

4.2 Class Lecture 10/10

Is the following set of vectors a subspace of \mathbb{R}^n ?

$$\{(2a, 3b - a, 5, c + 3a - 1) \mid a, b, c \text{ are in } \mathbb{R}\}$$

No, $(0, 0, 0, 0)$ is not in this set.

Example 4.2.1. $\mathbb{P}_5[x]$; are any of these subspaces?

1. All polynomials of the form ax^4 , $a \in \mathbb{R}$. **Yes** $\vec{0} \checkmark \vec{v} + \vec{w} \checkmark, c\vec{v} \checkmark$.
2. All polynomials of the form $bx^3 - cx^2$, where $b, c \in \mathbb{R}$. **Yes**, $b = c = 0, \therefore \vec{0} \checkmark$; $\vec{v} + \vec{w} \checkmark; c\vec{v} \checkmark$.
3. All polynomials of the form $x^4 + \ell x$, where $\ell \in \mathbb{R}$. **No**, there is no closed vector, there is no sum of vectors, and there is no scalar multiple of vector v .
4. All polynomials in $\mathbb{P}_5[x]$ with integer coefficients. **No**, if c is real (non-rational) $c(f)$ will not have an integer coefficient.

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The *kernel* of a linear transformation $T : V \rightarrow W$ is the subspace of vectors \vec{v} in V for what $T(\vec{v}) = \vec{0}$. The kernel can be considered the same thing as the null space. However, the null space refers to matrix language, while a kernel refers to functional language.

$$\text{eval}_a : \mathbb{P}_n[x] \rightarrow \mathbb{R} \quad \text{eval}_a(f) : f(a)$$

For our fixed value a , what is the kernel of the evaluation of a ($\text{Ker}(\text{eval}_a)$)? It's all polynomials with a root (zero) at a . That is, $\{f \in \mathbb{P}_n[x] \mid f = (x - a)g, \text{ where } g \in \mathbb{P}_n[x]\}$.

Example 4.2.2. Consider $\mathbb{P}_3[x]$

1. Is the set of all polynomials with roots at ± 1 a subspace? $(x-1) + (x+1) = 2x \therefore$ **No**.

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Change of Basis and Relative Coordinates

If we are given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$ for a vector subspace V .

$$[\vec{v}]_{\mathcal{B}} = \sum_{i=1}^n \mathcal{B}_i \vec{b}_i$$

$$[\vec{v}]_C = \sum_{i=1}^n C_i \vec{c}_i$$

How do we convert between bases? We need to take all the b vectors and re-write them according to c .

$${}_C P_B = \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C & \cdots & [\vec{b}_n]_C \end{bmatrix}$$

If there is some standard of fixed basis \mathcal{U}

$$[[\vec{c}_1]_{\mathcal{U}} | [\vec{c}_2]_{\mathcal{U}} | \cdots | [\vec{c}_n]_{\mathcal{U}}] \rightarrow [I \mid {}_C P_B]$$

Example 4.2.3. Find ${}_C P_B$ on $\mathbb{P}_2[x]$ for

$$\mathcal{B} = \{2, x-1, x^2+2x-2\}$$

$$\mathcal{C} = \{x, 2x+1, x^2-x\}$$

Method 1:

$$2 = -4(x) + 2(2x+1)$$

$$x-1 = 3(x) + (-1)(2x+1)$$

$$x^2+2x = 6(x) - 2(2x+1) + 1(x^2-x)$$

$$\begin{bmatrix} -4 & 3 & 6 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = {}_C P_B$$

Method 2:

$$\begin{bmatrix} 0 & 1 & 0 & 2 & -1 & -2 \\ 1 & 2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -4 & 3 & 7 \\ 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{U} = \{1, x, x^2\}$$

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Given a linear transformation $T : V \rightarrow V$ where the dimension of $V = n$, we might ask a slightly odd question: Are there bases \mathcal{B} and \mathcal{C} for V so that *any* given matrix A is just $T[\vec{v}]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{C}}$. Probably *not*. $\text{Ker}(T)$ has got to be $\text{Null}(A)$. Those are independent of bases, they are part of subspaces. Similarly $\text{Range}(T)$ has to be $\text{Col.Sp.}(A)$.

Example 4.2.4. Let $T : V \rightarrow V$, where T is n -dimensional. What $n \times n$ matrices represent T as we consider all possible bases for V ? That is, we should consider A and B to be "equivalent" if there are bases \mathcal{A} and \mathcal{B} such that $A[\vec{v}]_{\mathcal{A}} = T(\vec{v}) = B[\vec{v}]_{\mathcal{B}}$

Really what we want to do is

$$B[\vec{v}]_{\mathcal{B}} = P_{\mathcal{A}} A_{\mathcal{A}} P_{\mathcal{B}}[\vec{v}]_{\mathcal{B}}$$

$$B = P A P^{-1}$$

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