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Chapter Five: Submanifolds

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5.2 Boundary is embedded submanifold. By Theorem 5.8, it suffices to prove that δM satisfies the local k-slice condition. Take $p \in \delta M$ and let (U, ϕ) be a smooth boundary chart such that $\phi(p) \in \delta\mathbb{H}^n$. By the topological invariance of the boundary, $\phi(\delta M) = \phi(U) \cap \{x^n = 0\}$. Therefore (U, ϕ) is the necessary slice chart.

5.4 Figure 8 is not an embedded submanifold of $\mathbb{R}^2$. Let S be the figure 8. Since small open subsets of S can be easily seen to homeomorphic to open intervals, it follows from the topological invariance of dimension (Theorem 1.2) that if S is an embedded submanifold, it must have dimension 1. Now, let (U, ϕ) be a chart containing $(0, 0)$ but not $(0, 1)$ or $(0, -1)$. Assume that U is an open ball centered at $(0, 0)$. Since U is connected, $\phi(U)$ must be an interval, so ϕ induces a homeomorphism between an "x" shape and an interval. But then ϕ restricts to a homeomorphism between $U \setminus (0, 0)$ and two open intervals. This is a contradiction, since $U \setminus (0, 0)$ has four connected components while the intervals have two.

5.6 Tangent sphere to a submanifold of $\mathbb{R}^n$. If i is the inclusion $M \hookrightarrow \mathbb{R}^n$, define $\Phi : TM \to \mathbb{R}$

$$\Phi(p, v_p) = \|Di_p(v_p)\|_{\mathbb{R}^n}.$$

Then Φ is smooth as the composition of smooth maps. It is a submersion, because if $N : \mathbb{R}^n \to \mathbb{R}$ is the norm map, then $DN_p \neq 0$ for $p \neq 0$, and therefore since $D\Phi_p = DN_{i(p)} \circ Di_p$, 1 is a regular value of Φ. But $UM = \Phi^{-1}(1)$, so be Corollary 5.14, UM is a smooth $2m - 1$ dimensional manifold.

5.8 Tangent sphere to a submanifold of $\mathbb{R}^n$. It is clear that $F(p) = \|p\|^{-1}$ is a smooth immersion $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, and therefore by Proposition 5.47, $D = \mathbb{R}^n \setminus \mathbb{B}_1(0) = F^{-1}(-\infty, 1]$ is an embedded submanifold with boundary of $\mathbb{R}^n \setminus \{0\}$, and therefore of $\mathbb{R}^n$. By Proposition 5.46, the boundary of D is the S^{n-1}.

Now, the definition of a regular coordinate ball B gives that $B \subset B'$ where (B', ϕ) is a chart, $\bar{B} \subset B'$, and the image of B and B' under ϕ are nested balls in $\mathbb{R}^n$. The above shows that $\phi(B') \setminus \phi(B)$ is a manifold with boundary, so $B' \setminus B$ is a manifold with boundary δB. Since the condition is local, this implies that $M \setminus B$ is a manifold with boundary δB. Additionally, ϕ restricted to δB gives the necessary diffeomorphism with S^{n-1}.