1. $DFT[x(n)] = \sum_{n=0}^{3} x(n)\cdot e^{-j\frac{\pi n k}{2}}$, $k=0,1,2,3,\cdots N-1$.

当 $N=4$ 时, $\begin{cases} X(0) = 1+2+3+4=10, & X(1) = -1-2+3j = 2j-2 \\ X(2) = -2, & X(3) = -2j-2. \end{cases}$

当 $N=8$ 时, $\begin{cases} X(0)=10, \quad X(1) = 1-\sqrt{2}-3(1+\sqrt{2})j. \quad X(2) = -2+2j \\ X(3) = (1+\sqrt{2})+3(1-\sqrt{2})j, \quad X_4 = -2, \quad X(5) = (1+\sqrt{2})+3(\sqrt{2}-1)j \\ X_6 = -2-2j, \quad X_7 = (1-\sqrt{2})+3(\sqrt{2}+1)j \end{cases}$

2. 假设采样序列为 $x(n)$.

$DFT[x(n)]$ 为 $x(n)$ 的 $N$ 点 DFT. 则有

$x(t) = IDFT[DFT[x(n)]] = \frac{1}{N}\sum_{k=0}^{N-1} DFT[x(n)]\cdot e^{j\frac{2nk\pi}{N}}$.

设周期为 $T$, 有 $x(n) = f(\frac{nT}{N})$, 故 $f(n\frac{T}{N}) = \sum_{k=0}^{N-1} \frac{DFT[x(n)]}{N} e^{j\frac{2nk\pi}{N}}$

故 $f(t) = \sum_{k=0}^{N-1} \frac{D[x(n)]}{N} e^{j\frac{2k\pi}{N}t}$.

而 $f(t) = \sum_{k=0}^{\infty} F_k e^{j\frac{2\pi k}{T}t} = \sum_{k=0}^{k_m} F_k e^{j\frac{2k\pi}{T}t}$ (因 $f$ 有频率上限).

故可知 $k_m = N-1$ 且 $NF_k = DFT[x(n)]$.

3. $$\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-jn\omega_k} = \sum_{n=0}^{N-1} \left( \sum_{m=0}^{r} x(mN+n) \right) e^{-j\frac{2nk\pi}{N}}.$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{r} x(mN+n)\, e^{-j\, 2k\pi \cdot \frac{(mN+n)}{N}}.$$

而 $L=r+1$，有 $$\sum_{n=0}^{N-1} \sum_{m=0}^{r} x(mN+n)\, e^{-\underbrace{2k\pi \cdot j \cdot \frac{(mN+n)}{N}}} \Rightarrow (mN+n)\omega_k$$

$$= \sum_{n=0}^{L-1} x(n)\, e^{-jn\omega_k}$$

故原式成立

4. $$X(k) = \sum_{n=0}^{N} x(n)\, W_n^{kn}.$$

$$G(k) = \sum_{m=0}^{\frac{N}{2}} x(2m)\, W_{\frac{N}{2}}^{km} = \sum_{m=0}^{\frac{N}{2}} x(2m)\, W_N^{2km}$$

$$H(k) = \sum_{m=0}^{\frac{N}{2}} x(2m+1)\, W_{\frac{N}{2}}^{km} = \sum_{m=0}^{\frac{N}{2}} x(2m+1)\, W_N^{2km}$$

故 $G(k) + W_N^{k} H(k)$

$$= \sum_{m=0}^{\frac{N}{2}} \left[ x(2m)\, W_N^{2km} + x(2m+1)\, W_N^{(2m+1)k} \right]$$

$$= \sum_{m=0}^{N} x(m)\, W_N^{km} = X(k).$$