WHAT IS...

a Young Tableau?

Alexander Yong

Young tableaux are ubiquitous combinatorial objects making important and inspiring appearances in representation theory, geometry, and algebra. They naturally arise in the study of symmetric functions, representation theory of the symmetric and complex general linear groups, and Schubert calculus of Grassmannians. Discovering and interpreting enumerative formulas for Young tableaux (and their generalizations) is a core theme of *algebraic combinatorics*.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$ be a **partition** of size $|\lambda| = \lambda_1 + \ldots + \lambda_k$, identified with its **Young diagram**: a left-justified shape of k rows of boxes of length $\lambda_1, \ldots, \lambda_k$. For example, $\lambda = (4,2,1)$ is drawn $\begin{smallmatrix}\square&\square&\square&\square\\ \square&\square&&\\ \square&&&\end{smallmatrix}$. A **(Young) filling** of λ assigns a positive integer to each box of λ, e.g., $\begin{smallmatrix}2&1&1&4\\ 6&2&&\\ 4&&&\end{smallmatrix}$. A filling is **semistandard** if the entries weakly increase along rows and strictly increase along columns. A semistandard filling is **standard** if it is a bijective assignment of $\{1,2\ldots,|\lambda|\}$. So $\begin{smallmatrix}1&2&2&4\\ 2&3&&\\ 4&&&\end{smallmatrix}$ is a **semistandard Young tableau** while $\begin{smallmatrix}1&3&4&6\\ 2&7&&\\ 5&&&\end{smallmatrix}$ is a **standard Young tableau**, both of **shape** λ.

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The author thanks Sergey Fomin, Victor Reiner, Hugh Thomas, Alexander Woo, and the editors for helpful suggestions. The author was partially supported by NSF grant DMS 0601010 and an NSERC postdoctoral fellowship.

We focus on the enumeration and generating series of Young tableaux. Frame-Robinson-Thrall's elegant (and nontrivial) **hook-length formula** states that the number of standard Young tableaux of shape λ is $f^{\lambda} := \frac{|\lambda|!}{\prod_b h_b}$, where the product in the denominator is over all boxes b of λ and h_b is the **hook-length** of b, i.e., the number of boxes directly to the right or below b (including b itself). Thus, $f^{(4,2,1)} = \frac{7!}{6\cdot 4\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1} = 35$.

A similar *hook-content formula* counts the number of semistandard Young tableaux, but we now consider instead their generating series: fix λ and a bound N on the size of the entries in each semistandard tableau T. Let $\mathbf{x}^T = \prod_{i=1}^{N} x_i^{\#i\text{'s in }T}$. The **Schur polynomial** is the generating series $s_\lambda(x_1,\ldots,x_N) := \sum_{\text{semistandard }T} \mathbf{x}^T$. For example, when $N = 3$ and $\lambda = (2,1)$ there are eight semistandard Young tableaux:

$$\begin{smallmatrix}1&1\\2&\end{smallmatrix},\ \begin{smallmatrix}1&2\\2&\end{smallmatrix},\ \begin{smallmatrix}1&3\\2&\end{smallmatrix},\ \begin{smallmatrix}1&2\\3&\end{smallmatrix},\ \begin{smallmatrix}1&1\\3&\end{smallmatrix},\ \begin{smallmatrix}1&3\\3&\end{smallmatrix},\ \begin{smallmatrix}2&2\\3&\end{smallmatrix},\ \begin{smallmatrix}2&3\\3&\end{smallmatrix}.$$

The corresponding Schur polynomial, with terms in the same order, is $s_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$. In general, these are **symmetric polynomials**, i.e., $s_\lambda(x_1,\ldots,x_N) = s_\lambda(x_{\sigma(N)},\ldots,x_{\sigma(N)})$ for all σ in the symmetric group $\mathfrak{S}_N$ (the proof is a "clever trick" known as the *Bender-Knuth involution*).

Both the irreducible complex representations of $\mathfrak{S}_n$ and the irreducible degree n polynomial representations of the general linear group $GL_N(\mathbb{C})$ are indexed by partitions λ with $|\lambda| = n$. The associated irreducible $\mathfrak{S}_n$-representation has dimension equal to f^λ, while the irreducible $GL_N(\mathbb{C})$ representation has character $s_\lambda(x_1,\ldots,x_N)$. These facts can be proved with an explicit construction of the respective representations having a basis indexed by the appropriate tableaux.

In algebraic geometry, the Schubert varieties, in the complex Grassmannian manifold $Gr(k, \mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$, are indexed by partitions λ contained inside a $k \times (n-k)$ rectangle. Here, the Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ represents the class of the Schubert variety under a natural presentation of the cohomology ring $H^\star(Gr(k, \mathbb{C}^n))$.

Schur polynomials form a vector space basis (say, over $\mathbb{Q}$) of the ring of symmetric polynomials in the variables $x_1, \ldots, x_N$. Since a product of symmetric polynomials is symmetric, we can expand the result in terms of Schur polynomials. In particular, define the **Littlewood-Richardson coefficients** $C_{\lambda,\mu}^{\nu}$ by

$$\begin{aligned} s_\lambda(x_1, \ldots, x_N) \cdot s_\mu(x_1, \ldots, x_N) = \\ \sum_{\nu} C_{\lambda,\mu}^{\nu} s_\nu(x_1, \ldots, x_N). \end{aligned} \tag{1}$$

In fact, $C_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$! These numbers count tensor product multiplicities of irreducible representations of $GL_N(\mathbb{C})$. Alternatively, they count *Schubert calculus* intersection numbers for a triple of Schubert varieties in a Grassmannian. However, neither of these descriptions of $C_{\lambda,\mu}^{\nu}$ is really a means to calculate the number.

The **Littlewood-Richardson rule** combinatorially manifests the positivity of the $C_{\lambda,\mu}^{\nu}$. Numerous versions of this rule exist, exhibiting different features of the numbers. Here is a standard version: take the Young diagram of ν and remove the Young diagram of λ, where the latter is top left justified in the former (if λ is not contained inside ν, then declare $C_{\lambda,\mu}^{\nu} = 0$); this **skew-shape** is denoted ν/λ. Then $C_{\lambda,\mu}^{\nu}$ counts the number of semistandard fillings T of shape ν/λ such that (a) as the entries are read along rows from right to left, and from top to bottom, at every point, the number of i's appearing always is weakly less than the number of $i-1$'s, for $i \geq 2$; and (b) the total number of i's appearing is μ_i. Thus $C_{(2,1),(2,1)}^{(3,2,1)} = 2$ is witnessed by

$$\begin{ytableau} {} & {} & 1 \\ {} & 1 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} {} & {} & 1 \\ {} & 2 \\ 1 \end{ytableau}.$$

Extending the $GL_N(\mathbb{C})$ story, a generalized Littlewood-Richardson rule exists for all complex semisimple Lie groups, where *Littelmann paths* generalize Young tableaux. In contrast, the situation is much less satisfactory in the Schubert calculus context, although in recent work with Hugh Thomas, we made progress for the *(co)minuscule* generalization of Grassmannians.

No discussion of Young tableaux is complete without the **Schensted correspondence**. This associates each $\sigma \in \mathfrak{S}_n$ bijectively with pairs of standard Young tableaux (T, U) of the same shape λ, where $|\lambda| = n$. This can be used to prove the Littlewood-Richardson rule but is noteworthy in its own right in geometry and representation theory.

Given a permutation (in one-line notation), e.g., $\sigma = 21453 \in S_5$, at each step i we add a box into some row of the current **insertion tableau** $\widetilde{T}$: initially insert $\sigma(i)$ into the first row of $\widetilde{T}$. If no entries y of that row are larger than $\sigma(i)$, place $\sigma(i)$ in a new box at the end of the row and place a new box containing i at the same place in the current **recording tableau** $\widetilde{U}$. Otherwise, let $\sigma(i)$ replace the leftmost $y > \sigma(i)$ and insert y into the second row, and so on. This eventually results in two tableaux of the same shape; Schensted outputs (T, U) after n steps. In our example, the steps are

$$(\varnothing, \varnothing), \left(\begin{ytableau} 2 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}\right), \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 \\ 2 \end{ytableau}\right), \left(\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\right),$$

$$\left(\begin{ytableau} 1 & 4 & 5 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}\right), \left(\begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}\right) = (T, U).$$

It is straightforward to prove well-definedness and bijectivity of this procedure. Also, T and U encode interesting information about σ. For example, it is easy to show that λ_1 equals the length of the longest increasing subsequence in σ (see e.g. work of Baik-Deift-Johansson for connections to random matrix theory). A sample harder fact is that if σ corresponds to (T, U) then σ^{-1} corresponds to (U, T).

An excellent source for more on the combinatorics of Young tableaux is [Sta99], whereas applications to geometry and representation theory are developed in [Ful97]. For a survey containing examples of Young tableaux for other Lie groups, see [Sag90]. Active research on the topic of Young tableaux continues. For example, recently in collaboration with Allen Knutson and Ezra Miller, we found a simplicial ball of semistandard tableaux, together with applications to Hilbert series formulae of determinantal ideals.

Further Reading

[Ful97] W. Fulton, *Young tableaux. With applications to representation theory and geometry*, London Mathematical Society Student Texts **35** (1997), Cambridge University Press, Cambridge.

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[Sta99] R. P. Stanley, *Enumerative combinatorics. Vol. 2.* With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, Cambridge Studies in Advanced Mathematics **62** (1999), Cambridge University Press, Cambridge.