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Chapter Fourteen: Differential Forms

Lee, *An Introduction to Smooth Manifolds*

14.1 Wedge product and linear independence. It is clear from anticommutativity (Proposition 14.11c) that $\omega^1 \wedge ... \wedge \omega^n = 0$ if $\omega_i = \omega_j$ for $i \neq j$. Now say $\sum_{k=1}^n c_k \omega^k = 0$, and assume without loss of generality that $c_1 \neq 0$. Then $\omega^1 = \sum_{k=2}^n c'_k \omega^k$. Therefore

$$\omega^1 \wedge ... \wedge \omega^n = \sum_{k=2}^n c'_k \omega^k \wedge \omega^2 \wedge ... \wedge \omega^n = 0$$

since every term is zero.

On the other hand, say $\omega^1, ..., \omega^n$ are independent. Extend this collection to a basis (ω_k) for V^* and then let (e_k) be a dual basis for E, so that $\omega^i(e_j) = \delta_{ij}$. Then $(\omega^j(v_i))$ is the $n \times n$ identity matrix, so by Proposition 14.11,

$$\omega^1 \wedge ... \wedge \omega^n(e_1, ..., e_n) = \det(\omega^j(v_i)) = 1.$$

Therefore $\omega^1 \wedge ... \wedge \omega^n = 0$ implies that (ω^i) are dependent.

14.3 Abstract alternating tensor algebra.

1. is easily verified that if $\omega^i = \omega^j$ for $i \neq j$, then $\mathrm{Alt}(\omega^1 \otimes ... \otimes \omega^n) = 0$. One way of showing this is to break up Σ_n into A_n and $(i\,j)\,A_n$ and show cancellation between terms in this sum. Therefore if f is the isomorphism $V^* \otimes ... \otimes V^* \simeq T^k(V^*)$, then the composite map $\mathrm{Alt} \circ f$ is well-defined on the quotient $A^k(V^*)$ and so induces a unique homomorphism $F : A^k(V^*) \to \Lambda^k(V^*)$.

   To see that F is an isomorphism, we first observe that (in the notation of Proposition 14.8) $f^{-1}(e^I) \in V^* \otimes ... \otimes V^*$ and $\mathrm{Alt}(e^I) = e^I$ by Proposition 14.3, so $F(\pi \circ f^{-1}(e^I)) = e^I$. Therefore the image of F contains a basis for $\Lambda^k(V)$, so F is surjective.

   To see that F is injective, we modify the language of the proof of Proposition 14.8 that elements of the form $\omega^{i_1} \otimes ... \otimes \omega^{i_n} + \mathcal{A}$ with $i_1 < ... < i_n$ span $A^k(V^*)$, where (ω_n) is a basis for V^*. Injectivity then follows from dimension counting. Let $x = \sum \alpha_I e^{i_1} \otimes ... \otimes e^{i_n}$ be an arbitrary element of $V^* \otimes ... \otimes V^*$. Then x is congruent modulo $\mathcal{A}$ to a tensor x' where α_I is 0 whenever I contains a repeated index. But observe that

$$\begin{aligned} e^{i_1} \otimes e^{i_2} \otimes ... \otimes e^{i_n} = -e^{i_2} \otimes e^{i_1} \otimes ... \otimes e^{i_n} + \\ \left(e^{i_1} + e^{i_2} \otimes e^{i_1} + e^{i_2} \otimes ... \otimes e^{i_n} - e^{i_1} \otimes e^{i_1} \otimes ... \otimes e^{i_n} - e^{i_2} \otimes e^{i_2} \otimes ... \otimes e^{i_n} \right) \end{aligned} \quad (1)$$

   This argument is easily generalized to arbitrary pairs of indices. Therefore permuting indices multiplies a simple tensor by the sign of the permutation. Now tensor products of V^* act on V in a canonical way. Let (E_n) be the basis dual to (ω_n). As in the proposition, we therefore have that

$$\sum{}' \alpha_I e^I(E_{j_1}, ..., E_{j_n}) = \sum{}' \alpha_I \delta^I_J = \alpha_J.$$

Therefore x is equal as a function, and therefore equal as a tensor that is a sum of the desired form. This shows that every element of $V^* \otimes ... \otimes V^*$ is congruent modulo $\mathcal{A}$ to a linear combination of tensors of the desired form, so these tensors span the quotient $\mathcal{A}^k(V^*)$.

2. Let $\mathcal{A}^k$ denote the kernel of π when restricted to $(V^*)^{\otimes k}$. The fact that this wedge product is well defined follows from the fact that if $\omega \in \mathcal{A}^k$ and $\eta \in (V^*)^{\otimes \ell}$, then $\omega \otimes \eta \in \mathcal{A}^{k+\ell}$. That $F(\omega \wedge \eta) = \mathrm{Alt}(\tilde{\omega} \otimes \tilde{\eta})$ follows from the commutativity of the diagram.

14.5 Cartan's lemma. First, we show that the hypothesis implies that each α^i_p is contained in the span of (ω^j_p) for all $p \in M$. Say that this fails to be true at some fixed p for α^i_p. Extend (ω^j_p) to a basis for T_pM^*. Then $\alpha^i_p = \sum_{j=1}^n c^i_j \omega^j_p$ and $c^i_j \neq 0$ for some $j > k$. Then if (e_i) is the dual basis to (ω^i_p)

$$\sum_{i=1}^k \alpha^i_p \wedge \omega^i_p(e_j, e_i) = c^i_j \neq 0.$$

This proves that α^i_p is a linear combination of the (ω^j_p) for $j = 1, ..., k$ and for all i. To show that the coefficients are smooth functions of p, we can extend (ω^j_p) to a local frame in a neighborhood of p and simply take the matrix inverse, which is a smooth operation by Cramer's rule.

14.7 Verifying that $F^*(d\omega) = d(F^*\omega)$.

1.

$$F^*(d\omega) = d(st) \wedge d(e^t) = (t\, ds + s\, dt) \wedge e^t dt = te^t\, ds \wedge dt$$

$$d(F^*\omega) = d(st\, e^t\, dt) = te^t\, ds \wedge dt$$

2.

$$F^*(\omega) = \sin^2(\theta)\cos(\phi)(\cos(\phi) + 2)^2\, d\theta \wedge d\phi$$

$$d\omega = dy \wedge dz \wedge dz$$

$$d(F^*\omega) = F^*(d\omega) = 0 \text{ by dimension}$$

14.8 Naturality of the exterior derivative. We have already verified that Ω^k is a contravariant functor, since the pullback map satisfies the necessary properties. Sorting through the definitions (Exercise 11.18), naturality is given by Proposition 14.26.