ORBITS OF PLANE PARTITIONS OF EXCEPTIONAL LIE TYPE

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ABSTRACT. For each minuscule flag variety X, there is a corresponding minuscule poset, describing its Schubert decomposition. We study an action on plane partitions over such posets, introduced by P. Cameron and D. Fon-der-Flaass (1995). For plane partitions of height at most 2, D. Rush and X. Shi (2013) proved an instance of the cyclic sieving phenomenon, completely describing the orbit structure of this action. They noted their result does not extend to greater heights in general; however, when X is one of the two minuscule flag varieties of Lie type E, they conjectured explicit instances of cyclic sieving for all heights.

   We prove their conjecture in the case that X is the Cayley plane of type E_6. For the other exceptional minuscule flag variety, the Freudenthal variety of type E_7, we establish their conjecture for heights at most 4, but show that it fails generally. We further give a new proof of a unpublished cyclic sieving of D. Rush and X. Shi (2011) for plane partitions of any height in the case X is an even-dimensional quadric hypersurface. Our argument uses ideas of K. Dilks, O. Pechenik, and J. Striker (2017) to relate the action on plane partitions to combinatorics derived from K-theoretic Schubert calculus.

1. INTRODUCTION

The *minuscule posets* are a remarkable collection of partially-ordered sets that arise naturally from the representation theory of Lie algebras or alternatively from the Schubert calculus of certain generalized flag varieties. We study dynamic enumerative properties of plane partitions over these posets.

A special case of this situation is that of ordinary plane partitions fitting inside a fixed rectangular box. In this context, P. Cameron and D. Fon-der-Flaass [CFDF95] initiated the study of a combinatorially-natural operator Ψ. This operator is now generally known as *rowmotion* and has become a subject of intense study (cf., e.g., [Pan09, SW12, AST13, RS13, EP14, PR15, GR15, GR16, DPS17, Vor17]). We will define and describe in detail both minuscule posets and the operation of rowmotion in Sections 2 and 3, respectively.

For any poset P, let $\mathsf{PP}^k(P)$ denote the set of plane partitions of height at most k over P, that is to say the set of order ideals in the product $P \times \mathbf{k}$ of P with a chain poset of size k. Let f_P^k denote the generating function that enumerates the elements of $\mathsf{PP}^k(P)$ by cardinality. In the special case $k \leq 2$, D. Rush and X. Shi [RS13]

showed that f_P^k also encodes the orbit structure of rowmotion via an instance of the **cyclic sieving phenomenon** (introduced by V. Reiner, D. Stanton, and D. White [RSW04]). That is to say that for $k \leq 2$, the number of minuscule plane partitions fixed by the d-fold application of rowmotion $\Psi^{\circ d}$ is the evaluation of the polynomial f_P^k at ζ^d, where ζ is any primitive nth root-of-unity and n is the order of Ψ on $\mathsf{PP}^k(P)$. (Since some of our other operators have superscripts in their names, we will consistently denote d-fold composition of any operator τ by $\tau^{\circ d}$, instead of the more usual notation τ^d, which could be ambiguous.)

It was noted, however, in [RS13] that this instance of cyclic sieving does not extend to the case $k \geq 3$ for general minuscule posets. Nonetheless, D. Rush and X. Shi conjectured the following. (The minuscule posets in question are illustrated in Figure 1.)

Conjecture 1.1 ([RS13, Conjecture 11.1])**.** *Let P be one of the two minuscule posets associated to an exceptional Lie algebra of type E and let $k \in \mathbb{Z}_{\geq 0}$. Then f_P^k is a cyclic sieving polynomial for the action of Ψ on $\mathsf{PP}^k(P)$.*

Our main result is to completely resolve Conjecture 1.1.

Theorem 1.2. *Conjecture 1.1 holds for the E_6 minuscule poset (cf. Figure 1A) and all k, but holds for the E_7 minuscule poset (cf. Figure 1B) only when $k \leq 4$.*

Verification of Conjecture 1.1 in the the cases $k \leq 4$ for the E_6 minuscule poset and $k \leq 3$ for the E_7 minuscule poset was previously reported in [RS13]. Our new results are therefore:

- cyclic sieving for the E_6 minuscule poset when $k > 4$,
- cyclic sieving for the E_7 minuscule poset when $k = 4$, and
- failure of the conjectured cyclic sieving for the E_7 minuscule poset for $k > 4$.

Our approach to proving Theorem 1.2 is to extend the ideas of K. Dilks, O. Pechenik, and J. Striker [DPS17] to this minuscule setting, thereby relating the action of Ψ to the action of *K-promotion* on *increasing tableaux*. *K*-promotion was first studied in [Pec14] building on combinatorial tools for *K*-theoretic Schubert calculus due to H. Thomas and A. Yong [TY09]. We show that the action of *K*-promotion is controlled by its behavior on a finite subset of increasing tableaux. By understanding the orbit structure of this subset, we are able to determine the complete orbit structure, thereby establishing Theorem 1.2.

Having developed these methods, it becomes straightforward to prove the following additional fact.

Theorem 1.3. *Let P be a minuscule poset associated to any even-dimensional quadric of type D (cf. Figure 3C) and let $k \in \mathbb{Z}_{\geq 0}$. Then f_P^k is a cyclic sieving polynomial for the action of Ψ on $\mathsf{PP}^k(P)$.*

Theorem 1.3 was previously announced by D. Rush and X. Shi [RS13, Theorem 10.1]; however, they omitted their proof [RS11, §10] from their published paper.

(A) Cayley poset

(B) Freudenthal poset

FIGURE 1. The two minuscule posets associated to exceptional Lie algebras of type E. Here, we have drawn the posets to resemble Young diagrams in Cartesian orientation; the boxes are the elements of the poset and each box covers the box immediately below it and the box immediately to its left (if such boxes exist). Hence the minimal element of each poset is the box at the far left of the bottom row. The Cayley poset is associated to E_6, while the Freudenthal poset is associated to E_7. (There is no minuscule poset associated to E_8.)

We believe that our alternative proof of Theorem 1.3 via K-theoretic combinatorics provides a different understanding.

Often instances of the cyclic sieving phenomenon can be proven using representation-theoretic techniques [RSW04, Rho10], and when instances are established in a more direct fashion (as we do here), it may be a clue toward as yet undeveloped algebra (cf. [Rho17]). The results in Theorems 1.2 and 1.3 perhaps suggest the existence of new symmetric group module structures on the sets $\mathsf{PP}^k(P)$; however, we do not currently understand how to build such structures with the requisite properties in these cases.

This paper is organized as follows. In Section 2, we define the minuscule posets, recalling their classification and certain other properties we will use. In Section 3, we collect the facts we need about the rowmotion operator Ψ. Section 4 concerns the operation of K-promotion on increasing tableaux, the key tool in our proofs of Theorems 1.2 and 1.3. In Section 5, we demonstrate the equivariant bijection between

$\mathsf{PP}^k(P)$ under the action of Ψ and certain increasing tableaux under K-promotion, extending the main observation of [DPS17]. Finally, Section 6 is devoted to study of these increasing tableaux; our results in this last section prove Theorems 1.2 and 1.3 via the equivariant bijection of Section 5.

2. Minuscule posets

Let G be a complex connected reductive Lie group with maximal torus T. Denote by W the Weyl group $N_\mathsf{G}(\mathsf{T})/\mathsf{T}$. The root system Φ of G may be partitioned $\Phi^+ \sqcup \Phi^-$ into positive and negative roots according to a choice Δ of simple roots. There is a natural poset structure on Φ^+ obtained as the transitive closure of the covering relation $\alpha \lessdot \beta$ if and only if $\beta - \alpha \in \Delta$. The choice of bipartition of Φ into positive and negative roots further specifies a choice of a Borel subgroup $\mathsf{B} \subset \mathsf{G}$ and an opposite Borel subgroup $\mathsf{B}_- \subset \mathsf{G}$ with $\mathsf{B} \cap \mathsf{B}_- = \mathsf{T}$.

We say $\delta \in \Delta$ is **minuscule** if for every $\alpha \in \Phi^+$, $\delta^\vee$ appears with multiplicity at most 1 in the simple coroot expansion of $\alpha^\vee$. The classification of minuscule roots is well known and is illustrated in Figure 2 in terms of Dynkin diagrams.

FIGURE 2. In each of the finite-type Dynkin diagrams above, each minuscule root is marked as a pale blue circle, while the non-minuscule simple roots are marked in black. In type A_n, every node is minuscule, while in the other types only the indicated leaves are minuscule. The remaining finite-type Dynkin diagrams are omitted as they have no minuscule roots.

For each minuscule simple root δ, there is an associated **minuscule poset** P_δ obtained as the subposet of Φ^+ induced on those positive roots α where δ appears with nonzero coefficient in the simple root expansion of α.

Alternatively, one may obtain the minuscule posets via the geometry of certain generalized flag varieties. If $\mathsf{P}_\delta \subset \mathsf{B}$ denotes the maximal parabolic subgroup of G

associated to the minuscule simple root δ, then the space $X = \mathsf{G}/\mathsf{P}_\delta$ is called a **minuscule variety**. The minuscule varieties are smooth projective varieties with various nice geometric properties (cf. [BL00] for details). The natural action of the Borel subgroup B on X has finitely-many orbits, whose Zariski closures are the **Schubert varieties**. Given two Schubert varieties in X, it is known that they are either disjoint or else one is a subset of the other. Indeed the poset Y_X of Schubert varieties of X with respect to inclusion is a distributive lattice and its corresponding poset of join irreducibles is the minuscule poset P_δ for the minuscule simple root δ.

The minuscule posets are completely classified. They are illustrated in Figures 1 and 3. Our focus will be on the **propellers**, the **Cayley poset**, and the **Freudenthal poset** as shown in Figures 3C, 1A, and 1B, respectively. The propeller with k elements ($k \geq 6$) is associated to the minuscule node that is not adjacent to the trivalent node in the $D_{\frac{k+2}{2}}$ Dynkin diagram, the Cayley poset is associated to the unique minuscule simple root for E_6, and the Freudenthal poset is associated to the unique minuscule simple root for E_7 (cf. Figure 2). The corresponding minuscule varieties are, respectively, even-dimensional quadric hypersurfaces, the octonionic projective plane (or Cayley plane), and the Freudenthal variety. The remaining minuscule roots yield minuscule posets that are *rectangles* or *shifted staircases* (illustrated in Figure 3A and B). These correspond respectively to type A Grassmannians and to maximal orthogonal Grassmannians; we will not consider them further in this paper.

A poset that is linearly ordered is called a **chain**; we denote the chain of size k by $\mathbf{k}$. By a **plane partition** of height at most k over a poset P, we mean an order ideal of the product poset $P \times \mathbf{k}$. Clearly, a such a plane partition may be identified with a weakly order-reversing map $P \to \mathbf{k}$. The poset P is called **Gaussian** if the generating function f_P^k enumerating plane partitions of height at most k over P according to cardinality may be expressed in the form

$$\frac{(1 - x^{h_1+m})(1 - x^{h_2+m}) \cdots (1 - x^{h_t+m})}{(1 - x^{h_1})(1 - x^{h_2}) \cdots (1 - x^{h_t})},$$

for some nonnegative integers $t, h_1, h_2, \ldots, h_t \in \mathbb{Z}_{\geq 0}$ independent of k. R. Proctor showed that every minuscule poset is Gaussian [Pro84]. Indeed, it is conjectured that there are no other Gaussian posets. Given the beautiful form of these generating functions, it is perhaps not too surprising that f_P^k should in some instances play a role in cyclic sieving phenomena, as in Theorems 1.2 and 1.3.

3. The rowmotion operator Ψ

If P is any poset, let $J(P)$ denote the set of all its order ideals. For $\mathcal{I} \in J(P)$ an order ideal of P, we define $\Psi(\mathcal{I}) \in J(P)$ to be the order ideal generated by the minimal elements of the complement $P - \mathcal{I}$. This action is closely related to that described in [BS74, Duc74, CFDF95] in slightly different contexts. It was first considered as an action on order ideals by J. Striker and N. Williams [SW12]; we follow them in referring to Ψ as **rowmotion**.

(A) Rectangle   (B) Shifted staircase

(C) Propeller

FIGURE 3. Together with the two posets shown in Figure 1, these are exemplars of all five families of minuscule poset, shown in Cartesian orientation. The elements of each poset are the boxes, and each box is covered by any box immediately above it or immediately to its right. Rectangles may have arbitrary height and width. Shifted staircases have arbitrary width, and height equal to their width. Propellers consist of two rows of arbitrary but equal length, overlapping by two boxes in the center. The Cayley and Freudenthal posets of Figure 1 are exceptional, forming singleton families.

Here, we recall the facts that we will need about rowmotion. For $x \in P$, define the **toggle at** x to be the operator $t_x : J(P) \to J(P)$ defined by

$$t_x(\mathcal{I}) = \begin{cases} \mathcal{I} \cup \{x\}, & \text{for } x \notin \mathcal{I} \text{ and } \mathcal{I} \cup \{x\} \in J(P); \\ \mathcal{I} - \{x\}, & \text{for } x \in \mathcal{I} \text{ and } \mathcal{I} - \{x\} \in J(P); \\ \mathcal{I}, & \text{otherwise.} \end{cases}$$

A **3-dimensional lattice projection** of a finite ranked poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^3$, where the partial order on $\mathbb{Z}^3$ is given by componentwise comparison and $\mathbb{Z}^3$ is ranked by the sum of the coordinates. Considering the classification of minuscule posets in Figures 1 and 3, observe that for every minuscule P and any $k > 0$, the poset $P \times \mathbf{k}$ has an injective 3-dimensional lattice projection. For $v = \{\pm 1\}^3$ and $i \in \mathbb{Z}$, let

$$\tau^i_{\pi,v} = \prod_{\langle v, \pi(x) \rangle = i} t_x.$$

That is, $\tau^i_{\pi,v}$ is the composition of the toggles t_x for all x with $\pi(x)$ lying on the affine plane defined by $\langle v, \cdot \rangle = i$. Since the toggles t_{x_1} and t_{x_2} clearly commute whenever neither x_1 nor x_2 covers the other, this composition is well-defined. Then (π, v)-**motion** on P is the operator

$$\mathsf{Mot}_{\pi,v} := \prod_{i \in \mathbb{Z}} \tau^i_{\pi,v},$$

where the factors are taken in reverse of the natural order. Note that in this product, only finitely-many factors act nontrivially, since $\pi(P)$ intersects only a finite number of the affine planes defined by $\langle v, \cdot \rangle = i$. We recall the following pair of results, which are fundamental to our further analysis of rowmotion on minuscule plane partitions.

Proposition 3.1 ([CFDF95, Lemma 1]; [DPS17, Proposition 3.18]). *For any finite ranked poset P with a 3-dimensional lattice projection π, we have*

$$\Psi = \mathsf{Mot}_{\pi,(1,1,1)}$$

as operators on $J(P)$. $\square$

Proposition 3.2 ([DPS17, Theorem 3.25]). *For any finite ranked poset P with a 3-dimensional lattice projection π and any vectors $v, w \in \{\pm 1\}^3$, there is an equivariant bijection between $J(P)$ under $\mathsf{Mot}_{\pi,v}$ and $J(P)$ under $\mathsf{Mot}_{\pi,w}$. In particular, $\mathsf{Mot}_{\pi,v}$ and $\mathsf{Mot}_{\pi,w}$ induce the same multiset of orbit cardinalities on $J(P)$.* $\square$

4. K-PROMOTION OF INCREASING TABLEAUX

4.1. **Definitions.** A classical approach to the cohomological Schubert calculus of complex type A Grassmannians is to use the jeu de taquin on standard Young tableaux introduced by M.-P. Schützenberger [Sch77]. (See [Ful97, Man01] for modern expositions of the classical jeu de taquin theory.) A theme of modern Schubert calculus has been extending such theories into richer generalized cohomology theories and in particular into the K-theory ring of algebraic vector bundles. For a partial survey of recent work related to K-theoretic Schubert calculus, see [PY17].

The first K-theoretic Littlewood-Richardson rule was discovered by A. Buch [Buc02]; this rule, however, was not based on jeu de taquin. H. Thomas and A. Yong [TY09] later found a different Littlewood-Richardson rule for the same structure coefficients by directly extending M.-P. Schützenberger's jeu de taquin. This latter rule was conjectured in [TY09] to extend to the K-theoretic Schubert structure coefficients of all minuscule varieties, as was proven by a combination of [BR12, CTY14, BS16]. In this Thomas-Yong theory, the role of standard Young tableaux is filled by *increasing tableaux*.

Definition 4.1. Let λ be a order ideal in a minuscule poset, considered as an generalized Young diagram in Cartesian orientation as in Figures 1 and 3. An **increasing tableau** of shape λ is an assignment of positive integers to the boxes of λ, such that entries strictly increase from left to right along rows and strictly increase from bottom to top going up columns. Let $\mathrm{Inc}^q(\lambda)$ denote the set of all increasing tableaux of shape λ with all entries at most q. We identify $\mathrm{Inc}^q(\lambda)$ with the set of all strictly order-preserving maps from the poset λ to the chain poset $\mathbf{q}$ of size q.

Using the K-jeu de taquin of H. Thomas and A. Yong [TY09], one has a K-promotion operator $\mathfrak{pro}^q$ on $\mathrm{Inc}^q(\lambda)$, directly extending M.-P. Schützenberger's classical definition of promotion [Sch72]. K-promotion was first studied in [Pec14], and has been further investigated in [BPS16, PSV16, Rho17, DPS17, Pec17, Vor17]. Since studying the action of $\mathfrak{pro}^q$ will be fundamental to our proof of Theorems 1.2 and 1.3,

we must recall the definition of $\mathfrak{pro}^q$. We will indeed use both the original formulation from [Pec14], as well as an equivalent formulation from [DPS17, Proposition 2.4].

Let $T \in \mathrm{Inc}^q(\lambda)$. The **$K$-Bender-Knuth operator** ρ_i acts on T by swapping the letters i and $i+1$ everywhere in T *where doing so would not violate the increasingness conditions.* More precisely, ρ_i looks at the set of boxes labeled i or $i+1$, decomposes that set into edge-connected components, swaps the labels i and $i+1$ in each component that is a single box, and acts trivially on each non-trivial connected component. Note that each ρ_i is an involution. **K-promotion** may then be defined by

$$\mathfrak{pro}^q(T) := \rho_{q-1} \circ \ldots \circ \rho_1(T).$$

Examples of the action of K-Bender-Knuth operators are shown in Figure 4, while an example of the full process of K-promotion is shown in Figure 5. We will relate K-Bender-Knuth operators to poset toggles in Section 5.

FIGURE 4. The three increasing tableaux on the right are obtained from the increasing tableau at left by applying the indicated K-Bender-Knuth operators. The arrows point in both directions since each K-Bender-Knuth operator is involutive.

Suppose $T \in \mathrm{Inc}^q(\lambda)$ and consider T as a strictly order-preserving map from λ to the chain poset $\mathbf{q}$. We say that T is **gapless** if this map is surjective; otherwise, T is **gappy**. We write $\mathrm{Inc}_{\mathrm{gl}}^q(\lambda)$ for the set of all gapless tableaux in $\mathrm{Inc}^q(\lambda)$.

FIGURE 5. The calculation of the K-promotion $\mathfrak{pro}^{13}(T)$ of the increasing tableau T (on the Cayley poset) through the action of a sequence of K-Bender-Knuth operators $\rho_1, \ldots, \rho_{12}$.

For $T \in \mathrm{Inc}^q(\lambda)$, let $N_T := |\mathrm{range}(T)|$ be the number of distinct labels in T. For each q, we define the **deflation** map

$$\mathsf{Defl}^q : \mathrm{Inc}^q(\lambda) \to \coprod_{0 \leq k \leq q} \mathrm{Inc}_{\mathrm{gl}}^k(\lambda)$$

by

$$[\mathsf{Defl}^q(T)](\mathsf{x}) = \#\{h \in \mathrm{range}(T) : h \leq T(\mathsf{x})\},$$

for $T \in \mathrm{Inc}^q(\lambda)$ and $\mathsf{x} \in \lambda$. Note that $\mathsf{Defl}^q(T) \in \mathrm{Inc}_{\mathrm{gl}}^{N_T}(\lambda)$ and that $N_T \leq q$.

For a positive integer $j \in \mathbb{Z}_{>0}$, write $[j]$ for the set $\{1, 2, \ldots, j\}$. Let $\binom{[j]}{k}$ denote the collection of binary vectors of length j with k 1's. We now define the **content**

vector function

$$\mathsf{Con}^q : \mathrm{Inc}^q(\lambda) \to \{0,1\}^q = \coprod_{0 \le k \le q} \binom{[q]}{k}$$

by

$$\mathsf{Con}^q(T) = (c_1, \dots, c_q),$$

where $c_k = 1$ if $k \in \mathrm{range}(T)$ and $c_k = 0$ if $k \notin \mathrm{range}(T)$.

Example 4.2. If $T = \begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array} \in \mathrm{Inc}^7(2 \times 3)$, then the deflation of T is

$$\mathsf{Defl}^7(T) = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 & 4 \\ \hline \end{array} \in \mathrm{Inc}_{\mathrm{gl}}^5(2 \times 3).$$

The content vector of T is $\mathsf{Con}^7(T) = (1,1,0,1,1,1,0) \in \binom{[7]}{5} \subset \{0,1\}^7.$ $\diamondsuit$

Clearly, the product map

$$\mathsf{DeflCon}^q = (\mathsf{Defl}^q, \mathsf{Con}^q) : \mathrm{Inc}^q(\lambda) \to \coprod_{0 \le k \le q} \left(\mathrm{Inc}_{\mathrm{gl}}^k(\lambda) \times \binom{[q]}{k} \right)$$

is injective. Indeed, a left inverse for $\mathsf{DeflCon}^q$ is given as follows. For a binary vector $v \in \{0,1\}^q$, let $N_v := \langle v, (1, ..., 1) \rangle$ be the number of 1s in v, and define a **vector inflation** map

$$\mathsf{VecInfl}_v^q : [N_v] \to [q]$$

by

$$\mathsf{VecInfl}_v^q(k) = \min_{n \in [q]} \left(\sum_{\ell=1}^{n} v_\ell = k \right).$$

Then, $\mathsf{VecInfl}_{\mathsf{Con}^q(T)}^q \circ \mathsf{Defl}^q(T) = T$. Therefore, the **tableau inflation** map

$$\mathsf{Infl}^q : \coprod_{0 \le k \le q} \left(\mathrm{Inc}_{\mathrm{gl}}^k(\lambda) \times \binom{[q]}{k} \right) \to \mathrm{Inc}^q(\lambda)$$

given by

$$\mathsf{Infl}^q(S, v) := \mathsf{VecInfl}_v^q \circ S$$

is a left inverse for $\mathsf{DeflCon}^q$.

Example 4.3. Let $v = (1, 1, 0, 1, 1, 1, 0) \in \{0, 1\}^7$. Then, $N_v = 5$ and $\mathsf{VecInfl}_v^7 : [5] \to [7]$ is given by

$$\mathsf{VecInfl}_v^7(1) = 1,$$
$$\mathsf{VecInfl}_v^7(2) = 2,$$
$$\mathsf{VecInfl}_v^7(3) = 4,$$
$$\mathsf{VecInfl}_v^7(4) = 5,$$
$$\mathsf{VecInfl}_v^7(5) = 6.$$

Now, for $S = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 & 4 \\ \hline \end{array}$, we have $\mathsf{Infl}^7(S, v) = \mathsf{VecInfl}_v^7 \circ S = \begin{array}{|c|c|c|} \hline 4 & 5 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array}$. Note that this process has recovered the tableau T of Example 4.2 from $S = \mathsf{Defl}^7(T)$ and $v = \mathsf{Con}^7(T)$. $\diamondsuit$

4.2. **Interaction between K-promotion and deflation.** By determining the relation between the maps $\mathsf{DeflCon}^q$ and $\mathfrak{pro}^q$, we will show that the behavior of $\mathfrak{pro}^q$ is controlled by its action on the subset of gapless increasing tableaux. Much of the power of this observation comes from the fact that there are only finitely-many gapless increasing tableaux of any fixed shape λ, and that therefore the action of $\mathfrak{pro}^q$ on the set of all increasing tableaux of shape λ is governed by finite data.

To establish this relation, we first need to introduce notation for the steps in a different characterization of $\mathfrak{pro}^q$. Let $\mathsf{Rep}_{1 \to \bullet}$ be the operator on $\mathrm{Inc}^q(\lambda)$ that replaces each instance of 1 by $\bullet$. Let swap_n be the operator that finds all edge-connected components containing $\bullet$ and n, leaves trivial components unchanged, and switches n and $\bullet$ in nontrivial connected components. Let $\mathsf{Rep}_{\bullet \to q+1}$ replace each instance of $\bullet$ with $q + 1$. Finally, let Decr be the operator that decrements each label by 1. Then, [DPS17, Proposition 2.4] gives the following alternative characterization of the operation of K-promotion:

$$\mathfrak{pro}^q = \mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to q+1} \circ \mathsf{swap}_q \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}.$$

(This was, in fact, the original definition of K-promotion introduced in [Pec14].) An example of computing K-promotion via this alternative process is shown in Figure 6.

Using this characterization of $\mathfrak{pro}^q$, we give names to the intermediate products of this process. For $T \in \mathrm{Inc}^q(\lambda)$, we define the tableau $T^{(N)} : \lambda \to \{2, \ldots, q, \bullet\}$ by

$$T^{(N)} := \mathsf{swap}_N \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1 \to \bullet}(T)$$

and define the order ideal $\lambda(T^{(N)}) := (T^{(N)})^{-1}(\{2, \ldots, N\})$ to be the set of boxes of $T^{(N)}$ containing entries from $\{2, \ldots, N\}$. Define $T_N := T^{(N)}|_{\lambda(T^{(N)})}$. Intuitively, $T^{(N)}$ shows T at the point in K-promotion at which the boxes labeled 2 through N have been acted upon by swap's, while T_N is the restriction of $T^{(N)}$ to these boxes. Examples of these intermediate tableaux appear in Figure 7. The following lemmas

FIGURE 6. The calculation of the K-promotion $\mathfrak{pro}^{13}(T)$ of the increasing tableau T through the action of a sequence of swaps.

use these intermediate products to show that K-promotion acts on a gappy tableau in essentially the same way and as on its deflation, provided that the gappy tableau contains at least one 1.

FIGURE 7. The tableau $T^{(3)}$ is obtained from T by applying in sequence the operations $\mathsf{Rep}_{1\to\bullet}$, swap_2, and swap_3. We then obtain T_3 from $T^{(3)}$ by restricting to those boxes with labels less than or equal to 3. The order ideal $\lambda\left(T^{(3)}\right)$ is the shape of the tableau T_3.

Lemma 4.4. Let $T \in \mathrm{Inc}^q(\lambda)$ with $1 \in \mathrm{range}(T)$. Let $1 = i_1 < \cdots < i_{N_T}$ be the ordered list of indices such that $\mathsf{Con}^q(T)(i_\ell) = 1$. Then, for all $\mathsf{x} \in \lambda$, we have

(4.1) $$\mathsf{Defl}^q(T)^{(k)}(\mathsf{x}) = \bullet \text{ if and only if } T^{(i_k)}(\mathsf{x}) = \bullet,$$

for $1 \leq k \leq N_T$.

Proof. We proceed by induction on k. Fix $T \in \mathsf{Inc}^q(\lambda)$ with $1 \in \mathrm{range}(T)$. In the base case $k = 1$, $T^{(i_1)} = T^{(1)} = \mathsf{Rep}_{1 \to \bullet}(T)$ has $\bullet$ in exactly those boxes $\mathsf{x} \in \lambda$ such that $T(\mathsf{x}) = 1$. On the other hand, $\mathsf{Defl}^q(T)^{(1)} = \mathsf{Rep}_{1 \to \bullet}(\mathsf{Defl}^q(T))$ has $\bullet$ in exactly those boxes $\mathsf{x} \in \lambda$ such that $\mathsf{Defl}^q(T)(\mathsf{x}) = 1$. But since $1 \in \mathrm{range}(T)$, $\mathsf{Defl}^q(T)(\mathsf{x}) = 1$ if and only if $T(\mathsf{x}) = 1$.

Now, say the lemma holds for all tableaux T for $k < t$. First consider the case where $T^{(i_t)}(\mathsf{x}) = \bullet$ and $T^{(i_{t-1})}(\mathsf{x}) \neq \bullet$. Then, $T^{(i_{t-1})}(\mathsf{x}) = i_t$ and $T^{(i_{t-1})}(\mathsf{x}') = \bullet$ for some x' adjacent to x. But since $\mathsf{Rep}_{1 \to \bullet}$ and swap_j for $j \leq i_{t-1}$ only act on elements of T labeled with values less than or equal to i_{t-1}, having $T^{(i_{t-1})}(\mathsf{x}) = i_t$ is equivalent to having $T^{(i_{t-1})}(\mathsf{x}) = T(\mathsf{x})$. This is, in turn, equivalent to $\mathsf{Defl}^q(T)(\mathsf{x}) = t$ and to $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}) = t$. Additionally, by the inductive hypothesis, $T^{(i_{t-1})}(\mathsf{x}') = \bullet$ if and only if $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}') = \bullet$. Therefore, the conditions of this case are equivalent to requiring $\mathsf{Defl}^q(T)^{(t)}(\mathsf{x}) = \bullet$ and $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}) \neq \bullet$.

Consider the other possible case, where $T^{(i_t)}(\mathsf{x}) = \bullet$ and $T^{(i_{t-1})}(\mathsf{x}) = \bullet$. Here, there is no x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = i_t$. By similar reasoning as in the previous case, these conditions are equivalent to the statement that there is no x' adjacent to x with $T(\mathsf{x}') = i_t$. This statement is, in turn, equivalent to saying that there is no x' adjacent to x with $\mathsf{Defl}^q(T)(\mathsf{x}') = \mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}') = t$. Since $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}) = \bullet$ by the inductive hypothesis, the conditions of this case are finally equivalent to requiring that $\mathsf{Defl}^q(T)^{(t)}(\mathsf{x}) = \bullet$ and $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}) = \bullet$. $\qquad\square$

Lemma 4.5. *Let $T \in \mathsf{Inc}^q(\lambda)$ with $1 \in \mathrm{range}(T)$. Let $1 = i_1 < \cdots < i_{N_T}$ be the ordered list of indices such that $\mathsf{Con}^q(T)(i_\ell) = 1$. Then, for $1 \leq k \leq N_T$, we have*

$$(4.2) \qquad \mathsf{Infl}^q\Big(\mathsf{Defl}^q(T)_k, \mathsf{Con}^q(T)\Big) = T_{i_k}.$$

That is, the two functions are identical tableaux on the same domain.

Proof. We proceed by induction on k. The base case $k = 1$ is trivial, since if $T \in \mathsf{Inc}^q(\lambda)$ is any tableau, then $\lambda\big(\mathsf{Defl}^q(T)^{(1)}\big)$ and $\lambda\big(T^{(1)}\big)$ are both the empty order ideal.

Now, assume that the lemma holds for all increasing tableaux T in the case $k < t$. We first analyze the left side of Equation (4.2). If $\mathsf{x} \in \lambda\big(\mathsf{Defl}^q(T)^{(t-1)}\big)$, then $[\mathsf{Defl}^q(T)^{(t-1)}](\mathsf{x}) \leq t - 1$ and hence

$$\big[\mathsf{swap}_t(\mathsf{Defl}^q(T)^{(t-1)})\big](\mathsf{x}) = \mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}).$$

If instead

$$\mathsf{x} \in \lambda\big(\mathsf{Defl}^q(T)^{(t)}\big) \setminus \lambda\big(\mathsf{Defl}^q(T)^{(t-1)}\big),$$

then either $[\mathsf{Defl}^q(T)^{(t-1)}](\mathsf{x}) = \bullet$ or $[\mathsf{Defl}^q(T)^{(t-1)}](\mathsf{x}) = t$. In the former case, there must be a box x' adjacent to x such that $[\mathsf{Defl}^q(T)^{(t-1)}](\mathsf{x}') = t$. In the latter case, we must have that for all boxes x' adjacent to x, $[\mathsf{Defl}^q(T)^{(t-1)}](\mathsf{x}') \neq \bullet$.

Now, we analyze the right side of Equation (4.2). If $\mathsf{x} \in \lambda\big(T^{(i_{t-1})}\big)$, then $T^{(i_{t-1})}(\mathsf{x}) \leq i_{t-1}$ and so we have

$$\big[\mathsf{swap}_{\ell+1}\big(T^{(\ell)}\big)\big](\mathsf{x}) = T^{(\ell)}(\mathsf{x})$$

for all $i_{t-1} \leq \ell$. Therefore, by the previous paragraph and the inductive hypothesis, this shows that Equation (4.2) holds at all such x. If $\mathsf{x} \in \lambda\left(T^{(i_t)}\right) \setminus \lambda\left(T^{(i_{t-1})}\right)$, then $i_{t-1} < T_{i_t}(\mathsf{x}) \leq i_t$. Thus, we must have that $T_{i_t}(\mathsf{x}) = i_t$, since swaps do not change the content of a tableau. Therefore, either $T^{(i_{t-1})}(\mathsf{x}) = \bullet$ or $T^{(i_{t-1})}(\mathsf{x}) = T(\mathsf{x}) = i_t$. In the former case, there must be a box x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = T(\mathsf{x}') = i_t$. This occurs exactly when $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}') = \mathsf{Defl}^q(T)(\mathsf{x}') = t$. In the latter case, there is no x' adjacent to x such that $T^{(i_{t-1})}(\mathsf{x}') = \bullet$. Since we have assumed that $1 \in \mathrm{range}(T)$, Lemma 4.4 applies, implying that this occurs exactly when $\mathsf{Defl}^q(T)^{(t-1)}(\mathsf{x}') \neq \bullet$ for every box x' adjacent to x. This proves that Equation (4.2) holds for the remaining boxes.

This shows that $\lambda\left(T^{(i_k)}\right) = \lambda\left(\mathsf{Defl}^q(T)^{(k)}\right)$ for all k, and that moreover, if $h = \mathsf{Defl}^q(T)_k(\mathsf{x})$, then $T_k(\mathsf{x}) = \mathsf{VecInfl}^q_{\mathsf{Con}^q(T)}(h)$. $\qquad\square$

Remark 4.6. As a corollary to Lemma 4.5, one can show that each T_k is, in fact, in $\mathrm{Inc}^q\left(\lambda\left(T^{(k)}\right)\right)$. However, we do not need this observation here.

Lemma 4.7. *Let* $T \in \mathrm{Inc}^q(\lambda)$ *with* $1 \in \mathrm{range}(T)$. *Let* $1 = i_1 < \cdots < i_{N_T}$ *be the ordered list of indices such that* $\mathsf{Con}^q(T)(i_\ell) = 1$. *Then*

$$(4.3) \qquad \mathsf{Defl}^q \circ \mathfrak{pro}^q(T) = \mathfrak{pro}^{N_T} \circ \mathsf{Defl}^q(T).$$

Proof. First, consider $\mathsf{x} \in \lambda$ with $\mathsf{x} \notin \lambda\left(T^{(q)}\right)$. Then, $T^{(q)}(\mathsf{x}) = \bullet$ and so

$$\mathfrak{pro}^q(T)(\mathsf{x}) = [\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to q+1}(T^{(q)})](\mathsf{x}) = q.$$

Now by [DPS17, Lemma 2.1], the content vector of $\mathfrak{pro}^q(T)$ is a cyclic rotation of the content vector of T. This implies that both content vectors contain the same number of 1s. Since q is the greatest element of $\mathrm{range}(\mathfrak{pro}^q(T))$, we then have

$$\mathsf{Defl}^q \circ \mathfrak{pro}^q(T)(\mathsf{x}) = N_T.$$

By Lemma 4.4, we observe on the other hand that $\mathsf{x} \notin \lambda\left(\mathsf{Defl}^q(T)^{(N_T)}\right)$. Hence, $\mathsf{Defl}^q(T)^{(N_T)}(\mathsf{x}) = \bullet$, and thus,

$$[\mathfrak{pro}^{N_T} \circ \mathsf{Defl}^q(T)](\mathsf{x}) = \left[\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to N_T+1}\left(\mathsf{Defl}^q(T)^{(N_T)}\right)\right](\mathsf{x}) = N_T.$$

Thus, the two sides of Equation (4.3) agree at such a box $\mathsf{x} \notin \lambda\left(T^{(q)}\right)$.

Now, consider $\mathsf{x} \in \lambda$ with $\mathsf{x} \in \lambda\left(T^{(q)}\right)$. We have $T^{(q)}(\mathsf{x}) = \ell$ for some $2 \leq \ell \leq q$. Indeed, $\ell = i_k$ for some $2 \leq k \leq N_T$. Then,

$$\mathfrak{pro}^q(T)(\mathsf{x}) = \left[\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to q+1}\left(T^{(q)}\right)\right](\mathsf{x}) = i_k - 1.$$

Because, by [DPS17, Lemma 2.1], the content vector of $\mathfrak{pro}^q(T)$ is a cyclic rotation of the content vector of T, this implies that

$$[\mathsf{Defl}^q \circ \mathfrak{pro}^q(T)](\mathsf{x}) = k - 1.$$

On the other hand, observe that since swap_ℓ acts trivially on $T^{(i_{N_T})}$ for $\ell > i_{N_T}$, we have $T_{i_{N_T}} = T_q$. Hence, $T_{i_{N_T}}(\mathsf{x}) = T_q(\mathsf{x}) = i_k$. By Lemma 4.5 with $k = N_T$,

$$T_{i_{N_T}} = \mathsf{Infl}^q\Big(\mathsf{Defl}^q(T)_{N_T}, \mathsf{Con}^q(T)\Big).$$

Thus, we must have

$$[\mathsf{Defl}^q(T)_{N_T}](\mathsf{x}) = k,$$

and so

$$[\mathfrak{pro}^{N_T} \circ \mathsf{Defl}^q(T)](\mathsf{x}) = [\mathsf{Decr} \circ \mathsf{Rep}_{\bullet \to N_T+1}(\mathsf{Defl}^q(T)_{N_T})](\mathsf{x}) = k - 1.$$

Thus, the two sides of Equation (4.3) also agree at such boxes $\mathsf{x} \in \lambda\left(T^{(q)}\right)$. □

> Need an easy lemma about promotion when there is no 1 in T.
>
> — Oliver

4.3. Computation of periods.

Define a **cyclic shift** map $\mathsf{sh} : \{0,1\}^q \to \{0,1\}^q$ on binary vectors by

$$\mathsf{sh}(v)_i = \begin{cases} v_{i+1}, & \text{for } i \neq q; \\ v_1, & \text{for } i = q. \end{cases}$$

Define $\mathfrak{pro}^q_* : \coprod_{0 \leq k \leq q}\left(\mathsf{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}\right) \to \coprod_{0 \leq k \leq q}\left(\mathsf{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}\right)$ by

$$\mathfrak{pro}^q_*(T, v) = \begin{cases} \Big(\mathfrak{pro}^k(T), \mathsf{sh}(v)\Big), & \text{if } 1 \in \mathrm{range}(T); \\ \Big(\mathsf{Defl}(T), \mathsf{sh}(v)\Big), & \text{otherwise.} \end{cases}$$

Then, $\mathsf{DeflCon}^q \circ (\mathfrak{pro}^q)^{\circ n}(T) = \mathfrak{pro}^{\circ n}_* \circ \mathsf{DeflCon}^q(T)$. Now, since $\mathsf{DeflCon}$ is an injection, $\mathfrak{pro}^n(T) = T$ exactly when $\mathsf{DeflCon} \circ \mathfrak{pro}^n(T) = \mathsf{DeflCon} \circ T$. So we are reduced to looking at what powers n of K stabilize $\mathsf{DeflCon}(T)$. Clearly we must have $\Sigma^n(\mathsf{Con}(T)) = \mathsf{Con}(T)$. In addition, we must have that $\mathfrak{pro}^{n'}(\mathsf{Defl}(T))$, where n' is n minus the number of times that $K^\ell(W(T))$ does not have 1 in its content vector for $\ell \in 1, ..., n$. Now say that the period of $\mathsf{Con}(T)$ is ℓ and say there are r 0's in the first ℓ entries of $\mathsf{Con}(T)$ which then repeat q/ℓ times. Then if $n = \ell$, $n' = \ell - r$. Now say we know that the period of $\mathsf{Defl}(T)$ is τ. This reasoning shows that $K^n(\mathsf{DeflCon}(T)) = \mathsf{DeflCon}(T)$ if and only if $n = e\ell$ for some positive integer e and $n = s\tau + er$. Therefore the period of T is equal to the minimum integer that can be written of the form $e\ell$ and $s\tau + er$ for the same e. But $e\ell = s\tau + er$ implies that $e(\ell - r) = s\tau$. Since $\ell > r \geq 0$ the minimum of $e\ell$ is clearly achieved when e is minimal, which occurs when $e(\ell - r)$ is minimal, which is the least common multiple of $\ell - r$ and τ. Thus the minimal n can be expressed as

$$n = e\ell = \frac{\ell \operatorname{lcm}(\ell - r, \tau)}{\ell - r} = \frac{\ell \tau}{\gcd(\ell - r, \tau)}.$$

This paragraph probably goes away Let λ be a shape of maximum rank N. We establish an injection, which we call $W(T) = (W_1(T), W_2(T))$, from increasing tableaux of height $q > N$ to pairs of increasing tableaux of height less than or equal to

$N + 1$ and binary vectors of length q. In addition, let Σ be a function that permutes vectors of length n by moving entry n to entry $n - 1$ modulo n, We then show that for $T \in \mathrm{Inc}^q(\lambda)$

(1) if $T(0, 0) = 1$, then $\mathfrak{pro}(T) = W^{-1}(\mathfrak{pro}(W_1(T)), \Sigma(W_2(T)))$
(2) if $T(0, 0) > 1$, then $\mathfrak{pro}(T) = W^{-1}(W_1(T), \Sigma(W_2(T)))$.

We then use this description to compute the period of an inflated orbit. The basic idea is that K-Promotion acts in two separate ways on a tableau: it changes the relative order of certain boxes and permutes the content vector. These two actions are independent, unless 1 is not an entry in the tableau, in which case only the content vector permutation occurs. **End this paragraph probably goes away**

5. The equivariant bijection

5.1. **Bijection between plane partitions and increasing tableaux.** Let e_i be the ith standard basis vector in $\mathbb{Z}^2$ and let $\pi_i : \mathbb{Z}^2 \to \mathbb{Z}$ be the ith coordinate projection map. We say that a subset $\lambda \subset \mathbb{Z}^2$ is **connected** if for all $\mathsf{x}, \mathsf{x}' \in \lambda$, there exist $r \in \mathbb{Z}_{\geq 0}$, $d_1, ..., d_r \in \{1, 2\}$, and signs $n_1, ..., n_r \in \{\pm 1\}$ such that

$$\left\{ \mathsf{x} + \sum_{k=1}^{\ell} n_k \mathsf{e}_{d_k} : 1 \leq j \leq r \right\} \subseteq \lambda$$

and $\mathsf{x} + \sum_{k=1}^{r} n_k \mathsf{e}_{d_k} = \mathsf{x}'$. Given these data, define $\mathsf{x}_0 = \mathsf{x}$ and $\mathsf{x}_\ell = \mathsf{x} + \sum_{k=1}^{\ell} n_k \mathsf{e}_{d_k}$ for $1 \leq \ell \leq r$. Then the sequence $(\mathsf{x}_\ell)_{0 \leq \ell \leq r}$ is called a **path of length** r. If $n_k = 1$ for all k, then the path is called a **chain**. We call a finite nonempty connected subset of $\mathbb{Z}_{\geq 0}^2$ a **shape**. Shapes inherit a partial order from the natural componentwise partial order on $\mathbb{Z}^2$. A shape poset is ranked by the rank function $\mathrm{rk}(x_1, x_2) = x_1 + x_2$.

We say that the shape λ is **convex** if $\mathsf{x} \in \lambda$ and $\mathsf{x} + k\mathsf{e}_d \in \lambda$ for some $d \in \{1, 2\}$ and $k \in \mathbb{Z}_{>0}$ together imply that $\{\mathsf{x} + j\mathsf{e}_d : 1 \leq j \leq k - 1\} \subseteq \lambda$.

For any shape λ, let

$$a_i(\lambda) = \max\{\pi_i(\mathsf{x}) : \mathsf{x} \in \lambda\}$$

and

$$\mathsf{a}_\lambda = (a_1(\lambda), a_2(\lambda)) \in \mathbb{Z}^2.$$

We say the shape λ is **justified** if $(0, 0) \in \lambda$ and $\mathsf{a}_\lambda \in \lambda$. Define

$$\lambda^\star := \{\mathsf{a}_\lambda - \mathsf{x} : \mathsf{x} \in \lambda\}.$$

It is clear that $\lambda^\star$ is a shape and that if λ is convex (resp. justified), then $\lambda^\star$ is convex (resp. justified). Define the **rank** of λ

$$\mathrm{rk}(\lambda) = \max_{\mathsf{x} \in \lambda} \mathrm{rk}(\mathsf{x}).$$

The following elementary lemmas determine useful characteristics of the shapes considered in this paper.

Lemma 5.1. *Let λ be a convex shape and let $\mathsf{x}, \mathsf{x}' \in \lambda$ be distinct. If $\mathsf{x} \leq \mathsf{x}'$, then λ contains a chain from x to x'.*

Proof. Let $\mathsf{x} = (i, j)$ and $\mathsf{x}' = (i', j')$. Because $\mathsf{x} \leq \mathsf{x}'$, we have $i \leq i'$ and $j \leq j'$. Since λ is a shape, there is at least one path from x to x'. Take such a path $P = (\mathsf{x}_\ell)_{0 \leq \ell \leq r}$ of minimal length, characterized by directions $d_1, ..., d_r \in \{1, 2\}$ and signs $n_1, ..., n_r \in \{\pm 1\}$. We claim that P is a chain.

If not, then $n_t = -1$ for some $1 \leq t \leq r$. Let t be the least such index. For $0 \leq \ell \leq r$, define $(i_\ell, j_\ell) = \mathsf{x}_\ell$, so that in particular $(i_0, j_0) = (i, j)$ and $(i_r, j_r) = (i', j')$.

Since $n_t = -1$, either $i_t = i_{t-1} - 1$ or $j_t = j_{t-1} - 1$. Without loss of generality, we may assume $i_t = i_{t-1} - 1$.

Suppose there is no $h < t$ with $d_h = d_t$ and $n_h = 1$. Then $i_t = i_0 - 1$. Furthermore, since $i' = i_r \geq i_0$, there must be a least $k > t$ such that $i_k = i_0 = i_{t-1}$. Let $q = j_k - j_{t-1}$. Then clearly the subpath $(\mathsf{x}_\ell)_{t-1 \leq \ell \leq k}$ of P has length at least $|q| + 2$. However by convexity, the path

$$\left((i_{t-1}, j_{t-1}), (i_{t-1}, j_{t-1} + \frac{q}{|q|}), (i_{t-1}, j_{t-1} + 2\frac{q}{|q|}), \ldots, (i_{t-1}, j_{t-1} + |q|\frac{q}{|q|}) \right)$$

of length $|q|$ from (i_{t-1}, j_{t-1}) to (i_k, j_k) lies entirely inside λ, contradicting the minimality of P.

Otherwise, consider the greatest $h < t$ with $d_h = d_t$ and $n_h = 1$. Then $i_{h-1} = i_t$. By minimality of t, we have $j_t \geq j_{h-1}$. Let $q = j_t - j_{h-1}$. Now the subpath $(\mathsf{x}_\ell)_{h-1 \leq \ell \leq t}$ of P has length $q + 2$. However by convexity, the path

$$((i_{h-1}, j_{h-1}), (i_{h-1}, j_{h-1} + 1), (i_{h-1}, j_{h-1} + 2), \ldots, (i_{h-1}, j_{h-1} + q))$$

of length q from (i_{h-1}, j_{h-1}) to (i_t, j_t) lies entirely inside λ, contradicting the minimality of P. $\qquad\square$

Lemma 5.2. *Let λ be a convex justified shape. Then every maximal chain has length $a_1(\lambda) + a_2(\lambda)$.*

Proof. Since for all $\mathsf{x} \in \lambda$ we have $(0, 0) \leq \mathsf{x}$ and $\mathsf{x} \leq (a_1(\lambda), a_2(\lambda))$, all maximal chains must start at $(0, 0)$ and end at $(a_1(\lambda), a_2(\lambda))$ by Lemma 5.1. But clearly any chain between these two elements has the desired length. $\qquad\square$

For a shape λ, an increasing tableau on the poset λ is equivalent to a map $T : \lambda \to \mathbb{Z}_{>0}$ such that whenever $\mathsf{x} \in \lambda$ and $\mathsf{x} + \mathsf{e}_i \in \lambda$ for $i \in \{1, 2\}$, then $T(\mathsf{x}) < T(\mathsf{x} + \mathsf{e}_i)$. The **height** of an increasing tableau T is the maximum value in its range. Recall that $\mathrm{Inc}^q(\lambda)$ is the set of all increasing tableaux of height at most q and base λ.

Fix a justified convex shape λ and $m \in \mathbb{Z}_{>0}$. Now, we define a poset

$$\lambda \times [m] := \{(\mathsf{x}, y) \in \mathbb{Z}^3 : \mathsf{x} \in \lambda, 0 \leq y \leq m - 1\},$$

with the partial order given by restricting the natural partial order on $\mathbb{Z}^3$, that is

$$\mathsf{p} \geq \mathsf{q} \text{ if and only if } \mathsf{p} - \mathsf{q} \in \mathbb{Z}^3_{\geq 0}.$$

Recall that for any poset P, we write $J(P)$ for the collection of all order ideals of P. Note that $J(\lambda \times [m])$ is exactly the set of all plane partitions of height at most m over the poset λ.

Define the map $\Omega : J(\lambda \times [m]) \to \mathrm{Inc}^{\mathrm{rk}(\lambda)+m+1}(\lambda^{\star})$ as follows. First, define the function

$$\mathrm{ht} : \mathbb{Z}_{\geq 0}^2 \times 2^{(\mathbb{Z}_{\geq 0}^3)} \to \mathbb{Z}_{\geq 0},$$

where $2^{(\mathbb{Z}_{\geq 0}^3)}$ denotes the power set of $\mathbb{Z}_{\geq 0}^3$ by

$$\mathrm{ht}(\mathsf{x}, S) := \#\{y \in \mathbb{Z}_{\geq 0} : (\mathsf{x}, y) \in S\}.$$

Intuitively, if an order ideal is viewed as a pile of blocks, then ht gives the number of blocks in a given column. Next, for $I \in J(\lambda \times [m])$, define $\Omega(I) := T$, where the increasing tableau

$$T : \lambda^{\star} \to [\mathrm{rk}(\lambda) + m + 1]$$

is given by

$$T(\mathsf{x}) = \mathrm{ht}(\mathsf{a}_\lambda - \mathsf{x}, I) + \mathrm{rk}(\mathsf{x}) + 1.$$

Proposition 5.3. *If λ is a justified convex shape, then*

$$\Omega : J(\lambda \times [m]) \to \mathrm{Inc}^{\mathrm{rk}(\lambda)+m+1}(\lambda^{\star})$$

is a bijection.

Proof. Since λ is justified and convex, $\lambda^{\star}$ has an element of rank 0 and all maximal chains in $\lambda^{\star}$ have equal length by Lemma 5.2. Hence the function Ω coincides with the map from [HPPW16, Proposition 8.2], and so by [HPPW16, Proposition 8.2] we have that Ω is a bijection between plane partitions over λ and increasing tableaux over $\lambda^{\star}$. $\qquad\square$

Note, in particular, that Proposition 5.3 applies to all of the minuscule posets, realized as justified convex shapes as in Figures 1 and 3.

5.2. **Equivariance of Ω.** The following is the key observation enabling our approach to proving Theorems 1.2 and 1.3. Proposition 5.5 is a mild extension of [DPS17, Lemma 4.2] and our proof follow the same outline. We however still give a somewhat detailed proof, as there are a few extra subtleties in our case.

The following gives an easy and useful way to determine whether a subset of $\lambda \times [m]$ is an order ideal.

Lemma 5.4. *The subset $Q \subseteq \lambda \times [m]$ is an order ideal if and only if both*

    (a) for each $\mathsf{x} \in \lambda$, $Q \cap (\{\mathsf{x}\} \times [m])$ is an order ideal (that is, each column of Q is an order ideal) and;

    (b) $(i, j) \geq (i', j')$ implies that $\mathrm{ht}((i, j), Q) \leq \mathrm{ht}((i', j'), Q)$.

Proof. The "only if" direction is clear.

Conversely, assume Q satisfies (a) and (b). Consider $(i, j, k) \in Q$ and $(i', j', k') \in \lambda \times [m]$ with $(i', j', k') \leq (i, j, k)$. We must show $(i', j', k') \in Q$.

Certainly, $(i', j') \leq (i, j)$. By condition (b), $\mathrm{ht}((i, j), Q) \leq \mathrm{ht}((i', j'), Q)$. Therefore, by condition (a), there is a $k'' \geq k$ with $(i', j', k'') \in Q$. But since $k' \leq k \leq k''$, condition (a) then implies $(i', j', k') \in Q$. $\qquad\square$

Proposition 5.5. *Let λ be a justified convex shape. Then Ω intertwines $\mathsf{Mot}_{id,(1,1,-1)}$ and $\mathfrak{pro}$. That is, for $P \in J(\lambda \times [m])$, we have $\Omega \circ \mathsf{Mot}_{id,(1,1,-1)}(P) = \mathfrak{pro} \circ \Omega(P)$.*

Proof. Let $a = a_1(\lambda)$ and let $b = a_2(\lambda)$, that is, $(a, b) = \mathsf{a}_\lambda$. Fix $P \in J(\lambda \times [m])$.

Recall that for $T \in \mathrm{Inc}^{a+b+m+1}(\lambda)$, we have by definition that

$$\mathfrak{pro}(T) = \rho_{a+b+m} \circ ... \circ \rho_1(T).$$

Since $-(m-1) \le \langle (1,1,-1), (x,y,z) \rangle \le a + b$ for $(x, y, z) \in \lambda \times [m]$, we have

$$\mathsf{Mot}_{id,(1,1,-1)}(P) = \tau_{id,(1,1,-1)}^{a+b-(a+b+m-1)} \circ ... \circ \tau_{id,(1,1,-1)}^{a+b}(P).$$

Therefore, it suffices to show

$$\Omega \left(\tau_{id,(1,1,-1)}^{a+b-(\ell-1)}(P) \right) = \rho_\ell \left(\Omega(P) \right)$$

for all ℓ.

Fix an index ℓ and an entry $\mathsf{y} = (i, j) \in \lambda$. Let $Y := P \cap (\mathsf{y} \times [m])$ be the column of y in P. We consider the effect on Y of applying $\tau_{id,(1,1,-1)}^{a+b-(\ell-1)}$ to P and the effect on the box $\mathsf{a}_\lambda - \mathsf{y}$ of applying ρ_ℓ to the increasing tableau $\Omega(P)$.

Define the function $H : \lambda \to [m]$ by $H(\mathsf{x}) = \mathrm{ht}(\mathsf{x}, P)$ for $\mathsf{x} \in \lambda$. By Lemma 5.4, observe that the function H must be nonincreasing along any chain in λ. There are now four cases to consider depending on the value of $H(\mathsf{y})$.

Case 1: $H(\mathsf{y}) > i + j - (a + b - (\ell - 1)) + 1$:

Here, $(\mathsf{y}, i+j-(a+b-(\ell-1))) \in Y$, so $\tau_{id,(1,1,-1)}^{a+b-(\ell-1)}$ either removes this element from Y or leaves Y unchanged. Since, by the case assumption, $(\mathsf{y}, i+j-(a+b+(\ell-1))+1) \in Y$, removing $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ from P does not yield an order ideal. Therefore the column Y is unchanged by $\tau_{id,(1,1,-1)}^{a+b-(\ell-1)}$. On the other hand, the case assumption implies that $[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y}) > \ell + 1$, so ρ_ℓ does not affect this entry of the increasing tableau $\Omega(P)$.

Case 2: $H(\mathsf{y}) = i + j - (a + b - (\ell - 1)) + 1$:

Here, $(\mathsf{y}, i + j - (a + b - (\ell - 1))) \in Y$, while $(\mathsf{y}, i + j - (a + b - (\ell - 1)) + 1) \notin Y$. We claim that $\tau_{id,(1,1,-1)}^{a+b-(\ell-1)}$ removes $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ from P exactly if $H(\mathsf{y} + \mathsf{e}_d) \le i + j - (a + b - (\ell - 1))$ for all $d \in \{1, 2\}$ with $\mathsf{y} + \mathsf{e}_d \in \lambda$.

For, on the one hand, if $\mathsf{y} + \mathsf{e}_d \in \lambda$ and $H(\mathsf{y} + \mathsf{e}_d) > i + j - (a + b - (\ell - 1))$, then by Lemma 5.4, removing $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ from P would not leave an order ideal. On the other hand, say that $H(\mathsf{y} + \mathsf{e}_d) \le i + j - (a + b - (\ell - 1))$ for all such d. To verify that removing $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ from P results in an order ideal, consider any fixed $\mathsf{y}' = (i', j') \in \lambda$ with $\mathsf{y}' > \mathsf{y}$. By Lemma 5.4, it is enough to check that $H(\mathsf{y}') \le i + j - (a + b - (\ell - 1))$. Now, by Lemma 5.1, since $\mathsf{y}' > \mathsf{y}$, there is a chain from y to y'. Clearly, this chain contains an element of the form $\mathsf{y} + \mathsf{e}_d \in \lambda$. But the function H is nonincreasing along this chain, so $H(\mathsf{y}') \le i + j - (a + b - (\ell - 1))$, as desired.

On the tableaux side, the case assumption implies that $[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y}) = \ell + 1$. The $\mathsf{a}_\lambda - \mathsf{y}$ entry of $\Omega(P)$ is decreased to ℓ by ρ_ℓ exactly when, for all d with $\mathsf{a}_\lambda - \mathsf{y} - \mathsf{e}_d \in \lambda^\star$,

$[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y} - \mathsf{e}_d) \le \ell - 1$. This is clearly equivalent to the stated condition on when $\tau^{a+b-(\ell-1)}_{\mathrm{id},(1,1,-1)}$ acts nontrivially on Y.

Case 3: $H(\mathsf{y}) = i + j - (a + b - (\ell - 1))$:

Here, $(\mathsf{y}, i + j - (a + b - (\ell - 1)) - 1) \in Y$ but $(\mathsf{y}, i + j - (a + b - (\ell - 1))) \notin Y$. We claim that $\tau^{a+b-(\ell-1)}_{\mathrm{id},(1,1,-1)}$ adds $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ to P exactly when $H(\mathsf{y} - \mathsf{e}_d) \ge i + j - (a + b - (\ell - 1)) + 1$ for all $d \in \{1, 2\}$ with $\mathsf{y} - \mathsf{e}_d \in \lambda$.

For, on the one hand, if $\mathsf{y} - \mathsf{e}_d \in \lambda$ and $H(\mathsf{y} - \mathsf{e}_d) < i + j - (a + b - (\ell - 1)) + 1$, then by Lemma 5.4, adding $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ to P does not yield an order ideal. On the other hand, say that $H(\mathsf{y} - \mathsf{e}_d) \ge i + j - (a + b - (\ell - 1)) + 1$ for all such d. To verify that adding $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ to P results in an order ideal, consider any fixed $\mathsf{y}' = (i', j') \in \lambda$ with $\mathsf{y}' < \mathsf{y}$. By Lemma 5.4, it is enough to check that $H(\mathsf{y}') \ge i + j - (a + b - (\ell - 1)) + 1$. Now, by Lemma 5.1, since $\mathsf{y}' < \mathsf{y}$, there is a chain from y' to y. Clearly, this chain contains an element of the form $\mathsf{y} - \mathsf{e}_d \in \lambda$. But the function H is nonincreasing along this chain, so $H(\mathsf{y}) \ge i + j - (a + b - (\ell - 1)) + 1$ as desired.

On the tableaux side, this case assumption implies that $[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y}) = \ell$. The $\mathsf{a}_\lambda - \mathsf{y}$ entry of $\Omega(P)$ is increased to $\ell + 1$ by ρ_ℓ exactly when, for all d with $\mathsf{a}_\lambda - \mathsf{y} + \mathsf{e}_d \in \lambda^\star$, $[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y} + \mathsf{e}_d)) \ge \ell + 2$. This is clearly equivalent to the stated condition on when $\tau^{a+b-(\ell-1)}_{\mathrm{id},(1,1,-1)}$ acts nontrivially on Y.

Case 4: $H(\mathsf{y}) < i + j - (a + b - (\ell - 1))$:

Here, $(\mathsf{y}, i + j - (a + b - (\ell - 1))) \notin Y$, so $\tau^{a+b-(\ell-1)}_{\mathrm{id},(1,1,-1)}$ either adds this element to Y or else leaves Y unchanged. Since, by the case assumption, $(\mathsf{y}, i + j - (a + b + (\ell - 1)) - 1) \notin P$, by Lemma 5.4, adding $(\mathsf{y}, i + j - (a + b - (\ell - 1)))$ to P does not yield an order ideal. Therefore, the column Y is unchanged by $\tau^{a+b-(\ell-1)}_{\mathrm{id},(1,1,-1)}$. On the other hand, the case assumption implies that $[\Omega(P)](\mathsf{a}_\lambda - \mathsf{y}) < \ell$, so ρ_ℓ does not affect this entry of $\Omega(P)$.

This completes the proof of Proposition 5.5 $\qquad\square$

Corollary 5.6. *Let λ be a justified convex shape. Then the multiset of orbit cardinalities of Ψ on $J(\lambda \times [m])$ and the multiset of orbit cardinalities of $\mathfrak{pro}$ on $\mathrm{Inc}^{\mathrm{rk}(\lambda)+m+1}(\lambda^\star)$ are equal.*

Proof. By Proposition 5.3, Ω is a bijection between $J(\lambda \times [m])$ and $\mathrm{Inc}^{\mathrm{rk}(\lambda)+m+1}(\lambda^\star)$. By Proposition 5.5, the orbit structure of $\mathfrak{pro}$ on $\mathrm{Inc}^{\mathrm{rk}(\lambda)+m+1}(\lambda^\star)$ is the same as the orbit structure of $\mathsf{Mot}_{\mathrm{id},(1,1,-1)}$ on $J(\lambda \times [m])$. By Proposition 3.1, the action of Ψ on $J(\lambda \times [m])$ is the same as the action of $\mathsf{Mot}_{\mathrm{id},(1,1,1)}$. By Proposition 3.2, $\mathsf{Mot}_{\mathrm{id},(1,1,1)}$ and $\mathsf{Mot}_{\mathrm{id},(1,1,-1)}$ induce the same orbit structure on $J(\lambda \times [m])$. $\qquad\square$

6. Proving the main theorems

We saw in Section 4.1 that $\mathsf{DeflCon}^q : \mathrm{Inc}^q(\lambda) \to \coprod_{0 \le k \le q} \mathrm{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}$ is injective. Conversely, every $(T, v) \in \mathrm{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}$ gives an element of $\mathrm{Inc}^q(\lambda)$ in the following

sense. Define the **tableau inflation** map

$$\mathsf{Infl}^q : \coprod_{1 \leq k \leq q} \mathrm{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k} \to \mathrm{Inc}^q(\lambda)$$

such that if $(T, v) \in \mathrm{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}$ then

$$\mathsf{Infl}^q(T, v) := \mathsf{VecInfl}^q_v \circ T \in \mathrm{Inc}^q(\lambda)$$

Note that

$$\mathsf{DeflCon}^q \circ \mathsf{Infl}^q(T, v) = (T, v).$$

In particular, Infl^q is injective. ! It is also surjective, since the image of $\mathsf{DeflCon}^q$ is contained in the domain of Infl^q and we showed in the previous section that for $T \in \mathrm{Inc}^q(\lambda)$, $\mathsf{Infl}^q \circ \mathsf{DeflCon}^q(T) = T$.

Fix a shape λ with maximum rank r and s boxes. A gapless tableau on λ must have its maximum label between $r + 1$ and s. The above shows that each tableaux of height q corresponds uniquely to an element of $\mathrm{Inc}^k_{\mathrm{gl}}(\lambda) \times \binom{[q]}{k}$ for some $k \leq q$. Further, each element of $\binom{[q]}{k}$ has a period (under cyclic permutation) which is a divisor of q. ! Therefore to determine the number of tableaux $S \in \mathrm{Inc}^q(\lambda)$ whose period divides a given positive integer L, we can execute the following algorithm.

Algorithm 1
Input: $L \in \mathbb{Z}_{>0}$ and the set $\coprod_k \mathrm{Inc}^k_{\mathrm{gl}}(\lambda)$ of gapless tableaux associated to λ.
Output: x, the number of $T \in \mathrm{Inc}^q(\lambda)$ whose period divides L.

Initialization: Set $x = 0$.
Procedure: Loop over gapless tableaux T of height between $r + 1$ and s.

$\Rightarrow$ For each T, loop over divisors d of L.
 (1) Compute the number i of content vectors with period exactly L/d.
 (2) Compute the period $\pi(L/d, T)$ associated with T and a content vector of period L/d, as described in Section 4.3.
 (3) If $\pi(L/d, T)$ divides L, add i to x.

This program can be simplified. First, since the numbered steps are identical for gapless tableaux from the same orbit, we can instead loop over *orbits* of gapless tableaux and replace i with (orbit size) $\times i$ in Step (3).

Second, we can determine in advance that only a subset of divisors of L will yield a nonzero number of content vectors. For the number of ones in such a content vector is equal to the full-content height of the tableaux. Call this height f. If the content vector divides into d identical segments, then $L - f$ must be divisible by d. Therefore, d must be a divisor of both L and f. While L is arbitrary, f is fixed for a given iteration of the loop and can take only finitely many values.

To execute Step (1), we use the Rush-Shi formula which has already been proven for a linear tableau (NOT CLEAR WHAT THIS MEANS) [RSW04, Theorem 1.1(b)].

! For a given d, this tells us the number of content vectors of length L with f ones whose period is a multiple of L/d. By the argument in the previous paragraph, this number will be zero for all but finitely many integers d. We can then recover the number of tableaux with period exactly L/d by subtracting off those whose periods strictly divide L/d.

Algorithm 2
Input: Integer L, decreasing list W of all divisors of both L and f.
Output: Array V, where $V[\ell]$ is the number of vectors of length T with f ones and with period exactly T/d_ℓ

Initialization: Let $k = \texttt{length}(W)$. Create a vector V of k zeros.
Procedure: Loop over $\ell = 1, \ldots, k$.

$\Rightarrow$ Let $d_\ell = W[\ell]$.
$\Rightarrow$ $V[\ell]+ =$ the number of content vectors of length L and f ones whose period divides ℓ.
$\Rightarrow$ For $m = 1, ..., \ell - 1$, if d_m divides d_ℓ then $V[\ell]+ = -V[m]$.

To summarize, we outline the code in `symbolic_cayley_plane.py`. We fix a tableaux λ and define a symbolic integer variable `m`. The predicted number of tableaux in $\text{Inc}^q(\lambda)$ will be a function of `m` which is interpreted as $m = q - (\text{rk}(\lambda) + 1)$. Let $B(\lambda)$ be the number of boxes in λ.

Algorithm 3: `symbolic_cayley_plane.py`
Input: Shape λ, full content tableaux for λ.
Output: Array V, where $V[\ell]$ is the number of vectors of length T with f ones and with period exactly T/d_ℓ

Initialization: Let `m` be a symbolic variable.
Procedure: Loop over all combinations C of prime numbers (with multiplicity) that can divide a number between $\text{rk}(\lambda) + 1$ to $B(\lambda)$ inclusive.

(1) Initialize empty array `dVals`. `For d` $\in [2, B(\lambda)]$, determine whether `d` divides a q value with the combination C of divisors. If so, append to `dVals`.
(2) Loop over the elements of `dVals` and compute the predicted period by the Rush-Shi formula. Save as `RushShiPeriods[L/d]`.
(3) Loop over full content orbit data.
   (a) `For` $d \in$ `dVals, if` d divides `fcHt`
      (i) Get exact number of vectors of length `q` with `fcHt` ones.
      (ii) Compute period
      (iii) number of tableaux to all periods divided by this one.
(4) Check if Rush-Shi and computed match.

q	orbit size	number of orbits
11	1	1
12	3	1
	12	1
13	13	6
14	7	2
	14	12
15	15	13
16	2	1
	4	1
	8	1
	16	4

TABLE 1. The number of orbits of full-content increasing tableaux in $\mathrm{Inc}^q(P)$ for each q for the E_6 poset.

Proof of Theorem 1.2.

> This should be probably very brief, given all the work done above. Mostly one just wants to call all the lemmas and chain them together. Not sure if we can prove the propeller theorem in the same proof environment, or whether it wants its own separate one.
>
> — Olive

$\square$

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q	orbit size	number of orbits
17	1	1
18	2	1
	18	2
19	19	30
20	20	228
21	7	3
	21	1044
22	22	3053
	66	2
23	23	5813
	69	13
24	8	7
	24	7195
	48	4
	72	26
25	25	5602
	50	8
	75	21
26	2	2
	26	2495
	52	4
	78	6
27	3	2
	9	4
	27	484

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