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Chapter Sixteen: Integration on Manifolds

Lee, *An Introduction to Smooth Manifolds*

16.1 Volume of a parallelipiped. Let $F:[0,1]^n \to P$ be the map $(x_1,...,x_n) \mapsto \sum_{i=1}^n x_i v_i$. Then it follows from the change of variables formula that

$$\int_P 1\,dV = \int_{[0,1]^n} |\det(DF)|\,dV = |\det(v_1,...,v_n)|.$$

16.2 Computing an integral. We use the strategy of Proposition 16.8. The map $F:(0,2\pi)^2 \to \mathbb{T}$ defined by $(s,t) \mapsto (\cos(s),\sin(s),\cos(t),\sin(t))$ gives a diffeomorphism onto a dense subset of $\mathbb{T}$, so Proposition 16.8 gives that $\int_{\mathbb{T}} \omega = \int_{(0,2\pi)^2} F^*\omega$. We compute

$$\begin{aligned}\int_{(0,2\pi)^2} F^*\omega &= \int_{(0,2\pi)^2} \sin^2(s)\sin^2(t)\cos(t)\,ds\,dt \\ &= \int_0^{2\pi} \sin^2(s)\,ds \quad \int_0^{2\pi} \sin^2(t)\cos(t)\,dt \\ &= 0.\end{aligned}$$

16.3 Integrating over a generalized covering space. (Part (i.) only) Let $D \subset M$ be the support of ω and let $U_n : n = 1,...,N$ be a cover of M with evenly covered charts (U_n,ϕ_n). For each n, let $U_n^m : m = 1,...,k$ be the components of $\pi^{-1}(U_n)$. Then (U_n^m) is an open cover of $\pi^{-1}(D)$ and each $(U_n^m, \phi_n \circ \pi)$ is a smooth chart. Let $\psi_{n,m}$ be the corresponding partition of unity. We compute

$$\begin{aligned}\int_E \pi^*\omega &= \sum_{n,m} \int_E \psi_{n,m}\,\pi^*\omega \\ &= \sum_{n,m} \int_{\phi_n(U_n)} ((\phi_n \circ \pi|_{U_n^m}^{-1})^{-1})^*\,\psi_{n,m}\,\pi^*\,\omega \\ &= \sum_{n,m} \int_{\phi_n(U_n)} \psi_{n,m} \circ (\phi_n \circ \pi|_{U_n^m}^{-1})^{-1} \cdot (\phi_n^{-1})^*\omega\end{aligned}$$

Since $\psi_{n,m}$ is supported within U_n^m, we can extend $\psi_{n,m} \circ \pi|_{U_n^m}^{-1}$ by zero to define a smooth function on M. Let $F_n = \sum_{m=1}^k \frac{1}{k} \cdot \psi_{n,m} \circ \pi|_{U_n^m}^{-1}$. We claim that (F_n) is a partition of unity on M subordinate to (U_n). The range of F_n is clearly between 0 and 1 and F_n is clearly supported within U_n. Since this collection is finite it is certainly locally finite. Finally,

$$\sum_{n=1}^N F_n(x) = \frac{1}{k}\sum_{n=1}^N \sum_{m=1}^K \psi_{n,m} \circ \pi|_{U_n^m}^{-1}(x) = \frac{1}{k}\sum_{m=1}^k \sum_{n=1}^N \psi_{n,m} \circ \pi|_{U_n^m}^{-1}(x) = \frac{1}{k}\sum_{m=1}^k 1 = 1.$$

Therefore

$$\begin{aligned}\frac{1}{k}\int_E \pi^*\omega &= \frac{1}{k}\sum_{n,m}\int_{\phi_n(U_n)} \psi_{n,m}\circ(\phi_n\circ\pi|_{U_n^m}^{-1})^{-1}\cdot(\phi_n^{-1})^*\omega \\ &= \sum_{n=1}^{M}\int_{\phi_n(U_n)}\left(\frac{1}{k}\sum_{m=1}^{k}\psi_{n,m}\circ\pi|_{U_n^m}^{-1}\right)\circ\phi_n^{-1}\cdot(\phi_n^{-1})^*\omega \\ &= \int_M \omega.\end{aligned}$$

16.4 A compact oriented manifold cannot retract smoothly onto its boundary. Let ω be an orientation form on δM, whose existence is guaranteed by Proposition 15.5. Say first that $F : M \to \delta M$ is a smooth retraction, so $F \circ i = \mathrm{Id}_{\delta M}$. Then

$$\int_M \delta\omega = \int_{\delta M}(F\circ i)^*\omega = \int_{\delta M} i^*F^*\omega = \int_M \delta(F^*\omega) = \int_M F^*\delta\omega,$$

where the second-to-last inequality follows from Stokes' theorem. But $F^*\delta\omega = 0$. For the Jacobian of DF_p has rank at most $n-1$, while $\delta\omega$ is an n form. On the other hand, in any oriented coordinate chart $(U,(x^i))$ on δM, ω has a coordinate representation $f(x_1,...,x_n)\cdot dx_1\wedge ...dx_n$ with $f(x_1,...,x_n)>0$. Therefore $\int_M \delta\omega > 0$. This contradiction proves that such an F cannot exist.

Now, say F is a merely continuous retraction $M \to \delta M$. By the Whitney Approximation Theorem (6.26) F is homotopic to a smooth map $G : M \to \delta M$. Further, since F is the identity map on δM, F is smooth on δM, so the homotopy can be taken relative to δM. But this implies that G is a smooth retraction $M \to \delta M$, so we are reduced to the first case.

16.5 Homotopic diffeomorphisms are either both orientation-preserving or both orientation-reversing. By Theorem 6.29, if F and G are homotopic then they are smoothly homotopic. Let H be the smooth homotopy, so $H(x,0) = F(x)$ and $H(x,1) = G(x)$. Note that $\delta(M\times I) = M\times\{0\}\cup M\times\{1\}$. Let ω be an orientation form on N. Then

$$\int_{\delta(M\times I)} H^*\omega = \int_{M\times\{0\}} H^*\omega + \int_{M\times\{1\}} H^*\omega.$$

Now if (U,ϕ) is a positively oriented chart for M, then $(U\times\{0\},\phi(x_1,...,x_n,0))$ is a negatively oriented chart and $(U\times\{1\},\phi(x_1,...,x_n,1))$ is a positively-oriented chart in the boundary orientation. For in the first case the outward-pointing normal vector points in the negatively oriented direction along $\mathbb{R}$, and in the second case it points in the positively-oriented direction.

Assume first that N is contained in the domain of a single chart oriented. We can take ω to be the corresponding n-form. Write $H_t(p) = H(t,p)$. Then M is as well, since it is diffeomorphic to N. Let (x^i) be coordinates on N and let $(y^i) = (x^i\circ H_0)$ be coordinates on $M\times\{0\}$. Examining the coordination expansion of $H^*\omega$, we see that

$$\begin{aligned} H^*dx^1 \wedge ... \wedge dx^n &= \sum \frac{dH^1}{dy_{j_1}}(p,0) \cdot ... \cdot \frac{dH^n}{dy_{j_n}}(p,0) dy^{j_1} \wedge ... \wedge dy^{j_n} \\ &= \det(dH(p,0)) dy^1 \wedge ... \wedge dy^n \\ &= \det(dF(p)) dy^1 \wedge ... \wedge dy^n \\ &= F^*dx^1 \wedge ... \wedge dx^n. \end{aligned}$$

We can remove that assumption that N be contained in a single chart by applying the above result to all the terms in the sum for the integral $\int_{M\times\{0\}} H^*\omega$. Combined with the above discussion of orientation, this shows that

$$\int_{M\times\{0\}} G^*\omega - \int_{M\times\{0\}} F^*\omega = \int_{M\times I} d(H^*\omega).$$

However, $d\omega = 0$ because it is an $n+1$ form on N. Therefore

$$\int_{M\times I} d(H^*\omega) = \int_{M\times I} H^* d\omega = 0.$$

16.6 The Hairy Ball Theorem.

1. $a \to b$: Let $V : \mathbb{S}^n \to \mathbb{S}^n$ be a nowhere-vanishing vector field. We identify tangent vectors to $\mathbb{S}^n$ with elements of $\mathbb{R}^{n+1}$ through the canonical isomorphism between T_pM and $\mathbb{R}^n$ when M is an n-dimensional embedded submanifold of $\mathbb{R}^k$. We have shown that $\langle v, x\rangle = 0$ for all $v \in T_x\mathbb{S}^n$. Therefore $V(x)$ is the desired map.

2. $b \to c$: ?

3. $c \to d$: This follows directly from the previous problem.

4. $d \to e$: Let $v_p^1, ..., v_p^n$ be an oriented collection of vectors in $T_p\mathbb{S}^n$. Then if ω is an orientation form on $\mathbb{R}^{n+1}$,

   $$\alpha^*(i_{N(p)}\omega)(v_p^1, ..., v_p^n) = \omega_{-p}(-p, D\alpha_p v_p^1, ..., D\alpha_p v_p^n) = (-1)^{n+1}\omega_{-p}(p, v_p^1, ..., v_p^n).$$

   This α is orientation-preserving only if n is odd.

5. $e \to a$: This was Problem 9-4.

16.7 Products of manifolds with corners. We note that if $A \subset X$ is open and $B \subset Y$ is open, then $A \times B \subset X \times Y$ is open by the definition of the product topology. Also, it is clear that $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^m_{\geq 0} = \mathbb{R}^{n+m}_{\geq 0}$. Now, let M and N be manifolds with corners and let (p,q) be a point in $M \times N$. If p and q are non-corner points, it is clear that (p,q) is a non-corner point. Say p is a corner point and q is not. Let (U,ϕ) be a corner chart for p and let (V,ψ) be a non-corner chart for q. Since we can choose a precompact chart for q, we can shift $\psi(V)$ so that it is an open subset of $\mathbb{R}^m_{\geq 0}$. Then $(U \times V, \phi \times \psi)$ is a corner chart for (p,q). It is smooth because transition maps are given by the product of the transition maps from the original charts, which were smooth. Clearly the argument is similar if q is a corner point and p is not. Finally, if p and q are both corner points, then the product of their corner charts is a corner chart without the need for a shift. Induction extends this argument from pairs to finite products.

The product of two manifolds with boundary is not necessarily a smooth manifold with boundary. For instance, $I \subset \mathbb{R}$ is a manifold with boundary but $I^2 \subseteq \mathbb{R}^2$ is not a smooth manifold with boundary. For say (U, ϕ) is a chart containing the point $(1,1) \in I^2$. We must have that ϕ maps interior points (as a subset of $\mathbb{R}^2$ of $\phi(U)$ to interior points of I^2, since ϕ^{-1} is a diffeomorphism. Therefore ϕ^{-1} gives a smooth map between $U \cap \{(x,y) : y = 0\}$ and δI^2. Let $\gamma(t) = (0,t)$ for $t \in (-\epsilon, \epsilon)$. Then $\phi^{-1} \circ \gamma(t)$ is a smooth curve $(-\epsilon, \epsilon) \to I^2$ with nonvanishing derivative that passes through $(1,1)$. A little work shows that this curve must enter the corner from one side of I^2 and pass up the adjacent side. (Maybe take a homeomorphism of this corner with the line and use connectedness.) But it is clear that the derivative of this curve cannot be continuous at 0.

16.9 A closed, not exact $n-1$ form on $\mathbb{R}^n \setminus \{0\}$. To see that $i^*_{\mathbb{S}^{n-1}}\omega$ is the Riemannian volume form on $\mathbb{S}^{n-1}$, note that a collection of orthogonal vectors $v_2, ..., v_n \in T_p\mathbb{S}^{n-1}$ is a positively oriented orthonormal basis for $T_p\mathbb{S}^{n-1}$ if and only if the matrix whose columns are $x, v_2, ..., v_n$ has determinant $+1$. But given such a collection,

$$i^*_{\mathbb{S}^{n-1}}\omega(v_2, ..., v_n) = \omega(p, v_2, ..., v_n)$$

which can be seen to be the same determinant.

A computation shows that $d\omega = 0$: given any term in the sum, only differentiation by $\frac{d}{dx_i}$ produces a nonzero covector. This derivative is computed as follows:

$$\frac{d}{dx_i}\left(\frac{x^i}{|x|^n}(-1)^{n-1}\right) = (-1)^{n-1}\frac{|x|^2 - n\,x_i^2}{|x|^{n+2}}$$

Therefore

$$\sum_{i=1}^{N} \frac{d}{dx_i}\left(\frac{x^i}{|x|^n}(-1)^{n-1}\right) dx^i \wedge dx^1 \wedge ... \wedge \hat{dx^i} \wedge ... \wedge dx^n = \sum_{i=1}^{N} \frac{|x|^2 - n\,x_i^2}{|x|^{n+2}}\, dx^1 \wedge ... \wedge dx^n = 0.$$

On the other hand, the integral of $i^*_{\mathbb{S}^{n-1}}\omega$ over $\mathbb{S}^{n-1}$ is nonzero by Proposition 16.28. (Of course, it is the volume of $\mathbb{S}^{n-1}$.) Therefore it follows from Corollary 16.15 that ω is not exact.