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1 Linear Equations in Linear Algebra

Systems of Linear Equations

Definition 1.1. A **linear equation** in the variables $x_1, \cdots, x_n$ is an equation that can be writen in the form

$$a_1, x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \cdots, a_n$ are real or complex numbers. $\diamond$

Definition 1.2. A **system of linear equations**, or a linear system, is a collection of one or more linear equations involving the same variables. $\diamond$

Example 1.3. The following is an example of a system of linear equations:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 1.5x_3 & = & 8 \\ x_1 & & & - & 4x_3 & = & -7 \end{array}$$

$\diamond$

Definition 1.4. A **solution** of the system is a list $(s_1, s_2, \cdots, s_n)$ of numbers that makes each equation a true statement when the values of $s_1, s_2, \cdots, s_n$ are substituted for $x_1, x_2, \cdots, x_n$ respectively. $\diamond$

Definition 1.5. A **solution set** is the set of all possible solutions of the linear system. $\diamond$

Definition 1.6. Two linear systems are **equivalent** if they have the same solution set. $\diamond$

We generally consider linear equations to have:

– No solution.

– Exactly on solution.

– Infinitely many solutions.

Definition 1.7. These systems are **consistent** if it has either one solution or infinitely many solutions, and **inconsistent** if it has no solution. $\diamond$

Matrix Notation

Example 1.8. These linear equations can be represented in matrix from. Given the following system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

we take the coefficients of each variable and align them in a matrix like so

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

We generally think of the above matrix as the **coefficient matrix** while the matrix below is known as the **augmented matrix**

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

◇

Solving a Linear System

Solving linear systems is almost identical in nature to how one solves matricies: row reduction. For a review of row reduction and echelon forms consult the WikiBooks article regarding the topic.

2 Matrix Algebra

Subspaces of $\mathbb{R}$

Abstract

The four fundamental concepts in this section are subspace, column space, null space, and basis. These techniques have been mechanically computed up until this point, however we will now understand the technical terminology behind them. These concepts will become essential in Chapters 5 & 7.

Definition 2.1. A **subspace** of $\mathbb{R}$ is any set H in $\mathbb{R}^n$ that has three properties:

- The zero vector is in H.
- For each u and v in H, the sume $u + v$ is in H.
- For each u in H and each scalar c, the vecture cu is in H.

$\diamond$

Example 2.2. If v_1 and v_2 are in $\mathbb{R}^n$ and $H = \text{Span}\{v_1, v_2\}$, then H is a subspace of $\mathbb{R}^n$. Note that the zero vector is in H since $0v_1 + 0v_2$ is a linear combination of v_1 and v_2. Given two arbitrary vectors in H:

$$u = s_1v_1 + s_2v_2 \quad \text{and} \quad v = t_1v_1 + t_2v_2$$

$$\text{then} \quad u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

this shows that $u+v$ is a linear combination of v_1 and v_2 and thus H. If v_1 is not zero and v_2 is a multiple of v_1, then v_1 and v_2 simply span a *line* through the origin. $\diamond$

Column Space and Null Space of a Matrix

Definition 2.3. The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A. If $A = [a_1, \cdots, a_n]$, with the columns of $\mathbb{R}^m$, then Col A is the same as $\text{Span}\{a_1, \cdots, a_n\}$. $\therefore$ the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. $\diamond$

Example 2.4. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. b is in the column space of $A \iff Ax = b$; that is, we need to row reduce $\begin{bmatrix} A & b \end{bmatrix}$.

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & 18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and from this we determine that $Ax = b$ is consistent and b is in Col A. $\diamond$

Definition 2.5. The **null space** of a matrix A is the set of Nul A of all solutions of the homogeneous equation $Ax = 0$. ◇

Theorem 2.6. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $Ax = 0$ of m homogenous linear equations in n unknowns is a subpsace of $\mathbb{R}^n$.*

Basis for a Subspace

Definition 2.7. A **basis** for a subspace of H of $\mathbb{R}^n$ is a linearly independent set in H that spans H. ◇

Example 2.8. Given $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ we want to find the basis of A.
Begin by writing the solution of $Ax = 0$ like so:

$$\begin{bmatrix} A & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{rcrcrcrcrcl} x_1 & - & 2x_2 & & & - & x_4 & + & 3x_5 & = & 0 \\ & & & & x_3 & + & 2x_4 & - & 2x_5 & = & 0 \\ & & & & & & & & 0 & = & 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w} = x_2 u + x_4 v + x_5 w$$

The example above shows that Nul A is generated by $\{u, v, w\}$, which automatically makes the set of linear equations linearly independent. According to theorem 12 all solutions of $Ax = 0$ are a subspace of $\mathbb{R}^n$, and in order for this to be true x_2, x_4, x_5 must be zero when $0 = x_2 u + x_4 v + x_5 w$. $\therefore$ $\{u, v, w\}$ is a *basis* for Nul A. ◇

Theorem 2.9. *The pivot columns of a matrix A form a basis for the column space of A.*

Dimension and Rank

Abstract

There are two fundamental concepts in this section, and unsurprisingly they are the dimension of a subspace and the rank of a matrix. The **Basis Theorem** and **Invertible Matrix Theorem** combines the concept of dimension. supbspace, linear independence, span, and basis.

2.1 Matrix Algebra – Class Lecture

Dimension and Rank

Definition 2.1.1. If W is a subspace of $\mathbb{R}^n$, any set of vectors X for the span $X = W$ is called a spanning set of W. If in addition X is linearly independent, X is called a base for W. $\diamond$

Definition 2.1.2. The dimension of W is the number of vectors in a basis for W

- dim(W) = |x|
- lemma Dim(W) is well defined, any subspace with a finite spanning set will have the same number of vectors in an basis

$\diamond$

Proof. Suppose X and Y are bases for W with ℓ and m vectors respectively; Without loss of generality suppose $\ell < m$

$$X = \{\vec{b_1}, \vec{b_2}, ...\vec{b_\ell}\}$$

$$Y = \{\vec{c_1}, \vec{c_2}, ...\vec{c_m}\}$$

$$\underbrace{\left(\vec{b_1}|\vec{b_2}|...|\vec{b_\ell}\right)}_{A}(\vec{x_1}|\vec{x_2}|...|\vec{x_m}) = (\vec{c_1}|\vec{c_2}|...|\vec{c_m})$$

Given an m x n matrix A

$$\vec{T}(x) = A\vec{x}$$

is a linear transformation

$$\vec{T} : \mathbb{R}^n \to \mathbb{R}^m$$

$\square$

Definition 2.1.3. The column space of A is the subspace of $\mathbb{R}^m$ spanned by the collumns of A. The rank of A is dim(col.sp.(A)). The null space of A is the subspace of $\mathbb{R}^n$ which is the solution set to $A\vec{x} = \vec{0}$ "nullspace(A)". The nullify of A is the null(A) = dim(nullspace(A)) $\diamond$

Theorem 2.1.4 (Rank Theorem)**.** *If A is an $m \times n$ matrix then rank(A) + null(A) $= n$.*

3.1 Determinants – Class Lecture

Determinants

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

We're trying to check that none of the rows are linear combinatons of the other. Find the least common multiple, which would be the most straightfoward. Take a bunch of determinants where we are checking that.

Example 3.1.1.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

◇

Definition 3.1.2. – A is an $n \times n$ matrix with a fixed k.

– det(A) = $|\ a_{ij}\ | = \sum_{k=1}^{n} a_{k\ell}$ (-1)$^{k+\ell}$ det(A$_{k\ell}$)

– A$_{k\ell}$ is A less the kth row and ℓth column.

◇

Example 3.1.3.

$$\begin{bmatrix} 3 & 5 & -1 \\ 7 & 2 & 0 \\ 0 & 4 & -4 \end{bmatrix}$$

$$(-1)\begin{vmatrix} 7 & 2 \\ 0 & 4 \end{vmatrix} + (0)\begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix}$$

$$-28 - 0 + 116 = 88$$

A is invertible $\iff$ $\det(A) \neq 0$ ◇

Properties of Determinants

Consider the three types of elemntary matricies:

$$E_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha \neq 0$$

$$E_k = \begin{pmatrix} 1 & 0 & k \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} k \neq 0$$

$$E_{ij} = Inversion\ row\ i,\ j$$

– $\det(\mathrm{E}_\alpha) = \alpha$

– $\det(\mathrm{E}_k) = 1$

– $\det(\mathrm{E}_{ij}) = ?$

Proof. When $n = 2$. $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$. Now suppose for some $n >= 2$ that any row intersection matrix I_{ij} satisfies $\det(I_{ij}) = -1$. Now fix $1 <= i < j <= n+1$ and consider $\det(I_{ij})$. If either $i >= 2$ or $j <= n$ then I_{ij} = first row and first column or last row and last column. $\det(\mathrm{I}_{i-1,j-1})$ $\det(\mathrm{I}_{1,n+1})$ = $(-1)^{1+n+1}$ $\det(\mathrm{B})$ = $(-1)^{1+n+1}(-1)^{1+n}$ = (-1) □

3.2 Determinants – Class Lecture

Stuff

Proof. Let A, B be an $n \times n$ matrix. Then $\det(AB) = \det(A)\det(B)$. If $\det(A)$ or $\det(B) = 0$, then one of A or B has linearly independent rows and columns. $\square$