Lie theory in Robotics

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In robotics, we often encounter groups such as SO(3) and SE(3). These groups are also smooth manifolds, making them *Lie groups*. If we want to optimize (e.g., gradient methods) along these groups, standard derivatives do not directly apply due to the curved nature of these spaces. However, we could define Lie derivatives with the help of Lie theory, allowing us to manipulate calculus. A Lie theory cheat sheet can be found here.

1 Lie theory

Before we start, we first show an excellent visualization by Sola *et al.* [5]. The groups we are dealing with can be intuitively represented as the (blue) sphere in the Figure 1.1, and its corresponding *Lie algebra* is the (red) tangent plane at the identity $\mathcal{E}$, which is a *vector space* under some operator.

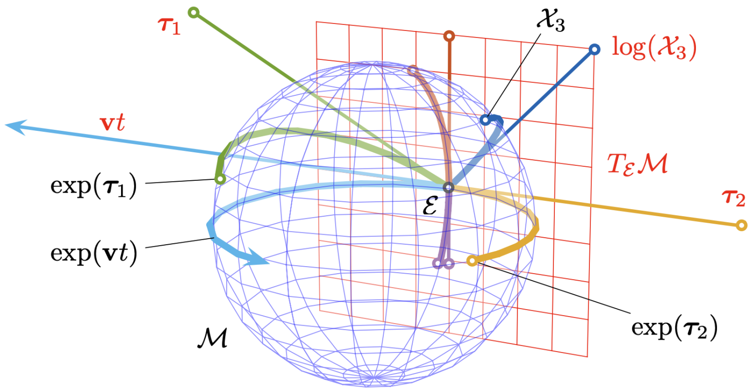

Figure 1.1: Representation of the relation between Lie group and Lie algebra [5].

1.1 Lie groups

A Lie group $\mathcal{G}$ is a smooth *manifold* whose elements satisfy the *group* axioms. A smooth manifold is a topological space that locally resembles a linear space, i.e., there exists a unique tangent plane at each point. A group is a set $\mathcal{G}$ with binary operator $\circ$ that satisfies group axioms, i.e., for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{G}$,

1. (closure) $\mathcal{X} \circ \mathcal{Y} \in \mathcal{G}$,
2. (associative) $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$,
3. (identity) there exists unique $\mathcal{E} \in \mathcal{G}$ such that $\mathcal{E} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{E} = \mathcal{X}$,
4. (inverse) there exists unique $\mathcal{X}^{-1} \in \mathcal{G}$ such that $\mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{E}$.

Note that by the closure property, group composition also lies on the manifold.

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Now given a Lie group $\mathcal{M}$ and a set V, we define $\mathcal{X} \cdot v$ the action of $\mathcal{X} \in \mathcal{M}$ on $v \in V$,

$$(\cdot) : \mathcal{M} \times V \to V \ ; \quad (\mathcal{X}, v) \mapsto \mathcal{X} \cdot v.$$

For $\cdot$ to be a group action, it must satisfy the following axioms:

1. (identity) $\mathcal{E} \cdot v = v$,
2. (compatibility) $(\mathcal{X} \circ \mathcal{Y}) \cdot v = \mathcal{X} \cdot (\mathcal{Y} \cdot v)$.

Notably, group composition is a group action itself. Here are some examples of group actions:

- Given a rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$ and a 3D point $\boldsymbol{p} \in \mathbb{R}^3$,
$$\mathbf{R} \cdot \boldsymbol{p} = \mathbf{R}\boldsymbol{p} \in \mathbb{R}^3.$$
- Given a transformation matrix $\mathbf{T} \in \mathrm{SE}(3)$ and a 3D point $\boldsymbol{p} \in \mathbb{R}^3$,
$$\mathbf{T} \cdot \boldsymbol{p} = \mathbf{R}\boldsymbol{p} + \boldsymbol{t} \in \mathbb{R}^3.$$

2 Lie algebra

Given $\mathcal{X}$ an element of a Lie group $\mathcal{M}$, the tangent space at $\mathcal{X}$ is denoted by $T_{\mathcal{X}}\mathcal{M}$. In particular, the tangent space at identity $T_{\mathcal{E}}\mathcal{M}$ is called the Lie algebra[1] of $\mathcal{M}$, denoted by $\mathfrak{m}$. Every Lie group $\mathcal{M}$ has an associated Lie algebra $\mathfrak{m}$ with the following properties:

- Lie algebra $\mathfrak{m}$ is a vector space.
- The exponential map $\exp(\cdot)$ maps elements of the Lie algebra to the Lie group.
- The logarithm map $\log(\cdot)$ is the inverse of the exponential map, which maps elements of the Lie group to the Lie algebra.
- Any tangent space $T_{\mathcal{X}}\mathcal{M}$ can be transformed to $\mathfrak{m} = T_{\mathcal{E}}\mathcal{M}$ via a linear transformation called the adjoint map.

We will discuss these properties in more detail in the following sections. Lie algebra is isomorphic to a Cartesian vector space, thus we denote $\boldsymbol{\tau}^\wedge$ for elements in tangent spaces where $\boldsymbol{\tau} \in \mathbb{R}^m$ and m is the degree of freedom (DoF) of the Lie group. A left superscript may also be added to specify the tangent space, e.g., ${}^{\mathcal{X}}\boldsymbol{\tau}^\wedge \in T_{\mathcal{X}}\mathcal{M}$ and ${}^{\mathcal{E}}\boldsymbol{\tau}^\wedge \in \mathfrak{m}$.

2.1 Cartesian vector space

Despite that Lie algebra is a vector space, it is often more convenient to work on an isomorphic Cartesian vector space, i.e., $\mathfrak{m} \cong \mathbb{R}^m$ where m is the DoF. Instead of directly manipulating elements in Lie algebra, we typically perform calculus on the Cartesian vector space. Therefore, we first define the maps between $\mathfrak{m}$ and $\mathbb{R}^m$: the *hat* operator $(\cdot)^\wedge$ and the *vee* operator $(\cdot)^\vee$:

$$(\cdot)^\wedge : \mathbb{R}^m \to \mathfrak{m} \ ; \ \ \boldsymbol{\tau} \mapsto \boldsymbol{\tau}^\wedge = \sum_{i=1}^{m} \tau_i \mathbf{E}_i, \tag{2.1}$$

$$(\cdot)^\vee : \mathfrak{m} \to \mathbb{R}^m \ ; \ \ \boldsymbol{\tau}^\wedge \mapsto \boldsymbol{\tau} = \sum_{i=1}^{m} \tau_i \mathbf{e}_i, \tag{2.2}$$

where $\boldsymbol{e}_i$ are the basis of $\mathbb{R}^m$ and $\mathbf{E}_i = \boldsymbol{e}_i^\wedge$ are the basis of $\mathfrak{m}$. These mappings establish the isomorphism $\mathfrak{m} \cong \mathbb{R}^m$, and $\boldsymbol{\tau} \in \mathbb{R}^m$ is obviously more convenient in terms of manipulating linear algebra. We say $\boldsymbol{\tau}$ is the Cartesian vector in Cartesian space to differentiate it with the Lie algebra $\boldsymbol{\tau}^\wedge$.

[1] Some literature loosely refers to $T_{\mathcal{X}}\mathcal{M}$ as Lie algebra for any $\mathcal{X}$, but this is informal. Only $T_{\mathcal{E}}\mathcal{M}$ is closed under Lie bracket, the operation that defines the Lie algebra.

Example 2.1: The rotation group SO(3) and its Lie algebra $\mathfrak{so}(3)$

The rotation group is a set of 3×3 matrices that satisfies the orthogonal constraint $\mathbf{R}^\top\mathbf{R} = \mathbf{I}_3$. Since the tangent plane consists of all possible derivatives $\dot{\mathbf{R}}$ along a smooth trajectory passing through $\mathbf{R}$, we differentiate the orthogonal constraint:

$$\mathbf{R}^\top\dot{\mathbf{R}} + \dot{\mathbf{R}}^\top\mathbf{R} = 0.$$

For Lie algebra, we set the tangent space with respect to the identity, i.e., $\mathbf{R} = \mathbf{I}_3$. We then obtain $\dot{\mathbf{R}} = -\dot{\mathbf{R}}^\top$, which implies that $\dot{\mathbf{R}}$ is a skew-symmetric matrix. Skew-symmetric matrices are denoted as $[\boldsymbol{\tau}]_\times$, where

$$[\boldsymbol{\tau}]_\times = \begin{pmatrix} 0 & -\tau_z & \tau_y \\ \tau_z & 0 & -\tau_x \\ -\tau_y & \tau_x & 0 \end{pmatrix}. \tag{2.3}$$

By Equation (2.1) and Equation (2.2), we have the relation between the Lie algebra of rotation group $\mathfrak{se}(3)$ and its isomorphic vector space $\mathbb{R}^3$:

$$[\cdot]_\times : \mathbb{R}^3 \to \mathfrak{so}(3)\ ;\ \ \boldsymbol{\tau} \mapsto [\boldsymbol{\tau}]_\times, \tag{2.4}$$

$$(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3\ ;\ \ [\boldsymbol{\tau}]_\times \mapsto \boldsymbol{\tau}. \tag{2.5}$$

We provided an example of a Lie algebra and its corresponding vector space in Example 2.1. In addition, we give a more intuitive interpretation. Recall that rotations can be viewed as a transformation between coordinate systems. Figure 2.1 shows an example of applying a rotation onto a coordinate system, and the derivative (or velocity) of each axis.

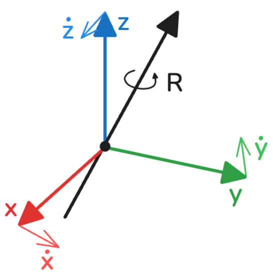


Figure 2.1: Derivative (or velocity) of a rotation.

Now if we take the black arrow in Figure 2.1 as the axis-angle representation $\boldsymbol{\theta} = \theta\boldsymbol{u}$, where $\boldsymbol{u}$ is the unit axis and θ is the angle. We have the following relations:

$$\dot{\boldsymbol{x}} = \boldsymbol{\theta} \times \boldsymbol{x},$$
$$\dot{\boldsymbol{y}} = \boldsymbol{\theta} \times \boldsymbol{y},$$
$$\dot{\boldsymbol{z}} = \boldsymbol{\theta} \times \boldsymbol{z}.$$

Let $\mathbf{R} = \mathbf{I}_3$ be the coordinate system, we can rearrange these equations into matrix form:

$$\dot{\mathbf{R}} = \begin{pmatrix} | & | & | \\ \boldsymbol{\theta}\times\boldsymbol{x} & \boldsymbol{\theta}\times\boldsymbol{y} & \boldsymbol{\theta}\times\boldsymbol{z} \\ | & | & | \end{pmatrix} = \boldsymbol{\theta}\times\mathbf{I}_3 = [\boldsymbol{\theta}]_\times\mathbf{I}_3 = [\boldsymbol{\theta}]_\times.$$

That is, the elements of rotation's Lie algebra $\mathfrak{so}(3)$ are skew-symmetric matrices, and more importantly, their corresponding Cartesian vectors are the axis-angle representations, also known as the *rotation vectors* or the *angular velocities* in robotics and kinematics.

2.2 Exponential map

The exponential map $\exp(\cdot)$ exactly transfers elements on Lie algebra to the Lie group. Intuitively (see Figure 1.1), $\exp(\cdot)$ wraps an element (lines $\boldsymbol{\tau}^\wedge$ on the tangent plane) around the manifold following the geodesic of the sphere (geodesics $\exp(\boldsymbol{\tau}^\wedge)$ around the sphere). For simplicity, the figure uses $\boldsymbol{\tau}$ informally since it is already a vector. We define the exponential map as follows:

$$\exp(\cdot) : \mathfrak{m} \to \mathcal{M} \ ; \quad \boldsymbol{\tau}^\wedge \mapsto \mathcal{X}. \tag{2.6}$$

Since we are primarily working with the Cartesian vectors, it is natural to extend Equation (2.6) to the Cartesian vector space:

$$\operatorname{Exp}(\cdot) : \mathbb{R}^m \to \mathcal{M} \ ; \quad \boldsymbol{\tau} \mapsto \mathcal{X}. \tag{2.7}$$

Example 2.2: Exponential map of rotation group

Let $\boldsymbol{\tau} = \theta \boldsymbol{u} = \boldsymbol{\theta}$ be the axis-angle representation, we compute the matrix exponential by power series:

$$\begin{aligned}
\exp([\boldsymbol{\theta}]_\times) &= \sum_{k=1}^{\infty} \frac{\theta^k}{k!}[\boldsymbol{u}]_\times^k \\
&= [\boldsymbol{u}]_\times^0 + \theta[\boldsymbol{u}]_\times + \frac{\theta^2}{2!}[\boldsymbol{u}]_\times^2 + \frac{\theta^3}{3!}[\boldsymbol{u}]_\times^3 + \frac{\theta^4}{4!}[\boldsymbol{u}]_\times^4 + \frac{\theta^5}{5!}[\boldsymbol{u}]_\times^5 + \frac{\theta^6}{6!}[\boldsymbol{u}]_\times^6 + \dots \\
&= \mathbf{I}_3 + \theta[\boldsymbol{u}]_\times + \frac{\theta^2}{2!}[\boldsymbol{u}]_\times^2 - \frac{\theta^3}{3!}[\boldsymbol{u}]_\times - \frac{\theta^4}{4!}[\boldsymbol{u}]_\times^2 + \frac{\theta^5}{5!}[\boldsymbol{u}]_\times + \frac{\theta^6}{6!}[\boldsymbol{u}]_\times^2 + \dots \\
&= \mathbf{I}_3 + [\boldsymbol{u}]_\times(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots) + [\boldsymbol{u}]_\times^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots) \\
&= \mathbf{I}_3 + [\boldsymbol{u}]_\times \sin\theta + [\boldsymbol{u}]_\times^2(1 - \cos\theta).
\end{aligned} \tag{2.8}$$

Equation (2.8) is also known as the Rodrigues' rotation formula.

2.3 Logarithm map

The logarithm map is the inverse of the exponential map:

$$\log(\cdot) : \mathcal{M} \to \mathfrak{m} \ ; \quad \mathcal{X} \mapsto \boldsymbol{\tau}^\wedge. \tag{2.9}$$

Again, we extend Equation (2.9) to the Cartesian vector space:

$$\operatorname{Log}(\cdot) : \mathcal{M} \to \mathbb{R}^m \ ; \quad \mathcal{X} \mapsto \boldsymbol{\tau}. \tag{2.10}$$

Example 2.3: Logarithm map of rotation group

Since the trace of any rotation matrix is $1 + 2\cos\theta$, we compute the angle by

$$\theta = \arccos\left(\frac{\operatorname{tr}(\mathbf{R}) - 1}{2}\right). \tag{2.11}$$

Then the axis can be computed by

$$\boldsymbol{u} = \frac{(\mathbf{R} - \mathbf{R}^\top)^\wedge}{2\sin\theta}. \tag{2.12}$$

We have a more detailed explanation about Equation (2.11) and Equation (2.12) in a separate note about rotation representations[2].

[2] https://dgbshien.com/docs/blogs/rotation-representation.pdf

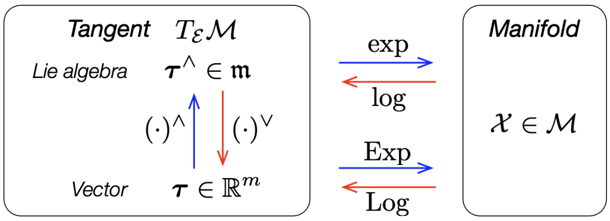


Figure 2.2: Mappings between Lie group and Lie algebra [5].

The capital logarithm map is commonly used to define a well-known distance metric, the *geodesic distance* or *angular distance*. To measure the distance between two rotation matrices, we first compute the relative rotation, then compute the norm of the Lie algebra or axis-angle representation $\boldsymbol{\theta} = \theta\boldsymbol{u}$ of the relative rotation matrix. Since $\boldsymbol{u}$ is a unit vector, the norm simplifies to $\|\boldsymbol{\theta}\| = \theta$. Thus, Equation (2.11) is also used to compute the geometric distance:

$$\mathrm{dist}(\mathbf{R}_1, \mathbf{R}_2) = \|\mathrm{Log}(\mathbf{R}_1^\top \mathbf{R}_2)\| = \left| \arccos\left(\frac{\mathrm{tr}(\mathbf{R}_1^\top \mathbf{R}_2) - 1}{2} \right) \right|.$$

2.4 Plus and minus operators

Plus and minus operators allow us to introduce *increments* between elements of a (curved) manifold, then express them in its (flat) Cartesian vector space by the capitalized maps. The right (local) operators are defined as follows:

$$\mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) = \mathcal{Y} \in \mathcal{M} \tag{2.13}$$

$$\mathcal{Y} \ominus \mathcal{X} = \mathrm{Log}(\mathcal{X}^{-1} \circ \mathcal{Y}) = {}^{\mathcal{X}}\boldsymbol{\tau} \in T_{\mathcal{X}}\mathcal{M} \tag{2.14}$$

Similarly, the left (global) operators are defined as

$${}^{\mathcal{E}}\boldsymbol{\tau} \oplus \mathcal{X} = \mathrm{Exp}({}^{\mathcal{E}}\boldsymbol{\tau}) \circ \mathcal{X} = \mathcal{Y} \in \mathcal{M}$$

$$\mathcal{Y} \ominus \mathcal{X} = \mathrm{Log}(\mathcal{Y} \circ \mathcal{X}^{-1}) = {}^{\mathcal{E}}\boldsymbol{\tau} \in T_{\mathcal{E}}\mathcal{M}$$

We use the right (local) operators if not specified. We next derive the relation between elements in (local) tangent spaces and the (global) Lie algebra.

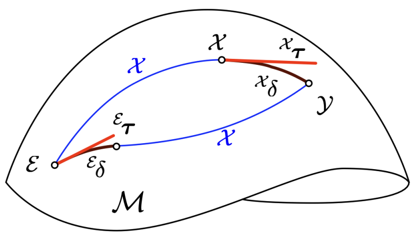


Figure 2.3: Relation between right and left operators [5].

2.5 Adjoint

Figure 2.3 shows the relation between right and left operators (clockwise cycle):

$$\begin{aligned}
&\mathrm{Exp}({}^{\mathcal{E}}\boldsymbol{\tau})\mathcal{X} = \mathcal{X}\mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \\
\Longrightarrow\ &\exp({}^{\mathcal{E}}\boldsymbol{\tau}^\wedge)\mathcal{X} = \mathcal{X}\exp({}^{\mathcal{X}}\boldsymbol{\tau}^\wedge) \\
\Longrightarrow\ &\exp({}^{\mathcal{E}}\boldsymbol{\tau}^\wedge) = \mathcal{X}\exp({}^{\mathcal{X}}\boldsymbol{\tau}^\wedge)\mathcal{X}^{-1} = \exp(\mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}) \\
\Longrightarrow\ &{}^{\mathcal{E}}\boldsymbol{\tau}^\wedge = \mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}.
\end{aligned}$$

We next define the *adjoint* of $\mathcal{M}$ at $\mathcal{X}$ as

$$\mathrm{Ad}_{\mathcal{X}} : T_{\mathcal{X}}\mathcal{M} \to \mathfrak{m} \; ; \quad {}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge} \mapsto \mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge}) = \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1} = {}^{\mathcal{E}}\boldsymbol{\tau}^{\wedge}, \tag{2.15}$$

and thus ${}^{\mathcal{E}}\boldsymbol{\tau}^{\wedge} = \mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge})$. The adjoint is a linear transformation, and we have an equivalent matrix operator on the Cartesian vector space, the *adjoint matrix*, defined as follows:

$$\mathbf{Ad}_{\mathcal{X}} : \mathbb{R}^m \to \mathbb{R}^m \; ; \quad {}^{\mathcal{X}}\boldsymbol{\tau} \mapsto \mathbf{Ad}_{\mathcal{X}} \cdot {}^{\mathcal{X}}\boldsymbol{\tau} = {}^{\mathcal{E}}\boldsymbol{\tau}. \tag{2.16}$$

For example, Table 2.1 shows the commutative relation of the adjoint on the rotation group.

Lie algebra		Cartesian space
$\boldsymbol{\theta}^{\wedge} = [\boldsymbol{\theta}]_{\times} \in \mathfrak{so}(3)$	$\xrightarrow{(\cdot)^{\vee}}$	$\boldsymbol{\theta} \in \mathbb{R}^3$
$\downarrow \mathrm{Ad}_{\mathcal{X}}$		$\downarrow \mathbf{Ad}_{\mathcal{X}}$
$\boldsymbol{\tau}^{\wedge} \in T_{\mathcal{X}}\mathrm{SO}(3)$	$\xleftarrow{(\cdot)^{\wedge}}$	$\boldsymbol{\tau} \in \mathbb{R}^3$

Table 2.1: Adjoint of the rotation group.

3 Lie derivatives

Since we have plus and minus operators to express *vector increments* $\boldsymbol{\tau}$, we define the right (local) derivative, analogous to standard derivative in Euclidean space:

$$\frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\boldsymbol{\tau}\to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n\times m}, \tag{3.1}$$

where$f : \mathcal{M} \to \mathcal{N}$ is a function acting on manifolds, m and n are the DoF of $\mathcal{M}$ and $\mathcal{N}$, respectively. Similarly, the left (global) derivative can be defined by

$$\frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\boldsymbol{\tau}\to 0} \frac{f(\boldsymbol{\tau} \oplus \mathcal{X}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n\times m}.$$

Again, we use the right (local) Jacobian if not specified, and denote it as $\mathbf{J}_{\mathcal{X}}^{f(\mathcal{X})}$.

Example 3.1: Derivative of rotation action

Consider the rotation action $f : \mathrm{SO}(3) \to \mathbb{R}^3$ with $f(\mathbf{R}) = \mathbf{R}\boldsymbol{p}$, that is, a rotation $\mathbf{R}$ acts on a 3D point $\boldsymbol{p}$. The Lie derivative of this function is

$$\begin{aligned}
\frac{Df(\mathbf{R})}{D\mathbf{R}} &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\mathbf{R} \oplus \boldsymbol{\theta}) \ominus \mathbf{R}\boldsymbol{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathbf{R}\mathrm{Exp}(\boldsymbol{\theta})\boldsymbol{p} - \mathbf{R}\boldsymbol{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathbf{R}(\mathbf{I} + [\boldsymbol{\theta}]_{\times})\boldsymbol{p} - \mathbf{R}\boldsymbol{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathbf{R}[\boldsymbol{\theta}]_{\times}\boldsymbol{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta}\to 0} \frac{-\mathbf{R}[\boldsymbol{p}]_{\times}\boldsymbol{\theta}}{\boldsymbol{\theta}} = -\mathbf{R}[\boldsymbol{p}]_{\times} \in \mathbb{R}^{3\times 3}.
\end{aligned}$$

We next discuss about some differentiation rules on manifolds. Although we consider only the right Jacobian, one can obtain the left Jacobian with the help of adjoint matrices:

$$\frac{{}^{\mathcal{E}}Df(\mathcal{X})}{D\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{{}^{\mathcal{X}}Df(\mathcal{X})}{D\mathcal{X}} \mathbf{Ad}_{\mathcal{X}}^{-1}.$$

3.1 Differentiation rules

For $\mathcal{Y} = f(\mathcal{X})$ and $\mathcal{Z} = g(\mathcal{Y})$, we have $\mathcal{Z} = g \circ f(\mathcal{X})$ and the chain rule:

$$\mathbf{J}^{\mathcal{Z}}_{\mathcal{X}} = \mathbf{J}^{\mathcal{Z}}_{\mathcal{Y}} \mathbf{J}^{\mathcal{Y}}_{\mathcal{X}}. \tag{3.2}$$

The Jacobian of inverse functions $f(\mathcal{X}) = \mathcal{X}^{-1}$ is

$$\begin{aligned} \mathbf{J}^{\mathcal{X}^{-1}}_{\mathcal{X}} &= \lim_{\boldsymbol{\tau} \to 0} \frac{(\mathcal{X} \oplus \boldsymbol{\tau})^{-1} \ominus \mathcal{X}^{-1}}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}(\mathcal{X}(\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau}))^{-1})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}(\mathcal{X}\mathrm{Exp}(-\boldsymbol{\tau})\mathcal{X}^{-1})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{(\mathcal{X}(-\boldsymbol{\tau})^{\wedge}\mathcal{X}^{-1})^{\vee}}{\boldsymbol{\tau}} = -\mathbf{Ad}_{\mathcal{X}}. \end{aligned} \tag{3.3}$$

The Jacobian of group compositions are

$$\begin{aligned} \mathbf{J}^{\mathcal{X} \circ \mathcal{Y}}_{\mathcal{X}} &= \lim_{\boldsymbol{\tau} \to 0} \frac{((\mathcal{X} \oplus \boldsymbol{\tau}) \circ \mathcal{Y}) \ominus (\mathcal{X} \circ \mathcal{Y})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}((\mathcal{X}\mathcal{Y})^{-1}\mathcal{X}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{Y})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}(\mathcal{Y}^{-1}\mathrm{Exp}(\boldsymbol{\tau})\mathcal{Y})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{(\mathcal{Y}^{-1}\boldsymbol{\tau}^{\wedge}\mathcal{Y})^{\vee}}{\boldsymbol{\tau}} = \mathbf{Ad}^{-1}_{\mathcal{Y}}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathbf{J}^{\mathcal{X} \circ \mathcal{Y}}_{\mathcal{Y}} &= \lim_{\boldsymbol{\tau} \to 0} \frac{(\mathcal{X} \circ (\mathcal{Y} \oplus \boldsymbol{\tau})) \ominus (\mathcal{X} \circ \mathcal{Y})}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}((\mathcal{X}\mathcal{Y})^{-1}\mathcal{X}\mathcal{Y}\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\tau}} \\ &= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}(\mathrm{Exp}(\boldsymbol{\tau}))}{\boldsymbol{\tau}} = \mathbf{I}. \end{aligned} \tag{3.5}$$

In addition, we define the *right Jacobian* as the derivative of $\mathrm{Exp}(\cdot)$, which maps a vector $\boldsymbol{\tau} \in \mathbb{R}^m$ to a local tangent space.

$$\mathbf{J}_r := \frac{D\mathrm{Exp}(\boldsymbol{\tau})}{D\boldsymbol{\tau}} \in \mathbb{R}^{m \times m}. \tag{3.6}$$

Several useful closed-form Jacobian exist but might not generalize to all manifolds. Group actions, capitalized exponential, logarithm maps of Lie groups such as SO(2), SO(3), SE(2), and SE(3) all have closed-forms. Please refer to [5] for further details.

3.2 Pose graph optimization

Here we demonstrate the application of Lie theory to solve a real-world robotics problem. Assuming the reader is already familiar with pose graph optimization (PGO), we focus primarily on deriving the Jacobian. We employ Gauss-Newton method, a widely used non-linear least squares solver thatapproximates the Hessian and Jacobian linearly. Since the variables in 2D PGO lies in SE(2), which is non-linear, we need some linear approximation. A common approach [3] is to vectorize $\mathbf{T} \in \mathrm{SE}(2)$ as follows:

$$\mathrm{vec}(\cdot) : \mathrm{SE}(2) \to \mathbb{R}^3 \; ; \quad \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}.$$

With the help of this vectorization, we define the error (or residual) as follows:

$$
\begin{aligned}
\boldsymbol{e}(\mathbf{T}_i, \mathbf{T}_j) &= \operatorname{vec}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j) \\
&= \operatorname{vec}\begin{pmatrix} \mathbf{R}_{ij}^\top \mathbf{R}_i^\top \mathbf{R}_j & \mathbf{R}_{ij}^\top(\mathbf{R}_i^\top(\boldsymbol{t}_j - \boldsymbol{t}_i) - \boldsymbol{t}_{ij}) \\ 0^\top & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}_{ij}^\top(\mathbf{R}_i^\top(\boldsymbol{t}_j - \boldsymbol{t}_i) - \mathbf{t}_{ij}) \\ -\theta_{ij} - \theta_1 + \theta_2 \end{pmatrix}. \qquad (3.7)
\end{aligned}
$$

Then we can compute the partial derivative of the error (3.7) with respect to $\operatorname{vec}(\mathbf{T}_i)$ by

$$
\begin{pmatrix} \frac{\partial \boldsymbol{e}(\mathbf{T}_i,\mathbf{T}_j)}{\boldsymbol{t}_i} & \frac{\partial \boldsymbol{e}(\mathbf{T}_i,\mathbf{T}_j)}{\theta_i} \end{pmatrix} = \begin{pmatrix} -\mathbf{R}_{ij}^\top \mathbf{R}_i^\top & \mathbf{R}_{ij} \frac{\partial \mathbf{R}_i^\top}{\partial \theta_i}(\boldsymbol{t}_j - \boldsymbol{t}_i) \\ 0^\top & 1 \end{pmatrix}, \qquad (3.8)
$$

and the partial derivative of the error (3.7) with respect to $\operatorname{vec}(\mathbf{T}_j)$ by

$$
\begin{pmatrix} \frac{\partial \boldsymbol{e}(\mathbf{T}_i,\mathbf{T}_j)}{\boldsymbol{t}_j} & \frac{\partial \boldsymbol{e}(\mathbf{T}_i,\mathbf{T}_j)}{\theta_j} \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{ij}^\top \mathbf{R}_i^\top & 0 \\ 0^\top & 1 \end{pmatrix}. \qquad (3.9)
$$

Although vectorization aligns well with linear approximation methods, it does not inherently capture the intrinsic geometry of the special Euclidean group. This is because naively treating SE(2) elements as vectors in $\mathbb{R}^3$ by extracting the translation and rotation components separately ignores the underlying structure of the manifold. Lie algebra, on the other hand, is another approach of linearization, which directly exploits the structure of the Lie group and its associated Lie algebra. This is essentially using the geodesic distance to measure the error, that is, we now use the error (or residual)

$$
\boldsymbol{e}(\mathbf{T}_i, \mathbf{T}_j) = \operatorname{Log}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j). \qquad (3.10)
$$

By chain rule (3.2), Equation (3.3), Equation (3.5), and the inverse of Equation (3.6), we have the partial derivative of the error (3.10) with respect to $\mathbf{T}_i$ as follows:

$$
\begin{aligned}
\mathbf{J}_{\mathbf{T}_i}^{\operatorname{Log}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j)} &= \underbrace{\mathbf{J}_{\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j}^{\operatorname{Log}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j)}}_{\mathbf{J}_r^{-1}(\boldsymbol{e}_{ij})} \cdot \underbrace{\mathbf{J}_{(\mathbf{T}_i^{-1}\mathbf{T}_j)}^{\mathbf{T}_{ij}^{-1}(\mathbf{T}_i^{-1}\mathbf{T}_j)}}_{\mathbf{I}} \cdot \underbrace{\mathbf{J}_{(\mathbf{T}_j^{-1}\mathbf{T}_i)}^{\mathbf{T}_i^{-1}\mathbf{T}_j}}_{-\mathbf{Ad}_{\mathbf{T}_j^{-1}\mathbf{T}_i}} \cdot \underbrace{\mathbf{J}_{(\mathbf{T}_i)}^{\mathbf{T}_j^{-1}(\mathbf{T}_i)}}_{\mathbf{I}} \\
&= -\mathbf{J}_r^{-1}(\boldsymbol{e}_{ij}) \cdot \mathbf{Ad}_{\mathbf{T}_j^{-1}\mathbf{T}_i}, \qquad (3.11)
\end{aligned}
$$

and the partial derivative of the error (3.10) with respect to $\mathbf{T}_j$ as follows:

$$
\mathbf{J}_{\mathbf{T}_j}^{\operatorname{Log}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j)} = \mathbf{J}_{\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j}^{\operatorname{Log}(\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j)} \mathbf{J}_{\mathbf{T}_j}^{\mathbf{T}_{ij}^{-1}\mathbf{T}_i^{-1}\mathbf{T}_j} = \mathbf{J}_r^{-1}(\boldsymbol{e}_{ij}). \qquad (3.12)
$$

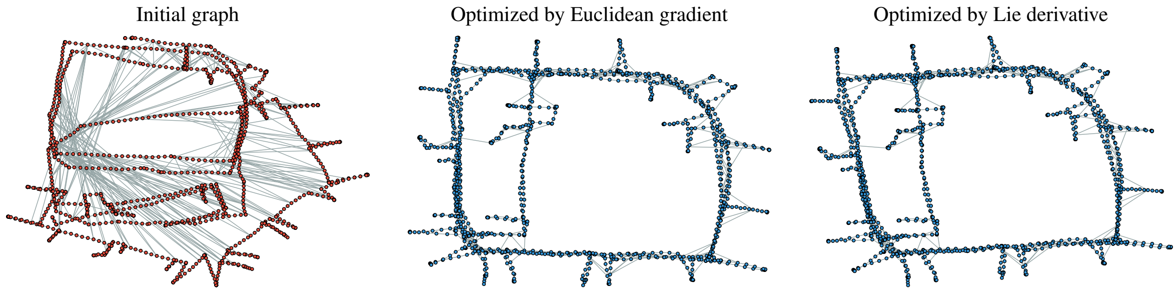
To actually implement this, we use the closed-forms:

$$
\begin{aligned}
\mathrm{SE}(2) &= \left\{ \mathbf{M} = \begin{pmatrix} \mathbf{R} & \boldsymbol{t} \\ 0^\top & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, \mathbf{R} \in \mathrm{SO}(2),\ \boldsymbol{t} \in \mathbb{R}^2 \right\}, && \text{[5, Equation (152)]} \\
\mathfrak{se}(2) &= \left\{ \boldsymbol{\tau}^\wedge = \begin{pmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0^\top & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, \boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\rho} \\ \theta \end{pmatrix} \in \mathbb{R}^3 \right\}, && \text{[5, Equation (153)]} \\
\operatorname{Exp}(\boldsymbol{\tau}) &= \begin{pmatrix} \mathbf{R} & \mathbf{V}(\theta)\rho \\ 0^\top & 1 \end{pmatrix} \;;\; \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, && \text{[5, Equation (156)]} \\
\operatorname{Log}(\mathbf{T}) &= \begin{pmatrix} \mathbf{V}^{-1}(\theta)\boldsymbol{t} \\ \theta \end{pmatrix} \;;\; \theta = \arctan\left(\frac{\mathbf{R}_{[1,0]}}{\mathbf{R}_{[0,0]}}\right), && \text{[5, Equation (157)]} \\
\mathbf{V}(\theta) &= \frac{\sin\theta}{\theta}\mathbf{I}_3 + \frac{1-\cos\theta}{\theta}[1]_\times,
\end{aligned}
$$

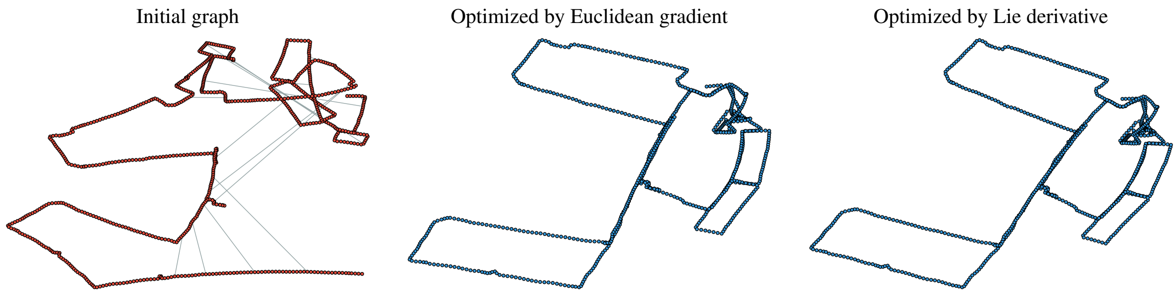
$$\mathbf{Ad}_{\mathbf{T}} = \begin{pmatrix} \mathbf{R} & -[1]_\times \boldsymbol{t} \\ 0^\top & 1 \end{pmatrix}, \qquad [5, \text{Equation (159)}]$$

$$\mathbf{J}_r(\boldsymbol{\tau}) = \begin{pmatrix} \frac{\sin\theta}{\theta} & \frac{(1-\cos\theta)}{\theta} & \frac{-\rho_2+\theta\rho_1-\rho_1\sin\theta+\rho_2\cos\theta}{\theta^2} \\ \frac{\cos\theta}{\theta} & \frac{\sin\theta}{\theta} & \frac{\rho_1+\theta\rho_2-\rho_1\cos\theta-\rho_2\sin\theta}{\theta^2} \\ 0 & 0 & 1 \end{pmatrix}. \qquad [5, \text{Equation (163)}]$$

We point out that during our implementation, [5, Equation (156)], [5, Equation (157)] and [5, Equation (163)] are numerically unstable if $\theta \to 0$. Thus, we employ Taylor approximation when θ is sufficient small. Figure 3.1 illustrates utilizing Euclidean gradient and Lie derivative to solve 2D pose graph optimization, respectively. Although the difference between the two methods may appear subtle, the Lie derivative approach generally outperforms the Euclidean gradient method in 3D cases. A complete implementation of the 2D pose graph optimization is available here: `https://github.com/doggydoggy0101/blog/tree/main/pgo`.



(a) Intel Research Lab.



(b) MIT Killian Court.

Figure 3.1: 2D pose graph optimization example.

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