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## Chapter Five: Submanifolds

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**5.2 Boundary is embedded submanifold.** By Theorem 5.8, it suffices to prove that $\delta M$ satisfies the local $k$-slice condition. Take $p \in \delta M$ and let $(U, \phi)$ be a smooth boundary chart such that $\phi(p) \in \delta \mathbb{H}^n$. By the topological invariance of the boundary, $\phi(\delta M) = \phi(U) \cap \{x^n = 0\}$. Therefore $(U, \phi)$ is the necessary slice chart.

**5.4 Figure 8 is not an embedded submanifold of $\mathbb{R}^2$.** Let $S$ be the figure 8. Since small open subsets of $S$ can be easily seen to homeomorphic to open intervals, it follows from the topological invariance of dimension (Theorem 1.2) that if $S$ is an embedded submanifold, it must have dimension 1. Now, let $(U, \phi)$ be a chart containing $(0, 0)$ but not $(0, 1)$ or $(0, -1)$. Assume that $U$ is an open ball centered at $(0, 0)$. Since $U$ is connected, $\phi(U)$ must be an interval, so $\phi$ induces a homeomorphism between an "x" shape and an interval. But then $\phi$ restricts to a homeomorphism between $U \setminus (0, 0)$ and two open intervals. This is a contradiction, since $U \setminus (0, 0)$ has four connected components while the intervals have two.

**5.6 Tangent sphere to a submanifold of $\mathbb{R}^n$.** If $i$ is the inclusion $M \hookrightarrow \mathbb{R}^n$, define $\Phi : TM \to \mathbb{R}$

$$\Phi(p, v_p) = \|Di_p(v_p)\|_{\mathbb{R}^n}.$$

Then $\Phi$ is smooth as the composition of smooth maps. It is a submersion, because if $N : \mathbb{R}^n \to \mathbb{R}$ is the norm map, then $DN_p \neq 0$ for $p \neq 0$, and therefore since $D\Phi_p = DN_{i(p)} \circ Di_p$, 1 is a regular value of $\Phi$. But $UM = \Phi^{-1}(1)$, so be Corollary 5.14, $UM$ is a smooth $2m - 1$ dimensional manifold.

**5.8 Removing a coordinate ball creates a boundary diffeomorphic to $S^{n-1}$.** It is clear that $F(p) = \|p\|^{-1}$ is a smooth immersion $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, and therefore by Proposition 5.47, $D = \mathbb{R}^n \setminus \mathbb{B}_1(0) = F^{-1}(-\infty, 1]$ is an embedded submanifold with boundary of $\mathbb{R}^n \setminus \{0\}$, and therefore of $\mathbb{R}^n$. By Proposition 5.46, the boundary of $D$ is the $S^{n-1}$.

Now, the definition of a regular coordinate ball $B$ gives that $B \subset B'$ where $(B', \phi)$ is a chart, $\bar{B} \subset B'$, and the image of $B$ and $B'$ under $\phi$ are nested balls in $\mathbb{R}^n$. The above shows that $\phi(B') \setminus \phi(B)$ is a manifold with boundary, so $B' \setminus B$ is a manifold with boundary $\delta B$. Since the condition is local, this implies that $M \setminus B$ is a manifold with boundary $\delta B$. Additionally, $\phi$ restricted to $\delta B$ gives the necessary diffeomorphism with $S^{n-1}$.

**5.10 A family of algebraic curves.** Define $F_a(x, y) = x(x - a)(x - 1) - y^2$. For $a \in \mathbb{R} \setminus \{0, 1\}$, $F_a$ has 0 as a regular value. For its differential is given by

$$D(F_a)\Big|_{(x,y)} = (3x^2 - 2(a + 1)x + a, -2y).$$

In order for $D(F_a)$ to vanish at a point such that $F_a(x, y) = 0$, we must have $y = 0$, so $x(x-1)(x-a) = 0$ as well. Checking the cases $x = 0, 1, a$ shows that this can only occur when $a \in \{0, 1\}$. It follows from Corollary 5.14 that $M_a$ is an embedded submanifold for $a \in \mathbb{R} \setminus \{0, 1\}$.

In the case where $a = 0$, $M_a$ is not an immersed submanifold. For let $S = M_0$, $S_+ = M_0 \cap \{y > 0\}$, $S_- = M_0 \cap \{y < 0\}$, and $\phi(x, y) = y$. Then $S$ is an embedded submanifold with smooth structure induced by the charts $(S_+, \phi|_{S_+})$ and $(S_-, \phi|_{S_-})$ (Lemma 1.35). For define $g_+ : \mathbb{R}_{>1} \to \mathbb{R}$ by $g_+(x) = +\sqrt{x^2(x-1)}$, so that $\phi^{-1}|_{y>0} = (g_+^{-1}(y), y)$. Now $g'(x) > 0$ for $x > 1$ so $g$ is a local diffeormorphism, and therefore a global diffeomorphism, between $\mathbb{R}_{>1}$ and $\mathbb{R}_{>0}$. Therefore the inclusion map $i$ has the coordinate representation on $S_+$

$$(g^{-1}(y), y)$$

which is a smooth immersion by what we have argued. A similar argument gives smoothness on $S_-$. It is clear that the inclusion map is a homeomorphism with respect to the topology defined by these charts.

Therefore, if $M_0$ is an immersed submanifold, its smooth structure must restrict to this smooth structure on $S$. NOT FINISHED

I believe that if $a = 1$, $M_a$ is an immersed submanifold, but I cannot figure out how to show it. NOT FINISHED

**5.10 Restricting a smooth covering map to a boundary component.** By Theorem 5.11, $\delta E$ is an embedded submanifold of $E$, and any component is an open subset of $\delta E$. Let $C$ be such a component. Then by Theorem 5.27 $\pi|_C$ is smooth. Now clearly $\pi(\delta E) = \delta M$. Since $\pi|_C$ is therefore continuous, $A = \pi|_C(C)$ is connected. By Proposition 4.33, $\pi$ is an open map, so $\pi_{\delta E}$ is an open map from $\delta E$ to its image $\delta E$ with the subspace topology. This implies that $\pi|_C(C)$ is open and closed in $\delta M$, so it is a component of $\delta M$.

It remains to show that $\pi|_C$ is a smooth covering map. Take $p \in A$ and $U$ an evenly covered set containing $p$. Let $B = \pi^{-1}(U \cap \delta M) \cap C$. Then $\pi(B)$ is also an open subset of $\delta M$ containing $x$ that is diffeomorphic to $B$ through $\pi$, again by Theorem 5.27. MORE WORK?