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Chapter Eight: Vector Fields

Lee, *An Introduction to Smooth Manifolds*

8.2 Euler vector field. We need to show that for all $x \in \mathbb{R}^n \setminus \{0\}$, $(Vf)_x = V_x(f) = cf(x)$. Take an arbitrary such x and define $\gamma_x(t) = (1+t)x$. Then $\gamma_x(0) = x$ and $\gamma_x'(0) = V_x$. Therefore

$$V_x(f) = \frac{d}{dt}\bigg|_{t=0} f((1+t)x) = \frac{d}{dt}\bigg|_{t=0} (1+t)^c f(x) = cf(x)$$

as desired.

8.4 Outward-pointing vector field on the boundary of a manifold. Take $p \in \delta M$, let (U, ϕ) be a smooth boundary chart, and consider $v_p = \sum_{i=1}^n v_i \frac{d}{dx_i}|_p$. It is clear from the characterization in terms of curves (page 118) that if $v_n < 0$, then the n^{th} coordinate is negative with respect to all coordinate representations. Now let $\{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{A}\}$ be a countable covering of δM with smooth boundary charts and let $\{f_\alpha\}$ be a partition of unity on the manifold δM subordinate to this cover. For all α and all $p \in U_\alpha$, let $X_\alpha = d(\phi_\alpha^{-1})_{\phi_\alpha(p)}(-\frac{d}{dx_n}|_{\phi_\alpha(p)})$. Then $X = \sum_\alpha f_\alpha X_\alpha|_{\delta M}$ is a smooth map $\delta M \to TM$ such that X_p is outward facing in T_pM for all $p \in \delta M$. Smoothness follows from the facts that each f_α is supported in the corresponding U_α and that the sum is locally finite. Outward-facingness follows because with respect to any boundary chart, each X_α has a strictly negative n^{th} coordinate and the f_αs sum to 1 at any point.

In fact, X is a smooth M vector field along δM. For each X_α is the restriction of a smooth function on an open subset of M, and since the above sum is locally finite, X is locally the restriction of a finite sum of smooth vector fields on M. Therefore Lemma 8.6 gives a global extension of X to M. Of course, then $-X$ (i.e. $p \mapsto (p, -X_p)$) is an inward-pointing vector field.

8.6-8 Skipped because didn't want to go back to the problem about quaternions.

8.10 Computing a pushforward. We can compute the differential of F:

$$DF_{(x,y)} = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

Applying this gives $F_*X = (2xy, 0)$ and $F_*Y = (xy, \frac{y}{x})$.

8.14 Vector field on product manifold. The subset $\Gamma = \{(x, f(x)) : x \in M\} \subset M \times N$ is a properly embedded submanifold (Proposition 5.7). Now the map $h : \Gamma \to T(M \times N)$ given by $(p, f(p)) \mapsto (X_p, Df_pX_p)$ is a smooth map from a closed subset of $M \times N$ to $T(M \times N)$. By Lemma 2.26, h extends to a smooth map on $M \times N$. It is clear that $DF_pX_p = Y_p$ for all $p \in M$.

8.15 Extending a vector field from an embedded submanifold. Let $S \subseteq M$ be an embedded submanifold with boundary, let (U_n, ϕ_n) be a countable covering of S with slice charts, and let (f_n) be a partition of unity of $\cup_{n=1}^{\infty} U_n \subseteq M$. Let $\pi_n : U_n \to U_n \cap S$ be the projection onto S whose coordinate represectation is $(x_1, ..., x_n) \mapsto (x_1, ..., x_k, 0, ..., 0)$. Define $X_n : U_n \to TM$ as the composition $X \cap \pi_n$ and define $Y = \sum_{n=1}^{\infty} f_n X_n$. Each X_n is smooth as the composition of smooth maps and Y is smooth because the sum is locally finite. It is also clear that $X_n|_{S \cap U_n} = X|_{S \cap U_n}$, so $Y \cap S = X$ as desired.

Recall that an embedded submanifold is properly embedded if and only if it is closed (Proposition 5.49). Now if S is properly embedded, then the above implies that $X \in \mathcal{X}(S)$ is a vector field along the closed set S, so by Lemma 8.6 X extends to a vector field on M. On the other hand, say that S is not properly embedded and therefore not closed. We construct a vector field $X \in \mathcal{X}(S)$ that cannot be extended to M. Let x be a limit point of S not contained in S and let (x_n) be a sequence of distinct points in S converging to x. Fix a chart (U, ϕ) containing x. Let $((U_n, \phi_n))$ be a sequence of slice charts for S in M containing each x_n but not x_m for $n \neq m$. An elementary argument shows that this is possible since x_n converges to a point different from each x_n. For each n let f_n be a smooth bump function supported in U_n and equal to 1 in a neighborhood of x_n. Now ϕ gives an isomorphism between T_pM and $\mathbb{R}^n$ for all $p \in U$, so for n large enough, we can talk about the length of of a vector $v_p \in T_{x_n}S$ by first applying the differential of the inclusion and then applying this isomorphism. For each n, let X_n be any smooth assignment of vectors of length n to all points in U_n, and define $X = \sum_{n=1}^{\infty} f_n X_n$. Then X is smooth, since at any point the sum only has one term, and $\|X_p\| \to \infty$ and $n \to \infty$. Therefore X cannot be extended to a map $M \to TM$ that is continuous at x.

8.17 Product of vector fields on product manifold. $X \oplus Y$ is smooth by Proposition 8.1. For given $(p, q) \in M \times N$, we can choose a smooth chart of the form $(U \times V, \phi \times \psi)$ where (U, ϕ) is a smooth chart on M containing p and (V, ψ) is a smooth chat on N containing q. But then if $X(p) = \sum_{i=1}^{m} X_i(p) \frac{d}{dx_i}|_p = (X_1(p), ..., X_m(p))$ and $Y(q) = \sum_{i=1}^{n} Y_i(q) \frac{d}{dy_i}|_q = (Y_1(q), ..., Y_n(q))$ in these coordinates, then $X \oplus Y(p, q)$ has the smooth coordinate representation $(X_1(p), ..., X_m(p), Y_1(q), ..., Y_n(q))$.

The second part of this question can be proven with a computation using Proposition 8.26. Computations are simplified by expanding the sum in terms whether $i, j \leq n$, or $i, j > n$, or $i \leq n < j$, or $j \leq n < i$. It is clear that the last two cases have all 0 terms and the first two cases are shown to be $[X_1, X_2]$ and $[Y_1, Y_2]$ in the coordinate representation described above.

8.18 Smooth lifts.

(a) Such a smooth submersion is a local diffeomorphism, so for every $q \in N$, there is a neighborhood $q \in V_q \subseteq N$ and a lift X^q defined on $U_q = F^{-1}(V_q)$ defined as the pushforward for $(F|_{U_q})^{-1}$. The set of all such U_q for $q \in N$ covers M since $x \in U_{F(x)}$. By uniqueness these lifts must agree on the overlaps of their domains, so the map $p \mapsto X^q_p$ for any q such that $p \in U_q$ is well defined and gives a global lift of Y.

(b) Let $m = \dim(M)$ and $n = \dim(N)$. Given $p \in M$, we can without loss of generality choose coordinates for M (say (U, ϕ)) and N such that the the upper left $n \times n$ submatrix of DF_p is invertible in this coordinate representation. Then the map $G : M \to N \times \mathbb{R}^{m-n}$ given by $x \mapsto (F(x), \phi^{n+1}(x), ..., \phi^m(x))$ is a smooth submersion but $N \times \mathbb{R}^{m-n}$ has the same dimension as M. By the previous bullet there is a lift X of the vector field $Y \oplus 0$ on $N \times \mathbb{R}^{m-n}$ under G. But the same X is clearly a lift of Y under F.

(c) It is clear that if X is the lift of any vector field, we must have $DF_pX_p = DF_qX_q$ whenever $F(p) = F(q)$, since both quantities must equal $Y_{F(p)}$. Conversely, if X satisfies these conditions, we can define $Y(q) = dF_pX_p$ for any $p \in F^{-1}(\{q\})$. Now by the Global Rank Theorem (Thm 4.14) and Theorem 4.26, for each $q \in N$, there exists a local section σ of F. Then for $f \in C^\infty(N)$, $Yf = X(f \circ F \circ \sigma)$, so Y is a smooth vector field F-related to X. Uniqueness follows clearly from surjectivity.

(d) The forward direction follows from Proposition 8.30. Conversely, say that $[V, X]$ is vertical whenever V is vertical. For $r \in N$, let $A = F^{-1}(\{r\})$ and pick $p \in A$. It is clear that A is closed, since it is the preimage of a point under the smooth map $p \mapsto DF_pX_p$. We will prove that

$$B = \{q \in A : DF_qX_q = DF_pX_p\}$$

is open in A. Since A is connected, this proves the criterion from (c).

Note that A is an embedded submanifold of M by the Constant-Rank Level Set Theorem (Theorem 5.12). Pick $x \in A$ and choose a slice chart (U, ϕ) for A in M containing x. We can assume without loss of generality that U is a coordinate ball centered at x, so for any $y \in U \cap A$, there is a linear path from x to y whose coordinate representation is $\gamma(t) = (1-t)x + ty$. Writing $x = (x_1, ..., x_k, 0, ...0)$ and $y = (y_1, ..., y_k, 0, ...0)$, define a vector field

$$V(z_1, ..., z_n) = \sum_{i=1}^{k} (y_i - x_i) \frac{d}{dx\,i}\bigg|_{(z_1,...,z_n)}.$$

It is clear that $\gamma'(t) = V_{\gamma(t)}$ for all $t \in [0, 1]$.

Multiplying V by a bump function that is identically 1 in a neighborhood of $\gamma(I)$ but supported in U allows us to extend V to a vector field on M, and this vector field (which we will just call V) is vertical. For instance, this can be seen by looking at the coordinate representations of dF_p and V. By the hypothesis, this implies that $V \circ X(f \circ F) = 0$ for all $f \in C^\infty(N)$. For

$$0 = [V, X](f \circ F) = V(X(f \circ F)) - X(V(f \circ F)) = V(X(f \circ F)) - X(0) = V(X(f \circ F)).$$

Now, we want to show that

$$X_{\gamma(0)}(f \circ F) = X_{\gamma(1)}(f \circ F) \quad \forall f \in C^\infty(N).$$

Since $g : t \mapsto X_{\gamma(t)}(f \circ F)$ is a smooth function $I \to \mathbb{R}$, it is enough to show that $g'(s) = 0$ for all $s \in I$. But

$$\frac{d}{dt}\bigg|_{t=s} X_{\gamma(t)}(f \circ F) = \frac{d}{dt}\bigg|_{t=s} (X(f \circ F))(\gamma(t)) = \gamma'(s)X(f \circ F) = V_{\gamma(s)}X(f \circ F) = 0$$

as desired. Since y was arbitrary, this proves there is a neighborhood in A of x on which DF_pX_p is constant, so B is open.