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Chapter Five: Submanifolds

Lee, *An Introduction to Smooth Manifolds*

6.2 Immersion of an n-dimensional manifold into $\mathbb{R}^{2n}$. We assume that M is an embedded submanifold of $\mathbb{R}^{2n+1}$. We saw in Problem 5.6 that UM is an $2n-1$ dimensional submanifold of $T\mathbb{R}^{2n+1}$. Since $\mathbb{R}P^{2n}$ has dimension $2n$, Corollary 6.11 to Sard's theorem implies that the image of the UM unde G has measure 0. Since $U_{2n+1} = \{[v_0, ..., v_{2n}] \in \mathbb{R}P^{2n} : v_{2n} \neq 0\}$ is open, this means that there exists $[w] \in U_{2n+1}$ not contained in $G(UM)$. Then w is a vector in $\mathbb{R}^{2n+1}$ not contained in $\mathbb{R}^{2n}$ such that $(p, w) \notin T_pM$ for any $p \in M$.

Let $F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the quotient map by w. Since F is linear, it is smooth and equal to its own differential under the canonical identification. We want to show that DF_p is injective for $p \in M$. Say otherwise that $DF_p(v) = 0$ for $v \in T_pM$. That implies that $v = \lambda w$, which implies that $(p, w) \in T_pM$, a contradiction. Therefore F is an immersion on M.

I don't see exactly where this fails if the boundary isn't empty. Does Problem 5.6 still hold? Otherwise, perhaps to issue is that the negative of w might be orthogonal to DF but not contained in UM.

6.3 Smooth lower approximation of a continuous function that vanishes on a closed set. Let f be a nonnegative smooth function such that $f^{-1}(0) = A$ (Theorem 2.29). Then $F = \frac{f}{f+1}$ is a function of the same type and $F < 1$. Let e be a smooth function such that $0 < e(x) < \delta(x)$ (Corollary 6.22). Then $\tilde{\delta} = F \cdot e$ is the desired function.

6.4 Smooth approximations to continuous functions.

(a) Take $\tilde{d}$ as an Problem 6.3. Let $\tilde{G}$ be a smooth $\tilde{d}$-close approximation to $F|_{M \setminus B}$ (Theorem 6.21) and define

$$\tilde{F}(x) = \begin{cases} \tilde{G}(x) & \text{if } x \in M \setminus B \\ F(x) & \text{if } x \in B \end{cases}.$$

Clearly $\tilde{F}$ is δ-close to F. To see that $\tilde{F}$ is continuous, take $p \in \delta M$ and let (U, ϕ) be a smooth chart such that $p \in U$. Now on the one hand, by the continuity of F there is a neighborhood $B_{\epsilon_1}(p)$ such that $|F(q) - F(p)| < \epsilon$ for $q \in B_{\epsilon_1}(p) \cap B$. On the other hand, since $\tilde{d}(p) = 0$, there is a neighborhood $B_{\epsilon_2}(p)$ such that $|\tilde{G}(q) - F(q)| < \epsilon$ for $q \in B_{\epsilon_2}(p) \cap M \setminus B$. Let $B = B_{\epsilon_1}(p) \cap B_{\epsilon_2}(p)$. It is clear that $|\tilde{F}(q) - \tilde{F}(p)| < \epsilon$ for $q \in B$. This shows that $\tilde{F}$ is continuous.

(b) Taking $\tilde{F}$ constructed here, the proof of Theorem 6.26 goes through unchanged, and clearly the homotopy is relative to B.

6.5 Tubular neighborhood of points with unique closest point. Let M be an embedded submanifold of $\mathbb{R}^n$ and let $y \in \mathbb{R}^n \setminus M$. If $y \in N$ satisfies $d(y, x) = \min_{x' \in N} d(y, x')$, then for any curve $\gamma : I \to M$

such that $\gamma(0) = x$, $F(t) = \|y - \gamma(t)\|^2$ must have a minimum at $t = 0$. Differentiating, this implies that $y - \gamma(t)$ is orthogonal to $\gamma'(t)$. Since γ was arbitrary, this implies that $y - \gamma(t)$ is orthogonal to T_xM.

Therefore, if $x \in M$ is a closest point to y, the $x - y \in N_xM$. Now, take some δ such that E gives a diffeomorphism between a neighborhood of x containing $\overline{B_\delta}(x) \subset \mathbb{R}^n$. By the slice chart condition, we can assume that δ is small enough that $S = \overline{B_\delta}(x) \cap M$ is closed in $\mathbb{R}^n$. Now consider $B_{\delta/2}(x)$. By the compactness of S, every element of $B_{\delta/2}(x)$ has a (not necessarily unique) closest element in S. But a closest element of S must be a closest element of M, since $y \in B_{\delta/2}(x)$ implies that $|y - x| < \delta/2$.

Further, we claim that for every $y \in B_{\delta/2}(x)$, there is exactly one such closest element. E restricts to a diffeomorphism between $E^{-1}(\overline{B_\delta}(x) \subset \mathbb{R}^n) \subseteq NM$ and $\overline{B_\delta}(x) \subset \mathbb{R}^n$. Now say there are two points x_1 and x_2 such that $y - x_1 \in N_{x_1}M$ and $y - x_2 \in N_{x_2}M$. Then since $\|y - x_i\| < \delta$ $(i = 1, 2)$, $(x_1, y - x_i) \in E^{-1}(\overline{B_\delta}(x) \subset \mathbb{R}^n)$ and $E(x_1, y - x_1) = E(x_2, y - x_2)$, a contradiction. Therefore the closest point is unique. This proves that for every $x \in M$, there is a radius ρ_x so that all point in $B_{\rho_x}(x)$ have a unique closest point in M.

Now, define $\rho' : M \to \mathbb{R}_{>0}$ to be the supremum of all such $\rho(x)$. An argument like the one in the text shows that ρ' is continuous. Then $\{(x, y) : y \in N_xM, \|y\| < \rho'(x)\}$ is the desired tubular neighborhood. It is clear from the above discussion of orthogonality that $r(y)$ gives the unique closest point in this case.

6.6 Points exactly ϵ away from an embedded submanifold. Let $T \subset \mathbb{R}^n$ be the tubular neighborhood of M satisfying the conditions of the previous problem. Since M is compact, there is $\epsilon > 0$ such that if $d(y, M) < \epsilon$, $y \in T$. Let $N_\epsilon M = \{(x, y) \in NM : y \in N_xM, \|y\| \leq \epsilon\}$ and $\delta N_\epsilon M = \{(x, y) \in NM : y \in N_xM, \|y\| = \epsilon\}$. Now $F(x, v) = \|v\|$ is a smooth function on NM such that d is a regular value for all $d > 0$. Therefore, by Proposition 5.47, $\overline{N_\epsilon M}$ is a regular domain. By Proposition 5.46, its boundary is the topological boundary $\delta N_\epsilon M$, and by Theorem 5.11 this boundary is an embedded codimension-1 submanifold. Since E restricts to a diffeormorphism on a neighborhood of $\overline{N_\epsilon M}$ and $\overline{M_\epsilon} = E(\overline{N_\epsilon M})$ and $\delta M_\epsilon = E(\delta N_\epsilon M)$, this completes the proof.

6.8 ?

6.10 Preimage of tangent space under transverse map. By Theorem 6.30, W is an embedded submanifold of N. Given $p \in W$ and $v \in T_pW$, let $\gamma : I \to W$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $F \circ \gamma$ is a smooth curve in X such that $(F \circ \gamma)'(0) = DF_pv$. This proves that $D_pW \subseteq (DF_p)^{-1}(T_{F(p)}X)$. On the other hand, let A be a complementary subspace to T_pW in T_pN. Then by the definition of transversality, $T_{F(p)}N = T_{F(p)}X + dF_p(T_pW) + dF_p(A) = T_{F(p)}X + dF_p(A)$, so by dimension counting (Theorem 6.30), we must have that $dF_p(A) \cap T_{F(p)}X = 0$.

Now if X and X' are embedded submanifolds of M that intersect transversally, then $i : X \hookrightarrow M$ is transverse to X'. But $X \cap X' = i^{-1}(X')$, so by the above $(di_p)^{-1}(T_p(X')) = T_p(X \cap X')$, and on the other hand since $T_pX = (di_p)^{-1}(X)$, $T_pX \cap T_pX' = (di_p)^{-1}(T_p(X'))$. (This is basically a question of notation – the intersections are taken to mean intersections under the inclusion.)

6.12 Approximation $M \to \mathbb{R}^N$ by smooth immersions when M is compact. Let $n = \dim M$. By Theorem 6.18, for $N \geq 2n$ there is an immersion $G : M \to \mathbb{R}^N$. Since M is compact, $G(M)$ is bounded, so $|G(x)| < K$ for all $x \in M$ and some K. Now since dG_p is injective for all $p \in M$, for all $p \in M$ and any $\epsilon' > 0$, there exists $0 < \epsilon < \epsilon'$ such that $d(F + \epsilon G)_p$ is injective. Let $H_\epsilon = F + \epsilon G$. Then $|H_\epsilon(x) - F(x)| \leq \epsilon K$.

6.14 Counterexample to Theorem 6.30 if transversality assumption removed. Let f be the smooth function $\mathbb{R}^{n-1} \to \mathbb{R}$ such that $f^{-1}(0) = A$, guaranteed by Theorem 2.29, and let $F : \mathbb{R}^n \to \mathbb{R}$ be the function $F(x, y) = f(x)+y$ where $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Then F is a smooth submersion, since $dF_p = (df_p, 1)$, so $F^{-1}(0)$ is a proper embedded hypersurface (Corollary 5.13). The contradiction here is that $S \cap S' = A$, S has codimension 1, S' has comdimension 1, but A can have any positive codimension.