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May 2017

Chapter Three: Tangent Vectors

Lee, *An Introduction to Smooth Manifolds*

3-2 Tangent space to a product manifold. Take $M = M_1 \times ... \times M_n$. The map α is clearly linear as the direct product of linear maps. To see that it is surjective, take $p = (p_1, ..., p_n) \in \text{Int } M$ and $(v_1, ..., v_n) \in \oplus_{i=1}^N T_{p_i} M_i$. By Proposition 3.23, there exist an $\epsilon > 0$ and a collection of curves $\gamma_i : (-\epsilon, \epsilon) \to M_i$ such that $\gamma(0) = p_i$ and $\gamma_i'(0) = v_i$. But then we can define a curve $\gamma = (\gamma_1, ..., \gamma_n) : (-\epsilon, \epsilon) \to M$ which is smooth in the natural charts on M. We have that $\gamma(0) = p$ and $\gamma'(0) = (v_1, ..., v_n)$. Therefore $\gamma'(0) \in T_p M$. But $(d\pi_i)_p \gamma'(0) = d(\pi_i \circ \gamma)_0 = d(\gamma_i)_0 = v_i$. This shows that $\alpha(\gamma'(0)) = v$. Injectivity then follows from dimension-counting (Example 1.8)

Now, if p is a boundary point of M, the proof goes through in the same way, only the curves in equation are defined on $(-\epsilon, 0]$ and derivatives are one-sided.

3-4 Tangent bundle to $\mathbb{S}^1$. Let $\gamma : [0, 2\pi] \to \mathbb{S}^1$ be the map $t \mapsto (\cos(t), \sin(t))$. Then $\gamma'(t_0)$ spans $T_{\gamma(t_0)}\mathbb{S}^1$, so a typical element of $T\mathbb{S}^1$ can be writen $(\gamma(t_0), a \cdot \gamma'(t_0))$. Define the map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ that sends $(\gamma(t_0), a \cdot \gamma'(t_0)) \mapsto (\gamma(t_0), a)$. Note that $\gamma'(0) = \gamma'(2\pi)$ so there is no ambiguity at this point. Recall the stereographic projection charts on S^1: $\phi_N(S^1 \setminus N) : (x, y) \mapsto \frac{x}{1-y}$ and $\phi_S(S^1 \setminus S) : (x, y) \mapsto \frac{x}{1+y}$. Now $\gamma'(t_0) = D(\phi^{-1})_{\phi \circ \gamma(t_0} \circ D(\phi \circ \gamma)|_{(t_0)}(\frac{d}{dt})$. A computation shows that $D(\phi \circ \gamma)|_{(t_0)} = \frac{1}{1-\sin(t_0)}$. But $\gamma(t_0) = (\cos(t_0), \sin(t_0))$. Therefore given $(x, v) \in \phi_N(S^1 \setminus N)$, if $\gamma(t_0) = \phi_N^{-1}(x)$, we can write

$$\phi_N^{-1}(x, v) = ((\frac{2x}{1+x^2}, \frac{x^2-1}{x^2+1}), vD\phi^{-1}(\frac{d}{dx})) = ((\frac{2x}{1+x^2}, \frac{x^2-1}{x^2+1}), \frac{2v}{(1+x^2)}\gamma'(t_0)).$$

Now there are charts on $S^1 \times \mathbb{R}$ given by $\phi_N \times \text{Id}$ and $\phi_S \times \text{Id}$. Using the first chart, we can represent F in coordinates on $\phi_N(S^1 \setminus N)$: $F(x, v) = (x, \frac{2v}{(1+x^2)})$. This is clearly smooth. A similar computation gives a smooth coordinate representation on $\phi_S(S^1 \setminus S)$.

The inverse of F is given by $G(x, a) = (x, a \cdot \gamma'(t_0))$, where t_0 is any point such that $\gamma(t_0) = x$. Calculations similar to those above give a smooth coordinate representation of G on $\phi_N \times \text{Id}(S^1 \setminus N \times \mathbb{R})$ as $(x, \frac{2(1+x^2}{v})$. This is also smooth. A similar computation gives a smooth coordinate representation on $\phi_S \times \text{Id}(S^1 \setminus S \times \mathbb{R})$. Therefore F is a diffeomorphism. CLEAN THIS UP

3-6 A smooth curve into $\mathbb{S}^3$ with nowhere-vanishing velocity. Write $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Now $\|(z_1, z_2)\| = 1$, so at most one of z_1 and z_2 can have unit length. First assume that $\|z_2\| < 1$. Then, we can write $\gamma'(t)$ in stereographic coordinates:

$$\gamma(t) = \frac{1}{1 - (a_2 \sin(t) + b_2 \cos(t))}(a_1 \cos(t) - b_1 \sin(t), a_1 \sin(t) + b_1 \sin(t), a_2 \cos(t) - b_2 \sin(t))$$

Since $a_2 \sin(t) + b_2 \cos(t) \neq 1$ under the assumption that $\|z_2\| < 1$, this gives a smooth coordinate represetation of γ on all of $\mathbb{R}$.

On the other hand, if $\|z_2\|= 1$, then $\|z_1\|\neq 1$ (it's zero), so a different choice of stereographic coordinates gives the same result.

To see that $\gamma'(t)$ is nowhere zero, we note that if $M \subset \mathbb{R}^n$ is a manifold and $v \in T_pM$, then we can identify v with a derivation on $\mathbb{R}^n$. For if $\eta(t)$ is a curve into M such that $\eta(0) = p$ and $\eta'(0) = v$, then η defines a curve into $\mathbb{R}^n$ and so $\eta'(0) \in T_p\mathbb{R}^n$. Under this identification, we can see that $v = 0$ as a derivation on M only if $v = 0$ as a derivation on $\mathbb{R}^n$. For if f is a smooth function on $\mathbb{R}^n$, then $f|_M$ is a smooth function on M, since for any smooth chart (U, ϕ) for M, $f \circ \phi^{-1}$ is a smooth function. In addition, if $\eta'(0)f \neq 0$, then $\eta'(0)f|_M\neq 0$. Therefore v is not the zero derivation.

Also, if $v \in T_pM$, then using the identification of Proposition 3.13, we see that $v = 0$ only if $\|v\|= 0$. For if $\|v\|> 0$, then there is some component $v^i \neq 0$, and then if $f(x^1, ..., x^n) = x^i$, then $v(f) = \frac{d}{dt}f(p+tv^i) = v^i$. These two facts show that $\gamma'(t) \neq 0$, for $\|\gamma'(t)\|= 1$ for all t.

3-8 Tangent vectors as isomorphism classes of velocity vectors. Ψ is well-defined. For say $\gamma_1 \sim \gamma_2$. Then since $\gamma_1'(0)f = \frac{d}{dt}|_{t=0}f \circ \gamma_1(t)$ and $\gamma_2'(0)f = \frac{d}{dt}|_{t=0}f \circ \gamma_2(t)$, we have that $\gamma_1'(0)f = \gamma_2'(0)f$ for all f, and therefore $\Psi(\gamma_1) = \Psi(\gamma_2)$. To see that it is injective, take a coordinate chart $\phi = (\phi^1, ..., \phi^n)$ containing p. Now if $\Psi(\gamma_1) = \Psi(\gamma_2)$, then $(\phi^i \circ \gamma_1)'(0) = (\phi^i \circ \gamma_2)'(0)$ for $i = 1, ..., m$. Then we can compute, in the coordinates induced by ϕ,

$$\gamma_1'(0) = \sum_{i=1}^{n} \frac{d(\phi^i \circ \gamma_1)}{dt}\bigg|_{t=0} \frac{d}{dx_i}\bigg|_p = \sum_{i=1}^{n} \frac{d(\phi^i \circ \gamma_2)}{dt}\bigg|_{t=0} \frac{d}{dx_i}\bigg|_p = \gamma_2'(0)$$

Finally, if $v \in T_pM$, then by Proposition 3.23, there is a curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and then $\Psi(\gamma) = v$. Therefore Ψ is surjective.