ORBITS OF PLANE PARTITIONS OF EXCEPTIONAL LIE TYPE

HOLLY MANDEL AND OLIVER PECHENIK

Abstract. For each minuscule flag variety X, there is a corresponding minuscule poset, describing its Schubert decomposition. We study an action on plane partitions over such posets, introduced by P. Cameron and D. Fon-der-Flaass (1995). For plane partitions of height at most 2, D. Rush and X. Shi (2013) proved an instance of the cyclic sieving phenomenon, completely describing the orbit structure of this action. They noted their result does not extend to greater heights in general; however, when X is one of the two minuscule flag varieties of exceptional Lie type E, they conjectured explicit instances of cyclic sieving for all heights.

We prove their conjecture in the case that X is the Cayley-Moufang plane of type E_6. For the other exceptional minuscule flag variety, the Freudenthal variety of type E_7, we establish their conjecture for heights at most 4, but show that it fails generally. We further give a new proof of an unpublished cyclic sieving of D. Rush and X. Shi (2011) for plane partitions of any height in the case X is an even-dimensional quadric hypersurface. Our argument uses ideas of K. Dilks, O. Pechenik, and J. Striker (2017) to relate the action on plane partitions to combinatorics derived from K-theoretic Schubert calculus.

1. Introduction

The *minuscule posets* are a remarkable collection of partially-ordered sets that arise naturally from the representation theory of Lie algebras or alternatively from the Schubert calculus of generalized flag varieties. We study dynamical enumerative properties of plane partitions over these posets.

A special case is the set of ordinary plane partitions that fit inside a fixed rectangular box. In this context, P. Cameron and D. Fon-der-Flaass [CFDF95] initiated the study of a combinatorially-natural operator Ψ. This operator is now generally known as *rowmotion* and has become a subject of intense study (cf., e.g., [Pan09, SW12, AST13, RS13, EP14, PR15, GR15, GR16, DPS17, Vor17, DSV17]). We will describe minuscule posets and the operation of rowmotion in Sections 2 and 3, respectively.

For any poset P, let $\mathsf{PP}^k(P)$ denote the set of plane partitions of height at most k over P, or equivalently, the set of order ideals in the product $P \times \mathbf{k}$ of P with a chain poset of size k. Let f_P^k denote the generating function that enumerates the elements of $\mathsf{PP}^k(P)$ by cardinality, so $f_P^k(q) := \sum_{\mathcal{I} \in \mathsf{PP}^k(P)} q^{|\mathcal{I}|}$. In the special case $k \leq 2$ and P

Date: December 21, 2017.

2010 *Mathematics Subject Classification.* Primary 05A15; Secondary 05E18, 06A07, 17B25.

Key words and phrases. plane partition, increasing tableau, promotion, rowmotion, exceptional Lie algebra, minuscule poset, cyclic sieving phenomenon.

minuscule, D. Rush and X. Shi [RS13] showed that f_P^k also encodes the orbit structure of rowmotion via an instance of the **cyclic sieving phenomenon** (introduced by V. Reiner, D. Stanton, and D. White [RSW04]). Thus for $k \leq 2$, the number of minuscule plane partitions fixed by the d-fold application of rowmotion $\Psi^{\circ d}$ is the evaluation of the polynomial f_P^k at ζ^d, where ζ is any primitive nth root-of-unity and n is the period of Ψ on $\mathsf{PP}^k(P)$. (Since some of our other operators have superscripts in their names, we denote the d-fold composition of an operator τ by $\tau^{\circ d}$.)

It was noted in [RS13] that this instance of cyclic sieving does not extend to the case $k \geq 3$ for general minuscule posets. However, D. Rush and X. Shi conjectured the following. (The posets in question are illustrated in Figure 1.)

Conjecture 1.1 ([RS13, Conjecture 11.1]). *Let P be one of the two minuscule posets associated to an exceptional Lie algebra of type E and let $k \in \mathbb{Z}_{\geq 0}$. Then f_P^k is a cyclic sieving polynomial for the action of Ψ on $\mathsf{PP}^k(P)$.*

Our main result is to completely resolve Conjecture 1.1.

Theorem 1.2. *Conjecture 1.1 holds for the E_6 minuscule poset P_{CM} (cf. Figure 1A) and all k, but holds for the E_7 minuscule poset P_F (cf. Figure 1B) only when $k \leq 4$.*

Verification of Conjecture 1.1 in the the cases $k \leq 4$ for P_{CM} and $k \leq 3$ for P_F was previously reported in [RS13]. Our new results are therefore:

- cyclic sieving for P_{CM} when $k > 4$,
- cyclic sieving for P_F when $k = 4$, and
- failure of the conjectured cyclic sieving for P_F when $k > 4$.

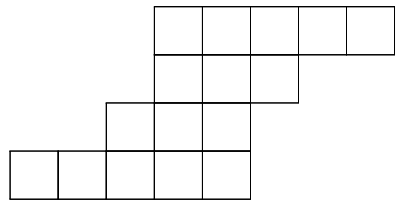
Our approach to proving Theorem 1.2 is to use the ideas of K. Dilks, O. Pechenik, J. Striker and C. Vorland [DPS17, DSV17] to relate the action of Ψ to the action of *K-promotion* on *increasing tableaux*. K-promotion was first studied in [Pec14], building on combinatorial tools for K-theoretic Schubert calculus due to H. Thomas and A. Yong [TY09]. We show that the action of K-promotion is controlled by its behavior on a finite subset of increasing tableaux. By understanding the orbit structure of this subset, we are able to determine the complete orbit structure, thereby establishing Theorem 1.2.

Having developed these methods, it becomes straightforward to prove the following additional result.

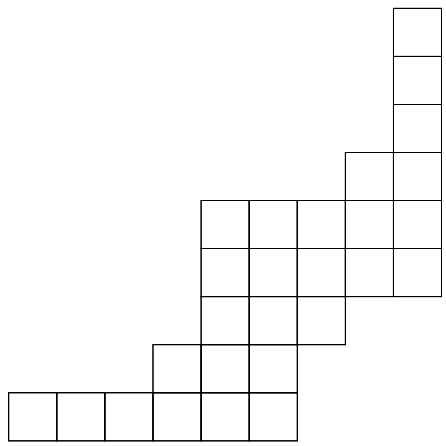
Theorem 1.3. *Let P be a minuscule poset associated to an even-dimensional quadric of type D_{p+1} (cf. Figure 3C) and let $k \in \mathbb{Z}_{\geq 0}$. Then f_P^k is a cyclic sieving polynomial for the action of Ψ on $\mathsf{PP}^k(P)$.*

Theorem 1.3 was previously announced by D. Rush and X. Shi [RS13, Theorem 10.1]; however, they omitted their proof [RS11, §10] from the published paper. We believe that our alternative proof of Theorem 1.3 via K-theoretic combinatorics provides a different understanding.

Often instances of cyclic sieving can be proven using representation-theoretic techniques [RSW04, Rho10], and when cyclic sieving is established in a more direct fashion, as we do here, it may be a clue toward new underlying algebra (cf. [Rho17]).



(A) Cayley-Moufang poset, P_{CM}



(B) Freudenthal poset, P_F

FIGURE 1. The two minuscule posets associated to exceptional Lie algebras of type E. Here, we have drawn the posets to resemble Young diagrams in Cartesian ("French") orientation; the boxes are the elements of the poset and each box covers the box immediately below it and the box immediately to its left (if such boxes exist). Hence the minimal element of each poset is the box at the far left of the bottom row. The Cayley-Moufang poset P_{CM} is associated to E_6, while the Freudenthal poset P_F is associated to E_7. (There is no minuscule poset associated to E_8.)

The results in Theorems 1.2 and 1.3 perhaps suggest the existence of new symmetric group module structures on the sets $\mathsf{PP}^k(P)$. Developing such representations would be an interesting direction for future work.

This paper is organized as follows. In Section 2, we define the minuscule posets, recalling their classification and other properties we will use. Section 3 gives the precise definition of rowmotion and, following [DPS17, DSV17], notes the close relation between rowmotion and K-promotion. We then develop new tools for understanding the orbit structure of K-promotion in Sections 4, 5, and 6. Specifically, Section 4 introduces the operations of *inflation* and *deflation*, retracting the set of increasing tableaux onto the finite subset of *gapless tableaux*. In Section 5, we recall the precise definition of K-promotion on increasing tableaux and show how K-promotion is governed by its restriction to gapless tableaux. Section 6 uses this information to determine the period of K-promotion on general increasing tableaux. Finally, Section 7 combines these ideas to prove Theorems 1.2 and 1.3.

2. Minuscule posets

Let G be a complex connected reductive Lie group with maximal torus T. Denote by W the Weyl group $N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. The root system Φ of G may be partitioned $\Phi^+ \sqcup \Phi^-$ into positive and negative roots according to a choice Δ of simple roots. There is a natural poset structure on Φ^+ obtained as the transitive closure of the covering relation $\alpha \lessdot \beta$ if and only if $\beta - \alpha \in \Delta$. The choice of bipartition of Φ into positive and negative roots further specifies a choice of a Borel subgroup $\mathsf{B}_+ \subset \mathsf{G}$ and an opposite Borel subgroup $\mathsf{B}_- \subset \mathsf{G}$ with $\mathsf{B}_+ \cap \mathsf{B}_- = \mathsf{T}$.

We say $\delta \in \Delta$ is **minuscule** if for every $\alpha \in \Phi^+$, $\delta^\vee$ appears with multiplicity at most 1 in the simple coroot expansion of $\alpha^\vee$. The classification of minuscule roots is well known and is illustrated in Figure 2 in terms of Dynkin diagrams.

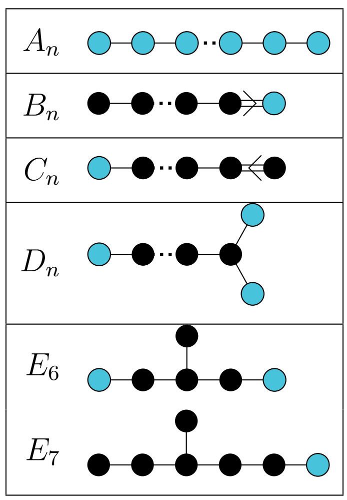


Figure 2. In each of the finite-type Dynkin diagrams above, each minuscule root is marked as a pale blue disk, while the non-minuscule simple roots are marked in black. In type A_n, every node is minuscule, while in the other types only the indicated leaves are minuscule. The remaining finite-type Dynkin diagrams are omitted because they have no minuscule nodes.

For each minuscule simple root δ, there is an associated **minuscule poset** P_δ obtained as the subposet of Φ^+ induced on those positive roots α where δ appears with nonzero coefficient in the simple root expansion of α.

Alternatively, one may obtain the minuscule posets via the geometry of certain generalized flag varieties. If $\mathsf{P}_\delta \supset \mathsf{B}_+$ denotes the maximal parabolic subgroup of G associated to the minuscule simple root δ, then the space $X = \mathsf{G}/\mathsf{P}_\delta$ is called a **minuscule variety**. The minuscule varieties are smooth projective varieties with many additional nice geometric properties (cf., e.g., [BL00] for details). The natural action of the Borel subgroup B_+ on X has finitely-many orbits, whose Zariski closures are the **Schubert varieties**. Given two Schubert varieties in X, it is known that they are either disjoint or else one is a subset of the other. Indeed the poset Y_X of Schubert varieties of X with respect to inclusion is a distributive lattice and its corresponding poset of join irreducibles is isomorphic to the minuscule poset P_δ for the minuscule simple root δ, as constructed above.

The minuscule posets are completely classified. We illustrate them here in Figures 1 and 3. Our focus will be on the **propellers**, the **Cayley-Moufang poset**, and the **Freudenthal poset** as shown in Figures 3C, 1A, and 1B, respectively. The propeller P_p with $2p$ elements ($p \geq 3$) is associated to the minuscule node that is not adjacent to the trivalent node in the D_{p+1} Dynkin diagram, the Cayley-Moufang poset P_{CM} is associated to the unique minuscule simple root for E_6, and the Freudenthal poset P_F is associated to the unique minuscule simple root for E_7 (cf. Figure 2). The corresponding minuscule varieties are, respectively, even-dimensional quadric hypersurfaces, the octonionic projective plane (or Cayley-Moufang plane), and the Freudenthal variety. The remaining minuscule roots yield minuscule posets that are *rectangles* or *shifted staircases* (illustrated in Figure 3A and B). These correspond respectively to type A Grassmannians and to maximal orthogonal Grassmannians; we will not consider these posets further in this paper, except as convenient examples.

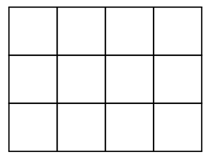
A poset that is linearly ordered is called a **chain**; we denote the chain of size k by $\mathbf{k}$. By a **plane partition** of height at most k over a poset P, we mean an order ideal of the product poset $P \times \mathbf{k}$. Clearly, such a plane partition may be identified with a weakly order-reversing map $\pi : P \to \mathbf{k}$ (i.e., a map π such that $\mathsf{x} \leq \mathsf{x}'$ implies $\pi(\mathsf{x}) \geq \pi(\mathsf{x}')$). The poset P is called **Gaussian** if the generating function $f_P^k(q)$ may be expressed in the form

$$f_P^k(q) = \frac{(1-q^{h_1+k})(1-q^{h_2+k})\cdots(1-q^{h_t+k})}{(1-q^{h_1})(1-q^{h_2})\cdots(1-q^{h_t})},$$

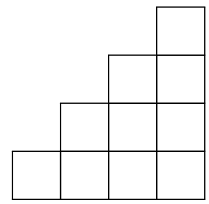
for some nonnegative integers $t, h_1, h_2, \ldots, h_t \in \mathbb{Z}_{\geq 0}$ independent of k. R. Proctor showed that every minuscule poset is Gaussian [Pro84]. Indeed, it is conjectured that there are no other Gaussian posets. In the case of a minuscule poset P, one can in fact take $t = |P|$ and, for $\mathsf{x} \in P$, take $h_{\mathsf{x}} = \mathrm{rk}(\mathsf{x}) + 1$, where $\mathrm{rk}(\mathsf{x})$ denotes the length of the largest chain in P with maximum element x. Given the beautiful form of these generating functions, it is perhaps not surprising that f_P^k should play a role in some instances of cyclic sieving, as in Theorems 1.2 and 1.3.

3. The rowmotion operator Ψ

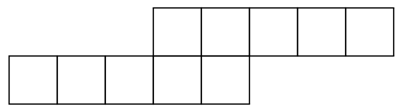
If P is any finite poset, let $J(P)$ denote the set of its order ideals. For $\mathcal{I} \in J(P)$, we define $\Psi(\mathcal{I}) \in J(P)$ to be the order ideal generated by the minimal elements of the complement $P - \mathcal{I}$. This operator Ψ is closely related to those described in [BS74, Duc74, CFDF95] in different contexts. It was first considered explicitly as



(A) Rectangle



(B) Shifted staircase



(C) Propeller, P_5

FIGURE 3. Together with the two posets shown in Figure 1, these are exemplars of all five families of minuscule poset, shown in Cartesian orientation. The elements of each poset are the boxes, and each box is covered by any box immediately above it or immediately to its right. Rectangles may have arbitrary height and width. Shifted staircases have arbitrary width, and height equal to their width. The propeller P_p consists of two rows of length p, overlapping by two boxes in the center. The Cayley-Moufang and Freudenthal posets of Figure 1 are exceptional, forming singleton families.

an action on order ideals by J. Striker and N. Williams [SW12]. We follow them in referring to Ψ as **rowmotion**.

We now observe that results of [DPS17] and [DSV17] enable us to study the rowmotion action Ψ via the action of K-promotion on increasing tableaux.

Definition 3.1. Let λ be an order ideal in a minuscule poset, considered as an generalized Young diagram in Cartesian orientation as in Figures 1 and 3. An **increasing tableau** of shape λ is an assignment of a positive integer to each box of λ, such that entries strictly increase from left to right along rows and strictly increase from bottom to top going up columns. Let $\mathrm{Inc}^m(\lambda)$ denote the set of increasing tableaux of shape λ with all entries at most m. We identify $\mathrm{Inc}^m(\lambda)$ with the set of strictly order-preserving maps from the poset λ to the chain poset $\mathbf{m} = \{1, 2, \ldots, m\}$ of size m. For an example of an increasing tableau, see Figure 4.

From the combinatorics of K-theoretic Schubert calculus one obtains a K-promotion operator $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(\lambda)$ that directly extends M.-P. Schützenberger's classical definition of promotion [Sch72]. We will define $\mathfrak{pro}^m$ in Section 5. For a finite poset P, let $\mathrm{rk}(P)$ denote the size of the longest chain in P.

Proposition 3.2. *Let $P \in \{P_{CM}, P_F, P_p\}$ and $k \in \mathbb{Z}_{\geq 0}$. There is an equivariant bijection between $\mathrm{Inc}^{k+\mathrm{rk}(P)}(P)$ under K-promotion and $J(P \times \mathbf{k})$ under Ψ.*

Proof. By [DSV17, Corollary 5.2], there is an equivariant bijection between the sets $\mathrm{Inc}^{k+\mathrm{rk}(P)}(P)$ under K-promotion and $J(\Gamma_1(P, k + \mathrm{rk}(P)))$ under Ψ, where $\Gamma_1(P, k +$

								23
								22
								21
							19	20
				11	12	15	18	19
				9	11	13	16	17
				8	10	11		
			6	7	9			
1	2	3	4	5	6			

FIGURE 4. A representative increasing tableau $T \in \mathrm{Inc}^{23}(P_F)$.

$\mathrm{rk}(P))$ is an auxilliary poset constructed from P in [DSV17, §3.1]. For any $\mathsf{x} \in P$, note that the maximal length of a chain through x is independent of x. (Specifically, this maximal length is $\mathrm{rk}(P)$; we have $\mathrm{rk}(P_{CM}) = 11$, $\mathrm{rk}(P_F) = 17$, and $\mathrm{rk}(P_p) = 2p - 1$.) Hence [DSV17, Corollary 3.28] applies, and in these cases we have the isomorphism of posets $\Gamma_1(P, k + \mathrm{rk}(P)) \cong P \times \mathbf{k}$. $\square$

Corollary 3.3. *Let $P \in \{P_{CM}, P_F, P_p\}$ and $k \in \mathbb{Z}_{\geq 0}$. Then the multiset of cardinalities of Ψ-orbits on $J(P \times \mathbf{k})$ and the multiset of cardinalities of $\mathfrak{pro}^{\mathrm{rk}(P)+k+1}$-orbits on $\mathrm{Inc}^{\mathrm{rk}(P)+k+1}(\lambda)$ are equal.* $\square$

4. Inflation and deflation

In light of Corollary 3.3, we now turn to a more thorough study of increasing tableaux. Let λ be an order ideal in a minuscule poset. Suppose $T \in \mathrm{Inc}^m(\lambda)$ and consider T as a strictly order-preserving map from λ to the chain poset $\mathbf{m}$. We say that T is **gapless** if this map is surjective and **gappy** otherwise. We write $\mathrm{Inc}^m_{\mathrm{gl}}(\lambda)$ for the subset of all gapless tableaux in $\mathrm{Inc}^m(\lambda)$. Notice that the set

$$\mathrm{Inc}_{\mathrm{gl}}(\lambda) \coloneqq \coprod_m \mathrm{Inc}^m_{\mathrm{gl}}(\lambda)$$

is finite. This fact will be critical to our proofs of Theorems 1.2 and 1.3.

For $T \in \mathrm{Inc}^m(\lambda)$, let m_T be the number of distinct labels in T. For each m, we define the **deflation** map

$$\mathsf{Defl}^m : \mathrm{Inc}^m(\lambda) \to \coprod_{0 \leq n \leq m} \mathrm{Inc}^n_{\mathrm{gl}}(\lambda)$$

by

$$[\mathsf{Defl}^m(T)](\mathsf{x}) = \#\{h \in \mathrm{range}(T) : h \leq T(\mathsf{x})\},$$

for $T \in \mathrm{Inc}^m(\lambda)$ and $\mathsf{x} \in \lambda$. Note that $\mathsf{Defl}^m(T) \in \mathrm{Inc}^{m_T}_{\mathrm{gl}}(\lambda)$ and that $m_T \leq m$.

For nonnegative integers j and k, let $\binom{[j]}{k}$ denote the collection of binary vectors of length j with k 1's. We now define the **content vector** function

$$\mathsf{Con}^m : \mathrm{Inc}^m(\lambda) \to \{0,1\}^m = \coprod_{0 \le n \le m} \binom{[m]}{n}$$

by

$$\mathsf{Con}^m(T) = (c_1, \ldots, c_m),$$

where $c_i = 1$ if $i \in \mathrm{range}(T)$ and $c_i = 0$ if $i \notin \mathrm{range}(T)$. (Note that this definition of content differs from the usual notion of content for semistandard tableaux in that here we do not care about the multiplicity of a label, but only its presence or absence.)

Example 4.1. If $T = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} \in \mathrm{Inc}^7(2 \times 3)$, then the deflation of T is

$$\mathsf{Defl}^7(T) = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 & 4 \\ \hline\end{array} \in \mathrm{Inc}^5_{\mathrm{gl}}(2 \times 3).$$

The content vector of T is $\mathsf{Con}^7(T) = (1,1,0,1,1,1,0) \in \binom{[7]}{5} \subset \{0,1\}^7$. $\diamondsuit$

Now if $\mathsf{Defl}^m(T) \in \mathrm{Inc}^n_{\mathrm{gl}}(\lambda)$, then $\mathsf{Con}^m(T) \in \binom{[m]}{n}$. We denote by $\mathsf{DeflCon}^m$ the product map

$$\mathsf{DeflCon}^m := (\mathsf{Defl}^m, \mathsf{Con}^m).$$

Proposition 4.2. *The map*

$$\mathsf{DeflCon}^m : \mathrm{Inc}^m(\lambda) \to \coprod_{0 \le n \le m} \left(\mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n} \right)$$

is bijective.

Proof. A two-sided inverse for $\mathsf{DeflCon}^m$ is given as follows. For any nonnegative integer j, let $[j]$ denote the set $\{1, 2, \ldots, j\}$. For a binary vector $v \in \{0,1\}^m$, let N_v be the number of 1's in v, and define a **vector inflation** map

$$\mathsf{VecInfl}^m_v : [N_v] \to [m]$$

by

$$\mathsf{VecInfl}^m_v(k) = \min \left\{ n \in [m] : \sum_{\ell=1}^{n} v\{\ell\} = k \right\}.$$

(We use curly braces "{}" throughout the paper to denote vector components.) An integer $j \in [m]$ is in the range of $\mathsf{VecInfl}^m_v$ if and only if $v\{j\} = 1$. Therefore

$$\begin{aligned}[\mathsf{VecInfl}^m_{\mathsf{Con}^m(T)} \circ \mathsf{Defl}^m(T)](\mathsf{x}) &= \min \left\{ n \in [m] : \#\{h \in \mathrm{range}(T) : h \le n\} = [\mathsf{Defl}^m(T)](\mathsf{x}) \right\} \\ &= T(\mathsf{x}).\end{aligned}$$

Now define the **tableau inflation** map

$$\mathsf{Infl}^m : \coprod_{0\le n\le m} \left(\mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n} \right) \to \mathrm{Inc}^m(\lambda)$$

by

$$\mathsf{Infl}^m(S, v) = \mathsf{VecInfl}^m_v \circ S.$$

Say $(S, v) \in \mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n}$. Since S is surjective onto $[n]$ and $\mathsf{VecInfl}^m_v$ maps $[n]$ onto the indices of nonzero components in v, $\mathsf{Con}^m \circ \mathsf{Infl}^m(S, v) = v$. Also,

$$\begin{aligned}
[\mathsf{Defl}^m(\mathsf{VecInfl}^m_v \circ S)](\mathsf{x}) &= \#\{h \in \mathrm{range}(\mathsf{VecInfl}^m_v \circ S) : h \le \mathsf{VecInfl}^m_v \circ S(\mathsf{x})\} \\
&= \#\{h : v\{h\} \neq 0 \text{ and } \sum_{\ell=1}^{h} v\{\ell\} \le S(\mathsf{x})\} \\
&= S(\mathsf{x}).
\end{aligned}$$

Therefore, Infl^m is a two-sided inverse for $\mathsf{DeflCon}^m$. □

Example 4.3. Let $v = (1, 1, 0, 1, 1, 1, 0) \in \{0, 1\}^7$. Then, $N_v = 5$ and the map $\mathsf{VecInfl}^7_v : [5] \to [7]$ is given by

$$\begin{aligned}
\mathsf{VecInfl}^7_v(1) &= 1, \\
\mathsf{VecInfl}^7_v(2) &= 2, \\
\mathsf{VecInfl}^7_v(3) &= 4, \\
\mathsf{VecInfl}^7_v(4) &= 5, \\
\mathsf{VecInfl}^7_v(5) &= 6.
\end{aligned}$$

Now, for $S = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 & 4 \\ \hline\end{array}$, we have $\mathsf{Infl}^7(S, v) = \mathsf{VecInfl}^7_v \circ S = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array}$. Note that this process has recovered the tableau T of Example 4.1 from $S = \mathsf{Defl}^7(T)$ and $v = \mathsf{Con}^7(T)$. ♢

5. Interaction between K-promotion and deflation

A classical approach to the cohomological Schubert calculus of type A Grassmannians is to use the jeu de taquin on standard Young tableaux introduced by M.-P. Schützenberger [Sch77]. (See [Ful97, Man01] for modern expositions of the classical jeu de taquin theory.) A theme of modern Schubert calculus has been extending such theories to richer generalized cohomologies and in particular into the K-theory ring of algebraic vector bundles. For a partial survey of recent work related to K-theoretic Schubert calculus and the associated combinatorics, see [PY17].

The first K-theoretic Littlewood-Richardson rule was discovered by A. Buch [Buc02]; this rule, however, was not based on jeu de taquin. H. Thomas and A. Yong [TY09] later found a different Littlewood-Richardson rule for the same structure coefficients by directly extending M.-P. Schützenberger's jeu de taquin. This latter rule was conjectured in [TY09] to extend to the K-theoretic Schubert structure coefficients of all minuscule varieties, as was proven by a combination of [BR12, CTY14, BS16]. In this Thomas-Yong theory, the role of standard Young tableaux is filled by increasing tableaux.

The K-jeu de taquin of H. Thomas and A. Yong [TY09] gives rise to a K-promotion operator $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(\lambda)$ that directly extends M.-P. Schützenberger's classical definition of promotion [Sch72]. K-promotion was first studied in [Pec14], and has been further investigated in [BPS16, PSV16, Rho17, DPS17, Pec17, Vor17].

In this section, we determine how $\mathfrak{pro}^m$ interacts with $\mathsf{DeflCon}^m$. Our results will show that $\mathfrak{pro}^m$ is controlled by its action on gapless increasing tableaux. The power of this observation is that there are only finitely-many gapless increasing tableaux of any fixed shape λ. Thus a finite amount of data governs the action of $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(\lambda)$ for all m.

First, we define $\mathfrak{pro}^m$, setting up notation we will need. Let $\mathsf{Rep}_{1\to\bullet}$ be the operator on $\mathrm{Inc}^m(\lambda)$ that replaces each instance of 1 by $\bullet$. Let swap_n be the operator that finds all edge-connected components containing $\bullet$ and n, leaves trivial components unchanged, and switches n and $\bullet$ in nontrivial connected components. Let $\mathsf{Rep}_{\bullet\to m+1}$ replace each instance of $\bullet$ with $m+1$. Finally, let Decr be the operator that decrements each label by 1. Then, following [Pec14], we define K-promotion on $\mathrm{Inc}^m(\lambda)$ by

$$\mathfrak{pro}^m = \mathsf{Decr} \circ \mathsf{Rep}_{\bullet\to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1\to\bullet}. \tag{5.1}$$

An example of computing K-promotion via this process is shown in Figure 5.

We give names to the intermediate products of $\mathfrak{pro}^m$. For $T \in \mathrm{Inc}^m(\lambda)$ and $n \leq m$, we define the tableau $T^{(n)} : \lambda \to \{2, \ldots, m, \bullet\}$ by

$$T^{(n)} := \mathsf{swap}_n \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1\to\bullet}(T).$$

(In particular, $T^{(1)} := \mathsf{Rep}_{1\to\bullet}(T)$.) We define the associated order ideal

$$\lambda\left(T^{(n)}\right) := \left(T^{(n)}\right)^{-1}(\{2, \ldots, n\})$$

to be the set of boxes of $T^{(n)}$ containing entries from $\{2, \ldots, n\}$. Finally, define

$$T_n := T^{(n)}|_{\lambda\left(T^{(n)}\right)}.$$

Intuitively, $T^{(n)}$ shows T at the point in K-promotion at which the boxes labeled 2 through n have been acted upon by swap operators, while T_n is the restriction of $T^{(n)}$ to this subset of boxes. Examples of these intermediate tableaux appear in Figure 6.

We use these intermediate products to characterize the action of K-promotion on gappy tableaux.

Proposition 5.1. *Let $T \in \mathrm{Inc}^m(\lambda)$. If $\mathsf{Con}^m(T)\{1\} = 1$, then*

$$\mathsf{Defl}^m \circ \mathfrak{pro}^m(T) = \mathfrak{pro}^{m_T} \circ \mathsf{Defl}^m(T). \tag{5.2}$$

If $\mathsf{Con}^m(T)\{1\} = 0$, then $\mathfrak{pro}^m(T) = \mathsf{Decr}(T)$.

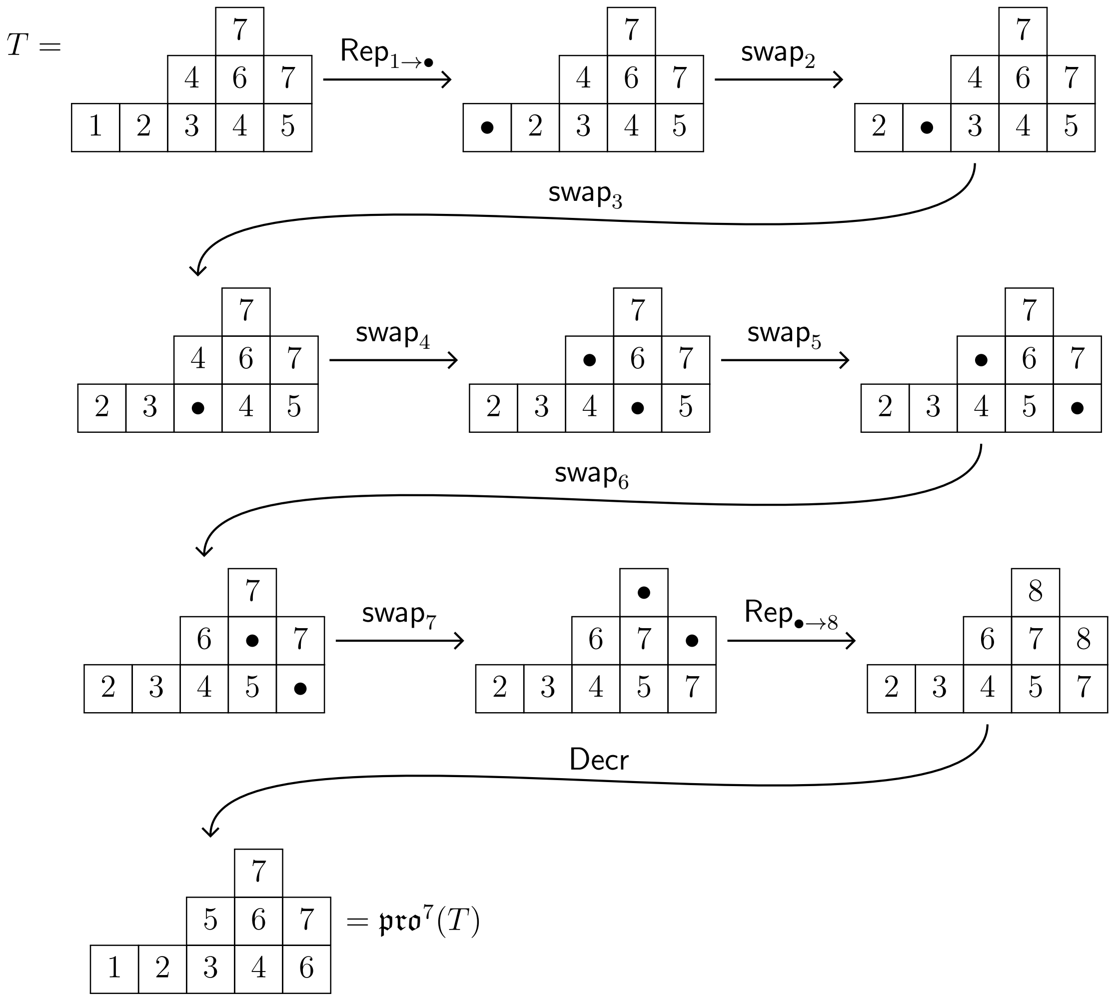


FIGURE 5. The calculation of the K-promotion $\mathfrak{pro}^7(T)$ of the increasing tableau $T \in \mathrm{Inc}^7(\lambda)$, where λ is the illustrated order ideal of P_{CM}.

In broad strokes, Proposition 5.1 is true because, after the initialization step that replaces 1s with $\bullet$s, the steps of K-promotion essentially depend only on the relative magnitudes of the labels. The following technical results (Lemmas 5.2 and 5.3) are necessary to make this argument rigorous.

Lemma 5.2. *Let $T \in \mathrm{Inc}^m(\lambda)$ with $\mathsf{Con}^m(T)\{1\} = 1$. Let $i_j = \mathsf{VecInfl}^m_{\mathsf{Con}^m(T)}(j)$ for $1 \le j \le m_T$. Then, for all $\mathsf{x} \in \lambda$ and $1 \le j \le m_T$, we have*

$$\mathsf{Defl}^m(T)^{(j)}(\mathsf{x}) = \bullet \text{ if and only if } T^{(i_j)}(\mathsf{x}) = \bullet. \tag{5.3}$$

Proof. We proceed by induction on j. Fix $T \in \mathrm{Inc}^m(\lambda)$ with $\mathsf{Con}^m(T)\{1\} = 1$. In the base case $j = 1$, $T^{(i_1)} = T^{(1)} = \mathsf{Rep}_{1\to\bullet}(T)$ has $\bullet$ in exactly those boxes $\mathsf{x} \in \lambda$ such that $T(\mathsf{x}) = 1$. On the other hand, $\mathsf{Defl}^m(T)^{(1)} = \mathsf{Rep}_{1\to\bullet}(\mathsf{Defl}^m(T))$ has $\bullet$ in

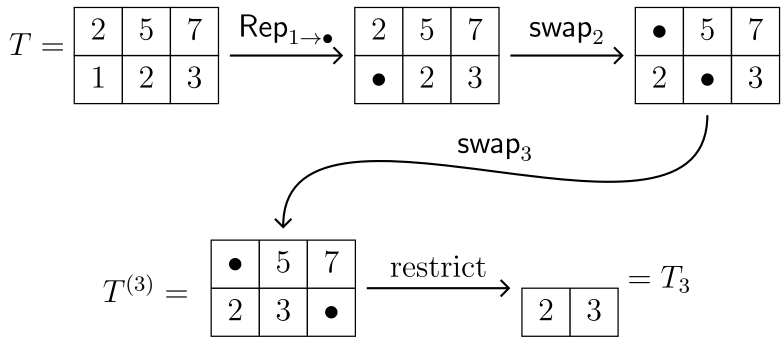


FIGURE 6. The tableau $T^{(3)}$ is obtained from T by applying in sequence the operations $\mathsf{Rep}_{1\to\bullet}$, swap_2, and swap_3. We then obtain T_3 from $T^{(3)}$ by restricting to those boxes with labels less than or equal to 3. The order ideal $\lambda\left(T^{(3)}\right)$ is the shape $\square\square$ of the tableau T_3.

exactly those boxes $\mathsf{x} \in \lambda$ such that $\mathsf{Defl}^m(T)(\mathsf{x}) = 1$. But since $\mathsf{Con}^m(T)\{1\} = 1$, $\mathsf{Defl}^m(T)(\mathsf{x}) = 1$ if and only if $T(\mathsf{x}) = 1$.

Now, say the lemma holds for all such T in the case $j < t$. First consider the case where $T^{(i_t)}(\mathsf{x}) = \bullet$ and $T^{(i_{t-1})}(\mathsf{x}) \neq \bullet$. This is equivalent to the statement that $T^{(i_{t-1})}(\mathsf{x}) = i_t$ and $T^{(i_{t-1})}(\mathsf{x}') = \bullet$ for some x' adjacent to x. But since $\mathsf{Rep}_{1\to\bullet}$ and swap_j for $j \leq i_{t-1}$ only act on elements of T labeled with $\bullet$'s or values less than or equal to i_{t-1}, $T^{(i_{t-1})}(\mathsf{x}) = i_t$ implies that $T(\mathsf{x}) = i_t$. This is, in turn, equivalent by Proposition 4.2 to the statement $\mathsf{Defl}^m(T)(\mathsf{x}) = t$, implying that $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}) = t$. Additionally, by the inductive hypothesis, $T^{(i_{t-1})}(\mathsf{x}') = \bullet$ if and only if $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}') = \bullet$. Therefore, the conditions of this case are equivalent to the conditions $\mathsf{Defl}^m(T)^{(t)}(\mathsf{x}) = \bullet$ and $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}) \neq \bullet$, as desired.

Consider the other possible case, where $T^{(i_t)}(\mathsf{x}) = \bullet$ and $T^{(i_{t-1})}(\mathsf{x}) = \bullet$. This scenario is equivalent to the statement that $T^{(i_{t-1})}(\mathsf{x}) = \bullet$ and there is no x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = i_t$. By similar reasoning as in the previous case, the second condition is equivalent to the statement that there is no x' adjacent to x with $T(\mathsf{x}') = i_t$. This statement is, in turn, equivalent to saying that there is no x' adjacent to x with $\mathsf{Defl}^m(T)(\mathsf{x}') = \mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}') = t$. Since $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}) = \bullet$ by the inductive hypothesis, the conditions of this case are finally equivalent to the conditions that $\mathsf{Defl}^m(T)^{(t)}(\mathsf{x}) = \bullet$ and $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}) = \bullet$, and we are done. $\square$

Lemma 5.3. *Take T and $i_1, \ldots, i_{m_T}$ as in Lemma 5.2. Then for $1 \leq j \leq m_T$,*

$$\mathsf{Infl}^m\Big(\mathsf{Defl}^m(T)_j, \mathsf{Con}^m(T)\Big) = T_{i_j}. \tag{5.4}$$

That is, the two sides are identical functions on the same domain.

Remark 5.4. The statement of the lemma involves a slight abuse of notation, since in general $\mathsf{Con}^m(T)$ has more nonzero entries than $\lambda(\mathsf{Defl}^m(T)^{(j)})$ has boxes. This does not present an issue, since Infl^m extends to a well-defined operator on vectors with excess 1s.

Proof. We proceed again by induction on j. The base case $j = 1$ is trivial, since for any $T \in \mathrm{Inc}^m(\lambda)$, $\lambda\left(\mathsf{Defl}^m(T)^{(1)}\right)$ and $\lambda\left(T^{(1)}\right)$ are both the empty order ideal.

Now, assume that the lemma holds for all increasing tableaux T in the case $j < t$. We first analyze the left side of Equation (5.4). Take $\mathsf{x} \in \lambda\left(\mathsf{Defl}^m(T)^{(t)}\right)$. If $\mathsf{x} \in \lambda\left(\mathsf{Defl}^m(T)^{(t-1)}\right)$, then $[\mathsf{Defl}^m(T)^{(t-1)}](\mathsf{x}) \leq t-1$ and hence

$$\left[\mathsf{swap}_t(\mathsf{Defl}^m(T)^{(t-1)})\right](\mathsf{x}) = \mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}).$$

If instead

$$\mathsf{x} \in \lambda\left(\mathsf{Defl}^m(T)^{(t)}\right) \setminus \lambda\left(\mathsf{Defl}^m(T)^{(t-1)}\right),$$

then either $[\mathsf{Defl}^m(T)^{(t-1)}](\mathsf{x}) = \bullet$ or $[\mathsf{Defl}^m(T)^{(t-1)}](\mathsf{x}) = t$. In the former case, there must be a box x' adjacent to x such that $[\mathsf{Defl}^m(T)^{(t-1)}](\mathsf{x}') = t$. In the latter case, we must have that for all boxes x' adjacent to x, $[\mathsf{Defl}^m(T)^{(t-1)}](\mathsf{x}') \neq \bullet$.

Now, we analyze the right side of Equation (5.4). Take $\mathsf{x} \in \lambda\left(T^{(i_t)}\right)$. If $\mathsf{x} \in \lambda\left(T^{(i_{t-1})}\right)$, then $\mathsf{x} \in \lambda(\mathsf{Defl}^m(T)^{(t-1)}) \subseteq \lambda(\mathsf{Defl}^m(T)^{(t)})$ by the inductive hypothesis. Further, this implies that $T^{(i_{t-1})}(\mathsf{x}) \leq i_{t-1}$, and so we have

$$\left[\mathsf{swap}_{\ell+1}\left(T^{(\ell)}\right)\right](\mathsf{x}) = T^{(\ell)}(\mathsf{x})$$

for all $i_{t-1} \leq \ell$. Therefore, by the previous paragraph and the inductive hypothesis,

$$T^{(i_t)}(\mathsf{x}) = T^{(i_{t-1})}(\mathsf{x}) = \mathsf{VecInfl}^m_{\mathsf{Con}^m(T)} \circ \mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}) = \mathsf{VecInfl}^m_{\mathsf{Con}^m(T)} \circ \mathsf{Defl}^m(T)^{(t)}(\mathsf{x}).$$

This shows that Equation (5.4) holds at x. If $\mathsf{x} \in \lambda\left(T^{(i_t)}\right) \setminus \lambda\left(T^{(i_{t-1})}\right)$, then $i_{t-1} < T^{(i_t)}(\mathsf{x}) \leq i_t$. Thus, we must have that $T^{(i_t)}(\mathsf{x}) = i_t$, since the swap operators do not change the content of a tableau. Therefore, either $T^{(i_{t-1})}(\mathsf{x}) = \bullet$ or $T^{(i_{t-1})}(\mathsf{x}) = T(\mathsf{x}) = i_t$. In the former case, there must be a box x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = T(\mathsf{x}') = i_t$. This occurs exactly when $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}') = \mathsf{Defl}^m(T)(\mathsf{x}') = t$. In the latter case, there is no x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = \bullet$. Since we have assumed that $\mathsf{Con}^m(T)\{1\} = 1$, Lemma 5.2 implies that this occurs exactly when $\mathsf{Defl}^m(T)^{(t-1)}(\mathsf{x}') \neq \bullet$ for every box x' adjacent to x. Therefore each of these conditions is equivalent to the corresponding condition from the previous paragraph.

Putting these facts together, we have shown that $\lambda\left(T^{(i_j)}\right) = \lambda\left(\mathsf{Defl}^m(T)^{(j)}\right)$ for all j, and that $\mathsf{Defl}^m(T)_j(\mathsf{x}) = h$ if and only if $T_j(\mathsf{x}) = \mathsf{VecInfl}^m_{\mathsf{Con}^m(T)}(h)$. □

Proof of Proposition 5.1. If $\mathsf{DeflCon}^m(T)\{1\} = 0$, then $\mathsf{Rep}_{1\to\bullet}(T) = T$. But then $\mathsf{swap}_\ell(T) = T$ for all ℓ, since there are no nontrivial short ribbons containing the labels $\bullet$ and ℓ. Also, $\mathsf{Rep}_{\bullet\to 1}(T) = T$, since there is no $\bullet$ label in T. Therefore, Decr is the only operator in Equation (5.1) that acts nontrivially on T.

Say otherwise that $\mathsf{DeflCon}^m(T)\{1\} = 1$. By Equation (5.1), we must show that

$$\begin{aligned}\mathsf{Defl}^m \circ \mathsf{Decr} \circ \mathsf{Rep}_{\bullet\to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1\to\bullet}(T) = \\ \mathsf{Decr} \circ \mathsf{Rep}_{\bullet\to m_T+1} \circ \mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1\to\bullet} \circ \mathsf{Defl}^m(T).\end{aligned}$$

Say $\mathsf{x} \notin \lambda(T^{(m)})$; that is, $T^{(m)}(\mathsf{x}) = \bullet$. Then,

$$[\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)](\mathsf{x}) = m.$$

Now by [DPS17, Lemma 2.1], the content vector of

$$\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T) = \mathfrak{pro}^m(T)$$

is a cyclic rotation by one of the content vector of T. This implies that there are the same number of distinct labels in the content vector of $\mathfrak{pro}^m(T)$, namely m_T, and, since m is included, it must be the greatest such label, so

$$[\mathsf{Defl}^m \circ \mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)](\mathsf{x}) = m_T.$$

On the other hand, since Lemma 5.2 implies that $\mathsf{x} \notin \lambda(\mathsf{Defl}^m(T)^{(m_T)})$, we have

$$[\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m_T+1} \circ \mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet} \circ \mathsf{Defl}(T)](\mathsf{x}) = m_T.$$

Now say $\mathsf{x} \in \lambda(T^{(m)})$ and let $i_1, \ldots, i_{m_T}$ be as in Lemma 5.2. We observe that since swap_ℓ does not affect the tableau $\mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)$ for $\ell > i_{m_T}$, we have that

$$(\mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T))_m = (\mathsf{swap}_{i_{m_T}} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T))_{i_{m_T}}.$$

Therefore, it follows from Lemma 5.3, taking $j = m_T$, that

$$(\mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T))_m = \mathsf{VecInfl}^m_{\mathsf{Con}^m(T)} \circ (\mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet} \circ \mathsf{Defl}^m(T)_{m_T}).$$

Now, since $\mathsf{x} \in \lambda(T^{(m)})$, we have that $T^{(m)}(\mathsf{x}) = \ell$ for $\ell \in \{2, 3, \ldots, m\}$. This implies that $\ell = i_j$ for some $2 \leq j \leq m_T$. Then,

$$\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)(\mathsf{x}) = i_j - 1.$$

Because $\mathsf{Con}^m(T)\{1\} = 1$ and the content vector shifts cyclically, this implies that

$$[\mathsf{Defl}^m \circ \mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m+1} \circ \mathsf{swap}_m \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)](\mathsf{x}) = j - 1.$$

On the other hand,

$$[\mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet} \circ \mathsf{Defl}^m(T)](\mathsf{x}) = j,$$

so

$$[\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to m_T+1} \circ \mathsf{swap}_{m_T} \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet} \circ \mathsf{Defl}^m(T)](\mathsf{x}) = j - 1. \qquad \square$$

6. Computation of period

In this section, we use Proposition 5.1 to relate the period of $T \in \mathrm{Inc}^m(\lambda)$ under $\mathfrak{pro}^m$ to data concerning $\mathsf{Defl}^m(T)$. Let

$$\Sigma^m : \{0,1\}^m \to \{0,1\}^m$$

be the cyclic rotation defined by

$$\Sigma^m(v_1, \ldots, v_m) = (v_2, \ldots, v_m, v_1)$$

for $(v_1, \ldots, v_m) \in \{0,1\}^m$.

Theorem 6.1. *Fix $T \in \mathrm{Inc}^m(\lambda)$. Let τ be the the period of $\mathfrak{pro}^{m_T}$ on $\mathsf{Defl}^m(T)$ and ℓ be the period of Σ^m on $\mathsf{Con}^m(T)$. Then, the period of $\mathfrak{pro}^m$ on T is*

$$\frac{\ell\tau}{\gcd(\ell m_T/m, \tau)}.$$

Proof. Define

$$K^m : \coprod_{n \leq m} \mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n} \to \coprod_{n \leq m} \mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n}$$

by

$$K^m(S, v) = \begin{cases} (\mathfrak{pro}^{N_S}(S), \Sigma^m(v)), & \text{if } v\{1\} = 1; \\ (S, \Sigma^m(v)), & \text{otherwise.} \end{cases}$$

We first show that, for $T \in \mathrm{Inc}^m(\lambda)$,

$$\mathsf{DeflCon}^m \circ \mathfrak{pro}^m(T) = K^m \circ \mathsf{DeflCon}^m(T). \tag{6.1}$$

Let π_1 be the projection onto the first factor of $\coprod_{n \leq m} \mathrm{Inc}^n_{\mathrm{gl}}(\lambda) \times \binom{[m]}{n}$ and π_2 the projection onto the second factor. It is immediate from the definitions that

$$\pi_2(K^m \circ \mathsf{DeflCon}^m(T)) = \Sigma^m \circ \mathsf{Con}^m(T).$$

But by [DPS17, Lemma 2.1],

$$\Sigma^m \circ \mathsf{Con}^m(T) = \mathsf{Con}^m \circ \mathfrak{pro}^m(T) = \pi_2(\mathsf{DeflCon}^m \circ \mathfrak{pro}^m(T)).$$

Thus, the two sides of Equation (6.1) are equal in the second factor.

We now show that Equation (6.1) holds in the first factor. Let

$$w \coloneqq \mathsf{Con}^m(T) = \pi_2(\mathsf{DeflCon}^m(T)).$$

By Proposition 4.2, we can write

$$T = \mathsf{Infl}^m(\mathsf{Defl}^m(T), w).$$

If $w\{1\} = 1$, then

$$\begin{aligned}
\pi_1(\mathsf{DeflCon}^m \circ \mathfrak{pro}^m(T)) &= \mathsf{Defl}^m \circ \mathfrak{pro}^m(T) \\
&= \mathfrak{pro}^{m_T} \circ \mathsf{Defl}^m(T) && \text{(by Proposition 5.1, } w\{1\} = 1) \\
&= \mathfrak{pro}^{m_T} \circ \pi_1 \circ \mathsf{DeflCon}^m(T) \\
&= \pi_1(K^m \circ \mathsf{DeflCon}^m(T)) && \text{(because } w\{1\} = 1).
\end{aligned}$$

On the other hand, if $w\{1\} = 0$, then

$$\begin{aligned}
\pi_1(\mathsf{DeflCon}^m \circ \mathfrak{pro}^m(T)) &= \mathsf{Defl}^m \circ \mathfrak{pro}^m(T) \\
&= \mathsf{Defl}^m(T) && \text{(by Proposition 5.1, } w\{1\} = 0) \\
&= \pi_1(\mathsf{DeflCon}^m(T)) \\
&= \pi_1(K^m \circ \mathsf{DeflCon}^m(T)) && \text{(because } w\{1\} = 0).
\end{aligned}$$

This completes the proof of Equation (6.1).

Now, since $\mathsf{DeflCon}^m$ is a bijection by Proposition 4.2, $(\mathfrak{pro}^m)^{\circ n}(T) = T$ exactly when $\mathsf{DeflCon}^m \circ (\mathfrak{pro}^m)^{\circ n}(T) = \mathsf{DeflCon}^m \circ T$. So, by Equation (6.1), we are reduced to determining which powers n of K^m stabilize $\mathsf{DeflCon}^m(T)$.

Say $(K^m)^{\circ n}(\mathsf{DeflCon}^m(T)) = \mathsf{DeflCon}^m(T)$. Applying π_2 to both sides, we have

$$(\Sigma^m)^{\circ n}(\mathsf{Con}^m(T)) = \mathsf{Con}^m(T),$$

so n is a multiple of ℓ. Hence, $n = t\ell$ for some integer $t \in \mathbb{Z}$. Applying instead π_1 to both sides, we have

$$(\mathfrak{pro}^m)^{\circ n'}(\mathsf{Defl}^m(T)) = \mathsf{Defl}^m(T),$$

where

$$\begin{aligned} n' &:= n - \#\{0 \leq j \leq n-1 : \pi_2((K^m)^{\circ j} \circ \mathsf{DeflCon}^m(T))\{1\} = 0\} \\ &= n - \#\{1 \leq i \leq n : \mathsf{Con}^m(T)\{i \mod m\} = 0\}. \end{aligned}$$

Let r be the number of 0s among the first ℓ entries of $\mathsf{Con}^m(T)$. Then, $n' = t(\ell - r)$. Since $(\mathfrak{pro}^m)^{\circ n'}$ fixes $\mathsf{Defl}^m(T)$, n' is a multiple of τ. Hence, $n' = s\tau$ for some integer s, and so $n = n' + tr = s\tau + tr$.

Thus, we have $(K^m)^{\circ n} \circ \mathsf{DeflCon}^m(T) = \mathsf{DeflCon}^m(T)$ if and only if $n = t\ell = s\tau + tr$ for some integers s and t. Hence, the period of $\mathfrak{pro}^m$ on T is the least positive integer h that can be simultaneously written in the forms $h = t\ell$ and $h = s\tau + tr$ for the same value of t. But $t\ell = s\tau + tr$ implies $t(\ell - r) = s\tau$. Since $0 \leq r < \ell$, the least such h is clearly achieved when t is minimal, i.e. when $t(\ell - r)$ is the least common multiple of $\ell - r$ and τ. Thus, h can be expressed as

$$h = t\ell = \frac{\ell \operatorname{lcm}(\ell - r, \tau)}{\ell - r} = \frac{\ell\tau}{\gcd(\ell - r, \tau)}. \tag{6.2}$$

Finally, observe that $r = \frac{\ell(m - m_T)}{m}$, so that

$$\ell - r = \frac{\ell m_T}{m}.$$

Therefore,

$$h = \frac{\ell\tau}{\gcd(\frac{\ell m_T}{m}, \tau)},$$

as desired. □

The following corollary will allow us to determine the period of $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(\lambda)$ in Section 7.

Corollary 6.2. *In the notation of Theorem 6.1, suppose j is a positive integer such that τ divides jm_T. Then, the period h of $\mathfrak{pro}^m$ on T divides jm.*

Proof. First, say $jm_T = \tau$. Then,

$$h = \frac{\ell\tau}{\gcd(\frac{\ell m_T}{m}, \tau)} = \frac{\ell \cdot jm_T}{\gcd(\frac{\ell m_T}{m}, jm_T)} = \frac{\ell \cdot jm_T}{\frac{\ell m_T}{m}} = jm. \tag{6.3}$$

Now, say τ divides jm_T. Then,

$$\frac{\tau}{\gcd(\frac{\ell m_T}{m}, \tau)} \quad \text{divides} \quad \frac{jm_T}{\gcd(\frac{\ell m_T}{m}, jm_T)}.$$

Thus, h divides jm. $\square$

7. Proof of Theorems 1.2 and 1.3

In this final section, we collect the above results into proofs of Theorems 1.2 and 1.3. Consider a poset P that is either the Cayley-Moufang poset, the Freudenthal poset, or one of the propellers. By Corollary 3.3, the multiset of Ψ-orbit cardinalities for $J(P \times \mathbf{k})$ equals the multiset of $\mathfrak{pro}^{\mathrm{rk}(P)+k+1}$-orbit cardinalities for $\mathrm{Inc}^{\mathrm{rk}(P)+k+1}(P)$. Therefore, we may prove Theorems 1.2 and 1.3 by studying increasing tableaux instead of plane partitions.

By Proposition 4.2, the $\mathfrak{pro}$-orbit structure of increasing tableaux is controlled by the data of gapless increasing tableaux and binary vectors. But for any fixed P, $\mathrm{Inc}_{\mathrm{gl}}(P)$ is a finite set. Using a computer, we found all 549 gapless increasing tableaux of shape P_{CM}, as well as all 624 493 gapless increasing tableaux of shape P_F. For each such tableau $T \in \mathrm{Inc}^m_{\mathrm{gl}}(P_{CM})$ or $\mathrm{Inc}^m_{\mathrm{gl}}(P_F)$, we then determined its period under $\mathfrak{pro}^m$ by direct calculation. These data are given in Table 1 for P_{CM} and in Table 2 for P_F.

m_T	period (τ)	number of orbits (N)
11	1	1
12	3	1
	12	1
13	13	6
14	7	2
	14	12
15	15	13
16	2	1
	4	1
	8	1
	16	4

Table 1. The distribution of $\mathfrak{pro}^{m_T}$-orbits of gapless increasing tableaux in $\mathrm{Inc}^{m_T}_{\mathrm{gl}}(P_{CM})$ for each m_T.

It is essentially trivial to observe that, for any p, $\mathrm{Inc}_{\mathrm{gl}}(P_p)$ consists of exactly three increasing tableaux. There are two tableaux in $\mathrm{Inc}^{2p}_{\mathrm{gl}}(P_p)$, forming a single $\mathfrak{pro}^{2p}$-orbit, and a single tableau in $\mathrm{Inc}^{2p-1}_{\mathrm{gl}}(P_p)$, which is necessarily fixed by $\mathfrak{pro}^{2p-1}$. These three tableaux and their orbits are illustrated in Figure 7 in the case $p = 4$. Table 3 records the observations of this paragraph.

We will use Theorems 6.1 and 7.2, together with the data of Tables 1, 2, and 3, to completely determine the multiset of $\mathfrak{pro}^m$-orbit cardinalities on set $\mathrm{Inc}^m(P)$ for

m_T	period (τ)	number of orbits (N)
17	1	1
18	2	1
	18	2
19	19	30
20	20	228
21	7	3
	21	1044
22	22	3053
	66	2
23	23	5813
	69	13
24	8	7
	24	7195
	48	4
	72	26
25	25	5602
	50	8
	75	21
26	2	2
	26	2495
	52	4
	78	6
27	3	2
	9	4
	27	484

TABLE 2. The distribution of $\mathfrak{pro}^{m_T}$-orbits of gapless increasing tableaux in $\mathrm{Inc}_{\mathrm{gl}}^{m_T}(P_F)$ for each m_T.

m_T	period (τ)	number of orbits (N)
$2p-1$	1	1
$2p$	2	1

TABLE 3. The distribution of $\mathfrak{pro}^{m_T}$-orbits of gapless increasing tableaux in $\mathrm{Inc}_{\mathrm{gl}}^{m_T}(P_p)$ for each m_T.

$P \in \{P_p, P_{CM}, P_F\}$ and any m. These tables, along with Corollary 6.2, immediately give the period of $\mathfrak{pro}^m$.

Theorem 7.1. *For $m \gg 0$, the period of $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(P)$ is*

- *m for $P = P_p$,*
- *m for $P = P_{CM}$, and*

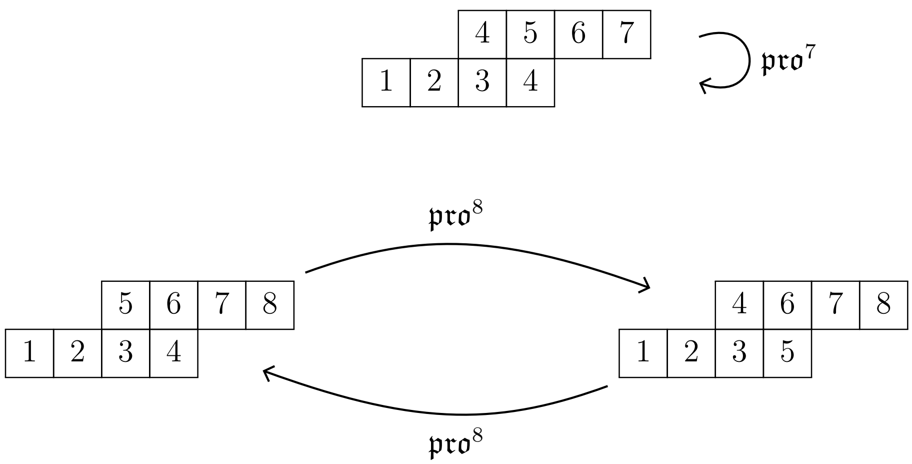


FIGURE 7. The set $\mathrm{Inc}_{\mathrm{gl}}(P_4)$ consists of the three illustrated gapless increasing tableaux. The unique element of $\mathrm{Inc}^7_{\mathrm{gl}}(P_4)$ forms a singleton $\mathfrak{pro}^7$-orbit, while $\mathfrak{pro}^8$ switches the two elements of $\mathrm{Inc}^8_{\mathrm{gl}}(P_4)$, as shown. The situation for $p \neq 4$ is exactly analogous.

- *$3m$ for $P = P_F$.*

(Here '$m \gg 0$' means $m \geq 2p$ for $P = P_p$, $m \geq 12$ for $P = P_{CM}$, and $m \geq 22$ for $P = P_F$.) For $m \in \{18, 19, 20, 21\}$, the period of $\mathfrak{pro}^m$ on $\mathrm{Inc}^m(P_F)$ is m.

Proof. Inspecting Table 1 shows that, for P_{CM}, the period of each gapless tableau divides its height (m_T). By Corollary 6.2, this implies that for any m, the $\mathfrak{pro}^m$-period of each tableau divides m. Clearly the period of $\mathfrak{pro}^{12}$ on $\mathrm{Inc}^{12}(P_{CM})$ is 12, and whenever $m > 12$, it is possible to find a tableau attaining period m by taking a gapless tableau $T \in \mathrm{Inc}^{12}_{\mathrm{gl}}(P_{CM})$ with period 12 and then inflating according to a content vector with Σ^m-period m (for example, by just considering $T \in \mathrm{Inc}^{12}_{\mathrm{gl}}(P_{CM})$ as a gappy tableau in $\mathrm{Inc}^m(P_{CM})$). The analogous calculation for $P = P_p$ is easy by inspection of Table 3.

The argument is identical for $P = P_F$ when $m \in \{18, 19, 20, 21\}$, but when $m \geq 22$, we do not have that the period of each gapless tableau divides its height. However, inspecting Table 2 shows that, for P_F and for every m, the period of each gapless tableaux of height m divides $3m$. Therefore, Corollary 6.2 gives that for every m, the period of each (possibly gappy) tableau divides $3m$. When $m \geq 22$, it is possible to find a tableau attaining period $3m$ by taking a gapless tableau $T \in \mathrm{Inc}^{22}_{\mathrm{gl}}(P_F)$ with period 66 and inflating according to a content vector of Σ^m-period m (for example, by considering T as a gappy tableau in $\mathrm{Inc}^m(P_{CM})$). $\square$

We will need the fact that the number of elements of $\binom{[i]}{j}$ of any fixed Σ^i-period is given explicitly by [RSW04, Theorem 1.1(b)]:

Theorem 7.2 (Reiner, Stanton, and White)**.** *Let*

$$f_{i,j}(q) := \begin{cases} \frac{[i]!_q}{[j]!_q \cdot [i-j]!_q} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

where $[\ell]!_q := \prod_{a=1}^{\ell} [a]_q$ *and* $[a]_q := \sum_{b=0}^{a-1} q^b = \frac{1-q^{a+1}}{1-q}$ *are the standard* q*-analogues. Then,*

$$\# \left\{ v \in \binom{[i]}{j} : (\Sigma^i)^{\circ s}(v) = v \right\} = f_{i,j}(\zeta^s),$$

where ζ *is any primitive* i*th root of unity.* □

For specified $P \in \{P_p, P_{CM}, P_F\}$, we denote by $m_T\{i\}$, $\tau\{i\}$, and $N\{i\}$ the ith element of the first, second, and third columns of the corresponding table, respectively.

Theorem 7.3. *Fix* $P \in \{P_p, P_{CM}\}$*,* $m \gg 0$*, and* d *dividing* m*. Let* $\mathcal{R}(P, m, d)$ *be the number of increasing tableaux* $T \in \mathrm{Inc}^m(P)$ *whose* $\mathfrak{pro}^m$*-period divides* m/d*. Then,*

$$\mathcal{R}(P, m, d) = \sum_{i \colon d \,\mid\, \frac{m_T\{i\}}{\tau\{i\}}} \tau\{i\}\, N\{i\}\, f_{m, m_T}(\zeta^{m/d}). \tag{7.1}$$

Proof. Proposition 4.2 gives that a tableau $T \in \mathrm{Inc}^m(P)$ corresponds bijectively to a pair consisting of a gapless tableau $U \in \mathrm{Inc}_{\mathrm{gl}}^{m_T}(P)$ and a content vector of length m with m_T ones. The table corresponding to P enumerates all gapless tableaux. We iterate over the rows of the table and determine, for each row, how many content vectors yield tableaux in $\mathrm{Inc}^m(P)$ whose periods divide m/d.

For each i, let $c\{i\} = m_T\{i\}/\tau\{i\}$. By Theorem 6.1, a gapless tableau $U \in \mathrm{Inc}_{\mathrm{gl}}^{m_T\{i\}}(P)$ and a content vector v with period $m\{i\}/d'$ inflate to a tableau V of period

$$\frac{\frac{m\{i\}}{d'} \frac{m_T\{i\}}{c\{i\}}}{\gcd(\frac{m_T\{i\}}{d'}, \frac{m_T\{i\}}{c\{i\}})} = \frac{\frac{m\{i\}}{d'} \frac{m_T\{i\}}{c\{i\}}}{\frac{m_T\{i\} \gcd(c\{i\}, d')}{d' c\{i\}}} = \frac{m\{i\}}{\gcd(c\{i\}, d')}.$$

Therefore, V has period dividing $m\{i\}/d$ if and only if d divides both d' and $c\{i\}$. Now, d divides d' if and only if the period of v divides $m\{i\}/d$. But, by Theorem 7.2, the number of content vectors of length $m\{i\}$ with $m_T\{i\}$ 1s and with period dividing $m\{i\}/d$ is precisely $f_{m,m_T}(\zeta^{m/d})$. □

The formula resulting from Theorem 7.3 is simple enough to check by hand. As in [RSW04, Proof of Theorem 7.1], we need the following elementary identity, which allows us to work with integers rather than q-integers.

Lemma 7.4. *Let* ζ *be a primitive* N*th root of unity and let* $d \mid N$*. Then* $[n]_{\zeta^{N/d}} = 0$ *if and only if* $n \equiv 0 \mod d$*. Moreover, if* $n_1 \equiv n_2 \mod d$*, then*

$$\lim_{q \to \zeta^{N/d}} \frac{[n_1]_q}{[n_2]_q} = \begin{cases} \frac{n_1}{n_2}, & \text{if } n_1 \equiv n_2 \equiv 0 \mod d; \\ 1, & \text{if } n_1 \equiv n_2 \not\equiv 0 \mod d. \end{cases} \tag{7.2}$$ □

For example, we evaluate $f_{P_p}(\zeta^{m/d})$ for some $d > 1$ dividing both m and p. Since the minuscule posets are Gaussian, we have

$$f_{P_p}^{m-2p+1}(q) = \frac{[m-(2p-2)]_q[m-(2p-2)+1]_q \cdots [m-(p-1)]_q^2 \cdots [m-1]_q[m]_q}{[1]_q[2]_q \cdots [p]_q^2 \cdots [2p-2]_q[2p-1]_q}.$$

Since $f_{P_p}(q)$ is, by definition, a polynomial, we clearly have $f_{P_p}(\zeta^{m/d}) = \lim_{q \to \zeta^{m/d}} f_{P_p}(q)$. Thus, Lemma 7.4 allows us to replace this ratio of polynomials with a ratio of integers by matching equivalence classes modulo d in the numerator and denominator. We see that the numerator and denominator have the same multiset of equivalence classes modulo d. Pairing equivalent terms and using Lemma 7.4, we have

$$f_{P_p}^{m-2p+1}(\zeta^{m/d}) = \frac{(m-(2p-d))\cdots(m)}{(d)\cdots(2p-d)p} = \frac{2p}{p}\binom{m/d}{2p/d} = 2\binom{m/d}{2p/d}.$$

An important special case allows us to extend Theorem 7.2. We have that for any $i, j \in \mathbb{Z}_{>0}$, if ζ is an ith root of unity and d divides $\gcd(i,j)$, then

$$f_{i,j}(\zeta^{i/d}) = \binom{i/d}{j/d} = \frac{(i-(j-d))(i-(j-2d))\cdots(i-d)(i)}{(d)(2d)\cdots(j-d)(j)}. \tag{7.3}$$

Equation (7.3) holds even when $j > i$, since in that case one of terms in the numerator is 0.

Proof of Theorem 1.3. Fix positive integers p and m, and suppose d divides m. We have that

$$f_{P_p}(\zeta) = \frac{[m]!_q\,[m-(p-1)]_q}{[2p-1]!_q\,[m-(2p-1)]!_q\,[p]_q}.$$

If $d = 1$, then by Theorem 7.3,

$$\begin{aligned}\mathcal{R}(P_p, m, d) = f_{m,2p-1}(1) + 2\,f_{m,2p}(1) &= \binom{m}{2p-1} + 2\binom{m}{2p} \\ &= \frac{(2m-2p+2)m!}{(2p)!\,(m-2p+1)!} = f_{P_p}(1).\end{aligned}$$

Now, for $d > 1$, we can have that d divides p, or d divides $2p-1$, or d divides neither. If d divides p, then by Equation 7.3,

$$\mathcal{R}(P_p, m, d) = 2\,f_{m,2p}(\zeta^{m/d}) = 2\binom{m/d}{2p/d} = f_{P_p}(\zeta^{m/d}).$$

If d divides $2p-1$, then

$$\mathcal{R}(P_p, m, d) = f_{m,2p-1}(\zeta^{m/d}) = \binom{m/d}{(2p-1)/d} = f_{P_p}(\zeta^{m/d}).$$

If d divides neither p nor $2p-1$, then on the one hand Theorem 7.3 predicts that $\mathcal{R}(P_p, m, d) = 0$, and on the other hand d divides $\lceil \frac{2p-1}{d} \rceil$ of the terms in the numerator of $f_{P_p}(\zeta^{m/d})$ and at most $\lfloor \frac{2p-1}{d} \rfloor$ of the terms in the denominator, so $f_{P_p}(\zeta^{m/d}) = 0$. $\square$

The verification for $P = P_{CM}$ is similarly straightforward.

Proof of Theorem 1.2. Theorem 7.5 proves the P_F cases. Hence, it remains to consider $P = P_{CM}$.

Inspecting Table 1, we see that the possible values of $\ell\{i\} = m_T\{i\}/\tau\{i\}$ are $1, 11, 2, 4$ and 8. Therefore we evaluate $\mathcal{R}(P_{CM}, m, d)$ for each d that divides one or more of these ℓ values. We verify that this matches the prediction given by f^m_{CM}, and that f^m_{CM} predicts $\mathcal{R}(P_{CM}, m, d) = 0$ for all d not dividing any of these integers.

- If d divides 1, then Theorem 7.3 predicts

$$\begin{aligned}\mathcal{R}(P_{CM}, m, 1) &= 1 \cdot 1 \cdot \binom{m}{11} + (3 \cdot 1 + 12 \cdot 1) \cdot \binom{m}{12} + 13 \cdot 6 \cdot \binom{m}{13} + (7 \cdot 2 + 14 \cdot 12)\binom{m}{14} \\ &\quad + 15 \cdot 13 \cdot \binom{m}{15} + (2 \cdot 1 + 4 \cdot 1 + 8 \cdot 1 + 16 \cdot 4)\binom{m}{16} \\ &= \frac{(m-10)(m-9)(m-8)(m-7)^2(m-6)^2(m-5)^2(m-4)^2(m-3)^2(m-2)(m-1)m}{1 \cdot 2 \cdot 3 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 7^2 \cdot 8^2 \cdot 9 \cdot 10 \cdot 11} \\ &= f^m_{CM}(1).\end{aligned}$$

- If d divides 11 but not 1, Theorem 7.3 predicts

$$\begin{aligned}\mathcal{R}(P_{CM}, m, 11) &= 1 \cdot 1 \cdot \binom{m/11}{11/11} \\ &= m/11 \\ &= f^m_{CM}(\zeta^{m/11}).\end{aligned}$$

- If d divides 8 but not 4, Theorem 7.3 predicts

$$\begin{aligned}\mathcal{R}(P_{CM}, m, 8) &= 2 \cdot 1 \cdot \binom{m/8}{16/8} \\ &= \frac{(m)(m-8)}{8^2} \\ &= f^m_{CM}(\zeta^{m/8}).\end{aligned}$$

- If d divides 4 but not 2, Theorem 7.3 predicts

$$\begin{aligned}\mathcal{R}(P_{CM}, m, 4) &= 3 \cdot 1 \cdot \binom{m/4}{12/4} + (4 \cdot 1 + 2 \cdot 1) \cdot \binom{m/4}{16/4} \\ &= \frac{(m-8)(m-4)^2 m}{4^2 \cdot 8^2} \\ &= f^m_{CM}(\zeta^{m/4}).\end{aligned}$$

- If d divides 2 but not 1, Theorem 7.3 predicts

$$\begin{aligned}\mathcal{R}(P_{CM}, m, 2) &= 3 \cdot 1 \cdot \binom{m/2}{12/2} + 7 \cdot 2 \cdot \binom{m/2}{14/2} + (2 \cdot 1 + 4 \cdot 1 + 8 \cdot 1) \cdot \binom{m/2}{16/2} \\ &= \frac{(m-10)(m-8)(m-6)^2(m-4)^2(m-2)m}{10 \cdot 8^2 \cdot 6^2 \cdot 4^2 \cdot 2} \\ &= f^m_{CM}(\zeta^{m/2}).\end{aligned}$$

This exhausts all possible d dividing one of the above integers. To see that the cyclic sieving polynomial predicts all other d values are 0, we first observe that $f^m_{CM}(\zeta^{m/d}) =$

0 if $d > 11$. For in this case, Lemma 7.4 implies that one of the numerator terms (namely, $[m]_{\zeta^{m/d}}$) is zero, while none of the denominator terms are.

Using the same identity to count zeros in the numerator and denominator, it is easily checked that the remaining values of $d \leq 11$ evaluate to 0. $\square$

Theorem 7.5. *Conjecture 1.1 holds for P_F only when $k \leq 4$.*

Proof. That Conjecture 1.1 holds for P_F in the case $k \leq 4$ can be checked numerically using Table 2 and Theorem 7.3.

It remains to show that the conjecture fails for $k \geq 5$. Hence, let $m \geq 22 = 5 + \mathrm{rk}(P_F) + 1$. By Theorem 7.1, $\mathfrak{pro}^m$ has period $3m$ on $\mathrm{Inc}^m(P_F)$.

Suppose $T \in \mathrm{Inc}^m(P_F)$ were fixed by $\mathfrak{pro}^m$. By Equation 6.2, the Σ^m-period of $\mathsf{Con}^m(T)$ is 1. Since $\mathsf{Con}^m(T)$ is certainly not a vector of all 0's, it must therefore be a vector of all 1's. Hence, $m_T = m$ and $\mathsf{Defl}(T) = T$. Therefore, T is gapless. However, Table 2 shows that no element of $\mathrm{Inc}^m_{\mathrm{gl}}(P_F)$ is fixed by $\mathfrak{pro}^m$, for any $m \geq 22$, so such a T does not exist.

Now, let ζ be a primitive $(3m)$th root of unity. By the above, Conjecture 1.1 claims that $f_{P_F}(\zeta) = 0$. But this is impossible, for if $f_{P_F}(\zeta) = 0$, then the minimal polynomial of ζ would divide the numerator of f_{P_F}, which is the product of polynomials of the form $1 - q^n$ where $n < 3m$. Thus, Conjecture 1.1 fails in these cases. $\square$

Acknowledgements

The authors thank Julianna Tymoczko for introducing them to each other. OP was partially supported by a Mathematical Sciences Postdoctoral Research Fellowship from the National Science Foundation. HM was supported by the Graduate Research Fellowship from the National Science Foundation.

References

[AST13] Drew Armstrong, Christian Stump, and Hugh Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4121–4151. 1

[BL00] Sara Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000. 5

[BPS16] Jonathan Bloom, Oliver Pechenik, and Dan Saracino, *Proofs and generalizations of a homomesy conjecture of Propp and Roby*, Discrete Math. **339** (2016), no. 1, 194–206. 10

[BR12] Anders Skovsted Buch and Vijay Ravikumar, *Pieri rules for the K-theory of cominuscule Grassmannians*, J. Reine Angew. Math. **668** (2012), 109–132. 10

[BS74] A. E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Mathematisch Centrum, Amsterdam, 1974, Mathematisch Centrum Afdeling Zuivere Wiskunde ZW 24/74. 5

[BS16] Anders Skovsted Buch and Matthew J. Samuel, *K-theory of minuscule varieties*, J. Reine Angew. Math. **719** (2016), 133–171. 10

[Buc02] Anders Skovsted Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. 9

[CFDF95] P. J. Cameron and D. G. Fon-Der-Flaass, *Orbits of antichains revisited*, European J. Combin. **16** (1995), no. 6, 545–554. 1, 5

[CTY14] Edward Clifford, Hugh Thomas, and Alexander Yong, *K-theoretic Schubert calculus for* $\mathrm{OG}(n, 2n+1)$ *and jeu de taquin for shifted increasing tableaux*, J. Reine Angew. Math. **690** (2014), 51–63. 10

[DPS17] Kevin Dilks, Oliver Pechenik, and Jessica Striker, *Resonance in orbits of plane partitions and increasing tableaux*, J. Combin. Theory Ser. A **148** (2017), 244–274. 1, 2, 3, 6, 10, 14, 15

[DSV17] Kevin Dilks, Jessica Striker, and Corey Vorland, *Increasing labelings, generalized promotion, and rowmotion*, preprint (2017), 26 pages, arXiv:1710.07179. 1, 2, 3, 6, 7

[Duc74] P. Duchet, *Sur les hypergraphes invariantes*, Discrete Math. **8** (1974), 269–280. 5

[EP14] David Einstein and James Propp, *Piecewise-linear and birational toggling*, 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), Discrete Math. Theor. Comput. Sci. Proc., AT, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014, pp. 513–524. 1

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. 9

[GR15] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. Combin. **22** (2015), no. 3, Paper 3.40, 49. 1

[GR16] ———, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. Combin. **23** (2016), no. 1, Paper 1.33, 40. 1

[Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001, Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3. 9

[Pan09] Dmitri I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. **30** (2009), no. 2, 586–594. 1

[Pec14] Oliver Pechenik, *Cyclic sieving of increasing tableaux and small Schröder paths*, J. Combin. Theory Ser. A **125** (2014), 357–378. 2, 10

[Pec17] ———, *Promotion of increasing tableaux: frames and homomesies*, Electron. J. Combin. **24** (2017), no. 3, Paper 3.50, 1–14. 10

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22** (2015), no. 3, Paper 3.4, 1–29. 1

[Pro84] Robert A. Proctor, *Bruhat lattices, plane partition generating functions, and minuscule representations*, European J. Combin. **5** (1984), no. 4, 331–350. 5

[PSV16] Timothy Pressey, Anna Stokke, and Terry Visentin, *Increasing tableaux, Narayana numbers and an instance of the cyclic sieving phenomenon*, Ann. Comb. **20** (2016), no. 3, 609–621. 10

[PY17] Oliver Pechenik and Alexander Yong, *Genomic tableaux*, J. Algebraic Combin. **45** (2017), no. 3, 649–685. 9

[Rho10] Brendon Rhoades, *Cyclic sieving, promotion, and representation theory*, J. Combin. Theory Ser. A **117** (2010), no. 1, 38–76. 2

[Rho17] ———, *A skein action of the symmetric group on noncrossing partitions*, J. Algebraic Combin. **45** (2017), no. 1, 81–127. 2, 10

[RS11] David B. Rush and XiaoLin Shi, *On orbits of order ideals of minuscule posets*, unpublished manuscript, available at `http://www-users.math.umn.edu/~reiner/REU/RushShi2011.pdf`, 2011. 2

[RS13] ———, *On orbits of order ideals of minuscule posets*, J. Algebraic Combin. **37** (2013), no. 3, 545–569. 1, 2

[RSW04] V. Reiner, D. Stanton, and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), no. 1, 17–50. 2, 19, 20

[Sch72] M.-P. Schützenberger, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94. 6, 10

[Sch77] ______, *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*, ch. La correspondance de Robinson, pp. 59–113. Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977. 9

[SW12] Jessica Striker and Nathan Williams, *Promotion and rowmotion*, European J. Combin. **33** (2012), no. 8, 1919–1942. 1, 6

[TY09] Hugh Thomas and Alexander Yong, *A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus*, Algebra Number Theory **3** (2009), no. 2, 121–148. 2, 9, 10

[Vor17] Corey Vorland, *Homomesy in products of three chains and multidimensional recombination*, preprint (2017), 17 pages, arXiv:1705.02665. 1, 10

(HM) Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720
E-mail address: holly.mandel@berkeley.edu

(OP) Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: pechenik@umich.edu