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Bruhat Lattices, Plane Partition Generating Functions, and Minuscule Representations*

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The Bruhat posets (arising from Weyl groups) which are lattices are classified. Seshadri's standard monomial result for miniscule representations is used to show that certain combinatorially defined generating functions associated to these lattices satisfy certain identities. The most interesting cases of these identities are known plane partition generating function identities. Independent combinatorial proofs of the other identities are given. Then the combinatorial proofs of these identities are used as a step in a simplified proof of Seshadri's standard monomial result. Partial results to the effect that the Bruhat lattices are the only distributive lattices with such generating function identities are quoted ('Gaussian poset' conjecture), a potential Dynkin diagram classification result. New proofs of the fact that Bruhat lattices are rank unimodal and strongly Sperner are given. Geometric interpretations (with respect to minuscule flag manifolds) of the combinatorial quantities studied are described.

1. INTRODUCTION

In this paper we study certain partially ordered sets, *minuscule posets* and *lattices*, which play a central role in the geometry of minuscule flag manifolds and related representations of Lie algebras. These posets have several nice combinatorial properties which can be (Section 7) or have been [34] used to solve ordinary combinatorial problems. Minuscule posets can be characterized in several ways. The definition we choose indexes these posets with a certain family of Dynkin diagrams which arises often in Lie theory. However, it may be possible to characterize this family of posets in purely combinatorial terms (Section 9). Our results are primarily obtained using the representation theory of complex semi-simple Lie algebras. But in Section 8 we use combinatorial methods to supply a step for a proof in representation theory.

Bruhat posets (defined in Section 2) are posets defined on Weyl or Coxeter groups, or on certain quotients of these groups. In Section 3 we determine which Bruhat posets arising from Weyl groups are lattices, and then give combinatorial descriptions of these lattices. Except for two exceptional cases, these lattices fall into three infinite families. The two interesting infinite families, often denoted $L(m, n)$ and $M(n)$, have already appeared in combinatorics [19], [20], [39].

The most important objects in this paper, minuscule representations of semisimple Lie algebras, are introduced in Section 4. It turns out that the weight diagrams of minuscule representations form the same set of lattices as do the Bruhat lattices. Henceforth these lattices are referred to as minuscule lattices. The posets of join irreducibles of these lattices are called *minuscule posets*.

Let $\underset{\sim}{\Lambda} = (\Lambda_1, \ldots, \Lambda_k)$ with $\Lambda_1 \geq \cdots \geq \Lambda_k$. Consider a set of integers $P = \{P_{ij}\}$, $1 \leq i \leq k$, $1 \leq j \leq \Lambda_i$ such that $0 \leq P_{ij} \leq m$. If $P_{i,j} \geq P_{i+1,j}$ and $P_{i,j} \geq P_{i,j+1}$ for all possible i and j, and if the sum of all the P_{ij} is N, then P is a *plane partition of N contained in $\underset{\sim}{\Lambda}$ with parts $\leq m$.* Given fixed $\underset{\sim}{\Lambda}$ and m, the generating function for such plane partitions is

$$G(\underset{\sim}{\Lambda}, m, x) = \sum_P x^{|P|},$$

* Some of the results in this paper are contained in the author's doctoral thesis written under the direction of R. P. Stanley at M.I.T., 1981.

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where the sum is over all such plane partitions and where $|P| = N$ if P is a plane partition of N.

A plane partition of shape contained in $\underset{\sim}{\Lambda}$ with parts $\leqslant m$ can be viewed as an increasing sequence of m order ideals in the poset $\mathbb{N}\times\mathbb{N}$, all contained in the fixed order ideal of shape $\underset{\sim}{\Lambda}$. This observation helps motivate the following definition from Stanley's thesis (published as [31], see p. 8).

DEFINITION. Given any finite poset P, the *generating function for m-flags of order ideals* is:

$$F(P, m, x) = \sum_{I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m} x^{|I_1|+|I_2|+\cdots+|I_m|},$$

where the I_i are order ideals in P.

Another concept introduced for purely combinatorial reasons in Stanley's thesis is the following [30, p. 173].

DEFINITION. A poset P is said to be *Gaussian* if for every $m \geqslant 0$ its generating function for m-flags of order ideals has the following form:

$$F(P, m, x) = \frac{(1-x^{h_1+m})(1-x^{h_2+m})\cdots(1-x^{h_r+m})}{(1-x^{h_1})(1-x^{h_2})\cdots(1-x^{h_r})},$$

where r and the h_i are non-negative integers independent of m.

The motivating examples for this definition were two families of plane partition generating function identities originally due to MacMahon, Bender, Knuth, Gordon, and Andrews. (Stanley actually used a more general definition of Gaussian and had additional motivating examples.) In Section 6 we use C. S. Seshadri's standard menomial theory for minuscule representations [29] to show that minuscule posets are Gaussian. Equivalently, we find identities for the m-flag generating functions of these posets.

For the two interesting infinite families of minuscule posets, these identities turn out to be the two families of plane partition generating function identities mentioned above. The third (uninteresting) infinite family of Gaussian posets/generating function identities was also already known (but unpublished). The two exceptional minuscule posets are new Gaussian posets. Section 7 uses the techniques of Section 6 to give new proofs of the two families of plane partition generating function identities. Sections 6 and 7 represent joint work with Richard Stanley.

Seshadri remarked that the proof of his result could be made more direct if certain generating function identities for multichains in minuscule lattices could be verified. These identities are just the identities mentioned above. Seshadri was unaware of the existing plane partition generating function versions of the identities. In Section 8 we use combinatorial methods to prove the identities for the third infinite family of minuscule posets as well as for the two exceptionals. We therefore give (or quote) two proofs for each of these identities: one algebraic and one combinatorial. Therefore these identities can be viewed as either lemmas or corollaries with respect to Seshadri's standard monomial result.

The minuscule posets are the only known Gaussian posets. Since minuscule posets are indexed by a kind of Dynkin diagram which also indexes other (sometimes apparently unrelated) families of mathematical objects, it is natural to conjecture that there are no other Gaussian posets. See Section 9.

Stanley has used algebraic geometric techniques to show that all Bruhat posets arising from Weyl groups are strongly Sperner and rank unimodal. We note in Section 10 how the representation machinery presented in Section 5 can be used to reproduce Stanley's result in the special case of Bruhat lattices. This proof can be converted into elementary linear algebra [26].

Section 11 lists several ways in which the minuscule posets can be characterized. One of these, due to Steinberg, is more natural in a representation theoretic setting than our original combinatorial definition.

Finally, in Section 12, we describe the geometric interpretations of the combinatorial quantities found for the minuscule lattices and posets. The geometric concepts discussed for the minuscule flag manifolds are: intersections of Schubert varieties with hyperplane sections, Hilbert series of the homogeneous coordinate rings, and the degrees of the manifolds when realized as projective varieties.

2. Definitions and Notation

Let P be a finite partially ordered set. It is *ranked* with $r+1$ ranks $P_0, P_1, \ldots, P_r$ if $P=\bigcup_{i=0}^{r} P_i$ and $y \in P_i$, y covers x, implies $x \in P_{i-1}$. If $k \geq 1$, then $\underset{\sim}{k}$ denotes the totally ordered set with k elements. The symbols $\oplus$ and $\times$ denote direct sum (disjoint union) and direct product of posets.

A subset $I \subseteq P$ is an *order ideal* of P if $y \in I$ and $x \leq y$ imply $x \in I$. The poset $J(P)$ of all order ideals of P is always a distributive lattice. Conversely, for any distributive lattice L there is a unique poset $P=j(L)$, the *poset of join irreducibles* of L, such that $L=J(P)$.

An *m-flag of order ideals* in P is a weakly increasing sequence of order ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m$ in P. The *weight* of an m-flag is $|I_1|+\cdots+|I_m|$. An *m-multichain* of elements of L is a weakly increasing subset $x_1 \leq x_2 \leq \cdots \leq x_m$. The *weight* of an m-multichain is $r(x_1)+\cdots+r(x_m)$, where $r(x_i)$ is the rank of x_i in L. If $L=J(P)$, then m-flags in P correspond to m-multichains of the same weights in L. A *path* in L from x to y is a sequence of elements $x=x_0<x_1<\cdots<x_n=y$ such that x_{i+1} covers x_i.

A ranked poset P is *rank symmetric* if $|P_i|=|P_{r-i}|$ and *rank unimodal* if there is some m such that $|P_0| \leq |P_1| \leq \cdots \leq |P_m| \geq |P_{m+1}| \geq \cdots \geq |P_r|$. It is *strongly Sperner* if for every $k \geq 1$ the largest union of k antichains is no larger than the largest union of k ranks.

We follow the notation of [15]. Let $\mathfrak{g}$ be a complex simple Lie algebra of type $\mathbf{X}$, $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$, and rank n. Fix a Cartan subalgebra H. Choose a set Φ^+ of positive roots α in $H_{\mathbb{R}}^*$ and let $\Delta=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote the positive simple roots. Call the corresponding simple reflections $s_1, s_2, \ldots, s_n$, and let W denote the group generated by these reflections, the Weyl group of $\mathfrak{g}$. Conjugates t of simple reflections are called reflections. The inner product on $H_{\mathbb{R}}^*$ is written $(\cdot,\cdot)$; set $\langle\lambda, \alpha\rangle = 2(\lambda, \alpha)/(\alpha, \alpha)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental weights of $\mathfrak{g}$. Dominant weights $\lambda=\sum m_i\lambda_i$, $m_i \geq 0$, exactly index all finite dimensional irreducible representations of $\mathfrak{g}$ via the notion of highest weight. If $\mathfrak{g}$ is of type $\mathbf{X}$ and rank n, let $\mathbf{X}_n(\lambda)$ denote the finite dimensional irreducible representation with highest weight λ. We will number the fundamental weights and simple reflections for the simple Lie algebra of type $\mathbf{X}_n$ as on page 58 of [15].

If $w \in W$, let $l(w)$ be the length of the shortest expression for w in terms of the elementary reflections. Define the Bruhat order on W as follows: The unique minimal element is e. And $u \geq v$ iff there are reflections $t_1, \ldots, t_k$ such that $u=t_k \cdots t_1 v$ and $l(t_{i+1} \cdots t_1 v) > l(t_i \cdots t_1 v)$ for $1 \leq i < k$. If $J \subseteq \{1, 2, \ldots, n\}$, let W_J be the 'parabolic' subgroup of W generated by the corresponding simple reflections. Set $W^J = W/W_J$. Each coset in W_J has a unique representative in W of minimum length; we will usually think of W^J in terms of these representatives. In particular, the Bruhat order on W^J is defined by restricting the original order on W to the subset of coset representatives.

If $J \subseteq \{1, 2, \ldots, n\}$, set $J^c=\{1, 2, \ldots, n\}-J$. The Weyl groups of types $\mathbf{B}$ and $\mathbf{C}$ are identical. If W is of type $\mathbf{X} \in \{\mathbf{A}, \mathbf{BC}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}\}$ and of rank n, let $\mathbf{X}_n(J^c)$ denote the Bruhat order W^J. If $J^c=\{j\}$, let $\mathbf{X}_n(j)$ denote the order W^J.

A Weyl group is *irreducible* if it cannot be expressed as a direct product of smaller Weyl groups. An irreducible Weyl group must be of one of the types above. An irreducible Bruhat poset (or lattice) is a Bruhat poset defined on W^J with W irreducible. Analogous concepts can be defined for the other objects considered in this paper: Lie algebras, Gaussian posets, representations, and minuscule lattices and posets. Because there is complete reducibility in each case, we will often consider only the irreducible objects.

3. Bruhat Lattices

This section does not use any representation theory. Only facts concerning Weyl groups, root systems, and reflections are needed.

PROPOSITION 3.1. *The following is a list of all irreducible Bruhat posets which are lattices*: $\mathbf{A}_{n-1}(j)$, $1\leq j\leq n-1$, $\mathbf{BC}_n(1)$, $\mathbf{BC}_n(n)$, $\mathbf{D}_n(1)$, $\mathbf{D}_n(n-1)$, $\mathbf{D}_n(n)$, $\mathbf{E}_6(6)$, $\mathbf{E}_7(7)$, $\mathbf{G}_2(1)$, $\mathbf{G}_2(2)$.

PROPOSITION 3.2. *The Bruhat lattices have the following combinatorial descriptions*:

$$\mathbf{A}_{n-1}(j)\cong J(J(\underset{\sim}{j-1}\oplus\underset{\sim}{n-j-1}))$$

$$\mathbf{BC}_n(1)\cong \underset{\sim}{2n}$$

$$\mathbf{BC}_n(n)\cong\mathbf{D}_{n+1}(n+1)\cong\mathbf{D}_{n+1}(n)\cong J(J(J(\underset{\sim}{1}\oplus\underset{\sim}{n-2})))$$

$$\mathbf{D}_n(1)\cong J^{n-1}(\underset{\sim}{1}\oplus\underset{\sim}{1})$$

$$\mathbf{E}_6(1)\cong\mathbf{E}_6(6)\cong J^4(\underset{\sim}{1}\oplus\underset{\sim}{2})$$

$$\mathbf{E}_7(7)\cong J^5(\underset{\sim}{1}\oplus\underset{\sim}{2})$$

$$\mathbf{G}_2(1)\cong\mathbf{G}_2(2)\cong\underset{\sim}{6}.$$

Hence every Bruhat lattice is a distributive lattice.

The Bruhat lattices $\mathbf{E}_6(6)$ and $\mathbf{E}_7(7)$ are shown in Figure 1. [The numbers appearing in this figure are explained in (5) of Section 11.] Stanley denotes the lattices $\mathbf{A}_{n-1}(j)$ and $\mathbf{BC}_n(n)$ by $L(j, n-j)$ and $M(n)$ respectively [34]. See Section 11 for an explanation of the close relationship of these lattices via the J operation.

A proof of the following lemma appears in [25].

LEMMA 3.1. *Let W be a Weyl group and let $\lambda_1,\ldots,\lambda_n$ be the fundamental weights in the Euclidean space spanned by the root system associated to W. Let $\lambda=\sum m_i\lambda_i$ with $m_i\geq 0$, and set $J^c=\{\mathrm{i}: m_i>0\}$. Define P to be the poset consisting of the weights $w\lambda$, $w\in W$, with order generated by the relations $u\lambda>v\lambda$ if $v\lambda-u\lambda=k\alpha$, where α is a positive root and $k>0$. Then P is isomorphic to the Bruhat order W^J. The unique minimal element of P is λ.*

LEMMA 3.2. *If there is an element u of W^J such that $u\lambda=\sum_{i=1}^n r_i\lambda_i$ with $r_j=p>0$, $r_k=q>0$ and $(\alpha_j,\alpha_k)<0$, then W^J is not a lattice.*

PROOF For convenient (albeit imprecise) notation, refer to an element w of W^J with the jth and kth coordinates of $w\lambda$ with respect to the basis of fundamental weights, e.g. $u=(p,q)$. Suppose that $(\alpha_k,\alpha_k)=2(\alpha_j,\alpha_j)$. Then $u\lambda$ is covered by $s_ju\lambda=(-p,p+q)$ and $s_ku\lambda=(p+2q,-q)$. In turn, $s_ju\lambda$ is covered by $s_ks_ju\lambda=(p+2q,-p-q)$ and $s_ku\lambda$ is covered by $s_js_ku\lambda=(-p-2q,p+q)$. Now $l(s_ks_ju)=l(s_js_ku)=l(s_ju)+1=l(s_ku)+1$. But $(s_ks_js_ks_js_k)s_ju=s_js_ku$ and $(s_js_ks_js_ks_j)s_ku=s_ks_ju$. Hence both s_ju and s_ku are covered by both s_ks_ju and s_js_ku, implying that W^J is not a lattice. The cases $(\alpha_k,\alpha_k)=(\alpha_j,\alpha_j)$ and $(\alpha_k,\alpha_k)=3(\alpha_j,\alpha_j)$ are similar.

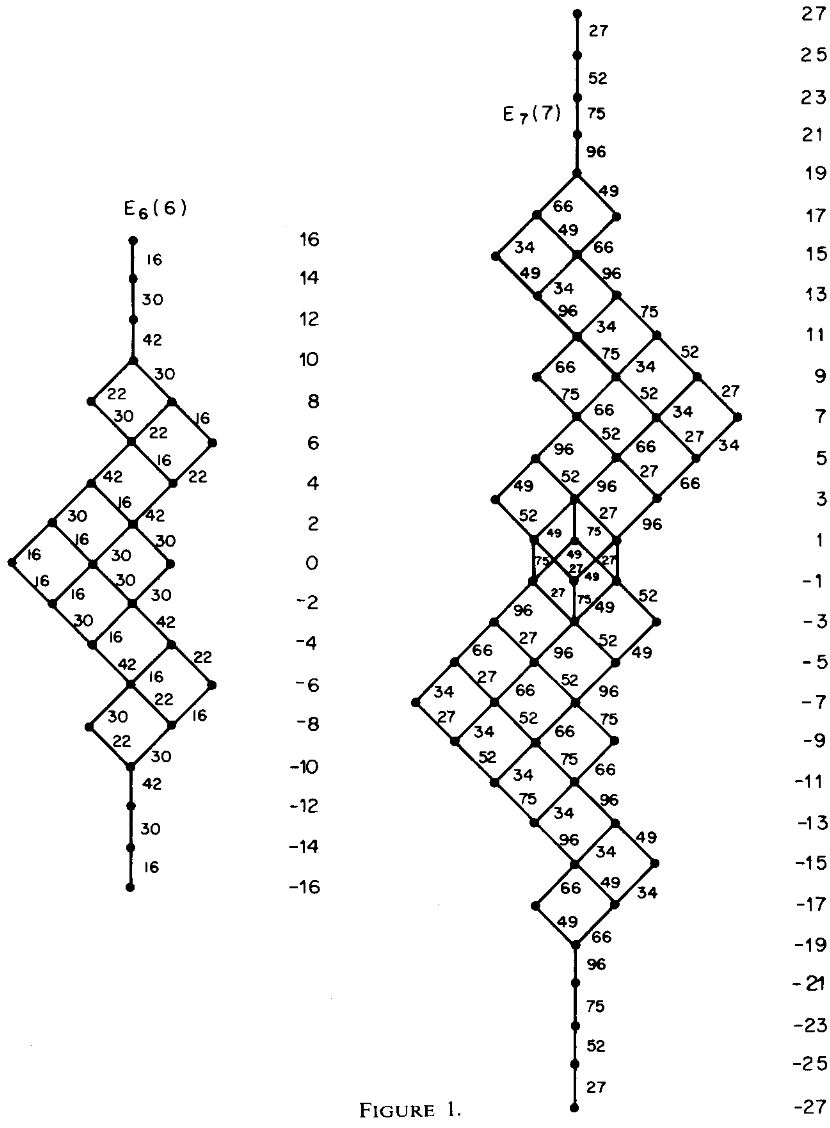

FIGURE 1.

PROOF OF PROPOSITION 3.1. Using tables [6, pp. 250-275], one may eliminate all other finite irreducible Bruhat posets in less than an hour with the following method. Take an X_n and J^c which do not appear in the list. Set $\lambda = \sum_{j\in J^c} \lambda_j$. Operate on λ with simple reflections until the situation of Lemma 3.2 is produced. That the posets listed are in fact lattices is shown in the next proof.

PROOF OF PROPOSITION 3.2. The classical cases A_{n-1}, BC_n, and D_n can be described with n-tuples of integers by combining Lemma 3.1 with the tables of [6] as in [25, theorems 4ABCD]. Elementary combinatorial manipulations complete the identification. The four exceptional cases can be drawn directly using Lemma 3.1 and tables; the posets $\mathsf{E}_6(6)$ and $\mathsf{E}_7(7)$ have 27 and 56 elements. Finally, it is well known that $J(P)$ is a distributive lattice for any poset P.

What about the Bruhat orders arising from finite Coxeter groups which are not Weyl groups? The posets $I_2^{(p)}(1)$ and $I_2^{(p)}(2)$ are obviously total orders with p elements. The posets $H_3(1)$ and $H_3(3)$ have 12 and 20 elements; one can readily draw them and see that they are distributive lattices with 11 and 13 ranks, respectively. Although $H_3(1) \cong J^5(\underset{\sim}{1} \oplus \underset{\sim}{1})$, the poset $j(H_3(3))$ is not a distributive lattice. The only other possibilities for distributive lattices are $H_4(1)$ and $H_4(4)$, with 120 and 600 elements, respectively.

4. Minuscule Representations, Lattices, and Posets

Definition Let ρ be a finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ. The representation ρ is a *minuscule representation* if every one of its weights is of the form $w\lambda$ for some $w \in W$.

Fact 3.1. [15, ex. 13.13]. The minuscule representations of complex simple Lie algebras are: $\mathbf{A}_{n-1}(\lambda_j)$, $1 \leq j \leq n-1$, $\mathbf{B}_n(\lambda_n)$, $\mathbf{C}_n(\lambda_1)$, $\mathbf{D}_n(\lambda_1)$, $\mathbf{D}_n(\lambda_{n-1})$, $\mathbf{D}_n(\lambda_n)$, $\mathbf{E}_6(\lambda_1)$, $\mathbf{E}_6(\lambda_6)$, $\mathbf{E}_7(\lambda_7)$.

Recall that a partial order is defined on the weights of any representation. We will reverse this order, namely: $\mu \geq \omega$ if and only if $\omega - \mu$ is a sum of positive roots.

Definition. An *irreducible minuscule lattice* is the set of weights of some minuscule representation of a simple Lie algebra, ordered as above.

The use of the word 'lattice' for these posets is justified by the following proposition.

Proposition 4.1. *Every irreducible minuscule lattice is an irreducible Bruhat lattice. In fact, the minuscule lattice corresponding to the minuscule representation $X_n(\lambda_j)$ is just the Bruhat lattice $X_n(j)$.*

Proof. Consider a minuscule lattice arising from a representation with highest weight λ. Let $W_J \subseteq W$ be the stabilizer of λ. All weights are of the form $u\lambda$, with $u \in W^J$. Suppose that $u\lambda > v\lambda$ under the order on weights. Then $u\lambda = v\lambda - \sum k_i\alpha_i$ with $k_i \geq 0$, $1 \leq i \leq n$. Now $\|v\lambda\| = \|u\lambda\|$ implies that $\langle v\lambda, \alpha_j\rangle > 0$ for some j. Lemma 3.1 of [25] then implies that $s_j v > v$ in W. Exercise 13.13 of [15] states that $\langle v\lambda, \alpha_j\rangle = +1$, 0, or -1, since λ corresponds to a minuscule representation. Thus $s_j v\lambda = v\lambda - \alpha_j$. Apply induction and Lemma 3.1 to conclude that $u > v$ in W^J. Conversely, by Lemma 3.1 of [25], $u > v$ implies $u\lambda > v\lambda$. Hence every minuscule lattice is just the corresponding Bruhat lattice.

On the other hand, $\mathbf{BC}_n(1)$ arises from $\mathbf{C}_n(\lambda_1)$, $\mathbf{BC}_n(n)$ arises from $\mathbf{B}_n(\lambda_n)$, and the other Bruhat lattices arise from the obvious minuscule representations except for $\mathbf{G}_1(1)$ and $\mathbf{G}_2(2)$. These are both 6 element total orders and hence are isomorphic to the minuscule lattices arising from $\mathbf{A}_5(\lambda_1)$ or $\mathbf{A}_5(\lambda_5)$ or $\mathbf{C}_3(\lambda_1)$.

The minuscule lattice arising from $\mathbf{X}_n(\lambda_j)$ can now be denoted by $\mathbf{X}_n(j)$. Since each minuscule lattice L is distributive, there is a unique poset P such that $L = J(P)$.

Definition. An *irreducible minuscule poset* is the partially ordered set of join irreducible elements of some irreducible minuscule lattice. The minuscule poset corresponding to $\mathbf{X}_n(j)$ will be denoted by $\mathbf{x}_n(j)$; i.e. $\mathbf{X}_n(j) = J(\mathbf{x}_n(j))$.

The following result is a consequence of Propositions 3.2 and 4.1.

Proposition 4.2. *The irreducible minuscule posets are*:

$$\mathbf{a}_{n-1}(j) \cong \underset{\sim}{j} \times \underset{\sim}{n-j} \cong \mathbf{A}_{j-1}(1) \times \mathbf{A}_{n-j-1}(1)$$

$$\mathbf{d}_n(n) \cong J(\underset{\sim}{2} \times \underset{\sim}{n-2}) \cong \mathbf{A}_{n-1}(2)$$

$$\begin{aligned}
\mathbf{e}_6(6) &\cong J^2(\underset{\sim}{2}\times\underset{\sim}{3}) \cong \mathbf{D}_5(5)\\
\mathbf{e}_7(7) &\cong J^3(\underset{\sim}{2}\times\underset{\sim}{3}) \cong \mathbf{E}_6(6)\\
\mathbf{d}_n(1) &\cong J^{n-3}(\underset{\sim}{2}\times\underset{\sim}{2})\\
\mathbf{b}_{n-1}(n-1) &\cong \mathbf{d}_n(n-1)\cong \mathbf{d}_n(n)\\
\mathbf{c}_n(1) &\cong \underset{\sim}{2n-2}\\
\mathbf{e}_6(1) &\cong \mathbf{e}_6(6).
\end{aligned}$$

The minuscule posets are shown in [27].

Just as one can more generally consider semisimple Lie algebras instead of just simple Lie algebras, one can also consider not necessarily irreducible minuscule lattices and minuscule posets. An arbitrary minuscule lattice is a direct product of irreducible ones, and an arbitrary minuscule poset is a direct sum of irreducible ones.

5. Principal Three-Dimensional Subalgebras

The proof of Proposition 4.1 shows that if $v\lambda$ is covered by $u\lambda$ in a minuscule lattice W^J, then $v\lambda - u\lambda = \alpha_j$, viz. some positive simple root. Hence if $u\lambda = \sum k_i\alpha_i$, then the poset rank of $u\lambda$ in $W\lambda$ is given by $\sum k_i$ up to an additive constant. Let $\delta^\vee$ denote the unique element of $H_{\mathbb{R}}^*$ such that $(\alpha_i, \delta^\vee) = 1$ for $1 \leq i \leq n$. It is easy to show that $\delta^\vee = \sum 2\lambda_i/(\alpha_i, \alpha_i)$. If w_0 is the unique element of the Weyl group which takes every positive root to a negative root [15, ex. 19.9], then by Lemma 3.1, $w_0\lambda$ is the unique maximal element of $W\lambda$. Since positive simple roots must pass to negative simple roots, $(w_0\lambda, \delta^\vee) = -(\lambda, \delta^\vee)$. Hence $W\lambda$ has $2(\lambda, \delta^\vee)+1$ ranks as a ranked poset.

There is a unique element h in the Cartan subalgebra of any simple Lie algebra $\mathfrak{g}$ such that $\rho(h)v = -(\mu, \delta^\vee)v$ for any representation ρ and weight vector v of weight μ for ρ. Let $\{x_i, y_i, h_i\}_{i=1}^n$ generate $\mathfrak{g}$ as in [15, pp. 37, 112]. In particular, $\lambda_i(h_j) = \delta_{ij}$. Then

$$h = -\sum_{i=1}^n \left[\sum_{j=1}^n \frac{2(\lambda_i, \lambda_j)}{(\alpha_j, \alpha_j)}\right] h_i.$$

(This is confirmed by applying λ_k to both sides.)

The weight basis for the representation space of a minuscule representation of highest weight λ is in one-to-one correspondence with the weights of the representation and hence also with the elements of the corresponding minuscule lattice. To determine the poset rank of an element of the lattice, act on the corresponding weight vector with h and add $(\lambda, \delta^\vee)$ to the eigenvalue observed.

Definitions. Set

$$x = \sum_{i=1}^n y_i$$

and

$$y = \sum_{i=1}^n \left[\sum_{j=1}^n \frac{2(\lambda_i, \lambda_j)}{(\alpha_j, \alpha_j)}\right] x_i.$$

Then $[h, x] = x$, $[h, y] = -y$, and $[x, y] = h$ and these three elements span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

This subalgebra is a *principal three-dimensional subalgebra* [17, p. 996]. (We choose h corresponding to $-\delta^\vee$ rather than $\delta^\vee$ and interchange x and y so that the minimal element of the Bruhat order λ will have minimal weight with respect to h.)

Any representation of $\mathfrak{g}$ produces a representation of $\mathfrak{sl}(2, \mathbb{C})$ via this embedding. Principal three dimensional subalgebras have been employed previously in combinatorics [14], [18] and [35].

6. Minuscule Posets are Gaussian

This section and the next section represent joint work with R. Stanley.

The following lemma is a special case of an important recent result in algebraic geometry and representation theory [21], [29].

LEMMA 6. (SESHADRI). *Let $\mathbf{X}_n(\lambda_j)$ be a minuscule representation of a simple Lie algebra. Then there is a weight basis for the representation $\mathbf{X}_n(m\lambda_j)$ which is indexed in a natural fashion by m-multichains in the minuscule lattice $\mathbf{X}_n(j)$.* The weight of a basis vector for $\mathbf{X}_n(m\lambda_j)$ *corresponding to $\mu_1 \leq \cdots \leq \mu_m$ in $\mathbf{X}_n(j)$ is $\mu_1 + \cdots + \mu_m$.*

Let $x_1, \ldots, x_n$ be generating function indeterminants. If $\mu \in H^*_{\mathbb{R}}$ and $\mu = (\mu_1, \ldots, \mu_n)$ with respect to some basis (usually the root basis), set $\underset{\sim}{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.

MULTIVARIATE WEYL DEGREE FORMULA. Let $\Pi(\lambda)$ be the set of weights of a finite dimensional irreducible representation of a complex semisimple Lie algebra with highest weight λ. Let $d(\mu)$ be the dimension of the weight space of weight μ. Then

$$\sum_{\mu \in \Pi(\lambda)} d(\mu) \underset{\sim}{x}^\mu = \frac{\sum_{w \in W} (-1)^{l(w)} \underset{\sim}{x}^{w(\lambda+\delta)-\delta}}{\prod_{\alpha \in \Phi^+} (1 - \underset{\sim}{x}^{-\alpha})},$$

where $\delta = \sum_1^n \lambda_i$.

The following one variable form of the Weyl degree formula was first used in combinatorics by Lepowsky [18, p. 180].

PRINCIPAL SPECIALIZATION OF WEYL'S FORMULA. Now consider the representation of highest weight λ restricted to the principal three-dimensional subalgebra defined in Section 5. Let $d(i)$ be the dimension of the weight space of weight i with respect to h. Then

$$x^{(\lambda,\delta^\vee)} \sum d(i) x^i = x^{(\lambda,\delta^\vee)} \sum_{\mu \in \Pi(\lambda)} d(\mu) x^{-(\mu,\delta^\vee)} = \frac{\prod_{\alpha \in \Phi^+} (1 - x^{(\lambda+\delta,\alpha)})}{\prod_{\alpha \in \Phi^+} (1 - x^{(\delta,\alpha)})}.$$

PROOF. Weight vectors of weight μ for the representation of $\mathfrak{g}$ have weights $-(\mu, \delta^\vee)$ with respect to h. Modify Jacobson's derivation of the total degree formula: Set $Z_\varphi(\underset{\sim}{x}^\mu) = x^{(\mu,\varphi)}$. Then [16, p. 256]

$$Z_\varphi\left[\sum_{w \in W} (-1)^{l(w)} \underset{\sim}{x}^{w\mu}\right] = Z_\mu\left[\sum_{w \in W} (-1)^{l(w)} \underset{\sim}{x}^{w\varphi}\right].$$

Apply $Z_{-\delta^\vee}$ to both sides of the multivariate formula, use the above rule in the numerator, and factor $\sum_{w \in W} (-1)^{l(w)} \underset{\sim}{x}^{\delta^\vee} = x^{-\delta^\vee} \prod_{\alpha \in \Phi^+} (1 - \underset{\sim}{x}^{\alpha^\vee})$ [15, p. 138], where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

The following theorem is a new result for the exceptional cases $\mathbf{e}_6(6)$ and $\mathbf{e}_7(7)$.

THEOREM 6. *Every minuscule poset is a Gaussian poset. In fact, if an irreducible minuscule poset P has ranks $P_1, P_2, \ldots, P_r$,* then

$$F(P, m, x) = \frac{\prod_{\alpha \in P} (1 - x^{m+r(\alpha)})}{\prod_{\alpha \in P} (1 - x^{r(\alpha)})},$$

where $r(\alpha) = i$ when $\alpha \in P_i$.

PROOF. Suppose P arises from the minuscule lattice L corresponding to a minuscule representation $X_n(\lambda_j)$. Consider the representation $X_n(m\lambda_j)$. An m-flag of order ideals in P of weight N corresponds to an m-multichain in L of weight N. By Lemma 6, the number of these is equal to the dimension of the weight space of weight $-(\mu, \delta^\vee) = N - (m\lambda_j, \delta^\vee)$ with respect to h. Thus $F(P, m, x)$ is obtained by choosing $\lambda = m\lambda_j$ in the right-hand side of the principal specialization of Weyl's formula. Now use tables. Cancel factors when $\langle \lambda_j, \alpha \rangle = 0$. Now $\langle \lambda_j, \alpha \rangle = 1$ for all other positive roots [15, ex. 13.13]. And note by direct calculation that the exponents $\langle \delta, \alpha \rangle$ are as claimed for these roots in each case. Arbitrary minuscule posets are Gaussian because $F(P \oplus Q, m, x) = F(P, m, x) F(Q, m, x)$.

7. PLANE PARTITION GENERATING FUNCTION IDENTITIES

Proposition 7.1 below is originally due to MacMahon [23]. Proposition 7.2 was conjectured by Bender and Knuth [3] and proved by Gordon [11] and Andrews [1]. Proposition 7.3 was conjectured by MacMahon and proved by Andrews [1]. Our proofs of Propositions 7.1 and 7.2 are closely related to MacDonald's [22, exs. I.5.13(b) and I.5.19], but were found independently. Our proof of Proposition 7.3 follows MacDonald's [22, ex. I.5.17], but is cast in the present context of representations of Lie algebras.

PROPOSITION 7.1. *Let $\underset{\sim}{\Lambda} = (q, \ldots, q)$ be a p-tuple. Let $G(\underset{\sim}{\Lambda}, m, x)$ be the generating function for plane partitions of shape contained in $\underset{\sim}{\Lambda}$ with parts $\leq m$. Then*

$$G(\underset{\sim}{\Lambda}, m, x) = \frac{\prod_1^p \prod_1^q (1 - x^{m+i+j-1})}{\prod_1^p \prod_1^q (1 - x^{i+j-1})}.$$

A plane partition of shape contained in $\underset{\sim}{\Lambda}$ is *column strict* if $P_{i,j} > P_{i+1,j}$ whenever $P_{i+1,j} > 0$.

PROPOSITION 7.2. *Let $\underset{\sim}{\Lambda} = (m, \ldots, m)$ be an n-tuple. Let $G'(\underset{\sim}{\Lambda}, n, x)$ be the generating function for column strict plane partitions of shape contained in $\underset{\sim}{\Lambda}$ with parts $\leq n$. Then*

$$G'(\underset{\sim}{\Lambda}, n, x) = \frac{\prod_1^n \prod_1^i (1 - x^{m+i+j-1})}{\prod_1^n \prod_1^i (1 - x^{i+j-1})}.$$

A plane partition of shape contained in $\underset{\sim}{\Lambda}$ is *symmetric* if $P_{i,j} = P_{j,i}$.

PROPOSITION 7.3. *Let $\underset{\sim}{\Lambda}=(n,\ldots,n)$ be an n-tuple. Let $G^*(\underset{\sim}{\Lambda},m,x)$ be the generating function for symmetric plane partitions of shape contained in $\underset{\sim}{\Lambda}$ with parts $\leq m$. Then*

$$G^*(\underset{\sim}{\Lambda},m,x)=\frac{\prod(1-x^{m+2i-1})\prod_{i<j}(1-x^{2(m+i+j-1)})}{\prod(1-x^{2i-1})\prod_{i<j}(1-x^{2(i+j-1)})}.$$

PROOFS OF PROPOSITIONS 7.1 AND 7.2. The plane partitions of Proposition 7.1 are m-flags of ideals in the minuscule poset $\underset{\sim}{p}\times\underset{\sim}{q}$. The plane partition weight is the same as the m-flag weight. Use the proof of Theorem 6 in the case $\mathbf{a}_{p+q-1}(p)$.

Now consider the minuscule poset $J(\underset{\sim}{2}\times\underset{\sim}{n-1})$. One can describe an order ideal $I_{\underset{\sim}{a}}$ of this poset with the n-tuple $\underset{\sim}{a}=(\mathbf{a}_1,\ldots,\mathbf{a}_n)$, where $n\geq\mathbf{a}_1>\cdots>\mathbf{a}_k>\mathbf{a}_{k+1}=\cdots=\mathbf{a}_n=0$ for some k between 0 and n. Then $I_{\underset{\sim}{a}}\supseteq I_{\underset{\sim}{b}}$ iff $\mathbf{a}_1\geq\mathbf{b}_1,\ldots,\mathbf{a}_n\geq\mathbf{b}_n$. Hence the plane partitions of Proposition 7.2 correspond to m-flags of ideals in $\mathbf{b}_n(n)$. The plane partition weight and the m-flag weight are the same. Use the proof of Theorem 6.

The multivariant analogs of Propositions 7.1 and 7.2 can also be obtained with these methods. The plane partitions of Proposition 7.1 correspond 1-1 with column strict plane partitions of shape $(p,\ldots,p)$ (m-tuple) with parts ≥ 1 and $\leq p+q$. These partitions are now analogous to those of Proposition 7.2: In both cases each column corresponds to an element of the minuscule lattice at hand. In fact, Propositions 4ABC of [25] enable one to read off the multivariant weight of the corresponding weight of the minuscule representation. If $\underset{\sim}{a}$ is an n-tuple as in the proof of Proposition 7.2, let $d_i=|\{\mathbf{a}_j:\mathbf{a}_j\geq n-i+1\}|$ for $1\leq i\leq n$. Then the corresponding weight of $\mathbf{B}_n(\lambda_n)$ is $\lambda_n-\sum d_i\alpha_i$, where the n-tuple $(0,\ldots,0)$ corresponds to λ_n. By Lemma 6, there is a basis of weight vectors for the representation $\mathbf{B}_n(m\lambda_n)$ which is indexed by the plane partitions of Proposition 7.2. Let $m\lambda_n-\beta$ be the weight of $\mathbf{B}_n(m\lambda_n)$ which corresponds to a plane partition P. Choose a basis $\{\varepsilon_i\}$ for $H^*_{\mathbb{R}}$ such that $\alpha_n=-\varepsilon_1$, $\alpha_{n-1}=\varepsilon_1-\varepsilon_2,\ldots,\alpha_1=\varepsilon_{n-1}-\varepsilon_n$. Now $m\lambda_n-\beta$ is the sum of the weights corresponding to the columns of P, and one can show that $-\beta=\sum f_i\varepsilon_i$ iff P has f_i parts equal to i. Set $\lambda=m\lambda_n=-(m/2)\sum\varepsilon_i$ and $\delta=-\frac{1}{2}\sum(2i-1)\varepsilon_i$ in the multivariate Weyl formula. Express the numerator and denominator as determinants and multiply both sides by $\underset{\sim}{x}^{-\lambda}$ to obtain MacDonald's expression [22, ex. I.5.16(2')]:

$$G'(\underset{\sim}{\Lambda},n,x_1,\ldots,x_n)=\frac{\det(x_j^{m+2n-i}-x_j^{i-1})}{\det(x_j^{2n-i}-x_j^{i-1})}$$
$$=\frac{\det(x_j^{j-i}(1-x_j^{m+2i-1}))}{\prod(1-x_i)\prod_{i<j}(1-x_ix_j)(1-x_i^{-1}x_j))}.$$

In the case of simple Lie algebras of type **A**, there is a well known more general version of Lemma 6 [5, theorem 5.3] [21]. Weight basis vectors for the representation $\mathbf{A}_{n-1}(\sum_1^{n-1}m_i\lambda_i)$ are indexed by column strict plane partitions with parts $\leq n$ and of exactly shape $\underset{\sim}{\Lambda}$, where $\underset{\sim}{\Lambda}$ has m_{n-1} columns of length $n-1,\ldots,m_1$ columns of length 1. This result can be combined with the specialized Weyl formula to obtain a quotient of two products form for the generating function of these plane partitions. But note that the denominator and numerator obtained in this manner are different from the hook length denominator and content numerator of Stanley [32, theorem 10.1].

PROOF OF PROPOSITION 7.3. View the plane partitions of Proposition 7.2 as solid partitions in an $n\times m\times n$ box. By reflecting these partitions about the line $x=z$, one can see that these partitions correspond 1-1 to the partitions of Proposition 7.3. If the original partition had f_i parts equal to i, then the corresponding symmetric partition has weight $\sum f_i(2i-1)$. Take the multivariate Weyl formula for $\mathbf{B}_n(m\lambda_n)$, multiply both sides by $\underset{\sim}{x}^{-m\lambda_n}$, and act on both sides with $Z_{-2\delta}$ instead of $Z_{-\delta^\vee}$. Since $-2\delta=\sum(2i-1)\varepsilon_i$, we have $(-\beta, -2\delta)=(\sum f_i\varepsilon_i, \sum(2i-1)\varepsilon_i)=\sum f_i(2i-1)$ for the partition corresponding to the weight $m\lambda_n-\beta$ and having f_i parts equal to i. Hence the left hand side is the generating function of Proposition 7.3. Use the same trick to factor the numerator as was used in the principal specialization of the Weyl formula. The final result is obtained after cancelling the same factors as in the proof of Proposition 7.2.

We know of no other dominant vectors η for which $\sum_{w\in W}(-1)^{l(w)}\underset{\sim}{x}^{w\eta}$ factors into a product besides $\eta=\delta$ and $\eta=\delta^\vee$. Since $\delta^\vee$ is δ for the dual root system, and since the root systems $\mathbf{A}_{n-1}$ and $\mathbf{D}_n$ are self-dual, there do not appear to be any other places to use this trick.

Despite their elegant forms, there are currently no completely satisfactory proofs of these identities. The original proofs consisted of manipulation of generating functions. Remmel and Whitney [28] have recently produced a bijective combinatorial proof of Stanley's hook length/content formula for column strict partitions, but they cannot describe their bijection in closed form. On the other hand, the proofs presented here have more of a combinatorial flavor than may be readily apparent. Verma [37] has shown that the multivariate Weyl formula can be interpreted as inclusion/exclusion over the Bruhat order (on the Weyl group) of vector spaces. Rota *et al.* [8] have provided a proof of Lemma 6 in case $\mathbf{A}_{n-1}$ which uses combinatorial reasoning for key steps. Perhaps it might be possible to combine these methods to provide a nice direct linear combinatorial proof of the identities, where the plane partitions would describe certain words subject to linear relations. More optimistically, one might seek to directly interpret the multivariate Weyl formula as inclusion/exclusion on $|w|$ infinite families of combinatorial objects counted by the factored denominator in the usual generating function manner. See also Lemma 3 of [9].

8. Combinatorial Proofs of Gaussianess

Seshadri remarked [29, pp. 207, 237-238] that his proof of Lemma 6 could be made more direct if he could show that the total number of 2-multichains in the minuscule lattice $X_n(j)$ was equal to the total dimension of $X_n(2\lambda_j)$. Instead, he used the Cohen–Macaulayness of the homogeneous coordinate ring of the associated flag manifold G/P_j, a result due to Demazure [7]. The identities of Theorem 6 are finer versions of the desired equalities for arbitrary m, instead of $m=2$. In Section 6 we proved these identities using Lemma 6. In this section we will reverse this process by providing or quoting proofs of the identities which use no representation theory. (In a later paper, Seshadri *et al.* were able to avoid using Demazure's result by proving the desired equality for the total dimension in the case $m=2$ with induction inside the minuscule lattice.)

The second part of this theorem is a restatement of the second part of Theorem 6. But the proof here uses only combinatorics and generating function manipulations.

Theorem 8. *Let $X_n(\lambda_j)$ be a minuscule representation of a simple Lie algebra. Then the principal graded dimensions of the representation $X_n(m\lambda_j)$ are equal to the rank weighted numbers of m-multichains in $X_n(j)$. Equivalently, the generating function for m-flags of order ideals in the minuscule poset $x_n(j)$ is as described in Theorem 6.*

Proof. The two statements are equivalent because m-multichains in the minuscule lattice correspond to m-flags of order ideals in the minuscule poset, and because, as was noted in the proof of Theorem 6, one can verify with Weyl's formula and tables that the desired dimensions are counted by the right-hand sides of the identities given in Theorem 6. We want to prove these identities without using representation theory.

As should be clear from Section 7, the desired identities for $\mathbf{a}_{n-1}(j)$ and $\mathbf{b}_n(n)$ ($\cong \mathbf{d}_{n+1}(n) \cong \mathbf{d}_{n+1}(n+1)$) are already known as plane partition generating function identities. And note that $\mathbf{c}_n(1) \cong \mathbf{a}_{2n-1}(1)$. We now use counting techniques from Stanley's thesis (published as [31]) to prove the identities for the remaining cases $\mathbf{d}_n(1)$, $\mathbf{e}_6(6)$ ($\cong \mathbf{e}_6(1)$), and $\mathbf{e}_7(7)$.

Let P be a poset with p elements. Fix a labelling of the elements of P with the integers $1, 2, \ldots, p$ which respects the order on P. For each order preserving bijection $f: P \to \underset{\sim}{p}$, consider the permutation $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(p)$. A *descent* of this permutation is an index i such that $f^{-1}(i) > f^{-1}(i+1)$. The *index* of this permutation is $\mathrm{ind}(f^{-1}) = \sum i$, where the sum is over all descents. Define p polynomials associated to P:

$$W_s(x) := \sum x^{\mathrm{ind}(f^{-1})}, \qquad 0 \leq s \leq p-1,$$

where the sum is over all order preserving bijections f such that f^{-1} has exactly s descents.

NOTATION. $[k] := (1 - x^k)$, $[k]! := [k][k-1] \cdots [1]$

$$[k]_i := \frac{[k]!}{[k-i]!}, \quad \begin{bmatrix} k \\ i \end{bmatrix} := \frac{[k]!}{[i]![k-i]!}$$

LEMMA 8. (Stanley [31, p. 24]). *For any poset P with p elements,*

$$F(P, m, x) = \sum_{s=0}^{p-1} W_s(x) \begin{bmatrix} p+m-s \\ p \end{bmatrix}.$$

Consider $P = d_n(1) \cong J^{n-2}(\underset{\sim}{1} \oplus \underset{\sim}{1})$. It is very easy to see that $W_0(x) = 1$, $W_1(x) = x^{n-1}$, and $W_s(x) = 0$ for $s \geq 2$. Hence

$$F(P, m, x) = \begin{bmatrix} m+2n-2 \\ 2n-2 \end{bmatrix} + x^{n-1} \begin{bmatrix} m+2n-3 \\ 2n-2 \end{bmatrix}.$$

Very easy manipulation transforms this to

$$F(P, m, x) = \begin{bmatrix} m+2n-3 \\ 2n-3 \end{bmatrix} \frac{[m+n-1]}{[n-1]},$$

the desired result. [Stanley was aware of this result in 1970 (unpublished).]

Now consider the two new Gaussian posets $\mathbf{e}_6(6)$ and $\mathbf{e}_7(7)$. The first one has 78 extensions to a total order; the second one has 13,110. Tabulating descents by hand for $\mathbf{e}_6(6)$ one obtains $W_0(x) = 1_0$, $W_1(x) = 1_4 11121111$, $W_2(x) = 1_{10}122334332211$, $W_3(x) = 1_{18}122334332211$, and $W_5(x) = 1_{28}11121111$, where the subscript indicates the lowest power of x with non-zero coefficient. Tabulating descents by computer for $\mathrm{e}_7(7)$ one obtains $W_0(x) = 1$, $W_1(x) = 1_5 11122222222221111, \ldots$, $W_5(x) = 2_{45}$, 2, 4, 6, 10, 14, 20, …, 202_{66}, 204_{67}, 204_{67}, 202_{68}, …, 20, 14, 10, 6, 4, 2, 2_{90}, …, $W_9(x) = x^{108} W_1(x)$, $W_{10}(x) = 1_{135}$. (For all i, $W_{10-i}(x) = x^{135} W_i(1/x)$, as might be expected from the self-duality of $\mathbf{e}_7(7)$. However, the coefficients of the $W_s(x)$ are also unimodal and symmetric about their middle non-zero powers.) Set $y = x^m$. The proof for $\mathbf{e}_6(6)$ is complete if one can show

$$\{W_0(x)[m+16]_5 + W_1(x)[m+15]_4[m] + W_2(x)[m+14]_3[m]_2 + \cdots + W_5(x)[m]_5\}[8]_5 = [m+8]_5[16]_5$$

is a polynomial identity in the two variables x and y. Both this and an analogous identity for $\mathbf{e}_7(7)$ were verified by computer computations. After collecting like terms, either side of the above identity for $\mathbf{e}_6(6)$ was

$$1-\cdots+16x^{40}y^2+\cdots-16x^{54}y^2-\cdots-16x^{46}y^3-\cdots+16x^{60}y^3+\cdots+x^{100}y^5$$

(276 non-zero terms altogether). Either side of the analogous identity for $\mathbf{e}_7(7)$ had 2442 non-zero terms, four of which were

$$1-\cdots+3912x^{157}y^5+3912x^{158}y^5+\cdots+x^{315}y^{10}.$$

The proof of Theorem 8 is complete.

9. The Gaussian Poset Conjecture

A poset is *connected* if its Hasse diagram is a connected graph. This notion coincides with the notion of irreducibility for minuscule posets.

The minuscule posets are the only known Gaussian posets. The minuscule posets can be characterized in several ways (see Section 11.) Furthermore, they are indexed by an ubiquitous family of Dynkin diagrams with special node. Hence it is natural to make the following conjecture:

Conjecture. Any connected Gaussian poset is one of the following posets: $\underset{\sim}{p}\times\underset{\sim}{q}$, $J(\underset{\sim}{2}\times\underset{\sim}{n})$, $J^r(\underset{\sim}{2}\times\underset{\sim}{2})$, $J^2(\underset{\sim}{2}\times\underset{\sim}{3})$, $J^3(\underset{\sim}{2}\times\underset{\sim}{3})$. Any Gaussian poset is a disjoint union of these posets.

I am indebted to Richard Stanley and Philip Hanlon for permitting me to report their following partial results (personal communications) which have been deduced from facts in [31]. Stanley has shown:

(1) Every connected component of a Gaussian poset is Gaussian. (The converse is obvious.)
(2) All maximal chains in a connected Gaussian poset have the same length.
(3) The dual of a Gaussian poset is Gaussian.
(4) If a Gaussian poset has ranks $P_1, P_2, \ldots, P_r$ with $|P_i| = p_i$, then

$$F(P, m, x) = \frac{\prod_1^r (1-x^{m+i})^{p_i}}{\prod_1^r (1-x^i)^{p_i}}.$$

(5) Any element of a Gaussian poset covers at most one minimal element.

Hanlon has shown:

(1) If P is a connected Gaussian poset, then $p_i = p_{r-i+1}$.
(2) Furthermore, $p_i \leq p_{i+1} \leq p_i + p_1$, where $1 \leq i < r/2$.

Hanlon has also used direct computation in conjunction with the above results to confirm the conjecture for connected posets with exactly one minimal element and 8 or fewer ranks or 25 or fewer elements.

10. Minuscule Lattices are Strongly Sperner

Stanley has shown that all Bruhat posets arising from Weyl groups are rank symmetric, rank unimodal, and strongly Sperner [34]. His proof combined a linear algebra/combinatorial technique with the hard Lefschetz theorem of algebraic geometry. We will reproduce this result here in the special case of Bruhat lattices (i.e. minuscule lattices) by replacing the use of the hard Lefschetz theorem with the machinery developed in Sections 4 and 5 for the proof of Gaussianess.

Associate to any ranked poset

$$P = \bigcup_{i=0}^{r} P_i$$

a graded complex vector space

$$\tilde{P} = \bigoplus_{i=0}^{r} \tilde{P}_i,$$

where $\tilde{P}_i$ is the complex vector space freely generated by vectors $\tilde{\mathbf{a}}$ corresponding to elements of P_i. A linear operator X on $\tilde{P}$ is a *lowering operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i-1}$. It is a *raising operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i+1}$. A raising operator defined by

$$X\tilde{\mathbf{a}} = \sum \theta(\mathbf{a}, \mathbf{b})\tilde{\mathbf{b}}$$

is an *order raising operator* if $\theta(\mathbf{a}, \mathbf{b}) \neq 0$ implies $\mathbf{b}$ covers $\mathbf{a}$. Define a linear operator H on $\tilde{P}$ by

$$H\tilde{\mathbf{a}} = \left(i - \frac{r}{2}\right)\tilde{\mathbf{a}}$$

when $\mathbf{a} \in P_i$. The poset P *carries a representation of* $\mathbf{sl}(2, \mathbb{C})$ if there exist a lowering operator Y and an order raising operator X on $\tilde{P}$ such that $XY - YX = H$.

The following lemma is Theorem 1 of [24]; it incorporates the linear algebra/combinatorial technique Lemma 1.1 of [34].

LEMMA 10. *A ranked poset is rank symmetric, rank unimodal, and strongly Sperner if and only if it carries a representation of* $\mathfrak{sl}(2, \mathbb{C})$.

THEOREM 10. *Minuscule lattices are rank symmetric, rank unimodal, and strongly Sperner.*

PROOF. Compose the corresponding minuscule representation of a Lie algebra $\mathfrak{g}$ with the principal embedding of $\mathfrak{sl}(2, \mathbb{C})$ defined in Section 5. Let X, Y, and H be the images of x, y, and h, and choose the usual weight basis for the minuscule representation. Since all weight spaces are one-dimensional, we can use the weights themselves to denote the weight vectors. From [15, p. 107], it is clear that $X\mu = \sum \theta(\mu, \nu)\nu$, $\theta(\mu, \nu) \neq 0$, only when $\nu = \mu - \alpha$ for some simple root α, viz. (by the definition of minuscule lattice) when ν covers μ in the lattice. The operator Y is a lowering operator for the minuscule lattice, and the operator H has the correct eigenvalues. Apply the lemma.

See (5) of Section 11 and the last paragraph of Section 12 for further comments.

11. THE UBIQUITY OF MINUSCULE POSETS

Minuscule posets can be characterized in several ways:

(1) (Definition) Posets of join irreducible elements of minuscule lattices.

The Dynkin diagrams with special node which index the minuscule lattices or representations can be described in several ways with root system language [15, ex. 13.13]. These diagrams with special node index various other objects, including all flag manifolds with easily described Schubert calculus (Section 12 and [12]), and all hermitian symmetric spaces [40, p. 289] [in this case $\mathbf{B}_n(n)$ and $\mathbf{C}_n(1)$ are replaced by $\mathbf{B}_n(1)$ and $\mathbf{C}_n(n)$].

(2) Posets of join irreducibles of all Bruhat posets which are lattices.
(3) All direct sums of posets of the form $J^r(\underset{\sim}{p} \oplus \underset{\sim}{q})$ where (p, q, r) is in the order ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ described by $r \leq 1$ or $pq \leq 1$ or $pr \leq 2$ or $qr \leq 2$ or ($pq \leq 2$ and $r \leq 4$).
(4) All known Gaussian posets.
(5) All posets P upon which a complex-valued function f can be defined satisfying

$$\sum_{\substack{\text{maximal elements}\\ \text{of } I}} f(x) - |I| = \sum_{\substack{\text{minimal elements}\\ \text{of } P-I}} f(y) - |P - I|$$

for every order ideal $I \subseteq P$ [27].

If such a function f exists, then $J(P)$ carries a representation of $\mathfrak{sl}(2, \mathbb{C})$ and is therefore rank symmetric, rank unimodal, and strongly Sperner. Here the coefficients for the order raising operator X are all unity and the order lowering operator Y has coefficients described by f. The coefficients are shown in Figure 1 for $\mathbf{E}_6(6)$ and $\mathbf{E}_7(7)$, where the coefficients for H appear off to the right.
(6) Robert Steinberg has observed empirically (personal communication) that any minuscule lattice $\mathbf{X}_n(j)$ is isomorphic to $J(U_j^-)$, where $U_j^- := \{\alpha^\vee \in \Phi^{\vee -} : (\lambda_j, \alpha) < 0\}$, a subset of the negative dual roots, is given the reverse of the usual ordering by simple dual roots.

We will now outline an á priori proof of this fact, thus providing a root theoretic interpretation of the combinatorial operation J in this context. Recall that $\mathbf{X}_n(j) := W^{\langle j \rangle}$, where $\langle j \rangle = \{1, \ldots, \hat{j}, \ldots, n\}$. The term *k-path* will refer to a path in $W^{\langle j \rangle}$ consisting of $k+1$ elements beginning with the identity. The term k-flag will refer to a sequence of k order ideals $I_1 \subset \cdots \subset I_k$ in U_j^- such that $|I_i| = i$.

THEOREM 11. *Let λ_j be a minuscule weight for a simple root system of type $\mathbf{X}_n$. Then $\mathbf{X}_n(j) \cong J(U_j^-)$. In fact, there is a map $F: U_j^- \to \Delta$ such that the correspondence above is given by $I \leftrightarrow \lambda_j - \sum_{\alpha \in I} F(\alpha)$, where $I \subseteq U_j^-$ is an order ideal.*

LEMMA 11.1. *If $\alpha < \beta$ in U_j^- and $w \in W^{\langle j \rangle}$, then $w\alpha < w\beta$ in Φ.*

LEMMA 11.2. *Let $s_{i_k} \cdots s_{i_1}$ be a k-path in $W^{\langle j \rangle}$ and let $I_j = I_{j-1} \cup \{-(s_{i_{j-1}} \cdots s_{i_1})^{-1} \alpha_{i_j}\}$. Then $I_1 \subset \cdots \subset I_k$ is a k-flag in U_j^-. This process defines a bijection between all k-paths in $W^{\langle j \rangle}$ and all k-flags in U_j^-.*

LEMMA 11.3. *Two k-flags corresponding respectively to two k-paths end at the same order ideal in U_j^- if and only if the two k-paths end at the same element in $W^{\langle j \rangle}$. Hence order ideals in U_j^- correspond to elements of $W^{\langle j \rangle}$.*

Given $\alpha \in U_j^-$ *let* $w \in W^{\langle j \rangle}$ correspond to the order ideal $\{\beta : \beta < \alpha\}$. Then $w\alpha = -\alpha_i$ for some $1 \leq i \leq n$. Set $F(\alpha) = \alpha_i$.

LEMMA 11.4. *If θ covers μ and ν, and μ and ν cover ω in $X_n(j)$, then $\omega - \mu = \alpha_i$ implies $\nu - \theta = \alpha_i$.*

PROOF OF THEOREM. Lemmas 11.2 and 11.3 show that $\mathbf{X}_n(j) \cong J(U_j^-)$. Parallel edges in $J(U_j^-)$ correspond to the same element of U_j^-. By Lemma 11.4, parallel edges in $\mathbf{X}_n(j)$ represent subtraction of the same simple root. Therefore the map F can be used to describe the correspondence between order ideals and elements of $\mathbf{X}_n(j)$.

Theorem 11 can be restated as $U_j^- = \mathbf{x}_n(j)$. An immediate consequence is that the number of paths in $\mathbf{X}_n(j)$ from the minimal element to the maximal element is equal to the number of order preserving bijections $f\colon U_j^- \to \underset{\sim}{p}$ (where $p := |U_j^-|$). It can thus be seen that Theorem 11 is an analog for minuscule $\tilde{W}^J$ to a recent result of Stanley's [36] concerning the number of paths in the weak Bruhat order on the symmetric group: Note that standard Young tableaux on a perfect staircase shaped Ferrers diagram are order preserving bijections from the poset of all positive (or negative) roots in the root system of type $\mathbf{A}_{n-1}$ to a total order. This observation led this author to the current conjectured analog for the hyperoctahedral group to Stanley's result. See equation (10) of Section 7 of [36] for the statement of the analog in general. It is possible to use the techniques above to also prove the analog in the 'quasiminuscule' cases. Empirical data show that the analog is true for $\mathbf{G}_2$, probably true for all $\mathbf{BC}_n$ cases, and probably false for all cases of type $\mathbf{D}$, $\mathbf{E}$, or $\mathbf{F}$ which are not minuscule or quasi-minuscule.

Steinberg (personal communication) has also noted (and provided an a priori proof of) the following fact: An irreducible root system $\mathbf{X}_n$ is of type $\mathbf{ADE}$ and has a minuscule weight λ_j if and only if $U_j^- = W_{\langle j\rangle}\cdot(-\alpha_j)$, i.e. if and only if $-\alpha_j$ is a minuscule weight with respect to the root system $\mathbf{X}_n\langle j\rangle$ of rank $n-1$. This fact together with Theorem 11 'explains' the sequence $\mathbf{E}_7(7) \cong J^5(\underset{\sim}{1}\oplus\underset{\sim}{2})$, $\mathbf{e}_7(7)\cong\mathbf{E}_6(6)\cong J^4(\underset{\sim}{1}\oplus\underset{\sim}{2})$, $\mathbf{e}_6(6)\cong\mathbf{D}_5(5)\cong J^3(\underset{\sim}{1}\oplus\underset{\sim}{2})$, $\mathbf{d}_5(5)\cong\mathbf{A}_4(3)\cong J^2(\underset{\sim}{1}\oplus\underset{\sim}{2})$, $\mathbf{a}_4(3)\cong\mathbf{A}_1(1)\times\mathbf{A}_2(2)\cong J(\underset{\sim}{1}\oplus\underset{\sim}{2})$, $\mathbf{a}_1(1)\oplus\mathbf{a}_2(2)\cong\underset{\sim}{1}\oplus\underset{\sim}{2}$.

12. Geometric Remarks

We will now describe two interrelated algebraic geometric contexts in which the combinatorial and representation theoretic constructions of this paper play interesting roles. These comments build upon remarks of Stanley regarding Schubert varieties in Grassmannians [33], and upon remarks of Hiller regarding minuscule flag manifolds [12]. In order to illustrate various interrelationships, we will compute the degree of embedded minuscule flag manifolds five different ways.

The first context concerns rational cohomology of flag manifolds. Let G be a complex simple algebraic group of type X_n, P a parabolic subgroup, and W_J the corresponding subgroup of W. The Bruhat poset $\mathbf{X}_n(J^c) = W^J$ indexes the 'Schubert varieties' of the flag manifold G/P, and thus it also indexes a basis for $H^*(G/P, \mathbb{C})$ consisting of $[U]$, U a Schubert variety [34]. If P is a maximal parabolic subgroup, then there is a unique codimension 1 Schubert variety T. If P is in addition minuscule (defined later), then $[T]\cup[U]$ is simply the sum (with coefficients 1) of all $[V]$ such that V covers U in W^J. Let p be the dimension of G/P as a complex manifold. Then $[T]^p = N[E]$, where e is the unique 0-dimensional Schubert variety and N is the 'self-intersection multiplicity' of T. As Hiller notes [12], if P is minuscule, then N is simply the number of paths in $W^J = X_n(j)$ from the minimal element to the maximal element. In the viewpoint of Section 8, N is the number of order preserving bijections $f\colon x_n(j)\to\underset{\sim}{p}$, since $X_n(j) = J(x_n(j))$.

The second context concerns ample line bundles M on $\tilde{G}/P$ and sheaf cohomology $H^*(G/P, M)$. This is the situation with which Seshadri *et al.* are mainly concerned [21]. Consult [4] for general facts. For any G/P, P maximal, there is an ample line bundle L analogous to the Plücker line bundle (or embedding) on the Grassmannian SL_{n+1}/P_j. Let $R = \bigoplus_{m\geq 0} R_m$ be the homogeneous coordinate ring for G/P under the embedding given by L. The Borel-Weil theorem implies that $R_m \cong H^0(G/P, \otimes^m L)$ as G-modules. Also recall that the degree of an embedded variety of dimension p is $p!\, b_p$, where b_p is the leading coefficient of the Hilbert polynomial $H(m) = \dim_{\mathbb{C}} R_m$. Let i be the Weyl involution on the Dynkin diagram $\mathbf{X}_n$. Then if maximal $P_j \subset G$ corresponds to $W_j \subset W$, we have $H^*(G/P_{i(j)}, \otimes^m L) \cong X_n(m\lambda_j)$ as G-modules [4]. Call $P_{i(j)}$ is minuscule iff $\mathbf{X}_n(\lambda_{i(j)})$ is minuscule. Note that $\mathbf{X}_n(\lambda_j)$ is minuscule iff $\mathbf{X}_n(\lambda_{i(j)})$ is minuscule, and $\mathbf{X}_n(j) = \mathbf{X}_n(i(j))$ as posets always.

We now compute the degree of G/P (with P minuscule maximal) in the embedding given by L (which is the generator of the Picard group of G/P), see Figure 2. The degree of an embedded projective variety is the self-intersection multiplicity of a hyperplane section. The unique codimension one Schubert variety T of G/P is a hyperplane section corresponding to L [29]. Hence the degree is N, the number of paths in $\mathbf{X}_n(j)$ (where $P=P_j$). As Hiller notes, this number can be found using the Frame-Robinson-Thrall hook formula for standard Young tableaux of rectangular shape in case $A_{n-1}(j)$; Schur's hook formula [22, ex. III.7.8] for shifted standard Young tableaux in cases $\mathbf{B}_n(n)$, $\mathbf{D}_{n+1}(n+1)$, and $\mathbf{D}_{n+1}(n)$; and direct counting in the other cases. (Hiller's count of 13,188 for $\mathbf{E}_7(7)$ is slightly off; the correct count is 13,110.)

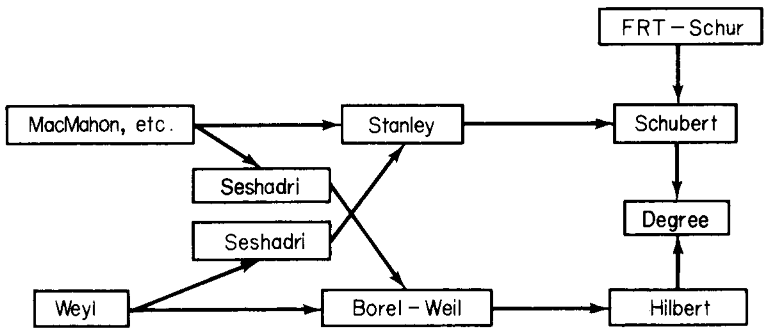

FIGURE 2.

The second and third methods go as follows. We first note three generating function identities of Stanley [31, propositions 8.3 and 8.4, corollary 5.3] which hold for *any* poset P with p elements:

$$\sum_{m\geq 0} F(m,x)t^m=\frac{\sum_{s=0}^{p-1} W_s(x)t^s}{(1-t)(1-tx)\cdots(1-tx^p)},$$

$$W_s(x)=\sum_{m=0}^{s}(-1)^m x^{\binom{m}{2}}\begin{bmatrix}p+1\\ m\end{bmatrix}F(s-m,x),$$

$$F(x)=\frac{W(x)}{(1-x)(1-x^2)\cdots(1-x^p)},$$

where $F(m,x):=F(P,m,x)$ as in Section 1, $W_s(x)$ is as in Section 8, $F(x):=\lim_{m\to\infty}F(m,x)$ and $W(x):=\sum_{s=0}^{p-1}W_s(x)$. Note that $F(m,1)$ is the total number of m-multichains in $J(P)$. The function $Z(n)=F(n-1,1)$ is often called the zeta polynomial of $J(P)$. Also note that $W(1)$ is the number N of order preserving maps from P to $\underset{\sim}{p}$ or the number of paths in $J(P)$ from $\emptyset$ to P. The third equation implies:

$$N=W(1)=(1-x)\cdots(1-x^p)F(x)|_{x=1}$$

Here the number of maximal paths in $J(P)$ is expressed in terms of the rank-weighted generating functions for m-multichains in $J(P)$. To apply this to $J(P)=\mathbf{X}_n(j)$, take $F(m,x)$ from either Section 8 (MacMahon, Gordon, computer, etc.) or from Section 6 (Seshadri and Weyl), and let $m\to\infty$. Thus

$$F(x)=\frac{1}{\prod_{\alpha\in P}(1-x^{r(\alpha)})}$$

where $r(\alpha)$ is the rank of the element α in $x_n(u)$ (minimal elements were assigned to rank 1). Therefore

$$N = \frac{p!}{\prod_{\alpha \in P} r(\alpha)}.$$

We have thus derived the hook product formulas in certain cases from the Weyl product formula.

Now for the fourth and fifth methods. As noted earlier, the degree can be found from knowledge of $\dim_{\mathbb{C}} R_m$ via the Hilbert polynomial. Since $R_m \cong \mathbf{X}_n(m\lambda_{i(j)})$ as G-modules, we need only find $\dim \mathbf{X}_n(m\lambda_{i(j)})$ as a function of m. We could use Seshadri's result, $\dim \mathbf{X}_n(m\lambda_j)$ = number of m-multichains in $\mathrm{X}_n(j)$, combined with setting $x=1$ in the combinatorially derived generating functions of Section 8. This is, of course, silly: Just use the Weyl character formula immediately. Hirzebruch used this method for certain G/P, obtaining $N=78$ and $N=13{,}110$ for $\mathbf{E}_6(6)$ and $\mathbf{E}_7(7)$, among others [13]. This last method can in fact be applied to any G/P with maximal.

It is interesting to note that the first generating function identity above, a central result of Stanley's purely combinatorial work, has a nice interpretation in the present algebraic context of homogeneous coordinate rings R of minuscule flag manifolds. The left hand side is the Hilbert series for R, with an additional grading (using the variable x) according to weight level within each homogeneous piece R_m. The Hilbert series for any projective variety of dimension p can be written in the form $N(t)/(1-t)^{p+1}$, with $N(t) \in \mathbb{Z}[t]$. Setting $x=1$ in (1), we see that the coefficients of $N(t)$ are the $W_s(1)$ of Section 8. The $N(t)/(1-t)^{p+1}$ form of the Hilbert series is particularly interesting when the ring is Cohen–Macaulay, as the coordinate rings at hand are known to be [7]. Baclawski and Garsia [2] have studied systems of elements in Cohen–Macaulay rings which generate the ring in a manner compatible with this expression for the Hilbert series. For the particular case of rings with straightening laws over distributive lattices, Garsia's theory of separators [10] is a ring-theoretic analog of the combinatorial constructions of Stanley by which the polynomials $W_s(x)$ are defined.

We now will provide a geometric interpretation of the positive integer edge labels of Figure 1. The representations of $\mathfrak{sl}(2,\mathbb{C})$ constructed in Section 10 can arise in two other independent ways. Once Lemma 10 has been formulated, one can ask for all distributive lattices L which carry a representation of $\mathfrak{sl}(2,\mathbb{C})$ such that $\theta(\mathbf{a},\mathbf{b})=1$ whenever $\mathbf{b}$ covers $\mathbf{a}$ for the X operator, and such that the Y operator respects the order on L [see (5) of Section 11]. All such possible lattices L and their representations of $\mathfrak{sl}(2,\mathbb{C})$ are constructed in [27]; they are exactly the minuscule lattices. The third context in which these representations of $\mathfrak{sl}(2,\mathbb{C})$ arise is that of $H^*(G/P,\mathbb{C})$ with P minuscule. As noted earlier, cup product with $[T]$ produces only Schubert varieties lying directly above the original variety in $\mathbf{X}_n(j)$, and with coefficients all 1. Recall that under the specified embedding, the Schubert variety T is a hyperplane section. Any complex projective variety is a Kähler manifold [38]. In this context, cup product with a hyperplane section is the same as wedge product with the fundamental form Ω. Call the linear operator on $H^*(G/P,\mathbb{C})$ induced by this action X. The Kähler picture provides two other operators, Y and H, which together with X define a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^*(G/P,\mathbb{C})$. The operator Y is the adjoint of the operator X with respect to the Hodge metric on $\Lambda^*T^*(G/P)$ [38, p. 183].

The representations of $\mathfrak{sl}(2,\mathbb{C})$ in the second and third contexts are concrete matrix representations: a basis indexed by $\mathbf{X}_n(j)$ has been specified in each case. Using techniques similar to those used in the proof of Theorem 11, it is possible to show that a weight basis may be chosen for the representation $\mathbf{X}_n(\lambda_j)$ of Section 10 in such a way that all of the coefficients for the operator X of Section 10 are unity, i.e. $\theta(\mu,\nu)=1$ whenever ν covers μ in $\mathbf{X}_n(j)$. Thus in all three contexts the operator X just produces the sum of elements of $\mathbf{X}_n(j)$ lying above a given element. The operator H is identical in all three situations. Now X and H uniquely determine Y in any representation of $\mathfrak{sl}(2, \mathbb{C})$ [24, Proposition 2]. Thus the Y coefficients are identical. These coefficients are computed in [27], where it is found that they are always positive integers. These numbers, shown in Figure 1 for $\mathbf{E}_6(6)$ and $\mathbf{E}_7(7)$, are thus the coefficients of the Hodge adjoint of cup product with a hyperplane section with respect to the Schubert cell basis for the cohomology of a minuscule flag manifold.

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