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Chapter Nine: Integral Curves and Flows

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9.2 Integral curves of a vector field tangent to a submanifold.

(a) Under the identification given by the inclusion $T_pS \hookrightarrow T_pM$, the restriction of V to S is a smooth vector field on S. (Smoothness is easily seen by taking a slice chart containing $p \in S$ and projecting onto the first k factors.) Therefore for all $p \in S$ there is $\epsilon > 0$ and a curve $\eta : I = (-\epsilon, \epsilon) \to S$ such that $\eta(0) = p$ and $\eta'(s) = V_p$ for all $s \in (-\epsilon, \epsilon)$. But $i \circ \eta$ is an integral curve for V starting at p in M. By uniqueness, if γ is any integral curve $J \to M$ for V starting at p, then $\eta = \gamma$ on $I \cap J$. This implies that any such γ is contained in S for an interval around 0.

(b) Let γ be an integral curve for V starting at $p \in S$ with domain I. Let $A = \{t \in I : \gamma(t) \in S$. The previous bullet proves that A is nonempty. A is closed, because if $(s_n) \to s$ and $\gamma(s_n) \in S$ for all n, then $\gamma(s) \in S$ since γ is continuous and S is closed. On the other hand, A is open, because if $s \in A$, then $\gamma(s) \in S$, so by Proposition 9.2 there is an integral curve η based at $\gamma(s)$ which extends γ by uniqueness. This proves that $A = I$.

(c) Let $U \subset \mathbb{R}$ be open and bounded, $V = \frac{d}{dx}$, and $\gamma : \mathbb{R} \to \mathbb{R}$ be the identity map.

9.3 Computing flows.

(a) $\theta((x_0, y_0), t) = (x_0 + y_0 t + \frac{t^2}{2}, y_0 + t)$

(b) $\theta((x_0, y_0), t) = (x_0 e^t, y_0 e^{2t})$

(c) $\theta((x_0, y_0), t) = (x_0 e^t, y_0 e^{-t})$

(d) $\theta((x_0, y_0), t) = (\frac{x_0+y_0}{2} e^t + \frac{x_0-y_0}{2} e^{-t}, \frac{x_0+y_0}{2} e^t - \frac{x_0-y_0}{2} e^{-t})$

9.4 A nonvanishing vector field on $\mathbb{S}^{2n-1}$. Under the standard injection $T_p\mathbb{S}^{n-1} \hookrightarrow T_p\mathbb{C}^n \simeq \mathbb{C}^n$, $V_z = (\theta^{(z)})'(0) = iz$.

9.6 Proof of the Escape Lemma. Say otherwise that there exists t_0 such that $\gamma([t_0, b)) \subseteq X \subseteq M$ for some X compact. Assume without loss of generality that M is properly embedded in $\mathbb{R}^N$, so in particular M is closed in $\mathbb{R}^N$ (Theorem 6.15, Proposition 5.5). Now

$$\lim_{t \to^+ b} \gamma(t) = z$$

for some $z \in X$, because $|\gamma'(s)|$ is bounded and X is closed and bounded in $\mathbb{R}^N$. (Take a convergent subequence $(\gamma(t_n))$ and use bounded derivative to show that if $|t_n - b|$ is small then $|\gamma(t_n) - z|$ is small.)

Let $Z = \gamma([t_0, b)) \cup z$. Considering each possible limit point of Z shows that Z is closed. For each $p \in Z$, let U_p be a coordinate ball centered at p such that $\bar{U_p}$ is compact, let $V_p = U_p/2$ be the ball of half the original radius, and let $W_p = U_p/4$ be the ball of one-quarter the original radius. Then there is a finite collection $W_1, ..., W_n$ such that $Z \subseteq W_1 \cup ... \cup W_n$. Then $W = \bar{W_1} \cup ... \cup \bar{W_n}$ is a compact set containing Z in its interior, $V = V_1 \cup ... \cup V_n$ is an open set containing W, and $U = \bar{U_1} \cup ... \bar{U_n}$ is a compact set containing Z. Now by Lemma 8.6, there is a vector field $\tilde{V}$ agreeing with V on W, and supported in V, which is contained in the compact set W. Clearly γ is an integral curve of $\tilde{V}$. By Theorem 9.16, this means that γ is defined on $\mathbb{R}$, so we can extend γ in a neighborhood of z contained in W. But since $\tilde{V}$ agrees with V in a neighborhood of this extension, this gives an extension of γ past b, contradicting maximality.

9.8 Flow neighborhood of compact submanifold is embedded. By Theorem 9.20, there is an open neighborhood $\mathcal{O}_\delta$ such that Θ restricted to $\mathcal{O}_\delta$ is an injective smooth immersion. Covering $\mathcal{O}_\delta$ with basis open sets of the form $(\epsilon_p, -\epsilon_p) \times U_p$, extracting a finite subcover, and taking the minimum ϵ_p in this subcover, we can assume that $\mathcal{O}_\delta$ is of the form $(-\epsilon, \epsilon) \times S$. Now $V = [-\epsilon/2, \epsilon/2] \times S$ is a compact subset of $\mathbb{R} \times M$ by Tychonoff's theorem, and since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, Φ_V is a homeomorphism. But then the restriction of Φ to $W = (-\epsilon, \epsilon) \times S$ is a homeomorphism as well, so Φ is an embedding $W \hookrightarrow M$.

9.10 Flow charts.

(a) $U = \mathbb{R}^2$, $\phi(x, y) = (x - \frac{y^2}{2}, x)$.

(b)

(c)

(d) $U = B_1(1, 0)$, $\phi(x, y) = \left(\frac{x}{2} (\sqrt{\frac{x+y}{x-y}} + \sqrt{\frac{x+y}{x-y}}^{-1})^{-1}, \frac{1}{2} \log(\frac{x+y}{x-y}) \right)$.

9.12 Properties of the connected sum. The smooth structure described in Theorem 9.29 has these properties. For let M_i be two manifolds, p_i the two points, U_i the coordinate balls, and W_i tubular neighborhoods of $M_i' = M_i \setminus U_i$. As in the notation of the proof of Theorem 9.29, we write $h : \delta M_1' \to \delta M_2'$ for the diffeomorphism identifying the two boundaries, $X = M_1' \coprod M_2'$ for the quotient space induced by h, $Y = \pi(W_1 \coprod W_2)$, and Ψ for the diffeomorphism $(-1, 1) \to \delta M_1 \to Y$ whose existence is shown during the proof of the theorem. Define $\tilde{M_1} = \pi(M_1' \cup W_2)$. It is clear that V_1 is open in the quotient topology on X. It is diffeomorphic to $M_1 \setminus \bar{U}_1$. For let $g : (-1, 1) \to (-1, 0)$ be a diffeomorphism that is equal to the identity for sufficiently small values. Define the map $F : \tilde{M_1} \to M_1 \setminus \bar{U}_1$

$$F(x) = \begin{cases} x & \text{if } x \in M_1 \setminus V_1 \\ (g(t), p) & \text{if } x \in Y \text{ and } \Psi(x) = (t, p) \end{cases}.$$

Then F is a diffeomorphism, since for every point in $\tilde{M_1}$ there is a neighborhood on which F is either the identity or the map $\Psi^{-1} \circ g \times \mathrm{Id} \circ \Psi$. But an elementary argument shows that $M_1 \setminus \bar{U}_1$ is diffeomorphic to $M_1 \setminus p_1$. Similar reasoning shows that $\tilde{M_2}$ is diffeomorphic to $M_2 \setminus p_2$, and it is clear that $\tilde{M_1} \cap \tilde{M_2}$ is diffeomorphic to $(-1, 1) \times \mathbb{S}^{n-1}$, since $\delta M_1 \simeq \mathbb{S}^{n-1}$.

9.14 Using the double to prove the Whitney Embedding Theorem for manifolds with nonempty boundary. There is a diffeomorphism ϕ mapping M into its double $2M$, and by the original Whitney Embedding Theorem, there is a smooth embedding $f : 2M \hookrightarrow \mathbb{R}^{2n+1}$. But then the composition $f \circ \phi$ is an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.

9.16 Vector fields that agree on closed subsets may have different Lie derivatives. Let $V = \delta_x + \sin(y)\,\delta_y$, $V' = \delta_x$, and $W = \delta y$. Taking Lie brackets (Theorem 9.38) shows that

$$L_V W(x, y) = -\cos(y)\,\delta y$$

which does not vanish at $(0, 0)$ but

$$L_{V'} W(x, y) = 0.$$