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Chapter Eleven: The Cotangent Bundle

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11.2 The algebraic dual to an infinite-dimensional vector space.

1. Let (v_n) be a basis for V and define w_n dual to v_n, so that $w_n(\sum_{k=1}^{\ell} c_{n_k} v_{n_k}) = c_n$. Let $\Phi : V \to V^*$ be defined
$$\Phi(\sum_{k=1}^{\ell} c_{n_k} v_{n_k}) = \sum_{k=1}^{\ell} c_{n_k} w_{n_k}.$$
Then if $v = \sum_{k=1}^{\ell} c_{n_k} v_{n_k}$ and $\Phi(v) = 0$, $[\Phi(v)](w_\ell) = c_\ell = 0$ for all ℓ, so $v = 0$. Therefore Φ is injective.

2. For $S \in \mathrm{Pow}((v_n))$, let $w_\ell = \sum_{i_s \in S} w_\ell$. Let (u_n) be a subset of w_ℓ that is linearly independent over $\mathbb{Z}$. RIGHT CARDINALITY?

3. Part one implies there is an injection $g : V^* \hookrightarrow V^{**}$. If there were such an isomorphism $f : V \to V^{**}$, then the composite map $f^{-1} \circ g$ would be an injection $V^* \to V$, contradicting the second part.

11.4 Functions that vanish at a point.

1. The important thing I didn't realize is that ℓ_p^2 is the *span* of products of function in ℓ_p. If I had a dollar for every time I didn't realize I could take the span...

   Given p, choose a local coordinate system such that $p = 0$. In these coordinates, say that $f(0) = 0$ and $\frac{df}{dx_i}(0) = 0$. Then by Taylor's theorem, there are finitely many functions f_α, determined by integrating higher-order derivatives of f, such that $f = \sum_\alpha x^i x^j f_\alpha(x_1, ..., x_k)$. Since each expression x^i vanishes at 0, f is therefore in ℓ_p^2.

   The converse is shown by simply calculating the derivative in arbitrary coordinates.

2. It is clear from the coordinate formula (11.10) that if all first-order derivatives vanish, the differential is 0, so by the first part of this question, Φ vanished on ℓ_p^2. Therefore Φ is well-defined (and clearly linear) on ℓ_p/ℓ_p^2. It is surjective because, again using coordinates centered at p, if $v = \sum_{i=1}^k v_i dx_i \in T_p^*(M)$, then $f(x) = \sum_{i=1}^k v_i x_i$ satisfies $\Phi(f) = v$ and $f \in \ell_p$. It is injective, because if $df_p = 0$ and $f(p) = 0$, then it is clear (11.10) that the first-order Taylor polynomial of f vanishes.

11.6 Some statements about when real functions define local coordinates.

1. We show that the map $x \mapsto (y^1(x), ..., y^k(x))$ is nonsingular at p. Let (U, ϕ) be a coordinate system for M such that $p \in U$ and let $\frac{d}{dx_i}|_p$ be the corresponding basis for T_pM. Now for any vector space V, if $e_1, ..., e_k$ is a basis for V and $f_1, ..., f_k$, then the matrix $(f_i(e_j))$ is linearly independent. For say there exist constants such that $\sum_{i=1}^k c_k f_k(e_j) = 0$ for all j. Then $\sum_{i=1}^k c_k f_k$ is the zero form, so $c_k = 0$ for all k. Since $dy_p^i(\frac{d}{dx_j}|_p) = \delta_j(y^i \circ \phi^{-1}(p))$, this shows that $F \circ \phi^{-1}$ is a regular function $\mathbb{R}^n \to \mathbb{R}^n$.

Therefore it maps a neighborhood of $\phi(p)$ to a neighborhood of $F(p)$. Since ϕ is a diffeomorphis, this suffices to show that F gives a coordinate chart for some neighborhood of $F(p)$.

2. Pick vectors $w_{k+1}, ..., w_n$ extending $dy_p^1, ..., dy_p^n$ to a basis for T_pM, and then define functions $y^{k+1}, ..., y^n$ in a neighborhood of p such that $d(y^\ell)_p = w_\ell$ for $\ell = k+1, ..., n$ (for instance by working in coordinates and multiplying a bump function constant at p by an appropriate linear function). Then apply the first part.

3. Select a basis from among the $\{dy_p^\ell\}$ and apply the first part.

11.8 A functor from a category of smooth manifolds to vector bundles.

1. The map $dF^* : T^*N \to T^*M$ is defined

$$(p, v) \mapsto (F^{-1}(p), dF^*_{F^{-1}(p)}v).$$

Since dF^* is a bundle homomorphism over F^{-1} by this definition, it only remains to show that it is smooth. Let (U, ϕ) be a chart for M containing p. Then since F is a diffeomorphism, $(F^{-1}(U), \phi \circ F)$ is a smooth chart for N containing q. But a computation shows that dF^* is the identity map in these coordinates, so it is smooth.

2. The functorial properties of this map follow directly from the corresponding properties of the dual map (Proposition 11.4).

11.10 Computing coordinate representations of differentials.

1. $df_{(x,y)} = \frac{y^2-x^2}{(x^2+y^2)^2}dx + \frac{-2xy}{(x^2+y^2)^2}dy$. Does not vanish on M.

2. $df_{(r,\theta)} = \frac{-\cos(\theta)}{r^2}dr + \frac{-\sin(\theta)}{r}d\theta$.

11.12 Connected manifolds are smoothly path-connected. Take $p \in M$ and let S be the set of points in M that can be connected to p by a smooth path. S is open because given $q \in S$, a path from p to q can be extended to a smooth path to any point in a neighborhood of of p. For say $\gamma(0) = p$, $\gamma(1) = q$, and γ is smooth. Take a coordinate ball centered at q and let q' be a point in this coordinate ball. Let $f : [0, 1] \to [0, 1]$ be a smooth increasing surjective function that is constant on a neighborhood of 1. Let $g : [1, 2] \to \mathbb{R}$ be a smooth increasing function such that $g(x) = 0$ for x in a neighborhood of 0 and $g(2) = 1$. Such functions can be constructed by integrating smooth bump functions. Then define $\eta : [0, 2] \to M$ as follows:

$$\gamma(t) = \begin{cases} \gamma \circ f(t) & t \in [0,1] \\ q' \cdot g(t) + q \cdot (1 - g(t)) & t \in [1,2] \end{cases}.$$

Then η is a path from p to q', and it is smooth. For both functions in the expansion of η are constant in a neighborhood of 2. It can be checked that if $\alpha : (-\epsilon, 0] \to \mathbb{R}$ and $\beta : [0, \epsilon) \to \mathbb{R}$ are smooth functions such that $\lim\limits_{t \to^+ 0} \alpha^{(k)}(t) = \lim\limits_{t \to^- 0} \beta^{(k)}(t)$ for all $k \le j$, then the concatenation of α and β is j-times continuously differentiable.

On the other hand, S is closed, because if $q \in \delta S$, any chart containing q contains an element q' of S, and then the same argument produces a smooth path from p to q by extending the path terminating in q'.

11.14 Line integrals

1. $\int_\gamma \omega = 2\log(2) - 1$; $\int_\gamma \eta = 1 + \log(2)$.
2. Since the vector fields are defined on $\mathbb{R}^3$, it suffices to check (11.21). ω is not exact because

$$\omega^1_z = \frac{-4}{(x^2+1)^2}$$

   while

$$\omega^3_x = \frac{2-2x^2}{(x^2+1)^2}.$$

   Checking each cross-term as in (11.21) shows that η is exact.

3. A computation as in Example 11.51 gives a potential function $\eta = df$,

$$f(x,y,z) = \frac{2z}{x^2+1} + \log(y^2+1).$$

   This agrees with the computed integral.

11.16 No nonvanishing exact covector fields on compact manifolds. Write $\omega = df = \frac{df}{dx_j} dx_j$ for some smooth function f. Now since M is compact, f attains a maximum and minimum on M. Either f is constant, or there is at least one point achieving the global maximum and one other point achieving the global minimum on M, and at these two points, all derivatives of f vanish, so by the coordinate formula (11.10), $\omega = 0$ at these two points.