Github/dogestoyevsky
May 2017

Chapter Four: Submersions, Immersions, and Embeddings

Lee, *An Introduction to Smooth Manifolds*

Exercise 4.9 We give a counterexample if M has boundary. Consider the inclusion $i : \mathbb{H}^n \to \mathbb{R}^n$. This map is clearly smooth since it has the identity as a smooth extension on any open set. Also, i is an immersion and a submersion. For the differential of i at a point p is given by the differential of any smooth extension, so $Di_p = D(\mathrm{Id})_p$. However, given $p \in \delta\mathbb{H}^n$, there cannot be an inverse diffeomorphism in a neighborhood of $i(p) \in \mathbb{R}^n$. For say there is such a diffeomorphism G. Then G maps an open subset of $\mathbb{R}^n$ into a subset of $\mathbb{H}^n$ that intersects the boundary in p. But G is an open map since $DG_{i(p)}$ has nonzero determinant, so this is a contradiction.

On the other hand, M is a manifold without boundary and N a manifold with boundary, a similar argument (in coordinates) shows that F maps M into the interior of N, in which case Proposition 4.8 applies. But if F is a local diffeomorphism $M \to \mathrm{Int}\ N$, then it is a local diffeomorphism $M \to N$, since $\mathrm{Int}\ N$ is an open subset of N.

Exercise 4.24 Let $A = [0, 1) \subseteq \mathbb{R}$ and consider the inclusion $i : A \hookrightarrow \mathbb{R}$. Then i is a topological embedding since the identity map restricts to a continuous inverse $i(A) \to A$, and it is smooth as a map from a manifold with boundary to a manifold in the obvious coordinates. Since i is the identity in these coordinates, the differential is injective. Therefore i is a smooth immersion. But $[.5, 1) \subseteq A$ is closed, while $i(A)$ fails to be closed, and on the other hand $[0, .5) \subseteq A$ is open, while $i(A)$ fails to be open.

4.2 A smooth map with surjective differential maps interior points to interior points. Take $p \in M$. Working in coordinates, we can assume that $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{H}^m \subset \mathbb{R}^n$. The assumption that dF_p is nonsingular means that F maps an open neighborhood of p to an open neighborhood of $F(p)$ in $\mathbb{R}^n$. By the smooth invariance of the boundary, this implies that $F(p)$ is an interior point of N.

4.4 A dense curve in $\mathbb{T}^2$. Let $f(x) = x - \lfloor x \rfloor$ as in the proof of Lemma 4.21. We have seen that for all N, there must be a pair of integers $0 \leq n, m \leq N$ such that $0 < |f(n\alpha) - f(m\alpha)| = \delta < 1/N$.

This implies that for all $t \in [0, 1]$, there exists k such that $|t - f(\alpha k(n - m))| \leq 1/N$. For if $t > f(nx)$, then let $d = t - f(nx)$ and $s = \lfloor d/\delta \rfloor$. Note that $f(a + b) = f(f(a) + f(b))$ and if $0 \leq u < 1/f(r)$ then $f(ur) = uf(r)$. Therefore

$$f(\alpha(n + s(m - n))) - t = f(f(\alpha n) + s\delta) - t < \epsilon.$$

The argument if $t < f(nx)$ is similar. This shows that the points $f(n\alpha) : n \in \mathbb{Z}$ are dense in $[0, 1]$, which shows that the points $e^{2\pi i\alpha n} : n \in \mathbb{Z}$ are dense in $\mathbb{S}^1$.

Now, take any point $p = (e^{2\pi ix}, e^{2\pi iy}) \in \mathbb{T}^2$. The above allows us to fine an integer n such that $|e^{2\pi i\alpha n} - e^{2\pi i(x-\alpha y)}| < \epsilon$. But then

$$e^{2\pi i\alpha(y+n)} - e^{2\pi ix} = e^{2\pi i(\alpha y+\alpha n)} - e^{2\pi i(\alpha y+(x-\alpha y))} = e^{2\pi i\alpha n} - e^{2\pi i(x-\alpha y)}$$

and

$$\gamma(y+n) = (e^{2\pi i\alpha(y+n)}, e^{2\pi i(y+n)}) = (e^{2\pi i\alpha(y+n)}, e^{2\pi iy})$$

so $\|\gamma(y+n) - p\| < \epsilon$, as desired.

4.6 No smooth submersion from a manifold to $\mathbb{R}^k$. Say there is a smooth submersion $\pi : M \to \mathbb{R}^k$ for any $k > 0$. By Proposition 4.28, π is an open map, so $\pi(M)$ is open in $\mathbb{R}^k$. But continuous mappings are compact mappings, so $\pi(M)$ is also a compact subset of $\mathbb{R}^k$. This contradicts the Heine-Borel theorem.

4.8 A map that preserves smoothness under composition but is not a smooth submersion. Clearly π is smooth and surjective. If $F : \mathbb{R} \to P$ is smooth then $F \circ \pi$ is smooth by composition, while if $F \circ \pi$ is smooth, then F can be written as the composition $F \circ \pi \circ \psi$ where $\psi : x \mapsto (x, 1)$ is smooth, so F is smooth. But π is not a smooth submersion, since its differential vanishes at $(0, 0)$.

4.10 S^n is a smooth twofold cover of $\mathbb{R}P^n$. Let

$$U_j^+ = \{(x_1, ..., x_{n+1}) \in \mathbb{S}^n : x_j > 0\}$$

and similarly for U_j^-. Also, let

$$W_j = \{[x_1, ..., x_{n+1}] \in \mathbb{R}P^n : x_j \neq 0\}.$$

It has already been shown that q is smooth and surjective, and we know that $\{W_j : j = 1, ..., n+1\}$ is a cover of $\mathbb{R}P^n$. We claim that for each j, the components of $q^{-1}(W_j)$ are $V_j^+ = U_j^+ \cap q^{-1}(W_j)$ and $V_j^- = U_j^- \cap q^{-1}(W_j)$ and that q restricts to a diffeomorphism of each component with W_j.

To produce a smooth local inverse, define $r^+ : W_j \to V_j^+$ by the equation

$$r^+([x_1, ..., x_{n+1}]) = \frac{x_j}{|x_j|\sqrt{\sum_{i=1}^{n+1} x_i^2}}(x_1, ..., x_{n+1}),$$

which is easily seen to be well-defined. The coordinate representation using the standard coordinates on $\mathbb{R}P^n$ and $\mathbb{S}^n$ (Examples 1.4, 1.5) is

$$(x_1, ..., x_n) \mapsto \frac{1}{\sqrt{1 + \sum_{i=1}^{n} x_i^2}}(x_1, ..., x_n)$$

which is clearly smooth. Now if $x = (x_1, ..., x_{n+1}) \in U_j^+ \cap q^{-1}(W_j)$, then $r^+(q(x)) = \frac{x_j}{|x_j|}(x_1, ..., x_{n+1}) = x$, while if $y = [y_1, ..., y_n] \in W_j$, then $q(r^+([y_1, ..., y_n])) = [y_1, ..., y_n]$. An analogous argument gives an inverse r^- on V_j^-.

To show that V_j^+ and V_j^- are the components of $q^{-1}(W_j)$, we note that W_j is connected, since its coordinate representation is $\mathbb{R} \setminus \{0\}$, and so $r^+(W_j)$ and $r^-(W_j)$ are connected. But these two sets are clearly disjoint, and since each point in $\mathbb{R}P^n$ has a two-element preimage, they must be all of $q^{-1}(W_j)$.

4.12 Embedding $\mathbb{T}^2$ in $\mathbb{R}^3$. Since a smooth covering map is a surjective sooth submersion, and since X is clearly constant on the fibers of ϵ^2, Theorem 4.30 gives that X descends to a smooth map $\tilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$. Since X is a submersion and ϵ^2 a local diffeomorphism, $\tilde{X}$ is a submersion. It is easily seen to be a biijection, so it is a homeomorphism as a continuous bijection from a compact space into a Hausdorff space. It is straightforward to see that the image of this map is the surface of revolution described on Page 79.