Github/dogestoyevsky
July 2017

Chapter Fifteen: Orientations

Lee, *An Introduction to Smooth Manifolds*

15.2 Local diffeomorphisms are either orientation-preserving or orientation-reversing on connected manifolds. Let $A \subseteq M$ be the set of points where F is orientation-preserving. We claim that A is both open and closed. To see that A is closed, let x be a limit point of A, let (x_ℓ) be a sequence of points in A converging to x, and let ω be an orientation form on N. There is a smooth oriented chart containing x and all but finitely many points of (x_ℓ). Let $E_1, ..., E_n$ be the corresponding oriented coordinate frame. Then $\omega(dF_{x_\ell}E_1(x_\ell), ..., dF_{x_\ell}E_n(x_\ell))$ is defined and positive for all sufficiently large ℓ. But then $\omega(dF_xE_1(x), ..., dF_xE_1) \geq 0$ by continuity. However, this quantity is nonzero since F is a local diffeomorphism and ω is nondegenerate. Therefore $\omega(dF_xE_1(x), ..., dF_xE_1) > 0$, so F is orientation-preserving at x. That A is open follows from Exercise 15.13: given a choice of oriented charts on M and N, if the Jacobian determinant of F is positive at $p \in M$ then it is positive in a neighborhood of p.

This shows that A is either $\emptyset$ or M. A similar argument shows that the set of points where F is orientation-reversing is either $\emptyset$ or M. Since F is a local diffeomorphism, at any point it is either orientation-reversing or orientation-preserving, so one of these two sets is nonempty and therefore all of M.

15.4 Flows are orientation-preserving. Let ω be a smooth orientation form on M. Given $p \in M$, let (v_i) be an oriented basis for T_pM. The function

$$G(t) = \omega((d\theta_t)_pv_1, ..., (d\theta_t)_pv_n)$$

defined for all t such that $p \in M^t$ is smooth, and since $\theta_0 = \mathrm{Id}_M$, $G(0) > 0$. But since θ_t is a diffeomorphism for all t, G is nonvanishing, so it positive for all t where it is defined. This proves that $(d\theta_t)_p$ sends an oriented basis for T_pM to an oriented basis for $T_{\theta_t(p)}M$. Since p and t were arbitrary, this shows that θ_t is orientation-preserving wherever it is defined.

15.5 The tangent and cotangent bundles are orientable. The standard coordinate charts on TM are consistenly oriented. For given $p \in M$, let $(U, (x^i))$ and $(V, (y^i))$ be charts on M containing p. if $(y^{-1}(p), v)$ is the coordinate representation of a point in TM, then the transition map between these charts takes the form $(x \circ y^{-1}(p), D(x \circ y^{-1})v)$. The Jacobian of this map can be written blockwise with two diagonal blocks $D(x \circ y^{-1})_p$ and zeros in the upper right block. This proves that the Jacobian has determinant $\mathrm{Det}(D(x \circ y^{-1})_p)^2$, which is positive. Therefore, these charts are consistenly oriented, so TM is orientable by Proposition 15.6.

The argument is similar for T^*M, except the lower right block is the inverse transpose of $D(x \circ y^{-1})_p$. Since the determinant of the inverse transpose of a matrix has the same parity as the original matrix, the proof goes through.

15.7 Unit normal vector field determining the orientation of an immersed hypersurface. Given $p \in S$, let $(U, (x^i))$ be a chart for S containing p and let $(W, (y^i))$ be a chart for M containing p. Assume that $i(U) \subseteq W$. Let ω be an orientation form for M and η an orientation form for S.

Now, $(\frac{d}{dy^i}|_p)$ are not contained in the span of $(di_p \frac{d}{dx^i}|_p)$.Therefore we can apply the Gram-Schmidt algorithm to one of the coordinate vectors fields of $(W, (y^i))$ to produce a smooth vector field $X(p)$ that is orthogonal to T_pS at each p. Now define

$$Y(p) = X(p) \cdot \frac{\eta_p(\frac{d}{dx^1}, ..., \frac{d}{dx^n})}{\omega_p(X(p), di_p \frac{d}{dx^1}, ..., di_p \frac{d}{dx^1})}$$

and define $Z(p) = \frac{Z(p)}{\|Z(p)\|}$. These quantities are well-defined because the given vectors are independent and ω and η are nonvanishing. Then

$$\omega_p(Y(p), \frac{d}{dx^1}|_p, ..., \frac{d}{dx^n}|_p) = \frac{\eta_p(\frac{d}{dx^1}, ..., \frac{d}{dx^n})}{\omega_p(X(p), di_p \frac{d}{dx^1}, ..., di_p \frac{d}{dx^1})} \cdot \omega_p(X(p), \frac{d}{dx^1}|_p, ..., \frac{d}{dx^n}|_p) = \eta_p(\frac{d}{dx^1}|_p, ..., \frac{d}{dx^n}|_p).$$

Therefore $i_S^*(N \rfloor \omega)$ is an orientation form for U. Since $Z(p)$ is a positive multiple of $Y(p)$, this shows that $Z(p)$ is the desired vector field on U. Covering S with such charts and using a partition of unity gives the desired vector field on S.

15.8 Immersed submanifolds with trivial normal bundles. Say S has a trivial normal bundle and let $\phi : S \times \mathbb{R}^k \to TS$ be the inverse of a global trivilization. Then $X_\ell(p) = \phi(p, e_\ell)$ defines a smooth normal vector field along S and the collection $\{(\Phi_\ell)_p : \ell = 1, ..., k\}$ is independent for each p. Let ω be a nondegenerate n-form on M (Proposition 15.5). Then interior multiplication by the X_ℓ defines a nondegenerate $n-k$-form on S, so S is orientable.

Part (b) follows from the previous question, since a global normal vector field gives a global trivialization.

15.10 Characteristic property of the orientation covering. Let ω be an orientation form on X. Given $p \in X$, let G be an inverse for F in some neighborhood of $F(p)$, and let $\mathcal{O}_p$ be the orientation defined on $T_{F(p)}M$ by $G^*\omega$, which is independent of the choice of G. define $\hat{F}(p) = (p, \mathcal{O}_p)$. It is clear that $F = \hat{pi} \circ \hat{F}$.

We show that $\hat{F}$ is orientation-preserving. Let $(v_1, ..., v_n)$ be a positively-oriented collection of vectors in T_pX. Then $(dF_pv_1, ..., dF_pv_n)$ is positively-oriented with respect to the orientation induced by $G^*\omega$ on the domain of G. Because $F = \hat{\pi} \circ \hat{F}$, this implies that $(d\hat{\pi}_p \circ d\hat{F}_pv_1, ..., d\hat{\pi}_p \circ d\hat{F}_pv_n)$ is positively oriented. But by definition $d\hat{\pi}_p$ is orientation-preserving, so this implies that $(d\hat{F}_pv_1, ..., d\hat{F}_pv_n)$ is orientation-preserving.

To see that $\hat{F}$ is a local diffeomorphism, we can for instance write $\hat{F}$ as the composition of F with a smooth local section of $\hat{\pi}$ that is chosen to have the right orientation at $F(p)$ (Theorem 4.26).

15.12 Orientation-reversing diffeomorphism of $\mathbb{R}$. Define $g(x) = f(x) - x$. Since $f'(x) < 0$, $g'(x) < -1$. Take any point in the image of g and integrate in the right direction until you have to cross the x axis.

15.16 The orientation covering of the Mobius bundle is a cylinder. We prove that the cylinder is an oriented twofold cover of E, which gives the result by Theorem 15.42. Recall that the smooth structure on E is induced (through Proposition 4.40) by the quotient map $p : \mathbb{R}^2 \to E$. Now let $\sim$ be the relation on $\mathbb{R}^2$ such that $(x, y) \sim (x', y')$ if and only if $y = y'$ and $x = x' + 2n$ for $n \in \mathbb{Z}$, let π be the topological covering map $\mathbb{R}^2 \to \mathbb{R}^2/\sim$, and give $\mathbb{R}^2/\sim$ the smooth structure such that π is a smooth covering map. On the other hand, let $F : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ send $(x, y) \mapsto (e^{\pi i x}, y)$. Then since F is constant on the fibers of π, so by Theorem 4.31 $\mathbb{R}^2/\sim$ is diffeomorphic to $S^1 \times \mathbb{R}$. Therefore we may refer to $\mathbb{R}^2/\sim$ as the cylinder.

A similar argument shows that the cylinder is a smooth cover of E: the projection $\mathbb{R}^2 \to E$ is constant on the fibers of π so it induces a smooth map $G : \mathbb{R}^2/\sim \to E$. It can be checked that this map is a two-sheeted smooth covering map.