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On orbits of order ideals of minuscule posets

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Abstract An action on order ideals of posets considered by Fon-Der-Flaass is analyzed in the case of posets arising from minuscule representations of complex simple Lie algebras. For these minuscule posets, it is shown that the Fon-Der-Flaass action exhibits the cyclic sieving phenomenon, as defined by Reiner, Stanton, and White. A uniform proof is given by investigation of a bijection due to Stembridge between order ideals of minuscule posets and fully commutative Weyl group elements. This bijection is proven to be equivariant with respect to a conjugate of the Fon-Der-Flaass action and an arbitrary Coxeter element.

If P is a minuscule poset, it is shown that the Fon-Der-Flaass action on order ideals of the Cartesian product $P \times [2]$ also exhibits the cyclic sieving phenomenon, only the proof is by appeal to the classification of minuscule posets and is not uniform.

Keywords Order ideals · Antichains · Minuscule posets · Minuscule representations · Fully commutative elements · Cyclic sieving phenomenon · Panyushev complement · Rowmotion

1 Introduction

The Fon-Der-Flaass action on order ideals of a poset has been the subject of extensive study since it was introduced in its original form on hypergraphs by Duchet in 1974 [5]. In this article, we identify a disparate collection of posets characterized by properties from representation theory—the minuscule posets—that exhibits consistent behavior under the Fon-Der-Flaass action. We illustrate the commonality via the

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cyclic sieving phenomenon of Reiner–Stanton–White [9], which provides a unifying framework for organizing combinatorial data on orbits arising from cyclic actions.

If P is a poset, and $J(P)$ is the set of order ideals of P, partially ordered by inclusion, the Fon-Der-Flaass action Ψ maps an order ideal $I \in J(P)$ to the order ideal $\Psi(I)$ whose maximal elements are the minimal elements of $P \setminus I$. Since Ψ is invertible, it generates a cyclic group $\langle \Psi \rangle$ acting on $J(P)$, but the orbit structure is not immediately apparent.

In [9], Reiner, Stanton, and White observed many situations in which the orbit structure of the action of a cyclic group $\langle c \rangle$ on a finite set X may be predicted by a polynomial $X(q) \in \mathbb{Z}[q]$.

Definition The triple $(X, X(q), \langle c \rangle)$ exhibits the *cyclic sieving phenomenon* if, for any integer d, the number of elements x in X fixed by c^d is obtained by evaluating $X(q)$ at $q = \zeta^d$, where n is the order of c on X and ζ is any primitive nth root of unity.

In the case when $X = J(P)$ and c is the Fon-Der-Flaass action, the natural generating function to consider is the rank-generating function for $J(P)$, which we denote by $J(P;q)$. Here the rank of an order ideal $I \in J(P)$ is given by the cardinality $|I|$ (so that $J(P;q) := \sum_{I \in J(P)} q^{|I|}$).

The minuscule posets are a class of posets arising in the representation theory of Lie algebras that enjoy some astonishing combinatorial properties. We give some background.

Let $\mathfrak{g}$ be a complex simple Lie algebra with Weyl group W and weight lattice Λ. There is a natural partial order on Λ called the *root order* in which one weight μ is considered to be smaller than another weight ω if the difference $\omega - \mu$ may be expressed as a positive linear combination of simple roots. If $\lambda \in \Lambda$ is dominant and the only weights occurring in the irreducible highest weight representation V^λ are the weights in the W-orbit $W\lambda$, then λ is called *minuscule*, and the restriction of the root order to the set of weights $W\lambda$ (which is called the *weight poset*) has two alternate descriptions:

- Let W_J be the maximal parabolic subgroup of W stabilizing λ, and let W^J be the set of minimum-length coset representatives for the parabolic quotient W/W_J. Then there is a natural bijection

$$W^J \to W\lambda,$$
$$w \mapsto w_0 w\lambda$$

(where w_0 denotes the longest element of W), and this map is an isomorphism of posets between the *strong Bruhat order* on W restricted to W^J and the root order on $W\lambda$.

- Let P be the poset of join-irreducible elements of the root order on $W\lambda$. Then P is called the *minuscule poset for* λ, P is ranked, and there is an isomorphism of posets between the weight poset and $J(P)$.

If P is minuscule, Proctor showed ([8], Theorem 6) that P enjoys what Stanley calls the *Gaussian* property (cf. [12], Exercise 25): There exists a function $f : P \to \mathbb{Z}$ such that, for all positive integers, m,

$$J\big(P \times [m]; q\big) = \prod_{p \in P} \frac{1 - q^{m+f(p)+1}}{1 - q^{f(p)+1}}.$$

This may be verified case-by-case, but it follows uniformly from the *standard monomial theory* of Lakshmibai, Musili, and Seshadri, as is shown in [8]. Furthermore, all Gaussian posets are ranked, and if P is Gaussian, we may take f to be the rank function of P.

Thus, for all positive integers m, we are led to consider the triple $(X, X(q), \langle \Psi \rangle)$, where $X = J(P \times [m])$, $X(q) = J(P \times [m]; q)$, and P is any minuscule poset. We are at last ready to state the first two of our main results, answering a question of Reiner.

Theorem 1.1 *Let P be a minuscule poset. If $m = 1$, $(X, X(q), \langle \Psi \rangle)$ exhibits the cyclic sieving phenomenon.*

Theorem 1.2 *Let P be a minuscule poset. If $m = 2$, $(X, X(q), \langle \Psi \rangle)$ exhibits the cyclic sieving phenomenon.*

It turns out that the claim analogous to Theorems 1.1 and 1.2 is false for $m = 3$; computations performed by Kevin Dilks[1] reveal that when $m = 3$ and P is the minuscule poset $[3] \times [3]$, the triple $(X, X(q), \langle \Psi \rangle)$ does not exhibit the cyclic sieving phenomenon. However, if P belongs to the third infinite family of minuscule posets (see the classification at the end of the introduction), the same triple exhibits the cyclic sieving phenomenon for all positive integers m. This was proved in our original REU report [10] but is omitted here. The rest of this introduction is devoted to a discussion of Theorems 1.1 and 1.2 and a brief overview of our approach to their proofs.

It should be noted that several special cases of Theorem 1.1 already exist in the literature. When P arises from a Lie algebra with root system of type A, for instance, Theorem 1.1 reduces to a result of Stanley in [13] coupled with Theorem 1.1(b) in Reiner–Stanton–White [9], and it is recorded as Theorem 6.1 by Striker and Williams in [16]. The case when the root system is of type B turns out to be handled almost identically, and it is recorded as Corollary 6.3 in [16]. That being said, our theorem is a generalization of these results, and, in relating Theorem 1.1 to a known cyclic sieving phenomenon for finite Coxeter groups (Theorem 1.6 in [9]), we expose the Fon-Der-Flaass action to new algebraic lines of attack.

If P is a finite poset, it is shown by Cameron and Fon-Der-Flaass in [4] that the Fon-Der-Flaass action Ψ may be expressed as a product of the involutive generators $\{t_p\}_{p \in P}$ for a larger group acting on the poset of order ideals $J(P)$. For all $p \in P$ and $I \in J(P)$, $t_p(I)$ is obtained by *toggling I at p*, so that $t_p(I)$ is either the symmetric

[1] Computed using code in the computer algebra package `Maple`. The authors also thank Dilks for allowing them the use of his code for subsequent computations.

difference $I \triangle \{p\}$, if this forms an order ideal, or just I, otherwise. In [16], Striker and Williams named this group the *toggle group*.

On the other hand, there is a natural labeling of the elements of a minuscule poset P by the Coxeter generators $S = \{s_1, s_2, \ldots, s_n\}$ for the Weyl group W, which is given by Stembridge in [15]. In particular, if P is a minuscule poset, there exists a labeling of P such that the linear extensions of the labeled poset (which is called a *minuscule heap*) index the reduced words for the fully commutative element of W representing the topmost coset $w_0 W_J$. This labeling is illustrated in Fig. 2 (as well as in Figs. 4, 5, 6, 7, 8, and 9 of the Appendix) and explained more thoroughly in Sect. 5. It has the following important properties.

First, it realizes the poset isomorphism $J(P) \cong W^J$ explicitly. Given an order ideal $I \in J(P)$ and a linear extension $(x_1, x_2, \ldots, x_t)$ of the partial order restricted to the elements of I, if the corresponding sequence of labels is $(i_1, i_2, \ldots, i_t)$, define $\phi(I)$ to be $s_{i_t} \ldots s_{i_2} s_{i_1}$. Then the map $\phi : J(P) \to W^J$ is an order-preserving bijection.

Second, it indicates a correspondence between Coxeter elements in W and sequences of toggles in $G(P)$: The choice of a linear ordering on the Coxeter generators $S = (s_{i_1}, \ldots, s_{i_n})$ yields a choice of the following.

- An element $t_{(i_1,\ldots,i_n)}$ in the toggle group that executes the following sequence of toggles: first toggle at all elements of P labeled by s_{i_n}, in any order; then toggle at all elements of P labeled by $s_{i_{n-1}}$, in any order; …; then toggle at all elements of P labeled by s_{i_2}, and, finally, toggle at all elements of P labeled by s_{i_1}, and
- A *Coxeter element* $c = s_{i_1} s_{i_2} \ldots s_{i_{n-1}} s_{i_n}$ in the Weyl group, which acts on cosets W/W_J by left translation (i.e., $c(wW_J) = cwW_J$), and thus also acts on W^J.

The theorems that reduce Theorem 1.1 to the cyclic sieving result of Reiner–Stanton–White [9] are as follows.

Theorem 1.3 *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, the actions Ψ and $t_{(i_1,\ldots,i_n)}$ are conjugate in $G(P)$.*

Theorem 1.4 *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, if $\phi : J(P) \to W^J$ is the isomorphism described above, then the following diagram is commutative*:

$$\begin{array}{ccc} J(P) & \xrightarrow{\phi} & W^J \\ t_{(i_1,\ldots,i_n)} \downarrow & & c \downarrow \\ J(P) & \xrightarrow{\phi} & W^J. \end{array}$$

To see that these theorems suffice to demonstrate Theorem 1.1, we quote Theorem 1.6 from [9].

Theorem 1.5 *Let W be a finite Coxeter group*; *let S be the set of Coxeter generators, and let J be a subset of S. Let W^J be the set of minimum-length coset representatives, and let $W^J(q) = \sum_{w \in W^J} q^{l(w)}$, where $l(w)$ denotes the length of w. If $c \in W$ is a* *regular element in the sense of Springer* [11], *then* $(W^J, W^J(q), \langle c\rangle)$ *exhibits the cyclic sieving phenomenon.*

In Theorem 1.5, if W^J is a distributive lattice, then the length function l also serves as a rank function, so $W^J(q)$ is the rank-generating function. Furthermore, if c is a Coxeter element of W, then $c \in W$ is regular (cf. [11]).

The proofs of Theorems 1.3 and 1.4 are carried out in Sect. 6; Sects. 2, 3, 4, and 5 provide the requisite background. We did not manage to adapt the techniques developed in these sections for the proof of Theorem 1.2, so in Sects. 7–11 we adopt a less theoretical approach, suppressing most of the details. Full proofs may still be found in the REU report [10]. In Sect. 7, we review ordinary and symmetric plane partitions, which provide a convenient framework for analyzing order ideals of $P \times [2]$. Then Sects. 8, 9, and 10 cover the cases corresponding to the first, second, and third infinite families, respectively. The claim of Theorem 1.2 for the two exceptional cases is checked by computer in Sect. 11, using the software developed by Dilks. While the proofs of the results which we assemble into Theorem 1.2 are purely combinatorial, we would like to see a uniform resolution of this problem that draws upon the more algebraic techniques of Sects. 2–6. That, in particular, may seem like a tall order, but it should be noted that, for minuscule posets P, Stembridge found an instance of the $q = -1$ phenomenon (a special case of the cyclic sieving phenomenon for actions of order 2) that holds uniformly for all Cartesian products $P \times [m]$ in [14]. Thus, even though the situation in the case of general cyclic sieving is considerably more complicated, there may still be reason to be optimistic.

We close the introduction with a description of the three infinite families and two exceptional cases of minuscule posets and the root systems associated to the Lie algebras from which they arise. Pictures may be found in the Appendix. The following facts are well known (cf. for instance, [14]).

- For the root systems of the form A_n, there are n possible minuscule weights, which lead to n associated minuscule posets, namely, all those posets of the form $[j] \times [n+1-j]$ such that $1 \le j \le n$. Posets of this form are considered to comprise the *first infinite family*. Examples are depicted in Fig. 4, parts (b), (c), (d), and (e).
- For the root systems of the form B_n, there is one possible minuscule weight, which leads to one associated minuscule poset, namely, $[n] \times [n]/S_2$. Posets of this form are considered to comprise the *second infinite family*. An example is depicted in Fig. 5, part (b).
- For the root systems of the form C_n, there is one possible minuscule weight, which leads to one associated minuscule poset, namely, $[2n-1]$. Posets of this form already belong to the first infinite family. An example is depicted in Fig. 6, part (b).
- For the root systems of the form D_n, there are three possible minuscule weights, which, because two of the minuscule weights lead to the same minuscule poset, only lead to two associated minuscule posets, namely, $[n-1] \times [n-1]/S_2$ and $J^{n-3}([2] \times [2])$. Posets of the latter form are considered to comprise the *third infinite family* (it should be clear that posets of the former form already belong to the second infinite family). An example is depicted in Fig. 7, part (d).
- For the root system E_6, there are two possible minuscule weights, which, because both minuscule weights lead to the same minuscule poset, only lead to one asso-

ciated minuscule poset, namely, $J^2([2] \times [3])$. This poset is called the *first exceptional case*. It is depicted in Fig. 8, part (b).

- For the root system E_7, there is one possible minuscule weight, which leads to one associated minuscule poset, namely, $J^3([2] \times [3])$. This poset is called the *second exceptional case*. It is depicted in Fig. 9, part (b).

No other root systems admit minuscule weights.

2 The Fon-Der-Flaass action

In this section, we introduce and analyze the Fon-Der-Flaass action. This action was introduced by Duchet [5] and first studied in its present form by Brouwer and Schrijver [3], but it was the late Dmitry Fon-Der-Flaass who first made substantial progress in the case of products of chains [6], and it was this work that brought the action to our attention. The action has no accepted name in the literature, so we are honored to dedicate it to his memory.

Let $P = (X, <)$ be a partially ordered set, and let $J(P)$ be the set of order ideals of P, partially ordered by inclusion. Following the notation of [4], for all order ideals $I \in J(P)$, let

$$Z(I) = \{x \in I : y > x \Rightarrow y \notin I\},$$

and let

$$U(I) = \{x \notin I : y < x \Rightarrow y \in I\}.$$

Then the *Fon-Der-Flaass action*, which we denote by Ψ, is defined as follows.

Definition 2.1 For all $I \in J(P)$, $\Psi(I)$ is the unique order ideal satisfying $Z(\Psi(I)) = U(I)$.

Remark 2.2 It is clear from Definition 2.1 that Ψ permutes the order ideals of P (Fig. 1).

This definition of the Fon-Der-Flaass action is global. We now give an equivalent definition that decomposes it into a product of local actions, which are more easily understood. Recall from the introduction that for all $p \in P$ and $I \in J(P)$, we let $t_p : J(P) \to J(P)$ be the map defined by $t_p(I) = I \setminus \{p\}$ if $p \in Z(I)$, $t_p(I) = I \cup \{p\}$ if $p \in U(I)$, and $t_p(I) = I$ otherwise. The following theorem is equivalent to Lemma 1 in Cameron–Fon-Der-Flaass [4].

Theorem 2.3 *Let P be a poset. For all linear extensions $(p_1, p_2, \ldots, p_n)$ of P and order ideals $I \in J(P)$, $\Psi(I) = t_{p_1} t_{p_2} \cdots t_{p_n}(I)$.*

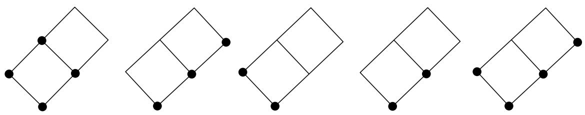


Fig. 1 An orbit of order ideals under the Fon-Der-Flaass action

The group $G(P) := \langle t_p \rangle_{p \in P}$ is named the *toggle group* by Striker and Williams [16]. Note that for all x and y, the generators t_x and t_y commute unless x and y share a covering relation.

In the case that the poset P is ranked, it is natural to consider the linear extensions label the elements of P by order of increasing rank. For the purposes of this paper, we shall say that P is ranked if there exists an integer-valued function r on X (called the *rank function*) such that $r(p) = 0$ for all minimal elements $p \in X$ and, for all $x, y \in X$, if x covers y, then $r(x) - r(y) = 1$.

If P is a ranked poset, let the maximum value of r be R. For all $0 \leq i \leq R$, let $P_i = \{p \in P : r(p) = i\}$, and let $t_i = \prod_{p \in P_i} t_p$. We see that t_i is always well defined because, for all i, t_x and t_y commute for all $x, y \in P_i$. By Theorem 2.3, $\Psi = t_0 t_1 \cdots t_R$. Note that t_i and t_j commute for all $|i - j| > 1$. The following theorem is also a result of Cameron–Fon-Der-Flaass [4].

Theorem 2.4 *For all permutations σ of $\{0, 1, \ldots, R\}$, $\Psi_\sigma := t_{\sigma(0)} t_{\sigma(1)} \cdots t_{\sigma(R)}$ is conjugate to Ψ in $G(P)$.*

Corollary 2.5 *The action Ψ_σ has the same orbit structure as Ψ for all σ.*

Let $t_{\text{even}} = \prod_{i \text{even}} t_i$, and let $t_{\text{odd}} = \prod_{i \text{ odd}} t_i$. It should be clear that t_{even} and t_{odd} are well defined, and it follows from Theorem 2.4 that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P)$, as noted in the second paragraph of Sect. 4 in [4]. This means that the action of toggling at all the elements of odd rank, followed by toggling at all the elements of even rank, is conjugate to the Fon-Der-Flaass action in the toggle group. As we shall see, this holds the key to demonstrating that the induced action of every Coxeter element of W on $J(P)$ under ϕ is conjugate to the Fon-Der-Flaass action as well. Striker and Williams made use of the same argument to obtain the conjugacy of promotion and rowmotion (their name for the Fon-Der-Flaass action) in Sect. 6 of [16], so it should be no surprise that our induced actions reduce to promotion in types A and B. In this sense, our proof of Theorem 1.1 may be considered to be a continuation of their work.

3 Minuscule posets

In this section, we introduce the primary objects of study for this paper—the minuscule posets. We begin with some notation, following Stembridge [14]. Let $\mathfrak{g}$ be a complex simple Lie algebra; let $\mathfrak{h}$ be a Cartan subalgebra; choose a set Φ^+ of positive roots α in $\mathfrak{h}^*$, and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots. Let $(\cdot, \cdot)$ be the inner product on $\mathfrak{h}^*$, and, for each root α, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Finally, let $\Lambda = \{\lambda \in \mathfrak{h}^* : \alpha \in \Phi \to (\lambda, \alpha^\vee) \in \mathbb{Z}\}$ be the weight lattice.

For all $1 \leq i \leq n$, let s_i be the simple reflection corresponding to the simple root α_i, and let $W = \langle s_i \rangle_{1 \leq i \leq n}$ be the Weyl group of $\mathfrak{g}$. If s is conjugate to a simple reflection s_i in W, we refer to s as an (abstract) reflection.

Let V be a finite-dimensional representation of $\mathfrak{g}$. For each $\lambda \in \Lambda$, let

$$V_\lambda = \{v \in V : h \in \mathfrak{h} \Rightarrow hv = \lambda(h)v\}$$

be the weight space corresponding to λ, and let Λ_V be the (finite) set of weights λ such that V_λ is nonzero. Recall that there is a standard partial order on Λ called the *root order* defined to be the transitive closure of the relations $\mu < \omega$ for all weights μ and ω such that $\omega - \mu$ is a simple root.

Definition 3.1 The *weight poset* Q_V of the representation V is the restriction of the root order on Λ to Λ_V.

If V is irreducible, Q_V has a unique maximal element, which is called the highest weight of V. This leads to the following definition.

Definition 3.2 Let V be a nontrivial, irreducible, finite-dimensional representation of $\mathfrak{g}$. V is a *minuscule representation* if the action of W on Λ_V is transitive. In this case, the highest weight of V is called the *minuscule weight*.

Theorem 3.3 *If V is minuscule, the weight poset Q_V is a distributive lattice.*

Remark 3.4 This result, due to Proctor (cf. [8], Propositions 3.2 and 4.1), was originally verified by exhaustive search, but it is also a consequence of Theorem 5.8, for which a case-free proof was given using Bruhat-theoretic techniques by Stembridge in [15].

Definition 3.5 If V is minuscule, let P_V be the poset of join-irreducible elements of the weight poset Q_V, so that P_V is the unique poset satisfying $J(P_V) \cong Q_V$. Then P_V is the *minuscule poset* of V, and posets of this form comprise the *minuscule posets*.

Remark 3.6 If V is a minuscule representation and λ is the highest weight of V, we refer to P_V as the *minuscule poset for* λ.

4 Bruhat posets

In this section, we develop the framework for the proofs of Theorems 1.3 and 1.4. We begin by discussing the Bruhat posets. Then we establish the connection between these objects and the weight posets of minuscule representations.

We continue with the notation of the previous section. Given a Weyl group W, we define a length function l on the elements of W as follows. For all $w \in W$, we let $l(w)$ be the minimum length of a word of the form $s_{i_1}s_{i_2}\dots s_{i_\ell}$ such that $w = s_{i_1}s_{i_2}\dots s_{i_\ell}$ and s_{i_j} is a simple reflection for all $1 \le j \le \ell$. This allows us to introduce a well-known partial order on W, known as the (strong) Bruhat order, for which l also serves as a rank function. The Bruhat order is defined to be the transitive closure of the relations $w <_B sw$ for all Weyl group elements w and (abstract) reflections s satisfying $l(w) < l(sw)$.

What is of interest is not the Bruhat order on W, but the restrictions of the Bruhat order to parabolic quotients of W, for these are the orders that give rise to the Bruhat posets.

Definition 4.1 If J is a subset of $\{1, 2, \ldots, n\}$, then $W_J := \langle s_i \rangle_{i \in J}$ is the *parabolic subgroup* of W generated by the corresponding simple reflections, and $W^J := W/W_J$ is the *parabolic quotient*.

It is well known that each coset in W^J has a unique representative of minimum length, so the quotient W^J may be regarded as the subset of W containing only the minimum-length coset representatives. This fact facilitates the definition of an analogous partial order on W^J.

Definition 4.2 The *Bruhat order* $<_B$ on the parabolic quotient W^J is the restriction of the Bruhat order on W to W^J. Posets of the form $(W^J, <_B)$ comprise the *Bruhat posets*.

We may also define the left (weak) Bruhat order on W to be the transitive closure of the relations $w <_L sw$ for all Weyl group elements w and *simple* reflections s satisfying $l(w) < l(sw)$. The analogous partial order on W^J is defined in precisely the same way: $(W^J, <_L)$ is the restriction of $(W, <_L)$ to the minimum-length coset representatives W^J. While the left Bruhat order is not necessary to establish the connection between the minuscule posets and the Bruhat posets, we introduce it here so that our work in this section may be compatible with the theory of fully commutative elements developed in Sect. 5 and exploited in Sect. 6.

We are now ready to state the following theorem, which appears as Proposition 4.1 in Proctor [8].

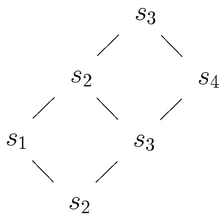
Theorem 4.3 *Let V be a minuscule representation with minuscule weight λ, and let $J = \{i : s_i\lambda = \lambda\}$. Then W_J is the stabilizer of λ in the Weyl group W, and the weight poset Q_V is isomorphic to the Bruhat poset $(W^J, <_B)$.*

Remark 4.4 There is a small subtlety in the proof Theorem 4.3 because the natural map $\varphi : W^J \to Q_V$ to consider, $w \mapsto w\lambda$, is order-reversing, rather than order-preserving. (In other words, for all $u, v \in W^J$, $u\lambda < v\lambda$ if and only if $v <_B u$.) However, composing φ with the order-reversing involution of Q_V given by $\omega \mapsto w_0\omega$, where w_0 is the unique longest element of W, yields a suitable isomorphism (as noted in the introduction). Alternatively, φ may be precomposed with the corresponding order-reversing involution of W^J given by $w \mapsto w_0 w (w_0^J)^{-1} w_0$, where w_0^J denotes the unique longest element of W^J. In Proctor's proof of Theorem 4.3, he circumvents this step by defining the partial order on the weights opposite to the root order. We avoid his approach here because it leads to unnecessary confusion over terms such as "highest weight."

Definition 4.5 The parabolic quotient W^J is *minuscule* if W_J is the stabilizer of a minuscule weight λ.

The assumption that $\mathfrak{g}$ be simple implies that λ is fundamental (recall that the fundamental weights $\omega_1, \omega_2, \ldots, \omega_n$ are defined by the condition $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for all $1 \le i, j \le n$, where δ_{ij} is the Kronecker delta). Hence if $\lambda = \omega_j$, then $s_i\lambda = \lambda$ for all

Fig. 2 If W is the Weyl group arising from the root system A_4, then the element $w := s_3 s_2 s_4 s_1 s_3 s_2$ is fully commutative, and the heap P_w is as displayed on the right



$i \neq j$. It follows that if W^J is minuscule, $J = \{1, 2, \dots, n\} \setminus \{j\}$, so W_J is a maximal parabolic subgroup of W. In general, a minuscule Bruhat poset is obtained precisely when the "missing" element of J is the index of a fundamental weight for which there exists a representation of $\mathfrak{g}$ in which that fundamental weight is minuscule.

We note that Bruhat posets W^J provide a natural setting for identifying instances of the cyclic sieving phenomenon because they come equipped with a group action, namely that of W, and a rank-generating function $W^J(q) := \sum_{w \in W^J} q^{l(w)}$, which is what motivated us to consider them in the first place. We now turn our attention to the labeling of the minuscule poset P_V and the construction of the isomorphism $\phi : J(P_V) \to W^J$, which lie behind the proofs of Theorems 1.3 and 1.4.

5 Fully commutative elements

In this section we borrow from Stembridge's theory of fully commutative elements of Weyl groups. In the next section, we shall see how the theory enables us to characterize the relationship between the action of the Weyl group on the elements of these lattices and the action of the toggle group on the order ideals of the corresponding minuscule posets.

Definition 5.1 Let W be a Weyl group, and let $S = \{s_1, s_2, \dots, s_n\}$ be the set of Coxeter generators. An element $w \in W$ is *fully commutative* if every reduced word for w can be obtained from every other by means of commuting braid relations only (i.e., via relations of the form $s_j s_{j'} = s_{j'} s_j$ for commuting Coxeter generators s_j and $s_{j'}$).

Given a fully commutative element w, we can define a labeled poset P_w that generates all the reduced words of w in the sense that putting labels in the place of poset elements gives a bijection between the linear extensions of P_w and the reduced words of w.

Definition 5.2 Let $s_{i_1} s_{i_2} \dots s_{i_\ell}$ be a reduced word for w. Let $P_w = (\{1, 2, \dots, \ell\}, <)$ be a partially ordered set, where the partial order on $\{1, 2, \dots, \ell\}$ is defined to be the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i_{j'}}$ do not commute. Then P_w is the *heap* of w, and, for all $1 \leq j \leq \ell$, s_{i_j} is the *label* of the heap element $j \in P_w$. An example is given in Fig. 2.

Let $\mathcal{L}(P_w) := \{\pi : \pi(1) \geq \pi(2) \geq \cdots \geq \pi(\ell)\}$ be the set of *reverse* linear extensions of P_w, and let $\mathcal{L}(P_w, w)$ be the set of labeled reverse linear extensions of P_w, i.e.,

$$\mathcal{L}(P_w, w) := \{s_{i_{\pi(1)}} s_{i_{\pi(2)}} \dots s_{i_{\pi(\ell)}} : \pi \in \mathcal{L}(P_w)\}.$$

As alluded to above, the set $\mathcal{L}(P_w, w)$ is significant for the following reason.

Proposition 5.3 *$\mathcal{L}(P_w, w)$ is the set of reduced words for w in W.*

Proof See Proposition 2.2 in [15]. □

Remark 5.4 We define the partial order on P_w to be the reverse of Stembridge's order and consider reverse linear extensions rather than linear extensions. Furthermore, Stembridge defines heaps for all words in W, whereas our definition is only correct for reduced words of fully commutative elements w. The implications for the theory are rather cosmetic; we make these deviations for the sake of convenience only.

It follows from Proposition 5.3 that, if w is fully commutative, the heaps of the reduced words for w are all equivalent, so we may refer to the heap of w unambiguously. This is also noted in [15]. The crucial claim is the next theorem.

Theorem 5.5 *Let $w \in W$ be fully commutative. Then $J(P_w) \cong \{x \in W : x \leq_L w\}$ is an isomorphism of posets.*

Proof A proof is found in [15] (cf. Lemma 3.1), but because our definitions are different from Stembridge's, and because the map between the two posets will be of importance in its own right for our proof of Theorem 1.4, we provide our own adaptation of Stembridge's proof.

For all $1 \leq k \leq n$, let $C_k := \{j : s_{i_j} = s_k\}$ be the set of all heap elements labeled by s_k. We first note that each C_k is a totally ordered subset of P_w. The proof is by contradiction. Suppose that there exist incomparable elements $j, j' \in P_w$ such that $s_{i_j} = s_{i_{j'}} = s_k$. Then there exists a reverse linear extension of P_w in which j and j' occur consecutively, which implies that the corresponding reduced word for w contains two consecutive instances of s_k. This is of course impossible. Thus, we may write C_k in the form $\{\gamma_{k,1} < \gamma_{k,2} < \cdots < \gamma_{k,\nu(k,w)}\}$, where $\nu(k, w)$ denotes the number of instances of s_k in a reduced word for w, and ν is well defined because w is fully commutative.

We are now ready to define the bijection between $J(P_w)$ and $\{x \in W : x \leq_L w\}$. Given an order ideal $I \in J(P_w)$, let ρ be a linear extension of P_w such that $\rho(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j) \notin I$ otherwise.

Definition 5.6

$$\phi : J(P_w) \to \{x \in W : x \leq_L w\}$$

is defined by

$$I \mapsto s_{i_{\rho(|I|)}} \dots s_{i_{\rho(2)}} s_{i_{\rho(1)}}.$$

Remark 5.7 The choice of the symbol ϕ to denote this map is deliberate, for when the heap P_w is minuscule (see Definition 6.1), ϕ is the map described in the introduction.

It is not immediately clear that ϕ is well defined. However, if ρ and ρ' are both linear extensions of P_w such that $\rho(j), \rho'(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j), \rho'(j) \notin I$ otherwise, then let $x = s_{i_{\rho(|I|)}} \dots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ and $x' = s_{i_{\rho'(|I|)}} \dots s_{i_{\rho'(2)}} s_{i_{\rho'(1)}}$. Since

$$\big(\rho(\ell), \dots, \rho(|I|+1), \rho(|I|), \dots, \rho(2), \rho(1)\big)$$

is a reverse linear extension of P_w,

$$s_{i_{\rho(\ell)}} \dots s_{i_{\rho(|I|+1)}} s_{i_{\rho(|I|)}} \dots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$$

is a reduced word for w, so $s_{i_{\rho(\ell)}} \dots s_{i_{\rho(|I|+1)}}$ is a reduced word for wx^{-1}. However,

$$\big(\rho(\ell), \dots, \rho(|I|+1), \rho'(|I|), \dots, \rho'(2), \rho'(1)\big)$$

is also a reverse linear extension of P_w. It follows that $s_{i_{\rho(\ell)}} \dots s_{i_{\rho(|I|+1)}}$ is a reduced word for wx'^{-1}, so $x = x'$, as desired.

To see that ϕ is bijective, we define the inverse map $\phi^{-1} : \{x \in W : x \leq_L w\} \to J(P_w)$ by $x \mapsto \bigcup_{k=1}^{n} \{\gamma_{k,h} : 1 \leq h \leq \nu(k, x)\}$. Because every reduced word for x is the final segment of a reduced word for w, it should be clear that $\phi^{-1}(x)$ is an order ideal of P_w for all $x \leq_L w$. It is a trivial matter to verify that $\phi^{-1}\phi$ is the identity on $J(P_w)$ and $\phi\phi^{-1}$ is the identity on $\{x \in W : x \leq_L w\}$, so this completes the proof. □

The following theorem demonstrates the relevance of the theory of fully commutative elements to our main results.

Theorem 5.8 *If W^J is minuscule, then the following three claims hold*:

(i) *If $w \in W^J$, w is fully commutative*;
(ii) *$(W^J, <_L)$ is a distributive lattice*;
(iii) *$(W^J, <_B) = (W^J, <_L)$.*

This theorem is a consequence of Theorems 6.1 and 7.1 in [15], for which Stembridge's proofs are uniform. We will see how it enables us to apply our knowledge of fully commutative heaps to the minuscule setting in the next section.

6 The main results

In this section, we prove Theorems 1.3 and 1.4. We start with the following definition and subsequent theorem.

Definition 6.1 If W^J is minuscule, and w_0^J is the longest element of W^J, then the heap $P_{w_0^J}$ is *minuscule*, and heaps of this form comprise the *minuscule heaps*.

Remark 6.2 Some of the minuscule heaps appear in Wildberger [17], but his construction differs from ours. In particular, he introduces a set of heaps that he calls *two-neighborly*, and he observes that these are precisely the minuscule heaps arising from complex simple Lie algebras whose root systems are simply laced.

Theorem 6.3 *Let V be a minuscule representation of a complex simple Lie algebra $\mathfrak{g}$ with minuscule weight λ and Weyl group W. If $S = \{s_1, s_2, \ldots, s_n\}$ is the set of Coxeter generators and W_J is the maximal parabolic subgroup stabilizing λ, then the following claims hold*:

(i) *If w_0^J is the longest element of W^J, then the poset $\{x \in W : x \leq_L w_0^J\}$ and the lattice $(W^J, <_L)$ are identical, and, furthermore, the minuscule heap $P_{w_0^J}$ and the minuscule poset P_V are isomorphic as posets.*

(ii) *The isomorphism $\phi : J(P_{w_0^J}) \to \{x \in W : x \leq_L w_0^J\} \cong (W^J, <_L) \cong (W^J, <_B)$ defined in Definition 5.6 satisfies the following property*: *For all $1 \leq k \leq n$, the induced action of the Coxeter generator s_k on $J(P_{w_0^J})$ in the toggle group $G(P_{w_0^J})$ may be expressed in the form*

$$\prod_{\substack{p \in P_{w_0^J} \\ p \text{ is labeled by } s_k}} t_p.$$

Example 6.4 In the case when the root system is A_4 and the minuscule weight is ω_2, Fig. 3 shows the minuscule heap $P_{s_3 s_2 s_4 s_1 s_3 s_2}$ (on the left) and the corresponding Bruhat poset $(W^J, <_B)$ (on the right). If I is the order ideal encircled by the solid line, then $\phi(I)$ is the coset representative encircled by the solid line, and $\prod_{p \in P_{s_3 s_2 s_4 s_1 s_3 s_2} \text{ is labeled by } s_2} t_p(I)$ is the order ideal encircled by the dotted line. Furthermore, $\phi(\prod_{p \in P_{s_3 s_2 s_4 s_1 s_3 s_2} \text{ is labeled by } s_2} t_p(I)) = s_2 \phi(I)$ is the coset representative encircled by the dotted line, thus illustrating the statement (ii) in Theorem 6.3.

Proof (i) By Proposition 2.6 in [15], w_0^J is the unique maximal element of $(W^J, <_L)$. It follows that if $x \in W^J$, $x \leq_L w_0^J$. To see that the converse also holds, let x_0 be the longest element of W_J, and note that $x \in W^J$ if and only if $x x_0$ is reduced (i.e. if and only if the product of a reduced word for x and a reduced word for x_0 is necessarily a reduced word for $x x_0$). If $x \leq_L w_0^J$, then there exists a reduced word for x that is the final segment of a reduced word for w_0^J. Since $w_0^J x_0$ is reduced, there exist a reduced word for x and a reduced word for x_0 such that their product is a reduced word for $x x_0$, and it follows from the fact that all reduced words for the same element are of the same length that $x x_0$ is reduced. We may conclude that $x \in W^J$, so, in general, $\{x \in W : x \leq_L w_0^J\} = (W^J, <_L)$. However, $J(P_{w_0^J}) \cong \{x \in W : x \leq_L w_0^J\}$, and $(W^J, <_L) = (W^J, <_B) \cong J(P_V)$ by Definition 3.5 and Theorems 4.3 and 5.8, so $J(P_{w_0^J}) \cong J(P_V)$, and it follows that $P_{w_0^J} \cong P_V$ is an isomorphism of posets, as desired.

(ii) Because $(W^J, <_L) = (W^J, <_B)$, it suffices to prove the claim with $(W^J, <_L)$ in place of $(W^J, <_B)$. Following the notation in the proof of Theorem 5.5, for all

$1 \le k \le n$, let C_k be the set of all heap elements labeled by s_k, and let t'_k be the toggle group element defined by $t'_k = \prod_{p \in C_k} t_p$. From Sect. 5, we know that C_k is totally ordered, and, by definition of $P_{w_0^J}$, no two elements of C_k share a covering relation, so it follows that t'_k is well defined for all k. Now let I be an order ideal in $J(P_{w_0^J})$, let $w = \phi(I)$, and let ℓ denote the length of the longest coset representative w_0^J. Consider the following lemmas:

Lemma 6.5 *The order ideal $t'_k(I)$ disagrees with I on at most one vertex of $P_{w_0^J}$.*

Proof It suffices to show that if there exists one vertex on which the two disagree, then there cannot exist any other such vertices. If there exists a vertex on which the two disagree, then there must exist a vertex p_0 labeled by s_k such that $p_0 \in Z(I)$ or $p_0 \in U(I)$. Without loss of generality, let $p_0 \in Z(I)$, and assume that the toggles t_p are applied to I in order of increasing p. Then for all $p \neq p_0$, the toggle at p has no effect, for p_0 is in the order ideal when t_p is applied if and only if $p < p_0$. □

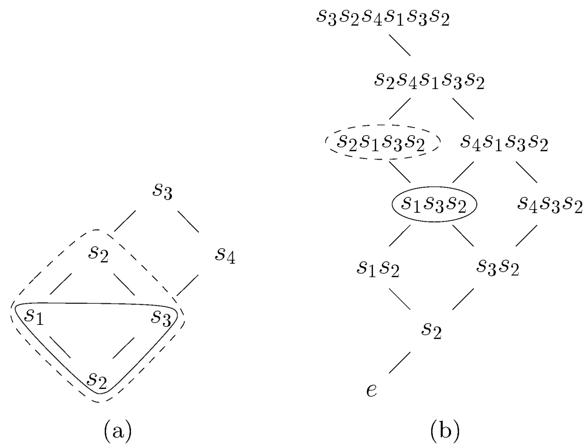
Lemma 6.6 *There exists an element $p_0 \in P$ such that $p_0 \in Z(I)$ if and only if $s_k w$ is not reduced. In this case, if $s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, then $s_k s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for w, and $\phi(I \setminus \{p_0\}) = s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$.*

Proof If $p_0 \in Z(I)$, let $s_{i_\ell} \dots s_{i_2} s_{i_1}$ be a reduced word for w_0^J, and assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word (recall that the heaps of every reduced word for w_0^J are equivalent). Since $p_0 \in Z(I)$, $I \setminus \{p_0\}$ is an order ideal of $P_{w_0^J}$. Let $(\rho(1), \rho(2), \dots, \rho(|I| - 1))$ be a linear extension of $I \setminus \{p_0\}$ (i.e. a linear extension of the poset with vertices in $I \setminus \{p_0\}$ and partial order given by the restriction of the partial order on $P_{w_0^J}$ to $I \setminus \{p_0\}$). Then $(\rho(1), \rho(2), \dots, \rho(|I| - 1), p_0)$ is a linear extension of I. Let $\rho(|I|) = p_0$, and extend this linear extension to a linear extension of $P_{w_0^J}$, $(\rho(1), \rho(2), \dots, \rho(\ell))$. By definition of ϕ, $s_{i_{\rho(|I|)}} s_{i_{\rho(|I|-1)}} \dots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ is a reduced word for w. Since p_0 is labeled by s_k, $s_{i_{p_0}} = s_k$, so it follows that $s_k w = s_{i_{\rho(|I|-1)}} \dots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$. This implies that $s_k w$ is not reduced.

If $s_k w$ is not reduced, let $s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ be a reduced word for $s_k w$. Note that $s_k s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for w, else $l(w) = l(s_k s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}) < l(s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}) = l(s_k w) < l(w)$, which is absurd. Let $i_{l(w)} = k$, and extend this word to a reduced word for w_0^J, $s_{i_\ell} \dots s_{i_2} s_{i_1}$. Without loss of generality, we may assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word, in which case the vertex corresponding to $s_{i_{l(w)}}$ is maximal over the vertices in the order ideal $\phi^{-1}(w) = I$. This implies that there exists an element of $P_{w_0^J}$ labeled by s_k that belongs to $Z(I)$ and $\phi(I \setminus \{p_0\}) = s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$. □

Lemma 6.7 *There exists an element $p_0 \in P$ such that $p_0 \in U(I)$ if and only if $s_k w$ is reduced and $s_k w \in W^J$. In this case, if $s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for w, then $s_k s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, and $\phi(I \cup \{p_0\}) = s_k s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$.*

Fig. 3 The map ϕ sends the indicated order ideals to the indicated coset representatives



Proof This result is analogous to Lemma 6.6. □

Now we take up the proof of Theorem 6.3.

- If $s_k w$ is not reduced, then, by Lemma 6.6, there exists an element $p_0 \in P$ labeled by s_k such that $p_0 \in Z(I)$, so, by Lemma 6.5, $\prod_{p\in C_k} t_p(I) = I \setminus \{p_0\}$. If $s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, it follows from Lemma 6.6 that $s_k s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for w and $\phi(I \setminus \{p_0\}) = s_{i_{l(s_k w)}} \dots s_{i_2} s_{i_1}$, as desired.
- If $s_k w$ is reduced and $s_k w \in W^J$, then, by Lemma 6.7, there exists an element $p_0 \in P$ labeled by s_k such that $p_0 \in U(I)$, so, by Lemma 6.5, $\prod_{p\in C_k} t_p(I) = I \cup \{p_0\}$. If $s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for w, it follows from Lemma 6.7 that $s_k s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$ and $\phi(I \cup \{p_0\}) = s_k s_{i_{l(w)}} \dots s_{i_2} s_{i_1}$, as desired.
- If $s_k w$ is reduced and $s_k w \notin W^J$, it follows from Lemmas 6.6 and 6.7 that no elements of $P_{w_0^J}$ labeled by s_k belong to $Z(I)$ or $U(I)$, so, by Lemma 6.5, $\prod_{p\in C_k} t_p(I) = I$. However, $s_k w$ covers w in the left Bruhat order on W, so, by Corollary 2.5.2 in Björner–Brenti [2], it follows that $s_k w = w s_j$, where $j \in J$. Since $s_j \in W_J$, we may conclude that $s_k w W_J = w s_j W_J = w W_J$, as desired.

Remark 6.8 In the proofs of Lemmas 6.5 and 6.7, we suppressed the cases in which elements of $P_{w_0^J}$ were given or shown to belong to $U(I)$ rather than $Z(I)$ because the conditions are symmetric, so the arguments are identical. However, we originally wrote out proofs that address both cases independently, and these may be found in the REU report [10].

We proceed to the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.4 Let P_V be a minuscule poset, and label each element of P_V by the label of the corresponding element of $P_{w_0^J}$. From Theorem 6.3, it follows that, for all $1 \le k \le n$, the following diagram is commutative:

$$\begin{array}{ccc} J(P_V) & \xrightarrow{\phi} & W^J \\ t'_k \downarrow & & s_k \downarrow \\ J(P_V) & \xrightarrow{\phi} & W^J. \end{array}$$

For any ordering of $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$, $c = s_{i_1} s_{i_2} \ldots s_{i_n}$ and $t_{(i_1, i_2, \ldots, i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$, so Theorem 1.4 follows immediately. □

Proof of Theorem 1.3 Let P_V be a minuscule poset, and again label each element of P_V by the label of the corresponding element of $P_{w_0^J}$. Theorem 6.3 embeds the Weyl group W as a subgroup of the toggle group $G(P_V)$. In light of Theorem 1.4, since the Coxeter elements are known to be pairwise conjugate in W, it suffices to exhibit a particular ordering $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that $t_{(i_1, i_2, \ldots, i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ is conjugate to Ψ in $G(P_V)$. However, in Sect. 2, we saw that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P_V)$. What we prove here is that there exists an ordering $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that the toggle group elements $t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ and $t_{\text{even}} t_{\text{odd}}$ are equal.

We start with two lemmas:

Lemma 6.9 *If P is a minuscule poset, then P is a ranked poset.*

Proof As discussed in the introduction, minuscule posets are Gaussian (cf. Proctor [8], Theorem 6). The fact that all Gaussian posets are ranked is recorded as Exercise 25(b) in Stanley [12]. □

Lemma 6.10 *If W is the Weyl group of a complex simple Lie algebra $\mathfrak{g}$, then the Dynkin diagram of the associated root system is acyclic and therefore bipartite.*

Proof See Chap. 1, Exercise 4 in Björner–Brenti [2]. □

Let r be the rank function for P_V. For all $1 \le k \le n$, we claim that the ranks of all the vertices labeled by s_k are of the same parity.

For the proof, the key observation is that each covering relation in the heap $P_{w_0^J}$ corresponds to an edge of the Dynkin diagram of the associated root system. (Recall that the partial order on $P_{w_0^J}$ is the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i_{j'}}$ do not commute; cf. Definition 5.2.) Assume for the sake of contradiction that there exists a k such that $j, j' \in P_{w_0^J}$ are both labeled by s_k but $r(j') - r(j)$ is odd. Without loss of generality, let $r(j') > r(j)$. Since C_k is totally ordered, it follows that $j' > j$, and that there exists a set of vertices $\{j_1, j_2, \ldots, j_{2u}\}$ such that j' covers j_1, j_{2u} covers j, and j_i covers j_{i+1} for all $1 \le i \le 2u - 1$. We may conclude that there exists a path of odd length in the Dynkin diagram from the vertex corresponding to the kth simple root to itself. However, by Lemma 6.10, the graph of the Dynkin diagram is bipartite, so this is impossible.

Note that if $p, p' \in P_V$ and $r(p) \equiv r(p') \pmod 2$, then t_p and $t_{p'}$ commute in $G(P_V)$. Let S_{odd} be the set of all k such that s_k is a simple reflection and the rank of p is odd for all vertices $p \in P_{w_0^J}$ labeled by s_k. Similarly, let S_{even} be the set of all k' such that $s_{k'}$ is a simple reflection and the rank of p is even for all vertices $p \in P_{w_0^J}$ labeled by $s_{k'}$. It follows that $t_{\text{even}} t_{\text{odd}} = \prod_{k' \in S_{\text{even}}} t'_{k'} \prod_{k \in S_{\text{odd}}} t'_k$. This completes the proof. $\square$

7 Plane partitions, preliminaries

For the remainder of the paper, we shift the focus from the representation-theoretic aspects of the minuscule posets to their combinatorial properties. We begin by recalling the Gaussian criterion from the introduction. Since all Gaussian posets are ranked (as noted in the proof of Lemma 6.9), we state it for ranked posets.

Definition 7.1 Let P be a ranked poset with rank function r. P is *Gaussian* if, for all positive integers m, the following equality holds:

$$J\bigl(P \times [m]; q\bigr) = \prod_{p \in P} \frac{1 - q^{m+r(p)+1}}{1 - q^{r(p)+1}}.$$

Remark 7.2 Not only are all minuscule posets Gaussian, but, interestingly enough, it is conjectured that all Gaussian posets are minuscule.

The two most important known families of Gaussian posets are the first two infinite families of minuscule posets. The order ideals of Cartesian products of these posets with chains may be identified with the combinatorial objects that we refer to as plane partitions. Thus, in establishing Theorem 1.2 for these cases, we are implicitly formulating new combinatorial identities for plane partitions that come already organized and explained.

Definition 7.3 A *plane partition* of an integer x is a two-dimensional array of nonnegative integers $\{x_{i,j}\}_{i,j \geq 1}$ satisfying $x = \sum_{i,j \geq 1} x_{i,j}$ and $x_{i,j} \geq x_{i,j+1}, x_{i+1,j}$ for all $i, j \geq 1$.

We say that a plane partition $\{x_{i,j}\}$ is inside $k \times n \times m$ if $0 \leq x_{i,j} \leq m$ for all $1 \leq i \leq k$, $1 \leq j \leq n$, and $x_{i,j} = 0$ otherwise. If we think of a plane partition as a polyhedron composed of stacks of unit cubes, with $x_{i,j}$ cubes stacked on the (i, j)th square for all (i, j), then it should be clear that any such plane partition corresponds to an order ideal of $[k] \times [n] \times [m]$. Conversely, any order ideal of $[k] \times [n] \times [m]$ corresponds to a plane partition inside $k \times n \times m$. In other words, there is a canonical bijection between the plane partitions inside $k \times n \times m$ and the order ideals of $[k] \times [n] \times [m]$.

Theorem 7.4 (MacMahon) *The following generating function counts plane partitions π inside $k \times n \times m$ by their cardinality*:

$$\sum_{\pi \subset k\times n\times m} q^{|\pi|} = \prod_{\substack{1\le i\le k\\ 1\le j\le n\\ 1\le l\le m}} \frac{[i+j+l-1]_q}{[i+j+l-2]_q}.$$

?

Remark 7.5 It follows that the MacMahon formula is also the rank-generating function for the poset of order ideals of $[k]\times[n]\times[m]$. Hence Theorem 7.4 is equivalent to the claim that the minuscule poset $[k]\times[n]$ is Gaussian.

Definition 7.6 A *symmetric plane partition* of an integer x is a plane partition as defined in Definition 7.3, subject to the additional condition $x_{i,j} = x_{j,i}$ for all $i, j \ge 1$.

As expected, there is a canonical bijection between the symmetric plane partitions inside $n \times n \times m$ and the order ideals of the poset $([n]\times[n])/S_2 \times [m]$. The analogue to the MacMahon formula for symmetric plane partitions is called the Bender–Knuth formula (cf. [1]).

Theorem 7.7 *The generating function that counts symmetric plane partitions π inside $n \times n \times m$ by their cardinality is as follows*:

$$\sum_{\substack{\pi\subset n\times n\times m\\ \pi\,\text{symmetric}}} q^{|\pi|} = \prod_{i=1}^{n}\Big(\frac{[m+2i-1]_q}{[2i-1]_q}\prod_{h=i+1}^{n}\frac{[2(m+i+h-1)]_q}{[2(i+h-1)]_q}\Big).$$

Remark 7.8 By similar reasoning, the Bender–Knuth formula is the rank-generating function for the poset of order ideals of $([n]\times[n])/S_2 \times [m]$. Hence Theorem 7.7 is equivalent to the claim that the minuscule poset $([n]\times[n])/S_2$ is Gaussian.

8 Proof of Theorem 1.2 for the first infinite family

Proof of Theorem 1.2 In the REU report [10], we presented a proof of Theorem 1.2 for the first infinite family by direct analysis of a collection of *bracket sequences* introduced by Cameron and Fon-Der-Flaass, which are shown in [4] to be in bijection with the order ideals of $[k]\times[n]\times[2]$. Inspired by our work, Striker and Williams obtained a bijection in [16] between these bracket sequences and non-crossing partitions of $\{1, 2, \ldots, k+n+1\}$ into $k+1$ parts. In Theorem 7.2 of Reiner–Stanton–White [9], it is proven that the latter exhibits the cyclic sieving phenomenon with respect to the q-analogue of the Narayana number, which is the same as the q-analogue of the MacMahon formula. It follows that Theorem 1.2 holds in this case. □

In light of the Striker–Williams simplifications, we omit our proof in this article. However, we remain hopeful that the techniques we used to analyze the bracket sequences can lead to a direct approach to establishing the cyclic sieving phenomenon in more complicated posets, e.g., $([n] \times [n])/S_2 \times [3]$, for which we conjecture that the cyclic sieving phenomenon holds. The original proof may still be found online in [10].

Substituting roots of unity into the MacMahon formula Knowing that the desired cyclic sieving phenomenon holds in the case $[k] \times [n] \times [2]$, we investigate some of its predictions. In [10], these data were used to verify that the cyclic sieving phenomenon holds for the first infinite family, but here their primary purpose is actually to help verify that the cyclic sieving phenomenon holds for the second infinite family, which will be shown in Sect. 9.

Letting $m = 2$ in Theorem 7.4, we see that the rank-generating function for $J([k] \times [n] \times [2])$ becomes

$$J([k] \times [n] \times [2]; q) = \begin{bmatrix} k+n+1 \\ n \end{bmatrix}_q \begin{bmatrix} k+n \\ n \end{bmatrix}_q \frac{[1]_q}{[n+1]_q}.$$

Lemma 8.1 *Let $n = n'd + r$ and $k = k'd + s$, where $0 \le r, s \le d-1$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}} = \binom{n'}{k'} \begin{bmatrix} r \\ s \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}}.$$

Proof See Proposition 2.1 in Guo-Zeng [7]. □

Proposition 8.2 *If ℓ is a proper divisor of $k+n+1$, let $d = \frac{k+n+1}{\ell}$. Then there are no orbits of order ℓ unless $d|n$ or $d|n+1$.*

Proof Let $q := (e^{\frac{2\pi i}{m+n+1}})^{\ell}$ be a primitive dth root of unity. Expanding the MacMahon formula, we have the following:

$$\begin{aligned} &\begin{bmatrix} k+n+1 \\ n \end{bmatrix}_q \begin{bmatrix} k+n \\ n \end{bmatrix}_q \frac{[1]_q}{[n+1]_q} \\ &\quad = \left(\frac{[k+n+1]_q \ldots [k+2]_q}{[n]_q \ldots [1]_q} \right) \left(\frac{[k+n]_q \ldots [k+1]_q}{[n]_q \ldots [1]_q} \right) \frac{[1]_q}{[n+1]_q}. \end{aligned}$$

Suppose $d \nmid n$ and $d \nmid n+1$; then, since $d|k+n+1$, by Lemma 8.1, it follows that

$$\begin{bmatrix} k+n+1 \\ n \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}} = \binom{\ell}{\ell'} \begin{bmatrix} r \\ r' \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}},$$

where $k+n+1 = \ell d + r$, $n = \ell' d + r'$, and $0 \le r, r' \le d-1$. Since $d|k+n+1$ and $d \nmid n$, it should be clear that $r = 0$ and $r' > 0$. Therefore, the expression evaluates to 0, as desired. □

What happens in $[k] \times [n] \times [m]$ when $m \ge 3$? It has been verified via Dilks's Maple code that cyclic sieving does not occur in the poset $[3] \times [3] \times [3]$. Furthermore, the order of the Fon-Der-Flaass action is 33 for the poset $[4] \times [4] \times [4]$, so, in particular, it is not true in general that the order of the Fon-Der-Flaass action is $k+n+m-1$ for the poset $[k]\times[n]\times[m]$. However, it is conjectured by Cameron and Fon-Der-Flaass that if $k+n+m-1$ is prime, then the order of the Fon-Der-Flaass action is divisible by $k+n+m-1$, and they have proved this in [4] for all posets in which m exceeds $(k-1)(n-1)$.

9 Proof of Theorem 1.2 for the second infinite family

In this section, we obtain the fact that the cyclic sieving phenomenon holds for all posets of the form $([n]\times[n])/S_2\times[2]$ as a consequence of the fact that the cyclic sieving phenomenon holds for all posets of the form $[n]\times[n]\times[2]$.

Note that the Fon-Der-Flaass action on ordinary plane partitions viewed as order ideals of $[n]\times[n]\times[2]$ restricts to the Fon-Der-Flaass action on symmetric plane partitions viewed as order ideals of $([n]\times[n])/S_2\times[2]$. Hence the order of Ψ on order ideals of $([n]\times[n])/S_2\times[2]$ divides $2n+1$. Since the orbit of the empty order ideal is of length $2n+1$, it follows that the order of Ψ is in fact equal to $2n+1$.

From the Bender–Knuth formula (Theorem 7.7), we see that the rank-generating function for $J(([n]\times[n])/S_2\times[2])$ is $\left[{2n+1 \atop n}\right]_q$. It follows by Lemma 8.1 that if q is a $(2n+1)$th root of unity, substituting q into the Bender–Knuth expression yields 0 unless $q=1$. Since we know that substituting $q=1$ gives the total number of order ideals of $([n]\times[n])/S_2\times[2]$, it suffices to show that all the orbits of order ideals of $([n]\times[n])/S_2\times[2]$ are free orbits of length $2n+1$.

Assume for the sake of contradiction that there exists an orbit of length $\frac{2n+1}{d}$, where $d>1$. As discussed, this orbit must also arise in the case $[n]\times[n]\times[2]$. However, it follows from Proposition 8.2 that $d|n$ or $d|n+1$. Since $\gcd(n,2n+1)=\gcd(n+1,2n+1)=1$, this contradicts the assumption $d>1$, so we may conclude that all the orbits are free orbits of length $2n+1$, as desired.

What happens in $([n]\times[n])/S_2\times[m]$ *when* $m\geq 3$? It has been verified via Dilks's Maple code that cyclic sieving does not occur in the poset $([6]\times[6])/S_2\times[4]$. However, every poset of the form $([n]\times[n])/S_2\times[3]$ that we tested was found to obey the cyclic sieving phenomenon, so it is tempting to conjecture that the cyclic sieving phenomenon holds for all such posets.

10 Proof of Theorem 1.2 for the third infinite family

Remarkably, the following theorem is true.

Theorem 10.1 *For all positive integers m, if $P:=J^{n-3}([2]\times[2])$ is a minuscule poset belonging to the third infinite family, then the triple $(J(P\times[m]), J(P\times[m];q),\Psi)$ exhibits the cyclic sieving phenomenon for all positive integers m.*

The proof of this result may be found in the REU report [10]. It is accomplished by a bijection between order ideals of $J^{n-3}([2]\times[2])\times[m]$ and the same Cameron–Fon-Der-Flaass bracket sequences that arise in the case $[k]\times[n]\times[2]$. To manipulate

Table 1 In each entry, the table displays the number of Fon-Der-Flaass action orbits of each size that occur in the indicated poset of order ideals. For instance, the poset of order ideals $J(J^3([2] \times [3]) \times [1])$ is composed of three orbits of order 18 and one orbit of order 2

	$P = J^2([2] \times [3])$	$P = J^3([2] \times [3])$
$m = 1$	$2 \times 12 + 1 \times 3$	$3 \times 18 + 1 \times 2$
$m = 2$	27×13	77×19

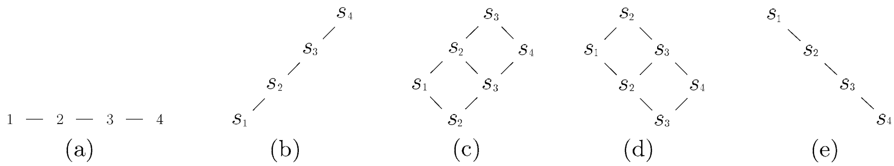


Fig. 4 Left to right: (**a**) the Dynkin diagram for root system A_4, (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_1, (**c**) the heap $P_{w_0^J}$ for minuscule weight ω_2, (**d**) the heap $P_{w_0^J}$ for minuscule weight ω_3, and (**e**) the heap $P_{w_0^J}$ for minuscule weight ω_4

these bracket sequences, we devised a rule analogous to that in Cameron–Fon-Der-Flaass [4] that differs in the details.

11 Proof of Theorem 1.2 for the exceptional cases

We verified via Dilks's code that, if P is the first exceptional poset, the cyclic sieving phenomenon holds for the triple $(J(P \times [m]), J(P \times [m]; q), \Psi)$ when $1 \leq m \leq 4$, and, if P is the second exceptional poset, the cyclic sieving phenomenon holds for the triple $(J(P \times [m]), J(P \times [m]; q), \Psi)$ when $1 \leq m \leq 3$. For reference, in Table 1 we provide the data on the orbit structures corresponding to both exceptional posets for the cases $m = 1$ and $m = 2$. These data are not required to establish Theorem 1.1 because the proof of Theorem 6.3 is uniform, but they are required to show that Theorem 1.2 holds in the exceptional cases and not just in the infinite families.

What happens in the exceptional cases for $m \geq 3$? It is tempting to propose the following conjecture.

Conjecture 11.1 *If P is an exceptional minuscule poset, then the triple $(J(P \times [m]), J(P \times [m]; q), \Psi)$ exhibits the cyclic sieving phenomenon for all positive integers m.*

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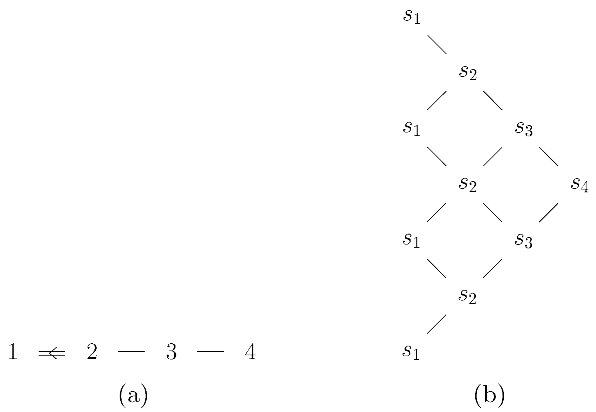

Fig. 5 Left to right: (**a**) the Dynkin diagram for root system B_4 and (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_1

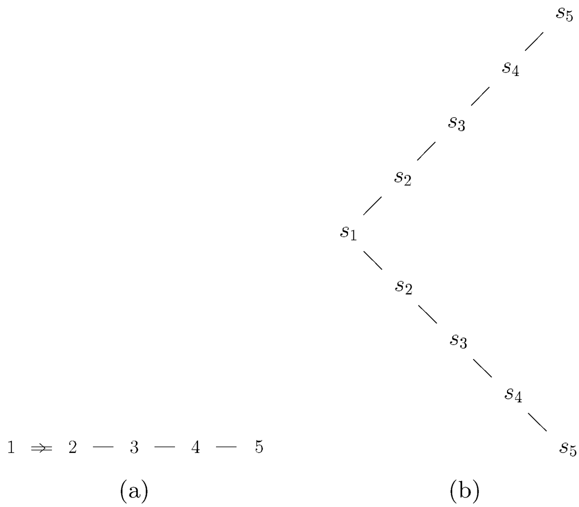

Fig. 6 Left to right: (**a**) the Dynkin diagram for root system C_5 and (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_5

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Appendix

In this section, we choose six example root systems, one of each classical type as well as E_6 and E_7, and we present an illustration of every minuscule heap arising from a Lie algebra whose root system is among these. The purpose is to provide an indication of all the possible shapes a minuscule heap may take on.

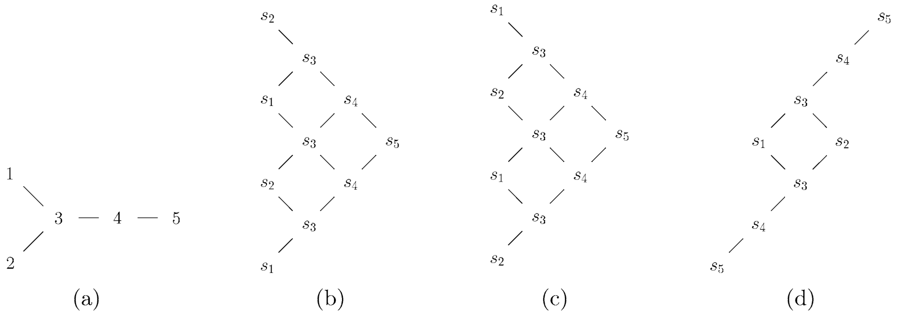

Fig. 7 Left to right: (**a**) the Dynkin diagram for root system D_5, (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_1, (**c**) the heap $P_{w_0^J}$ for minuscule weight ω_2, and (**d**) the heap $P_{w_0^J}$ for minuscule weight ω_5

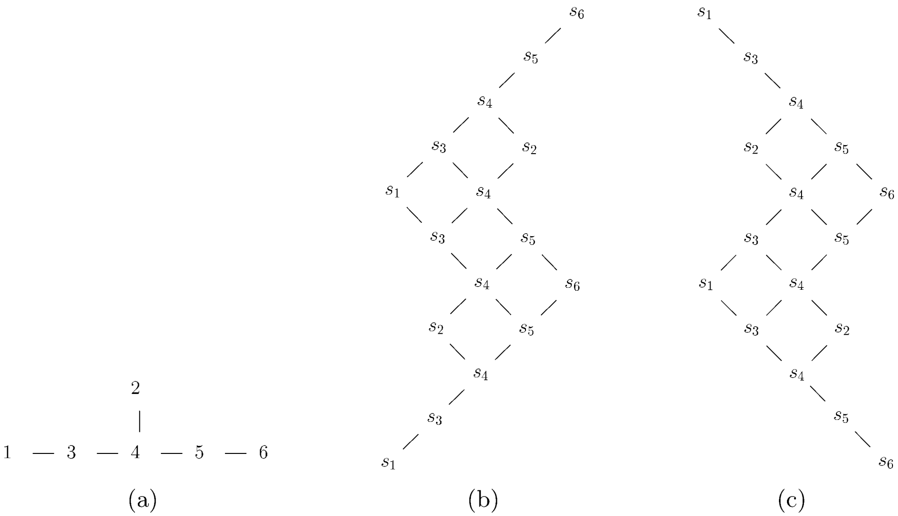

Fig. 8 Left to right: (**a**) the Dynkin diagram for root system E_6, (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_1, and (**c**) the heap $P_{w_0^J}$ for minuscule weight ω_6

A.1 The case A_n

For root systems of the form A_n, let $\alpha_j = \epsilon_{j+1} - \epsilon_j$ for all $1 \leq j \leq n$. The possible minuscule weights are $\omega_1, \omega_2, \dots, \omega_n$ (in other words, each fundamental weight may be minuscule), and the minuscule heaps arising from A_4 appear in Fig. 4.

A.2 The case B_n

For root systems of the form B_n, let $\alpha_1 = \epsilon_1$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weight is ω_1, and the minuscule heap arising from B_4 appears in Fig. 5.

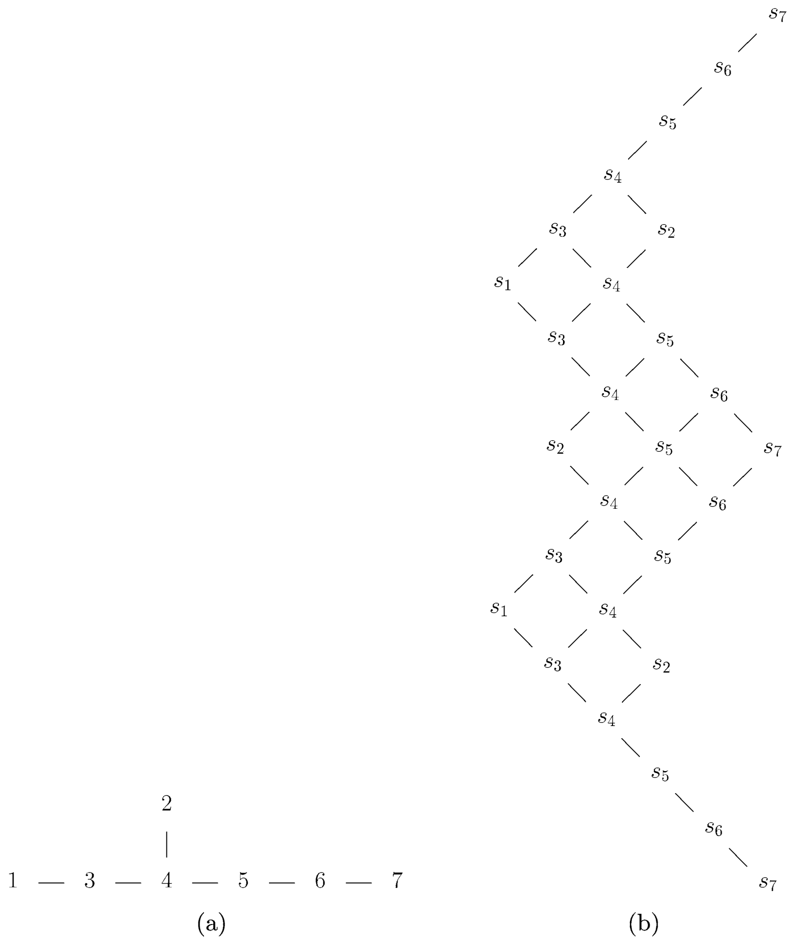

Fig. 9 Left to right: (**a**) the Dynkin diagram for root system E_7 and (**b**) the heap $P_{w_0^J}$ for minuscule weight ω_7

A.3 The case C_n

For root systems of the form C_n, let $\alpha_1 = 2\epsilon_1$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weight is ω_n, and the minuscule heap arising from C_5 appears in Fig. 6.

A.4 The case D_n

For root systems of the form D_n, let $\alpha_1 = \epsilon_1 + \epsilon_2$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weights are ω_1, ω_2, and ω_n, and the minuscule heaps arising from D_5 appear in Fig. 7.

A.5 The exceptional cases

For root systems E_6 and E_7, let the simple roots be chosen to obey the relationships depicted in the Dynkin diagrams in Figs. 8 and 9, respectively. For the case E_6, the only possible minuscule weights are ω_1 and ω_6, and the corresponding heaps appear in Fig. 8. For the case E_7, the only possible minuscule weight is ω_7, and the corresponding heap appears in Fig. 9.

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