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May 2017

Chapter Five: Submanifolds

Lee, *An Introduction to Smooth Manifolds*

6.2 Immersion of an n-dimensional manifold into $\mathbb{R}^{2n}$. We assume that M is an embedded submanifold of $\mathbb{R}^{2n+1}$. We saw in Problem 5.6 that UM is an $2n-1$ dimensional submanifold of $T\mathbb{R}^{2n+1}$. Since $\mathbb{R}P^{2n}$ has dimension $2n$, Corollary 6.11 to Sard's theorem implies that the image of the UM unde G has measure 0. Since $U_{2n+1} = \{[v_0, ..., v_{2n}] \in \mathbb{R}P^{2n} : v_{2n} \neq 0\}$ is open, this means that there exists $[w] \in U_{2n+1}$ not contained in $G(UM)$. Then w is a vector in $\mathbb{R}^{2n+1}$ not contained in $\mathbb{R}^{2n}$ such that $(p, w) \notin T_pM$ for any $p \in M$.

Let $F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the quotient map by w. Since F is linear, it is smooth and equal to its own differential under the canonical identification. We want to show that DF_p is injective for $p \in M$. Say otherwise that $DF_p(v) = 0$ for $v \in T_pM$. That implies that $v = \lambda w$, which implies that $(p, w) \in T_pM$, a contradiction. Therefore F is an immersion on M.

I don't see exactly where this fails if the boundary isn't empty. Does Problem 5.6 still hold? Otherwise, perhaps to issue is that the negative of w might be orthogonal to DF but not contained in UM.

6.3 Smooth lower approximation of a continuous function that vanishes on a closed set. Let f be a nonnegative smooth function such that $f^{-1}(0) = A$ (Theorem 2.29). Then $F = \frac{f}{f+1}$ is a function of the same type and $F < 1$. Let e be a smooth function such that $0 < e(x) < \delta(x)$ (Corollary 6.22). Then $\tilde{\delta} = F \cdot e$ is the desired function.

6.4 Smooth approximations to continuous functions.

(a) Take $\tilde{d}$ as an Problem 6.3. Let $\tilde{G}$ be a smooth $\tilde{d}$-close approximation to $F|_{M \setminus B}$ (Theorem 6.21) and define

$$\tilde{F}(x) = \begin{cases} \tilde{G}(x) & \text{if } x \in M \setminus B \\ F(x) & \text{if } x \in B \end{cases}.$$

Clearly $\tilde{F}$ is δ-close to F. To see that $\tilde{F}$ is continuous, take $p \in \delta M$ and let (U, ϕ) be a smooth chart such that $p \in U$. Now on the one hand, by the continuity of F there is a neighborhood $B_{\epsilon_1}(p)$ such that $|F(q) - F(p)| < \epsilon$ for $q \in B_{\epsilon_1}(p) \cap B$. On the other hand, since $\tilde{d}(p) = 0$, there is a neighborhood $B_{\epsilon_2}(p)$ such that $|\tilde{G}(q) - F(q)| < \epsilon$ for $q \in B_{\epsilon_2}(p) \cap M \setminus B$. Let $B = B_{\epsilon_1}(p) \cap B_{\epsilon_2}(p)$. It is clear that $|\tilde{F}(q) - \tilde{F}(p)| < \epsilon$ for $q \in B$. This shows that $\tilde{F}$ is continuous.

(b)