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July 2017

Chapter Twelve: Tensors

Lee, *An Introduction to Smooth Manifolds*

12.2 Tensoring with the base field. We show that $\mathbb{R}\otimes V\simeq V$; the proof for $V\otimes\mathbb{R}$ is identical. Define a map $\Phi_V:\mathbb{R}\times V\to V$ that sends $(r,v)\mapsto r\cdot v$. Then Φ_V is bilinear, so by the universal property (Prop 12.7) it descends to a linear map on $\mathbb{R}\otimes V$, which by the commutativity of the diagram (12.6) sends the simple tensor $r\otimes v$ to $r\cdot v$ and extends linearly. Now define $\Theta_V:V\to\mathbb{R}\otimes V$ that sends $v\mapsto 1\otimes v$. It is easily checked that Θ_V and Φ_V are inverse when Φ is restricted to simple tensors, which implies that they are inverse maps by linearity.

We show the sense in which this isomorphism is canonical. Let $\mathrm{Vec}_{\mathbb{R}}$ be the category of real vector spaces and linear maps. Let $\mathcal{F}$ be the identity functor and let $\mathcal{G}$ be the covariant functor that sends $V\mapsto\mathbb{R}\otimes V$ and $T\mapsto 1\otimes T$. If V and W are vector spaces and $T:V\to W$ a linear map, then $1\otimes T\circ\Phi_V(v)=1\otimes Tv$ while $\Phi_W\circ T(v)=1\otimes Tv$. Therefore the map $V\mapsto\Phi_V$ is canonical in the sense of Problem 11-18.

12.4 Canonical isomorphism $V_1^*\otimes...\otimes V_k^*\otimes W\simeq L(V_1,...,V_k;W)$. Define the map $\Phi_V:V_1^*\times...\times V_k^*\times W\to L(V_1,...,V_k;W)$

$$(f_1,...,f_k,w)\mapsto\left(v\mapsto\left(\prod_{n=1}^{k}f_n(v)\right)w\right).$$

This gives a multilinear map that descends to a linear map $V_1^*\otimes...\otimes V_k^*\otimes W\simeq L(V_1,...,V_k;W)$. The inverse is easily constructed by choosing a basis (e_i^n) for each V_n and w_i for W and then using the observation that functions of the form $(e^1_{i_1},...,e^n_{i_n})\mapsto w_\ell$ span $L(V_1,...,V_k,W)$, and the existence of this inverse proves that Φ_V is an isomorphism, regardless of the choice of basis we made to construct it.

It can be checked that the assignment $V\mapsto\Phi_V$ is canonical, where C is the category of products of $k+1$ vector spaces with products of linear maps, D is $\mathrm{Vec}_{\mathbb{R}}$, $\mathcal{F}$ sends $(V_1,...,V_k,W)$ to $V_1^*\otimes...\otimes V_k^*\otimes W$ and

$$(T_1,...,T_k,S)\mapsto T_1^*\otimes...\otimes T_k^*\otimes S$$

while $\mathcal{G}$ sends $(V_1,...,V_k,W)$ to $L(V_1,...,V_k;W)$ and

$$(T_1,...,T_k,S)\mapsto\Big(f(v_1,...,v_k)\mapsto S\cdot f(T_1v_1,...,T_kv_k)\Big).$$

This makes sense because our construction of Φ_V did not involve choosing a basis.

12.8 Transformation law for covariant tensors.

$$\begin{aligned}
A_{i_1,\dots,i_k} &= A\left(\frac{d}{dx_{i_1}} \otimes \dots \otimes \frac{d}{dx_{i_k}}\right) \\
&= \sum_{j_1,\dots,j_n} \tilde{A}_{j_1,\dots,j_k} d\tilde{x}^{j_1} \otimes \dots \otimes d\tilde{x}^{j_k} \left(\frac{d}{dx_{\ell_1}} \otimes \dots \otimes \frac{d}{dx_{\ell_k}}\right) \\
&= \sum_{j_1,\dots,j_n} \sum_{\ell_1,\dots,\ell_k} \left(\prod_{n=1}^{k} \frac{d\tilde{x}^{j_n}}{dx_{\ell_n}}\right) \tilde{A}_{j_1,\dots,j_k} d\tilde{x}^{j_1} \otimes \dots \otimes d\tilde{x}^{j_k} \left(\frac{d}{dx_{\ell_1}} \otimes \dots \otimes \frac{d}{dx_{\ell_k}}\right) \\
&= \sum_{j_1,\dots,j_n} \left(\prod_{n=1}^{k} \frac{d\tilde{x}^{j_n}}{dx_{i_n}}\right) \tilde{A}_{j_1,\dots,j_k}
\end{aligned}$$

12.10 Pullback and pushforward of mixed tensor fields by diffeomorphisms. Take $p \in N$ and $A \in \Gamma(T^{(k,l)}TN)$. For $\{v_i\} \in T_pM^*$ and $\{w_i\} \in T_pM$, let X_i be covector fields such that $X_i(p) = v_i$ and let Y_i be vector fields such that $Y_i(p) = w_i$. define

$$(F^*A)_{F^{-1}(p)}(v_1, \dots, v_k, w_1, \dots, w_\ell) = A_p((F^{-1*}X_1)_p, \dots, (F^{-1*}X_k)_p, (F_*Y_1)_p, \dots, (F_*Y_\ell)_p).$$

Then $p \mapsto (F_*A)_p$ is a rough section of $T^{(k,\ell)}TM$ and agrees with the pullback on covariant vector fields. On the other hand, take $B \in \Gamma(T^{(k,l)}TM)$. For $\{v_i\} \in T_pN^*$ and $\{w_i\} \in T_pN$, let X_i be covector fields such that $X_i(p) = v_i$ and let Y_i be vector fields such that $Y_i(p) = w_i$. Then define

$$(F_*B)_p(v_1, \dots, v_k, w_1, \dots, w_\ell) = B_{F^{-1}(p)}((F^*X_1)_p, \dots, (F^*X_k)_p, (F_*^{-1}Y_1)_p, \dots, (F_*^{-1}Y_\ell)_p).$$

Then $q \mapsto (F_*B)_q$ is a rough section of $T^{(k,\ell)}TN$ and agrees with the pullback on vector fields. These sections are smooth because they are the composition of the maps from points of M to a product of smooth vector and covectors fields, and the application of a smooth tensor field. I verified properties (a.)-(f.) on paper.

12.12 Alternative definition of the Lie derivative. We have already observed that the Lie derivative satisfies these properties (Proposition 12.32). To show uniqueness, say R satisfies these properties. Be show that $R = \mathcal{L}_V$ on $\mathcal{T}^k(M)$ by strong induction on k. The case where $k = 0$ is immediate from property (b.). The case $k = 1$ then follows from property (d.), since for $\omega \in \mathcal{T}^1(M)$,

$$(R(\omega))(X) = R(\omega(X)) - \omega([V, X]) = \mathcal{L}_V(\omega(X)) - \omega([V, X]) = \mathcal{L}_V(\omega)(X),$$

and two covector fields agree if they agree on all vector fields.

For the inductive step, take $p \in M$ and let (f_n) be a smooth covector field on M that restricts to a local frame for T^*M on a domain containing p. Then by the associativity of the tensor product, (c.), and the strong inductive hypothesis, we have that

$$\begin{aligned}
R(f_{i_1} \otimes \dots \otimes f_{i_k}) &= R(f_{i_1} \otimes \dots \otimes f_{i_{k-1}}) \otimes f_{i_k} + (f_{i_1} \otimes \dots \otimes f_{i_{k-1}}) \otimes R(f_{i_k}) \\
&= \mathcal{L}_V(f_{i_1} \otimes \dots \otimes f_{i_{k-1}}) \otimes f_{i_k} + (f_{i_1} \otimes \dots \otimes f_{i_{k-1}}) \otimes \mathcal{L}_V(f_{i_k}) \\
&= \mathcal{L}_V(f_{i_1} \otimes \dots \otimes f_{i_k})
\end{aligned}$$

(This is where (d.) is required – it is not enough to establish the base case $k = 0$, we need to be able to bring down basis tensors by a rank.) Now given an arbitrary section $A \in \mathcal{T}^kM$, we can write $A(p) =$

$\sum_{i_1,...,i_k} f_\alpha(p) f_{i_1} \otimes ... \otimes f_{i_k}$. Then by properties (a.) and (c.),

$$\begin{aligned}
R(\sum_{i_1,...,i_k} f_\alpha(p) f_{i_1} \otimes ... \otimes f_{i_k}) &= \sum_{i_1,...,i_k} R(f_\alpha(p) f_{i_1} \otimes ... \otimes f_{i_k}) \\
&= \sum_{i_1,...,i_k} \Big(R(f_\alpha(p)) \otimes f_{i_1} \otimes ... \otimes f_{i_k} + f_\alpha \otimes R(f_{i_1} \otimes ... \otimes f_{i_k}) \Big) \\
&= \sum_{i_1,...,i_k} \Big(\mathcal{L}_V(f_\alpha(p)) \otimes f_{i_1} \otimes ... \otimes f_{i_k} + f_\alpha \otimes \mathcal{L}_V(f_{i_1} \otimes ... \otimes f_{i_k}) \Big) \\
&= \mathcal{L}_V(\sum_{i_1,...,i_k} f_\alpha(p) f_{i_1} \otimes ... \otimes f_{i_k}).
\end{aligned}$$

This proves uniqueness.