

## 6 Class Lecture 10/10

### Homework 2.2

1. Show that if  $x, y \in \mathbb{Z}$ ;  $x, y$  both odd, there is **no**  $z \in \mathbb{Z}$  where  $x^2 + y^2 = z^2$ .

*Proof.* By contradiction. Let  $x, y, z \in \mathbb{Z}$ ;  $x, y$  are odd;  $x^2 + y^2 = z^2$ . Since  $x, y$  are odd,  $x = 2m + 1$ ,  $y = 2n + 1$  for  $n, m \in \mathbb{Z}$ .  $x^2 + y^2 = z^2$  so  $(2m + 1)^2 + (2n + 1)^2 = z^2$ .

$$- 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = z^2.$$

$$- 4 \underbrace{(m^2 + m + n^2 + n)}_p + 2 = z^2. \quad p \in \mathbb{Z}$$

$$- z^2 = 4p + 2 = 2(2p + 1).$$

$$- z = \sqrt{2}\sqrt{2p + 1}$$

$$- 2|z^2$$

$\therefore z$  is even.  $z = \sqrt{2}\sqrt{2p + 1}$  and  $z$  is even  $\therefore z = 2q$  such that  $q \in \mathbb{Z}$ .

$$- (2q)^2 = 4p + 2$$

$$- 4q^2 - 4p = 2$$

$$- 4(q^2 - p) = 2$$

$$- \therefore q^2 - p = \frac{1}{2} \notin \mathbb{Z}$$

This is a contradiction,  $\therefore$  there is **no**  $z \in \mathbb{Z}$  where  $x^2 + y^2 = z^2$  □

### GCD Continued

**Theorem 6.1** (Textbook 2.4.6). Let  $n, m, q \in \mathbb{Z}$ ;  $q|(nm)$  and  $\gcd(n, q) = 1$ . Then  $q|m$ .

*Proof.* Let  $n, m, q \in \mathbb{Z}$ . Assume  $q|(nm)$  and  $\gcd(q, n) = 1$ . (To Show:  $q|m$ ) Since  $\gcd(q, n) = 1$ ,  $1 = qx + ny$ .  $\therefore m = qmx + nmy$  and we know that  $q|(nm)$ .  $\therefore nm = qp$  such that  $p \in \mathbb{Z}$ . Hence,  $m = qmx + qpy = q \underbrace{(mx + py)}_{\text{for } x \in \mathbb{Z}}$ . So  $m = qs \therefore q|m$ . **QED.** □

**Theorem 6.2** (Corollary 2.5.2). If  $p, x, y \in \mathbb{Z}$  and  $p$  is prime, and  $p|(xy)$ , then  $p|x$  or  $p|y$ . (Example:  $4|12$ , so  $4|6 * 2$ ).

*Proof.* Let  $p, x, y \in \mathbb{Z}$ . Assume  $p$  is prime, and  $p|(xy)$ . (To Show:  $p|x \vee p|y$ ). By Contradiction  $\therefore p$  does not divide  $x$  or  $y$ .  $p$  is prime and  $p|(xy)$  and  $p$  does not divide  $x$ . The only divisor  $p$  are 1 and  $p$ , and  $p$  does not divide  $x$ .  $\therefore \gcd(p, x) = 1$ . By the previous theorem  $p|y$ . By assumption  $p$  does not divide  $y$ .  $\therefore$  there is a contradiction, and for  $p|(xy)$ , then  $p|x$  or  $p|y$ . **QED.** □

*Proof.* Same as above.

Case:  $p|x$  ✓

2. Case:  $p$  does not divide  $x$ .  $\therefore \gcd(p, x) = 1$  because  $p$  is prime. We assume that  $p|(xy)$ .  $\therefore$  by the previous theorem,  $p|y$ . ✓

3. **QED.**

□

**Theorem 6.3.** Let  $n, m, k \in \mathbb{Z}; n, m, k \geq 1$ . If  $n|k$  and  $m|k$ , and  $\gcd(n, m) = 1$ , then  $(nm)|k$ . (Example:  $3|24$  and  $4|24$  and  $(3 * 4)|24$ .)

*Proof.* Let  $n, m, k \in \mathbb{Z}$ . Assume  $n, m, k \geq 1$ ,  $n|k$ ,  $m|k$ , and  $\gcd(n, m) = 1$ .

–  $k = np$  and  $k = mq$ , where  $p, q \in \mathbb{Z}$ .

–  $1 = nx + my$

–  $k = knx + kmy$

–  $k = nm(qx + py)$

–  $\therefore (nm)|k$ .

– **QED.**

□