4.2 Class Lecture 10/10

Is the following set of vectors a subspace of $\mathbb{R}^n$?

$$\{(2a, 3b-a, 5, c+3a-1) \mid a, b, c \text{ are in } \mathbb{R}\}$$

No, $(0, 0, 0, 0)$ is not in this set.

Example 4.2.1. $\mathbb{P}_5[x]$; are any of these subpaces?

1. All polynomials of the form ax^4, $a \in \mathbb{R}$. **Yes** $\vec{0}$ ✓ $\vec{v}+\vec{w}$ ✓, $c\vec{v}$ ✓.
2. All polynomials of hte form $bx^3 - cx^2$, where $b, c \in \mathbb{R}$. **Yes**, $b = c = 0, \therefore \vec{0}$ ✓; $\vec{v}+\vec{w}$ ✓; $c\vec{v}$ ✓.
3. All polynomials of the form $x^4 + \ell x$, where $\ell \in \mathbb{R}$. **No**, there is no closed vector, there is no sum of vectors, and there is no scalar multiple of vector v.
4. All polynomials in $\mathbb{P}_5[x]$ with integer coefficients. **No**, if c is real (non-rational) $c(f)$ will not have an integer coefficient.

$\diamond$

The *kernel* of a linear transformation $T : V \to W$ is the subspace of vectors $\vec{v}$ in V for what $T(\vec{v}) = \vec{0}$. The kernel can be considered the same thing as the null space. However, the null space refers to matrix language, while a kernel refers to functional language.

eval_a:$\mathbb{P}_n[x] \to \mathbb{R}$ $\text{eval}_a(f)$:$f(a)$

For our fixed value a, what is the kernel of the evaluation of a $(Ker(eval_a))$? It's all polynomials with a root (zero) at a. That is, $\{f \in \mathbb{P}_n[x] \mid f = (x-a)g$, where $g \in \mathbb{P}_n[x]\}$.

Example 4.2.2. Consider $\mathbb{P}_3[x]$

1. Is the set of all polynomials with roots at ± 1 a subspace? $(x-1)+(x+1) = 2x \therefore$ **No.**

$\diamond$

Change of Basis and Relative Coordinates

If we are given a basis $\mathcal{B} = \left\{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}\right\}$ and $\mathcal{C} = \{\vec{c_1}, \vec{c_2}, \cdots, \vec{c_n}\}$ for a vector subspace V.

$$[\vec{v}]_{\mathcal{B}} = \sum_{i=1}^{n} \mathcal{B}_i \vec{b}_i$$

$$[\vec{v}]_{\mathcal{C}} = \sum_{i=1}^{n} \mathcal{C}_i \vec{c_i}$$

How do we convert between bases? We need to take all the b vectors and re-write them according to c.

$$_{\mathcal{C}}P_{\mathcal{B}} = \left[[\vec{b_1}]_{\mathcal{C}} | [\vec{b_2}]_{\mathcal{C}} | \cdots | [\vec{b_n}]_{\mathcal{C}} \right]$$

If there is some standard of fixed basis $\mathcal{U}$

$$[[\vec{c_1}]_{\mathcal{U}} | [\vec{c_2}]_{\mathcal{U}} | \cdots | [\vec{c_n}]_{\mathcal{U}} |] \to [\ I \mid {}_{\mathcal{C}}P_{\mathcal{B}}$$

Example 4.2.3. Find ${}_{\mathcal{C}}P_{\mathcal{B}}$ on $\mathbb{P}_2[x]$ for

$$\mathcal{B} = \{2, x-1, x^2+2x-2\}$$

$$\mathcal{C} = \{x, 2x+1, x^2-x\}$$

Method 1:

$$2 = -4(x) + 2(2x+1)$$

$$x - 1 = 3(x) + (-1)(2x+1)$$

$$x^2 + 2x = 6(x) - 2(2x+1) + 1(x^2 - x)$$

$$\begin{bmatrix} -4 & 3 & 6 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} =_{\mathcal{C}} P_{\mathcal{B}}$$

Method 2:

$$\begin{bmatrix} 0 & 1 & 0 & 2 & -1 & -2 \\ 1 & 2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -4 & 3 & 7 \\ 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{U} = \{1, x, x^2\}$$

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Given a linear transformation $T : V \to V$ where the dimension of $V = n$, we might ask a slightly odd question: Are there bases $\mathcal{B}$ and $\mathcal{C}$ for V so that *any* given matrix A is just $T[\vec{v}]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{C}}$. Probably *not*. $Ker(T)$ has got to be $Null(A)$. Those are independent of bases, they are part of subpsaces. Similarly $Range(T)$ has to be $Col.Sp.(A)$.

Example 4.2.4. Let $T : V \to V$, where T is n-dimensional. What $n \times n$ matricies represent T as we consider all possible bases for V? That is, we should consider A and B to be "equivalent" if there are bases $\mathcal{A}$ and $\mathcal{B}$ such that $A[\vec{v}]_{\mathcal{A}} = T(\vec{v}) = B[\vec{v}]\mathcal{B}$

Really what we want to do is

$$B[\vec{v}]_{\mathcal{B}} =_{\mathcal{B}} P_{\mathcal{A}} A_{\mathcal{A}} P_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}$$

$$B = PAP^{-1}$$

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