Differential Geometry Midterm

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1. $\mathbb{C}P^n$ **is a smooth complex manifold.** $\mathbb{C}P^n$ is defined as the quotient space $\mathbb{C}^{n+1} \setminus \{0\}/\sim$, where $x \sim \lambda x$ for all $\lambda \in \mathbb{C}^\times$. Let p be the projection map.

- $\mathbb{C}P^n$ is Hausdorff. For the quotient $X/\sim$ of a Hausdorff space X is Hausdorff exactly when the set $D = (x_1, x_2) : x_1 \sim x_2$ is a closed subset of $X \times X$. But $D = \{(z_0, ..., z_n, \lambda z_0, ..., \lambda z_n) : (z_0, ..., z_n) \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}\} \subset \mathbb{C}^{2n+2}$ is the intersection of the zero loci of the functions $P_{ij} : \mathbb{C}^{2n+2} \to \mathbb{C}$:

  $$P_{ij}(z_0, ..., z_n, w_0, ..., w_n) = z_i w_j - w_i z_j$$

  for $0 \leq i < j \leq n$. Therefore, D is closed, so $\mathbb{C}P^n$ is Hausdorff.

- $\mathbb{C}P^n$ is second-countable. For if $f : X \to Y$ is an open surjection and X is second countable, Y is second countable. The quotient map p is surjective by definition. We claim that p is an open map. For if $\mathcal{O}$ is an open set in $\mathbb{C}^{n+1}$, say $x \in p^{-1}(p(\mathcal{O}))$. Then there exists a nonzero λ such that $y = \lambda x \in \mathcal{O}$. By the openness of $\mathcal{O}$, there is some neighborhood $N_\eta(y) \subset \mathcal{O}$. But the function $x \mapsto \lambda \cdot x$ is a continuous function, so for sufficiently small ϵ, $\lambda \cdot N_\epsilon(x) \subset N_\eta(y) \subset \mathcal{O}$. Thus $p^{-1}(p(\mathcal{O}))$ is open, so by definition $p(\mathcal{O})$ is open. Therefore, p is an open map, and $\mathbb{C}P^n$ is second-countable.

An element of $\mathbb{C}P^n$ can be represented as $[(z_0, z_1, ..., z_n)]$, where $(z_0, ..., z_n) \in \mathbb{C}^{n+1}$ is a point on the line represented. To construct charts on $\mathbb{C}P^n$, Let $U_i = \{[(z_0, z_1, ..., z_n)] \in \mathbb{C}P^n : z_i \neq 0\}$ for $i = 0, ..., n$. Though an element of $\mathbb{C}P^n$ has many representatives, they will either all be zero in one coordinate or all not be zero in that coordinate, so this set is well-defined. Define the map $\phi_i : U_i \to \mathbb{C}^n$ as follows:

$$\phi_i([z_0, ..., z_n]) = (\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_i + 1}{z_i}, ..., \frac{z_n}{z_i})$$

Then ϕ_i is well defined on U_i because z_i is nonzero for any representative of $[z_0, ..., z_n]$, and all representatives map to the same point in $\mathbb{C}^n$. Since every element of $\mathbb{C}P^n$ is represented by points with at least one nonzero coordinate, the U_i cover $\mathbb{C}P^n$. The U_i are open, because the preimage of U_i under the quotient map is the open set $\mathbb{C}^{n+1} \setminus \{z_i = 0\}$. Therefore, the collection $\{(U_i, \phi_i) : i = 0, ..., n\}$ give an atlas for $\mathbb{C}P^n$.

Finally, we verify that transition maps given by these charts are analytic. Take $(w_1, ..., w_n) \in \phi_i(U_i \cap U_j)$. Assume first that $i > j$. Then

$$\begin{aligned}\phi_j \circ \phi_i^{-1}(w_1, ..., w_n) &= \phi_j([w_1, ..., w_{i-1}, 1, w_{i+1}, ..., w_n]) \\ &= (\frac{w_1}{w_{j+1}}, ..., \frac{w_j}{w_{j+1}}, \frac{w_{j+2}}{w_{j+1}}, ..., \frac{w_{i-1}}{w_{j+1}}, \frac{1}{w_{j+1}}, \frac{w_{i+1}}{w_{j+1}}, ..., \frac{w_n}{w_{j+1}})\end{aligned}$$

In this case, w_{j+1} is nonzero because it occupies that j^{th} position of $0, 1, ..., n$ in $\mathbb{C}^{n+1}$, and $\phi^{-1}(w_1, ..., w_n) \in U_j$. On the other hand, if $j > i$, then

$$\begin{aligned}\phi_j \circ \phi_i^{-1}(w_1, ..., w_n) &= \phi_j([w_1, ..., w_{i-1}, 1, w_{i+1}, ..., w_n]) \\ &= (\frac{w_1}{w_j}, ..., \frac{w_{i-1}}{w_j}, \frac{1}{w_j}, \frac{w_i}{w_j}, ..., \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, ..., \frac{w_n}{w_j})\end{aligned}$$

In this case, w_j is nonzero because it occupies the j^{th} position in $\mathbb{C}^{n+1}$. Therefore, in both cases the component functions of each of these maps are analytic in each variable, since they are quotients whose denominators do not vanish on the domain. This confirms that $\mathbb{C}P^n$ is a complex manifold.

2. Isometry between the plane and the punctured sphere.

(a.) By definition, the arc length of a curve $\gamma : I \to (M, g)$ is

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

In our example $(\mathbb{R}^n, \hat{g})$, we have $\gamma(t) = (t, 0, ..., 0)$ with velocity vector $\gamma'(t) = dx_1$. We compute

$$\begin{aligned}L(\gamma) &= \int_0^1 \sqrt{\frac{4}{(1+t^2)^2}}\, dt \\ &= \int_0^1 \frac{2}{1+t^2}\, dt \\ &= 2 \cdot \int_{\arctan(0)}^{\arctan(1)} d\theta \\ &= \pi/2\end{aligned}$$

(b.) As discussed in class, the orthogonal group $O(3)$ is a transitive group is isometries of $S^2 \subset \mathbb{R}^3$. Therefore if $N = (0, 0, 1)$ is the north pole of the sphere, then there is $A \in O(3)$ such that $Ap = N$, and $A : S^2 \setminus \{p\} \to S^2 \setminus \{N\}$ is an isometry. Since isometry is a transitive relation, we therefore may assume that $p = N$.

We have the stereographic projection $\phi : S^2 \setminus \{N\} \to \mathbb{R}^2$:

$$\phi(x_1, x_2, x_3) = \frac{1}{1 - x_1}(x_2, x_3)$$

We can compute the inverse $\phi^{-1}(y_1, y_2)$. For since $|x|^2 = 1$, we have the relation $y_1^2 + y_2^2 = \frac{1+x_1}{1-x_1}$. This implies that $x_1 = \frac{\|y\|^2-1}{\|y\|^2+1}$. Then $x_i = \frac{2y_i}{\|y\|^2+1}$ for $i = 2, 3$. Therefore

$$\phi^{-1}(y_1, y_2) = \left(1 - \frac{2}{(y_1^2 + y_2^2) + 1}, \frac{2y_1}{(y_1^2 + y_2^2) + 1}, \frac{2y_2}{(y_1^2 + y_2^2) + 1}\right)$$

Now we compute the differential of $\phi^{-1}(y_1, y_2)$:

$$D(\phi^{-1})_{(y_1,y_2)} = \begin{pmatrix} \frac{4y_1}{((y_1^2+y_2^2)+1)^2} & \frac{4y_2}{((y_1^2+y_2^2)+1)^2} \\ \frac{2}{(y_1^2+y_2^2)+1} + \frac{-4y_1^2}{((y_1^2+y_2^2)+1)^2} & \frac{-4y_1y_2}{((y_1^2+y_2^2)+1)^2} \\ \frac{-4y_1y_2}{((y_1^2+y_2^2)+1)^2} & \frac{2}{(y_1^2+y_2^2)+1} + \frac{-4y_2^2}{((y_1^2+y_2^2)+1)^2} \end{pmatrix}$$

Let δ_1 and δ_2 be the partial differential operators on $\mathbb{R}^2$ than span $T_y\mathbb{R}^2$ for all $y \in \mathbb{R}^2$. Then $D(\phi^{-1})_y\delta_1$ and $D(\phi^{-1})_y\delta_2$ form a basis for $T_x\mathbb{S}^2$, where $x = \phi^{-1}(y)$. Let $\langle \cdot, \cdot \rangle_x$ be the induced metric from $\mathbb{R}^3$ on S^2. Then

$$\langle D(\phi^{-1})_y\delta_1, D(\phi^{-1})_y\delta_2\rangle_x = 0$$
$$\langle D(\phi^{-1})_y\delta_1, D(\phi^{-1})_y\delta_1\rangle_x = \langle D(\phi^{-1})_y\delta_2, D(\phi^{-1})_y\delta_2\rangle_x = \frac{4}{(1+\|y\|^2)^2}$$

(See calculations attached). Therefore the map $\phi^{-1} : (\mathbb{R}^2, \hat{g}) \to (S^2, g_{S^n})$ is conformal. For given an arbitrary vector $v = v_1(\delta_1)_p + v_2(\delta_2)p$, on the one hand

$$\begin{aligned} \hat{g}_p(v,v) &= v_1^2\hat{g}_p((\delta_1)_p, (\delta_1)_p) + v_2^2\hat{g}_p((\delta_2)_p, (\delta_2)_p) + 2v_1v_2\hat{g}_p((\delta_1)_p, (\delta_2)_p) \\ &= \frac{4 \cdot (v_1^2 + v_2^2)}{1 + |p|^2} \end{aligned}$$

and on the other hand

$$\begin{aligned} (g_{S^n})_{\phi^{-1}(p)}(D\phi_p^{-1}(v), D\phi_p^{-1}(v)) &= v_1^2\langle D(\phi^{-1})_p\delta_1, D(\phi^{-1})_p\delta_1\rangle_p + v_1^2\langle D(\phi^{-1})_p\delta_2, D(\phi^{-1})_p\delta_2\rangle_p \\ &\quad + 2v_1v_2\langle D(\phi^{-1})_p\delta_1, D(\phi^{-1})_p\delta_2\rangle_p \\ &= \frac{4 \cdot (v_1^2 + v_2^2)}{1 + |p|^2} \end{aligned}$$

By the polarization identity, this proves that $\hat{g}_p(X_p, Y_p) = (g_{S^n})_p(D(\phi^{-1})_pX_p, D(\phi^{-1})_pY_p)$ for all $p \in \mathbb{R}^2$ and all $X_p, Y_p \in T_p\mathbb{R}^2$. Therefore ϕ^{-1} gives a conformal map between these two Riemannian manifolds with conformal factor $1 = e^0$.

3. Normal bundle is a smooth vector bundle. The normal space to a submanifold M^m of a Riemannian manifold (N^h, h) at a point p is

$$N_pM = \{v_p \in T_pM : h(v_p, w_p) = 0,\ \forall w_p \in T_pM\}$$

where T_pM is the image of the tangent space to M at p under the differential of the inclusion map. This differential is injective, since its coordinate representation with respect to a slice chart (Gundmundson Definition 2.8) is the upper $m \times m$ identity matrix. The normal bundle of M in N is defined as

$$NM = \{(p, v_p) : p \in M, v_p \in N_pM\}$$

Let $\pi : NM \to M$ be the projection map $(p, v_p) \mapsto p$. By the definiton of NM, π is surjective. In addition, $\pi^{-1}(\{p\}) = \{(q, w) \in NM : \pi(q, w) = p\} = \{p\} \times N_pM$. But $N_pM \simeq \mathbb{R}^{n-m}$, since T_pM has dimension m at every point, and the differential of the inclusion map is injective, so its orthogonal complement in T_pN has dimension $n - m$. This confirms Criterion (i) in Gundmundsson Def. 4.1.

By Gundmundsson Definition 2.8, for any $p \in M \subset N$ there is a chart (U, x) for $U \subset N$ such that $x(U \cap M) = x(U) \cap (\mathbb{R}^m \times \{0\}^{n-m})$. Given such a chart, let $X_i = D(x^{-1})_{x(p)}(\delta_i)$ be the pushforward of the i^{th} partial differential operator on $\mathbb{R}^n$, giving a basis of T_pN.

We produce a set of pointwise-orthonormal vector fields $Y^1, ..., Y^m$ with the property that $\text{Span}(Y_p^1, ..., Y_p^k) = \text{Span}(X_p^1, ..., X_p^k)$ for every $p \in U$ and $k \leq m$. First, set

$$Y_p^1 = \frac{X_p^1}{\sqrt{h_p(X_p^1, X_p^1)}}$$

The denominator is nonvanishing because h_p is positive definite and Dx^{-1} is nonsingular on $x(U)$. By the smoothness of h and X^1, Y^1 is a smooth vector field. Clearly the span of X_p^1 is the span of Y_p^1.

For the inductive step, say we are given $Y_1, ..., Y_k$ as described. Define $\hat{Y}^{k+1}$ inductively as

$$\hat{Y}_p^{k+1} = X_p^{k+1} - \sum_{i=1}^{k} h_p(X_p^{k+1}, Y_p^k)\, Y_p^k$$

Then $\hat{Y}^{k+1}$ has nonzero magnitude everywhere, because for all p, $X_p^{k+1} \notin \text{Span}(X_p^1, ..., X_p^k) = \text{Span}(Y_p^1, ..., Y_p^k)$. Therefore define

$$Y_p^{k+1} = \frac{\hat{Y}_p^{k+1}}{\sqrt{h_p(\hat{Y}_p^{k+1}, \hat{Y}_p^{k+1})}}$$

It is clear that Y_p^{k+1} is orthogonal to Y_p^j for $j \leq k$. But all Y^j for $j \leq k+1$ are linear combinations of $X^1, ..., X^{k+1}$, so we must have that $\text{Span}(Y_p^1, ..., Y_p^{k+1}) = \text{Span}(X_p^1, ..., X_p^{k+1})$. This process stops when $k = m$, completing the proof.

By our original choice of x, we now have that $Y_p^{m+1}, ..., Y_p^n$ form a basis for N_pM. For $Y_p^1, ..., Y_p^m$ have the same span as $X_p^1, ..., X_p^m$ by construction, and the latter set is a basis for T_pM. The remaining $\{Y_p^j\}$ are orthogonal to these vectors, and therefore to T_pM, and span a space of dimension $n - m$, so they span N_pM.

We can use this result to define a system of bundle charts on NM. For since $\{Y_p^i\}$ form a basis of T_pN for every p, every point in $\pi^{-1}(U)$ can be written uniquely as $(p, \sum_{i=m+1}^{n+m} a_i Y_p^i)$. Therefore we can define $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^{n-m}$ so that $(p, \sum_{i=m+1}^{n+m} a_i Y_p^i) \mapsto (p, (a_{m+1}, ..., a_{m+n}))$. Because we can choose such a (U, x) containing every point in M, the $\pi^{-1}(U)$ cover NM.

We show that transition maps in this atlas are smooth. Say that two charts $(\pi^{-1}(U_1), \psi_1)$ and $(\pi^{-1}(U_2), \psi_2)$ such that $U_1 \cap U_2 \neq \emptyset$. For any $p \in U_1 \cap U_2$, choose a chart (U, x) of N such that $p \in U$ and let $V = U_1 \cap U_2 \cap U$. (U, x) induces a canonical chart on TN. Let $\{Q^i\}$ be the orthonormal vectors corresponding to ψ_1 and $\{R^i\}$ be the orthonormal vectors corresponding to ψ_2. The vector fields Q^i and R^i have a smooth coordinate representatiosn with respect to (U, x):

$$Q^i(p) = \sum_{j=1}^{n} a_{ij}(p) \frac{d}{dx_i}\bigg|_p$$
$$R^i(p) = \sum_{j=1}^{n} b_{ij}(p) \frac{d}{dx_i}\bigg|_p$$

where $\frac{d}{dx_i}|_p$ are the pushforwards of the differential operator on V with respect to x^{-1} and the a_{ij} and b_{ij} are smooth functions $N \to \mathbb{R}$. Let $A(p) = (a_{ij}(p))$ and let $B(p) = (b_{ij}(p))$. By the Gram-Schmidt construction both matrices are upper-triangular and invertible, so $C(p) = B(p)A(p)^{-1}$ is an upper-triangular matrix. We write $R_p^i = \sum_{j=1}^n c_{ij} Q_p^j$, and calculate:

$$\sum_{i=m+1}^{n} v_i(p) R^i(p) = \sum_{i=m+1}^{n} v_i(p) (\sum_{j=1}^{n} c_{ij} Q_p^j) = \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} c_{ij} v_i(p) Q_p^j$$

where the last inequality follows because $c_{ij} = 0$ whenever $j > i$, and i is at least $m+1$. Therefore $\psi_1 \circ \psi_2^{-1} : U_1 \cap U_2 \times \mathbb{R}^{n-m} \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is the map

$$(p, (v_{m+1}, ..., v_n)) \mapsto (p, (\sum_{j=m+1}^{n} c_{1j}(p) v_j, ..., \sum_{j=m+1} c_{nj}(p) v_j))$$

By the smoothness of the c_{ij}, this a smooth function.

Under these charts, NM has a unique topology generated by $\psi^{-1}(V)$ for some ψ corresponding to a chart $(\pi^{-1}(U), \psi)$ and $V \subset U \times \mathbb{R}^{m-n}$ with respect to which it is a differentiable manifold ("Smooth Manifold Chart Lemma" 1.35, Lee *Introduction to Smooth Manifolds*). Each ψ is bijective, and is clearly a homeomorphism with respect to this topology, since the open sets are exactly the preimages of open sets in $U \times \mathbb{R}^n$. In addition, we can now confirm that $\pi : NM \to N$ is differentiable, since its coordinate representation with respect to a chart $(\pi^{-1}(U), \psi)$ on NM and (U, x) on M is just $(x_1, ..., x_n, a_1, ..., a_{n-m}) \mapsto (x_1, ..., x_n)$. Restricting ψ to $\pi^{-1}(\{p\})$ gives the linear transformation $\sum_{i=m+1}^{n+m} a_i Y_p^i \mapsto (a_{m+1}, ..., a_{m+n})$, which is a vector space isomorphism. This confirms Criterion (ii) in Gundmundsson Def. 4.1. This completes the proof.

Extra 1: Calculations for Q.2 calculations_1.pdf