Github/dogestoyevsky
May 2017

Chapter Two: Smooth Maps

Lee, *An Introduction to Smooth Manifolds*

2-2 Smooth maps to product manifolds. Let $M = M_1 \times ... \times M_k$. It is clear that if $F : N \to M$ is smooth, then each $\pi_i \circ F$ is smooth as the composition of smooth maps. On the other hand, say each $F_i = \pi_i \circ F$ is smooth. Then F is continuous as the direct product of continuous maps. Take $p \in N$ and consider $F(p) \in M$. By the definition of the product topology, there is an open subset of the form $W = W_1 \times ... \times W_k$ where each W_j is open and $F(p) \in W_1 \times ... \times W_k$. By restricting we can assume that (W_j, ϕ_j) is a smooth chart for M_j. Let $\phi = \Pi_{j=1}^k \pi_j$. Then (W, ϕ) is a smooth chart for M.

Now, take (U, ψ) any chart containing p (using the continuity of F, we can assume $F(U) \subseteq W$) and consider $\phi \circ F \circ \psi^{-1} : \psi(U) \to \psi(W)$. Then the jth component of $\phi \circ F \circ \psi^{-1}$ is equal to $\phi_j \circ \pi_j \circ F \circ \psi^{-1}$, which is smooth by the assumption that $\pi_j \circ F$ is smooth. Since a function $\mathbb{R}^n \to \mathbb{R}^m$ is smooth if all its partials are smooth, this shows that $\phi \circ F \circ \psi^{-1}$ is smooth. (If the jth component of M has boundary, then $\phi_j \circ \pi_j \circ F \circ \psi^{-1}$ has a local smooth extension, and the product of this extension with the other functions gives a local smooth extension of $\phi \circ F \circ \psi^{-1}$.) Since p was arbitrary, this shows that F is smooth.

2-4 Inclusion of a closed ball in Euclidean space. I think it makes more sense to come back to this after learning about smooth structures on submanifolds. Otherwise what, just choose any smooth structure for $\bar{\mathbb{B}^n}$?

2-6 Homogeneous smooth functions define smooth functions between projective spaces. Let f be a smooth homogeneous function $\mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$. Clearly F defined by $F([x]) = [f(x)]$ is well-defined, since if $[x] = [y]$ for $x, y \in \mathbb{R}^{n+1}$, then $x = \lambda y$ for $\lambda \neq 0$, so $f(x) = f(\lambda y) = \lambda^d f(y)$, and so $[f(x)] = [f(y)]$. Also, if $q_k : \mathbb{R}^{k+1} \to \mathbb{R}P^k$ is the quotient map, then $q_k \circ f$ is continuous, and since $q_k \circ f = F \circ q_n$, a basic property of quotient maps implies that F is continuous.

Now given $\mathbb{R}P^n$ and $\mathbb{R}P^k$ the smooth structure described in Chapter 1. Let (U_i, ϕ_i) be these charts on $\mathbb{R}P^n$ and (W_i, ψ_i) on $\mathbb{R}P^k$. Then on the set $\phi_i(U_i \cap F^{-1}(W_j))$, we can write

$$\psi_j \circ F \circ \phi_i^{-1}(x_1, ..., x_n) = \psi_j \circ F([x_1, ..., x_{i-1}, 1, x_i, ..., x_n])$$
$$= \psi_j([f(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)])$$
$$= (\frac{f_1(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}{f_j(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}, ..., \frac{f_{j-1}(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}{f_j(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}, \frac{f_{j+1}(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}{f_j(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}, ..., \frac{f_n(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}{f_j(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)}$$

Since f_j is nonzero on $U_i \cap F^{-1}(W_j)$, this is a smooth function. By Proposition 2.5(b), this implies that F is smooth.

2-8 A diffeomorphism from $\mathbb{K}^n$ to a dense subset of $\mathbb{K}P^n$. First let $\mathbb{K} = \mathbb{R}$. The map $\mathbb{R}^n \to \mathbb{R}P^n$ defined by $f(x_1, ..., x_n) = [x_1, ..., x_n, 1]$ is surjective onto the chart $U_{n+1} = \{[x_1, ..., x_{n+1}] : x_{n+1} \neq 0\}$. For given such an $[x_1, ..., x_{n+1}]$, we can fix a representative such that $x_{n+1} = 1$, and then $[x_1, ..., x_{n+1}] = f(x_1, ..., x_n)$. Also, U_{n+1} is dense in $\mathbb{R}P^n$. For given $j = 1, ..., n$, $\phi_j(U_{n+1} \cap U_j) = \{(x_1, ..., x_n) \in \mathbb{R}^n \backslash \{0\}, x_n \neq$

$0\}$. This is a dense subset of $\mathbb{R}^n$. Since ϕ_j is a homeomorphism, this implies that $U_{n+1} \cap U_j$ is dense in U_j for all j. Therefore U_{n+1} is dense in every element of an open cover of $\mathbb{R}P^n$, which implies that U_{n+1} is dense in $\mathbb{R}P^n$.

It is clear that f is smooth. For using the charts for $\mathbb{R}P^n$ described in the text, $f(\mathbb{R}^n) \subseteq U_{n+1}$ and $\phi_{n+1} \circ f \circ \mathrm{id}^{-1}(x_1, ..., x_n) = (x_1, ..., x_n)$. Now, let g be the inverse of f given by $[(x_1, ..., x_{n+1})] \mapsto (\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}})$. It is easily checked that this is well-defined on $U_{n+1} \subseteq \mathbb{R}P^n$ and that it is inverse to f. Then, g is smooth, since $g(U_{n+1}) \subseteq \mathbb{R}^n$, and $\mathrm{id} \circ g \circ \phi_{n+1}^{-1}(x_1, ..., x_n) = (x_1, ..., x_n)$.

If we substitute $\mathbb{C}$ throughout, the above argument shows that $f : \mathbb{C}^n \to \mathbb{C}P^n$ maps onto an open subset, and that f and g have holomorphic transition functions using the natural charts $\mathbb{C}P^n \to \mathbb{C}^n$. Composing these with the diffeomorphism $\mathbb{C}^n \to \mathbb{R}^{2n}$ gives the desired result.

2-10 Smooth maps define linear transformations of smooth functions.

(a) $F^*(f+g) = (f+g) \circ F = f \circ F + g \circ F$.

(b) First say that F is smooth and take $f \in C^\infty(N)$. Then for all $p \in M$, there exists (U, ϕ) such that $F(p) \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. Also, there exists (W, ψ) a chart for M containing p such that $W \subseteq F^{-1}(U)$ (by the continuity of F) and $\phi \circ F \circ \psi^{-1} : \psi(W) \to \phi(U)$ is smooth. But then

$$f \circ F \circ \psi^{-1} = f \circ \phi^{-1} \circ \phi \circ F \circ \psi^{-1}$$

is a smooth map $\psi(W) \to \mathbb{R}$. Since p was arbitrary, this proves that $F^*(f)$ is smooth.

On the other hand, say $F^*(C^\infty(M)) \subseteq C^\infty(N)$. Take $p \in M$ and $q = F(p) \in N$. Let (U, ϕ) be a chart on N such that $q \in U$. Now for each i, the component function ϕ^i is a smooth function on a closed neighborhood A_i of U containing q. By Lemma 2.26, ϕ^i extends to a smooth function $\tilde{\phi^i}$ on N. Then $F^*(\tilde{\phi^i})$ is smooth, $i = 1, ..., n$. Let $\tilde{\phi} = (\tilde{\phi^1}, ..., \tilde{\phi^n})$. Viewing $\tilde{\phi}$ as a smooth map $N \to \mathbb{R}^n$, it follows from Proposition 2.12 that $F^*\tilde{\phi} = \tilde{\phi} \circ F$ is smooth as a map $M \to \mathbb{R}^n$.

Now, take an open set $W \subseteq \cap_{i=1}^n A_i$ such that $q \in W$, and $\phi = \tilde{\phi}$ on W. Then $(W, \tilde{\phi})$ is a smooth chart for N, since all transition maps can be computed by restricting ϕ. But since $F^*\tilde{\phi}$ is a smooth function $M \to \mathbb{R}^n$, there is a chart (V, ψ) with $V \subseteq F^{-1}(W)$ such that $\mathrm{id}^{-1} \circ (\phi \circ F) \circ \psi^{-1} = \tilde{\phi} \circ F \circ \psi^{-1}$ is a smooth map $\psi(V) \to \phi(W)$, as desired.

(c) Let G be the inverse of F. Say F is a diffeomorphism, so G is smooth. Then G^* is a linear map and $G^*(C^\infty(N)) \subseteq C^\infty(M)$. But clearly $F^* \circ G^* = \mathrm{Id}(C^\infty(M))$ and $G^* \circ F^* = \mathrm{Id}(C^\infty(N))$. Thus F^* is a linear map $C^\infty(N) \to C^\infty(M)$ and with inverse G^*, so it is an isomorphism.

On the other hand, say F^* is an isomorphism $C^\infty(N) \to C^\infty(M)$. Note that for $h \in C(N)$, if $F^*(h) = 0$, then $h = 0$, since F is surjective. For $p \in N$, let (W, ψ) be a smooth chart on M containing $G(p)$. Then for $i = 1, ..., m$, $\psi^i \circ G \in C(N)$. But $F^*(\psi^i \circ G) = \psi^i \circ G \circ F = \psi^i \in C^\infty(M)$. Since F^* is injective, this implies that $\psi^i \circ G$ is smooth. By Proposition 2.12, this implies that $\psi \circ G$ is smooth, which by an argument used above implies that G is smooth.