I. Bijection between order ideals in $\lambda \times [n]$ and increasing tableaux.

Background. We call a nonempty finite subset $\lambda \subset \mathbb{Z}^{d-1}$ a *shape*. Let e_i be the standard basis vector in $\mathbb{R}^{d-1}$. Let $\pi_1(x)$ be the projection onto the i^{th} coordinate.

We say that λ is *convex* if whenever x and $x + k\, e_i$ are both contained in λ, than $x + je_i \in \lambda$ for $j = 1, ..., k-1$. We say that λ is *connected* if for all $x, x' \in \lambda$, there are sequences of coordinates $i_1, ..., i_s$ and parities $n_1, ..., n_s$ such that $x + \sum_{k=1}^{j} (-1)^{n_k} e_{i_k} \in \lambda$ for $j = 1, ..., s$ and $x + \sum_{k=1}^{s} e_{i_k} = x'$. We say that λ is *justified* if λ lies in the first quadrant and that $\min_{(i,j)\in\lambda} i = \min_{(i,j)\in\lambda} j = 0$.

Let $\lambda \times [n]$ be the partially ordered set $\{x \times y \in \mathbb{Z}^d : x \in \lambda,\ y \in [0, n-1]\}$, where $p \geq q$ if and only if $p - q \in \mathbb{Z}^d_{\geq 0}$. For any poset P, we write $J(P)$ to be the collection of order ideals of P.

Let $a_i(\lambda) = \max_{x\in\lambda} \pi_i(x)$ and $a_\lambda = (a_1(\lambda), ..., a_n(\lambda)) \in \mathbb{Z}^{d-1}$. Define $\lambda^* = \{a_\lambda - x : x \in \lambda\}$. It is clear if λ is convex/connected/justified, then λ^* is convex/connected/justified.

For a shape $\mu \subset \mathbb{Z}^{d-1}$, an increasing tableau of height n is a map $T : \mu \to \{1, ..., n\}$ such that for all $x \in \mu$ and $i = 1, ..., d-1$, $T(x) > T(x - e_i)$ whenever $x - e_i$ is contained in μ. Let $\mathrm{Inc}^n(\mu)$ be the set of increasing tableaux of height n with base shape μ.

Any subset of $\mathbb{Z}^d$ is ranked by the function $\mathrm{rk}(x_1, ..., x_d) = \sum_{i=1}^{d} x_i$.

The main bijection. Define the map $\Phi : J(\lambda \times [n]) \to \mathrm{Inc}^{\max \text{ rank}(\lambda)+n}$ as follows. First define the function $\mathrm{Ht} : \mathbb{Z}^{d-1}_{\geq 0} \times P(\mathbb{Z}^d_{\geq 0}) \to \mathbb{N}$ so that for $x \in \mathbb{Z}^{d-1}$, $(x, I) \mapsto \max_{(x,y)\in I} y$. Then define

$$\Phi(I) = \Big(a_\lambda - x \mapsto \mathrm{Ht}(x, I) + \mathrm{rk}(a_\lambda - x) + 1\Big)$$

II. Equivariance when d = 2. Following Lemma 4.2, we claim that Ψ intertwines $\mathrm{Pro}_{\mathrm{id},(1,1,-1)}$ and K-Pro. Let λ be a convex, connected, justified shape in $\mathbb{Z}^2$. Let $a = a_1$ and let $b = a_2$, that is, $(a, b) = a_\lambda$. Take $P \in J(\lambda \times [n])$.

By Proposition 2.4, *LIMITS ISSUES*

$$K\text{-Pro} = K\text{-BK}_{a+b+n-1} \circ ... \circ K\text{-BK}_1(T)$$

By definition,

$$\mathrm{Pro}_{\mathrm{id},(1,1,-1)} = T^{a+b-(a+b+n-1)}_{\mathrm{id},(1,1,-1)} \circ ... \circ T^{a+b}_{\mathrm{id},(1,1,-1)}$$

Therefore it suffices to show

$$\Psi(T^{a+b-l}_{\mathrm{id},(1,1,-1)}(P)) = K\text{-BK}_l(\Psi(P))$$

for all l and all $P \in J(\lambda \times [n])$.

$T^{a+b-l}_{\mathrm{id},(1,1,-1)}$ is by definition the product of toggles t_x where $x = (i, j, k) \in \lambda \times [n]$ and $\langle x, (1, 1, -1)\rangle = i + j - k = a + b - l$. Now if $(i, j, k) \in P$, then $(a - i, b - j)$ is an element of λ^* and is labeled at least $k + \mathrm{rk}(a - i, b - j) + 1 = k + (a - i) + (b - j) + 1 = l + 1$. Fix an element (i, j, k) on this hyperplane with $(i, j) \in \lambda$.

- Say $(i, j, k) \in P$.

  - On the one hand, if $(i,j,k+1)$ in P, then $t_x(P) = P$, since removing x from P would not result in an order ideal. In addition, K-BK$_l$ does not affect entry $(a-i, b-j)$ because $\Psi(P)(a-i,b-j) > l+1$.
  - On the other hand, if $(i,j,k+1)$ is not in P, then $(a-i, b-j)$ is labeled with $l+1$ so it is affected by K-BK$_l$. It will only be changed when it is part of a trivial short ribbon, meaning $\Psi(P)(u,v)$ is less than l for all $(u,v) \in \lambda^*$ covered by $(a-i, b-j)$. The convexity and connectedness assumptions imply that only the elements $(a-i-1, b-j)$ and $(a-i, b-j-1)$ can be covered by $(a-x)$ in λ^*.

    Say that $(a-i-1, b-j)$ or $(a-i, b-j-1)$ is contained in λ^* and labeled l. Then K-BK$_l$ does not change the $(a-i, b-j)$ entry. In addition, either $(i+1, j)$ or $(i, j+1)$ has height k in P. Therefore $P \setminus \{x\}$ is not an order ideal, so t_x does not alter P.

    On the other hand, say that neither element is contained in λ^* with label l. There are three possibilities:

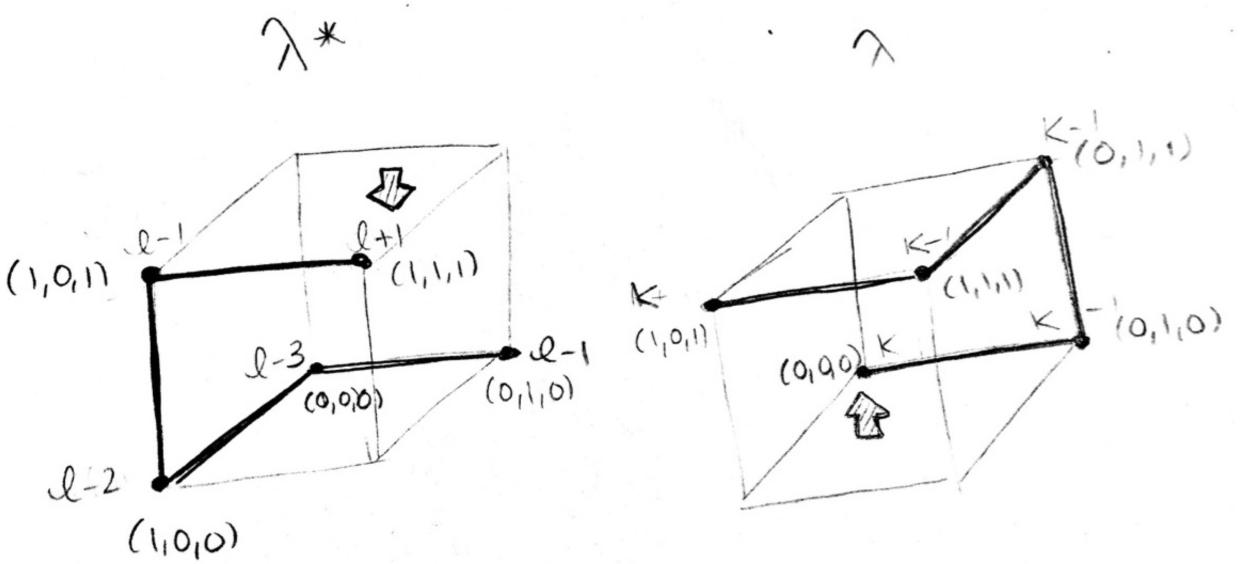
    * If both positions are part of λ^*, and $(a-i-1, b-j) < l$ and $(a-i, b-j-1) < l$, then we are reduced to the case covered in the paper.
    * If one entry, say $(a-i-1, b-j)$ is absent in λ^*, and the other has label less than l, then $P \setminus \{x\}$ defines an order ideal. For say $(u,v) > (i,j)$ in λ. By convexity, we must have $v > j$ since (u,v) cannot be contained on the line $\{(i+g, j) : g \in \mathbb{Z}_+\}$, and $u \geq i$. But then $(u,v) \geq (i, j+1)$, and $\Psi(P)(a-i, b-j-1) < l$, so $\mathrm{Ht}(u,v) \leq \mathrm{Ht}(i, j+1) < k$. Since all elements in the column over (i,j) below (i,j,k) are less than x, this proves that there is no element in P greater than x, so $P \setminus \{x\}$ is an ideal.
    * Say that both entries are absent in λ^*. Then (i,j) is maximal in λ. For say $(r,s) > (i,j)$ for some $(r,s) \in \lambda$. By connectedness, there is a path from (i,j) to (r,s). By convexity, no element of this path can be contained in the lines $\{(i+g, j) : g \in \mathbb{Z}_+\}$ or $\{(i, j+g) : g \in \mathbb{Z}_+\}$. Let (m,n) be the first element hit by this path that is greater than (i,j). Such a square must exist because $(r,s) > (i,j)$. Then $(m,n) - (i,j) = (x, 0)$ or $(0, x)$ for some $x > 0$. But this point is contained on one of the two excluded lines, a contradiction

- Say $(i,j,k) \notin P$.
  - If $(i,j,k-1) \notin P$, then $P \cup \{x\}$ is not an ideal so t_x does not affect P. In addition, $\Psi(P)(a-i, b-j) < l$ so K-BK$_l$ does not affect $(a-i, b-j)$.
  - If $(i,j,k-1) \in P$, then $\Psi(P)(a-i, b-j) = l$. K-BK$_l$ will change entry $(a-i, b-j)$ exactly when it is not part of a trivial short ribbon, meaning that all elements that cover it are labeled at least $l+2$. If either $(a-i+1, b-j)$ or $(a-i, b-j+1)$ is an element of λ^* and is labeled $l+1$, then $(a-i, b-j)$ is unchanged. In addition, this implies that $(i-1, j)$ or $(i, j-1)$ has height $k-1$ respectively, in which case $P \cup \{x\}$ is not an ideal. Otherwise, we have three possibilities as above:

    * If both $(a-i+1, b-j)$ and $(a-i, b-j+1)$ are contained in λ^* and labeled at least $l+2$, then we are reduced to the case from the paper.

* If one label, say $(a-i+1, b-j)$ is absent from λ^* and the other $(a-i, b-j+1)$ is labeled at least $l+2$, then $P \cup \{x\}$ is an ideal. For say $(u,v) < (i,j)$ in λ. By convexity, we must have $v < j$ since (u,v) cannot be contained on the line $\{(i-g, j) : g \in \mathbb{Z}_+\}$, and $u \leq i$. But then $(u,v) \leq (i, j-1)$, so $\mathrm{Ht}(u,v) \geq k$, and so (i,j,k).
* Finally, if both labels are absent, then (i,j) is minimal as above, so $P \cup \{x\}$ is an ideal.

III. Counterexample with $d = 3$.



This counterexample shows that K-BK_l and T_x will not act in the same way on the tableau of these vertices. For K-BK_l will decrement entry $(1,1,1)$ to l, but T_x cannot decrease the height of $(0,0,0)$ because $(1,0,1)$ has height k. My guess is that we could successfully replace the convexity condition with the requirement that λ be "twistless," defined as the condition that every two elements are connected by a strictly increasing path.