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Chapter Seven: Lie Groups

Lee, *An Introduction to Smooth Manifolds*

7.2 Differential of the multiplication map at the identity. For $i = 1, 2$, let $\gamma_i : I \to G$ be curves such that$\gamma_i(0) = e$. Let $\gamma : I \to G \times G$ be the product $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. The bijection of Proposition 3.14 can be described as the map $T_eG \times T_eG \to T_{(e,e)}G \times G$ that sends $(\gamma_1'(0), \gamma_2'(0))$ to $\gamma'(0)$. Now, if $X = \gamma_1'(0) \in T_eG$, then by this bijection $(X, 0) = (\gamma_1, e)'(0)$, and we can compute for any $f \in C^\infty(G)$

$$[dm_{(e,e)}(X,0)](f) = \frac{d}{dt}\bigg|_{t=0} f(m(\gamma_1(t), e)) = \frac{d}{dt}\bigg|_{t=0} f(\gamma_1(t)) = X(f).$$

Similarly, $dm_{(e,e)}(0, Y) = Y$. By linearity, therefore, $dm_{(e,e)}(X, Y) = X + Y$.

Now, define a map $H : G \to G \times G$ by $H(g) = (g, i(g))$. Then $m \circ H(g)$ is the constant map $g \mapsto e$, so $dm_{e,e} \circ dH_e = 0$. By the previous paragraph, $dm_{(e,e)} \circ dH_e(X) = X + di_e(X)$, so $di_e(X) = -X$.

7.4 Differential of the determinant function on $\mathbf{GL}_n(\mathbb{R})$. Since $\det(I + tA) = \det(B^{-1}(I + tA)B) = \det(I+tB^{-1}AB)$, we can assume without loss of generality that A is triangular. Then $\det(I+tA) = \prod_{n=1}^N (a_{nn}t - 1) = 1 + (a_{11} + ... + a_{nn})t + ... + a_{11} \cdot ... \cdot a_{nn}t^n$. The derivative of this expression at $t = 0$ is $\mathrm{Tr}(A)$.

Now, if $X \in \mathrm{GL}_n\mathbb{R}$, then for small e the curve $t \mapsto \phi^{-1}(X + tB)$ is a curve $(-\epsilon, \epsilon) \to \mathrm{GL}_n(\mathbb{R})$, where (U, ϕ) is the standard global chart on $\mathrm{GL}_n(\mathbb{R})$. Then

$$[d(\det \circ \phi)_X(B)](f) = \frac{d}{dt}\bigg|_{t=0} f(\det \circ \phi \circ \gamma(t)) = f'(\det(X))\det(X)\mathrm{Tr}(X^{-1}B)$$

so $d(\det \circ \phi)_X = \det(X)\mathrm{Tr}(X^{-1}B)$ as desired.

7.6 Bounded products. Let $V' = m^{-1}(U)$. Since m is a smooth map $G \times G \to G$, V' is open. Take a basis element of the form $A \times B$ containing (e, e), where A and B are each open in G, and let $V = A \cap B$.

7.8 Action of a connected group on a discrete space. For $k \in K$, let $\theta^{(k)}(g) = \theta(g, k)$. Then $\theta^{(k)}$ is a smooth map $G \to K$. Now clearly $\theta^{(k)}(e) = k$. But $(\theta^{(k)})^{-1}(k)$ and $(\theta^{(k)})^{-1}(K \setminus \{k\})$ are open in G, and clearly $G = (\theta^{(k)})^{-1}(k) \cup (\theta^{(k)})^{-1}(K \setminus \{k\})$. Since the first set is nonempty, $G = (\theta^{(k)})^{-1}(k)$, so $g \cdot k = k$ for all $g \in G$. Since k was arbitrary, this proves that the action of G is trivial on K.

7.12 Proof that smooth Lie group homomorphisms have constant rank using equivariant rank theorem. Let $F : G \to H$ be a smooth Lie group homomorphism. Now G acts smoothly on G by $g' \cdot g = g'g$ and on H by $g' \cdot h = F(g')h$. This action is clearly transitive on G, and $g' \cdot F(g) = F(g')F(g) = F(g' \cdot g)$, so F is equivariant under this action. Therefore F has constant rank by the equivariant rank theorem (Theorem 7.25).