=3, b=2			

4. Is the average sale the highest in the middle height?
Consider a simultaneous confidence interval with Bonferroni procedure at 0.95 level.

$$\hat{D}_1 = \hat{Y}_{1.} - \hat{Y}_{2.} = (\hat{Y}_{11}+\hat{Y}_{12})/2 \; - (\hat{Y}_{21}+\hat{Y}_{22})/2 = -23$$

$$s^2\{\hat{D}_1\} = \frac{2MSE}{bn} = \frac{2(10.3)}{2(2)} = 5.15, \qquad so\ s\{\hat{D}_1\} = 2.27$$

$$\hat{D}_2 = \hat{Y}_{3.} - \hat{Y}_{2.} = (\hat{Y}_{31}+\hat{Y}_{32})/2 \; - (\hat{Y}_{21}+\hat{Y}_{22})/2 = -25$$

$$s^2\{\hat{D}_2\} = \frac{2MSE}{bn} = \frac{2(10.3)}{2(2)} = 5.15, \qquad so\ s\{\hat{D}_2\} = 2.27$$

The $1-\alpha$ confidence interval for D are

$\hat{D} \pm Bs\{\hat{D}\}$ Where $B = t\left(1-\frac{\alpha}{2g}; (n-1)ab\right) = t(0.9875; 6)$=2.97

$\hat{D}_1 \pm Bs\{\hat{D}_1\}$= $-23 \pm 2.97 * 2.27 = -23 \pm 6.74 = (-29.74, -16.26)$

$\hat{D}_2 \pm Bs\{\hat{D}_2\}$= $-25 \pm 2.97 * 2.27 = -25 \pm 6.74 = (-31.74, -18.26)$

Example: Teaching method (significant interaction)

A junior college system studies the effects of teaching method (factor A) and student's quantitative ability (facto B) on learning of College mathematics.

- Factor A (teaching methods) Abstract and Standard, a=2
- Factor B (quantitative ability) Excellent, Good, and Moderate, b=3
- n=42 students were selected and randomly placed into classes, with each class containing equal numbers of students of each quantitative ability level.
- Y is the amount of learning of college mathematics, measured by a standard mathematics achievement test.

(a) Mean Learning Scores ($n = 21$)

Teaching Method i	Quantitative Ability (j) Excellent	Good	Moderate
Abstract	92 ($\bar{Y}_{11\cdot}$)	81 ($\bar{Y}_{12\cdot}$)	73 ($\bar{Y}_{13\cdot}$)
Standard	90 ($\bar{Y}_{21\cdot}$)	86 ($\bar{Y}_{22\cdot}$)	82 ($\bar{Y}_{23\cdot}$)

(b) ANOVA Table

Source of Variation	*SS*	*df*	*MS*
Factor *A* (teaching methods)	504	1	504
Factor *B* (quantitative ability)	3,843	2	1,921.5
AB interactions	651	2	325.5
Error	3,360	120	28
Total	8,358	125	

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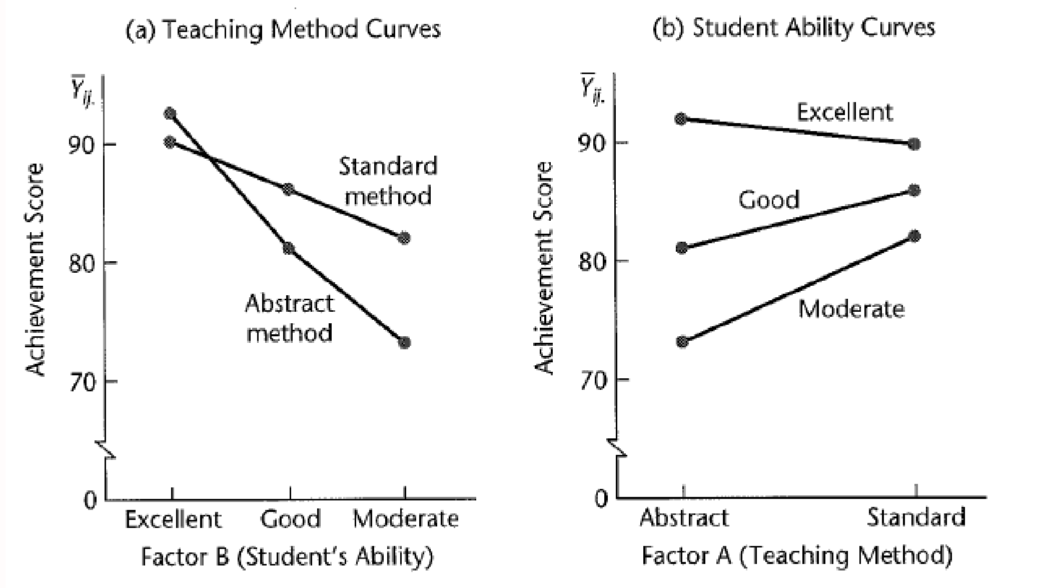

Investigate the nature of the interaction effects: estimating separately for students with excellent, good, and moderate quantitative abilities, how large is the difference in mean learning for the two teaching methods.

$$D_1 = \mu_{11} - \mu_{21} \qquad \hat{D}_1 = 92 - 90 = 2$$
$$D_2 = \mu_{12} - \mu_{22} \qquad \hat{D}_2 = 81 - 86 = -5$$
$$D_3 = \mu_{13} - \mu_{23} \qquad \hat{D}_3 = 73 - 82 = -9$$

$$s^2\{\hat{D}_1\} = s^2\{\hat{D}_2\} = s^2\{\hat{D}_3\} = \frac{2(28)}{21} = 2.667$$
$$s\{\hat{D}_1\} = s\{\hat{D}_2\} = s\{\hat{D}_3\} = 1.633$$

Consider Bonferroni procedure, $B = t\left(1 - \frac{\alpha}{2g}, ab(n-1)\right) = 2.428$

and the 95 percent confidence intervals for the family of comparisons are:

$$-1.96 \le \mu_{11} - \mu_{21} \le 5.96$$
$$-8.96 \le \mu_{12} - \mu_{22} \le -1.04$$
$$-12.96 \le \mu_{13} - \mu_{23} \le -5.04$$