Github/dogestoyevsky
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Chapter Ten: Vector Bundles

Lee, *An Introduction to Smooth Manifolds*

10.1 A nontrivial vector bundle.

(a) $\mathbb{Z}$ acts on $\mathbb{R}^2$ in the following way:

$$n \cdot (x, y) = (x + n, (-1)^n y).$$

Giving $\mathbb{Z}$ the discrete topology, this is a continuous action, since the map $(x, y) \mapsto n\cdot(x, y)$ is continuous for each n. It is clear that $\mathbb{Z}$ acts freely discontinuously. Since $\mathbb{R}^2$ is connected and locally path-connected, it follows that the quotient map $q : \mathbb{R}^2 \to E$ is a topological covering map.

To define a smooth strucure on E, take $p \in E$ and let V_p be an evenly covered neighborhood containing p. Assume that $V = N_\epsilon(p)$ for $\epsilon < 1/2$. Let U_p be the component of $q^{-1}(U)$ intersecting $[0, 1] \times \mathbb{R}$ such that $\inf_{(x,y)\in U} |x|$ is minimal. Then $q|_{U_p}^{-1} \colon V_p \to U_p$ is a homeomorphism of U_p with an open subset of $\mathbb{R}^2$. A subset of the U_p can be chosen to satisfy the countability and separation hypotheses of the Smooth Manifold Chart Lemma, and transition maps are either the identity or a shift. Therefore these charts define a smooth structure on E (Lemma 1.35). It is clear that q is smooth with respect to this smooth structure since its coordinate representation is either the identity or a shift and/or flip in $\mathbb{R}^2$. Also, any smooth structure on E making q a smooth covering map must contain all these charts, since the requirement that q must be a local diffeomorphism forces these local inverses to be smooth.

(b) Let $\tilde{\phi} : \mathbb{R}^2 \to \mathbb{S}^1$ be the map $(x, y) \mapsto e^{2\pi i x}$. Then $\tilde{\phi}$ is smooth and constant on the fibers of p, so ϕ descends to a smooth map $\phi : E \to \mathbb{S}^1$ (Theorem 4.30). Now for $x \in \mathbb{S}^1$, for sufficiently small $\epsilon > 0$, $\phi^{-1}(N_\epsilon(x)) = q(\tilde{\phi}^{-1}(\{x\}))$ is evenly covered by a disjoint union of components $I_k \times \mathbb{R} \subset \mathbb{R}^2$ for $I_k \subset \mathbb{R}$ an open interval, $k \in \mathbb{Z}$. Let V be the component that is either contained on $(0, 1) \times \mathbb{R}$ or intersects $0 \times \mathbb{R}$. Then $q|_V$ gives a diffeomorphism between $V = I \times \mathbb{R}$ and $\phi^{-1}(N_\epsilon(x))$. Let $F(x, y) = (e^{2\pi i x}, y)$. Then $\Phi = F \circ q|_V^{-1}$ is the desired local trivialization of E over $N_\epsilon(x)$ and clearly satisfies the projection conidition (ii-1). Identifying points of $\phi^{-1}(U)$ with their preimages in $\mathbb{R}^2$ gives a vector space structure on E_x and Φ clearly restricts to a linear map on each fiber.

(c) E contains a subset homeomorphic to a Mobius strip so it is not orientable, but $\mathbb{S}^1 \times \mathbb{R}$ is.

10.2 Projection from a vector bundle is a homotopy equivalence. Let $\sigma : M \to E$ be the zero section. Then σ is continuous and $\pi \circ \sigma = \mathrm{Id}_M$. On the other hand, $F = \sigma \circ \pi$ is homotopic to Id_E. To see this, define $G : E \times [0, 1] \to E$ that sends

$$(x, t) \mapsto (1 - t)x.$$

Then $G(p,0) = p$, and $G(p,1) = \sigma \circ \pi(p)$. To see that G is continuous, cover E by open neighborhoods $\pi^{-1}(U)$ such that there are local trivializations $\Phi_U : \pi^{-1}(U) \simeq U \times \mathbb{R}^k$. Define $H : U \times \mathbb{R}^k \times [0,1] \to U \times \mathbb{R}^k$:

$$H((p,v),t) = (p,(1-t)v).$$

Then

$$G|_U = [(x,t) \mapsto \Phi_U^{-1} \circ H(\Phi_U(x),t)]$$

and so $G|_U$ is continuous as the composition of continuous maps. This proves that G is continuous.

10.4 Transitions between local trivializations give smooth maps into $\mathbf{GL}_k(\mathbb{R})$. Let $e_1,...,e_k$ be the standard basis for $\mathbb{R}^k$. In the coordinates on $\mathrm{GL}_k(\mathbb{R}) \subset \mathbb{R}^{k^2}$ corresponding to this basis, we can write $\tau(p) = (\tau_{ij}(p))$. Now if $\pi_2 : (U \cap V) \times \mathbb{R}^k$ is the projection onto the second factor, then $F(p,v) = \pi_2 \circ \Phi \circ \Psi^{-1}(p)$ is a smooth map $U \cap V \to \mathbb{R}^k$, and $F(p,e_\ell) = (\tau_{1\ell}(p),...,\tau_{k\ell}(p))$. This proves that each τ_{ij} is a smooth function $U \cap V \to \mathbb{R}$, which proves that τ is smooth.

10.6 Vector bundle construction theorem. Let

$$E = \left(\coprod_{\alpha \in \mathcal{A}} U_\alpha \times \mathbb{R}^k \right) / \sim$$

where $(p,v) \in U_\alpha \times \mathbb{R}^k \sim (q,w) \in U_\beta \times \mathbb{R}^k$ if $p = q$ and $w = \tau_{\beta\alpha} v$. This is an equivalence relation: (10.7) gives transitivity directly, and it implies that $\tau_{\alpha\alpha} = \mathrm{Id}$, giving reflexivity, and $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$, giving symmetry. Then projection π onto the first factor is well-defined on the equivalence classes of E, so $E_p = \pi^{-1}(\{p\})$ is a well-defined subset of E. Further, E_p is a vector space for each p. For let $[(p,v)]$ and $[(p,w)]$ be two equivalence classes in E_p. We can choose representatives (p,v) and (p,w) contained in $U_\alpha \times \mathbb{R}^k$ for some α and define $[(p,v)] + [(p,w)] = [(p,v+w)] \in E_p$. This operation is well-defined, because if we had chosen some other chart (U,β), then we would have evaluated $(p,\tau_{\alpha\beta} v) + (p,\tau_{\alpha\beta} w) = (p,\tau_{\alpha\beta}(v+w)) \in U_\beta \times \mathbb{R}^k$ and these two points define the same class.

Now, for each α, $\pi^{-1}(U_\alpha)$ is represented by $U_\alpha \times \mathbb{R}^k \subseteq U$. Define the map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that

$$\Phi_\alpha([(p,v)]) = (p,v)$$

where (p,v) is the representative in U_α. Now for $(q,v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$,

$$\Phi_\beta \circ \Phi_\alpha^{-1}(q,v) = \Phi_\beta([(q,v) \in U_\alpha]) = \Phi_\beta([(q,\tau_{\beta\alpha} v) \in U_\beta]) = (q,\tau_{\beta\alpha} v).$$

The map $p \mapsto \tau_{\beta\alpha}$ is certainly smooth because it is constant. Therefore, the hypotheses of Lemma 10.6 are satisfied, and E is a smooth vector bundle over M.

10.8 Image of a vector bundle under covariant functor. The new vector bundle is $\coprod p \times \mathcal{F}(E_p)$, and the rest of the problem is using the smoothness of $\mathcal{F}$ to verify Lemma 10.6.

10.10 Ask someone for help with this one!

10.12 Conditions on transition maps guaranteeing the existence of a smooth bundle isomorphism. Say that the collection $\{\sigma_\alpha\}$ exists as described in the problem. Define the map $E \to \tilde{E}$ as follows: given $x \in E$, let $(p, v) = \Phi_\alpha(x) \in U_\alpha$, and let $F(x) = y$ where y is the unique point such that $\tilde{\Phi}_\alpha(y) = (p, \sigma_\alpha(v))$. Then F is well-defined. For if $x \in U_\alpha \cap U_\beta$, then $\Phi_\beta(x) = (p, \tau_{\beta\alpha} v)$, and $\tilde{\Phi}_\beta(y) = \tilde{\tau}_{\beta\alpha}\sigma_\alpha v$, which is equal to $\sigma_\beta \tau_{\beta\alpha} v$ by the relation. It is clear that F defined this way is a bundle homomorphism over M. It is smooth because in a neighborhood of each point it can be written as the composition of smooth maps. Since the inverse is constructed the same way, interchanging the role of E and $\tilde{E}$, it is also a smooth bundle homomorphism, so F is a smooth bundle isomorphism.

Conversely, say a smooth bundle isomorphism exists. For each α define the smooth map $\sigma_\alpha(p) : v \mapsto \pi_2 \circ \tilde{\Phi}_\alpha \circ F \circ \Phi_\alpha^{-1}(p, v)$. If $w = \tau_{\beta\alpha} v$, then $\phi_\beta^{-1})(p, w) = \phi_\alpha^{-1}(p, v)$, so $\sigma_\beta(p) w = \tilde{\tau_{\beta\alpha}} \sigma_\alpha(p) v$, which gives the desired relation.

10.14 Tangent bundle to an embedded submanifold is a subbundle. Apply Lemma 10.32 to the local frame corresponding to a slice chart at each point.