

## Guide to Orlov's theorem

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- $[\mathbb{A}^n/\mathbb{Z}_d]$  has a  $d$ -torsion orbifold line bundle  
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- The canonical bundle of  $[\mathbb{A}^n/\mathbb{Z}_d]$  is  $d$ -torsion  
 $\implies$  Kuznetsov components are fractional Calabi–Yau  
(for Fano hypersurfaces / complete intersections)

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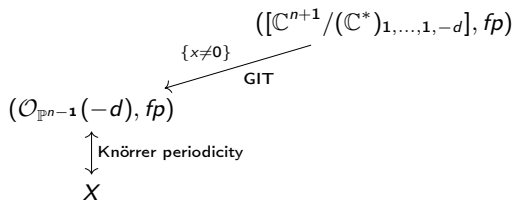
Hypersurface  $X \subset \mathbb{P}^{n-1}$ ,  $f \in H^0(\mathcal{O}(d))$ .

$$(\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp)$$

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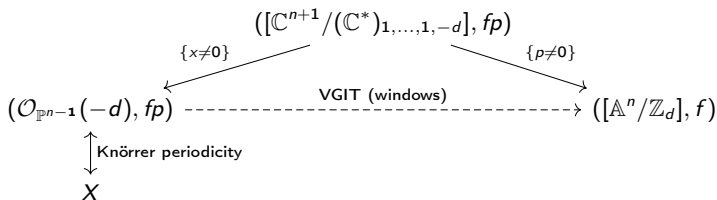
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$$\begin{array}{ccccc}
 & & ([\mathbb{C}^{n+1}/(\mathbb{C}^*)_{1,\dots,1,-d}], fp) & & \\
 & \swarrow \{\mathbf{x} \neq 0\} & & \searrow \{p \neq 0\} & \\
 (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) & \xrightarrow{\text{VGIT (windows)}} & & & ([\mathbb{A}^n/\mathbb{Z}_d], f) \\
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 X & & & & 
 \end{array}$$

Two steps:

- ① KP - easy geometric functor (push-pull).
- ② VGIT - mildly computationally annoying functor (iterate cones to move something into a window).

# Tower of Babel



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Physics jargon:

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- *Closed sector* of B-model  $\rightsquigarrow HH_\bullet$  of B-brane dg-category  
(HKR isomorphism  $\rightsquigarrow = H^\bullet(\Omega_X^\bullet, dW) = \text{Jac}(W)$ )

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- Landau–Ginzburg *A*-model  $\rightsquigarrow (X, W)$ ; manifold + global function (symplectic Lefschetz fibration)
- Superpotential  $\rightsquigarrow$  global function  $W$
- *A*-branes  $\rightsquigarrow$  thimbles / vanishing cycles
- Open sector of *A*-model  $\rightsquigarrow A_\infty$ -category of *A*-branes = Fukaya–Seidel category
- Closed sector of *A*-model  $\rightsquigarrow HH_\bullet$  of *A*-brane  $A_\infty$ -category (GPS, Abouzaid–Seidel, ...  $\rightsquigarrow = SH^\bullet(\text{Milnor fiber}) = \text{Jac}(W)$ )

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Curved dg-algebra:  $(A, d, w)$  with  $|w| = 2$ ,  $d^2 = [w, -]$  and  $dw = 0$ .

Curved dg-module:  $(M, d)$  with  $d^2(-) = w(-)$ .

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$X$  variety with  $G$  action,  $w \in \Gamma(X, L)$  for  $G$ -equivariant line bundle  $L$ .

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Orlov:  $X = \mathrm{Spec} A$ ,  $G = \mathbb{G}_m$ ,  $L = \mathcal{O}(\chi_{\mathbb{G}_m}^n)$  [like  $\mathrm{Proj} A$  with  $\mathcal{O}(n)$ ]

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Grading on  $\mathcal{O}_X$ :  $\mathbb{C}^*$ -action on  $X$ ,  $w \in \Gamma(\mathcal{O}_X)$  weight 2.

Sheaves of curved dg-modules over  $(\mathcal{O}_X, w)$ :

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## Fact

This is a special case of BFK definition with  $G = \mathbb{C}^*$ ,  $L = \mathcal{O}(\chi_{\mathbb{C}^*}^2)$ .

Conversely, BFK definition is equivalent to taking  $X = [U(L)/G]$  here.



## Derived categories: Objects

E.g.  $D^b(\mathbb{P}^2)$ , chain complexes,  $d^2 = 0$ :

$$\left( \mathcal{O}(-2)[2] \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}(-1)^2[1] \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O} \right) \simeq \mathcal{O}/(x, y)$$

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## Example (Koszul resolutions)

Exact sequence:

$$\begin{array}{ccccc}
 \mathcal{O}(3) & \xrightarrow{p} & \mathcal{O} & \xrightarrow{1} & \mathcal{O}/p. \\
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where  $\partial_x \lrcorner dW = yz = 0$ ,  $\partial_y \lrcorner dW = xz = 0$ .



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# Matrix factorizations: derived Homs

## Fact (Koszul resolutions)

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$$\begin{array}{ccc} \mathcal{O}_X(-d) & \xhookrightarrow{j} & (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) \\ \downarrow \pi & & \downarrow \pi \\ X & \xhookrightarrow{j} & \mathbb{P}^{n-1}. \end{array}$$

Compute that  $j_*\pi^* : D^b(X) \rightarrow \text{MF}(\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp)$  is an equivalence. □

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## VGIT and windows

Recall:

$$\begin{array}{ccc} & ([\mathbb{C}^{n+1}/\mathbb{C}_{1,\dots,1,-d}^*], fp) & \\ j_L^* \swarrow \quad \{x \neq 0\} & & \searrow j_R^* \\ (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) & & ([\mathbb{A}^n/\mathbb{Z}_d], f) \end{array}$$

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## Definition

The *window category* for the interval  $[k : k + l - 1]$  is

$$\mathcal{W}_{k,l} = "\langle \mathcal{O}(k), \mathcal{O}(k+1), \dots, \mathcal{O}(k+l-1) \rangle" \subset \text{MF}([\mathbb{C}^{n+1}/\mathbb{C}^*], fp).$$

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## Corollary

- If  $d = n$ ,  $D^b(X) \simeq \mathrm{MF}([\mathbb{A}^n/\mathbb{Z}_d])$ .
- If  $d < n$ ,  $\mathrm{MF}([\mathbb{A}^n/\mathbb{Z}_d]) \subset D^b(X)$ .
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The normal bundle of the unstable locus tells you the window size. □

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This comes from a length 3 window, so just restrict to  $\{p \neq 0\}$ .

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$$\mathcal{O}_p \mapsto \mathcal{O}_{\pi^{-1}(p)} \simeq \left( \mathcal{O}(-2)[2] \begin{array}{c} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} yzp & (x^2+z^2)p \end{pmatrix}} \end{array} \mathcal{O}(-1)^2[1] \begin{array}{c} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} -(x^2+z^2)p \\ yzp \end{pmatrix}} \end{array} \mathcal{O} \right).$$

This comes from a length 3 window, so just restrict to  $\{p \neq 0\}$ .

$$\mathcal{O}_p \mapsto \left( \mathcal{O}(2)[2] \oplus \mathcal{O} \begin{array}{c} \xrightarrow{\begin{pmatrix} y & -(x^2+z^2) \\ -x & yz \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} yz & x^2+z^2 \\ x & y \end{pmatrix}} \end{array} \mathcal{O}(-1)^2[1] \right) \in \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_3], f).$$



## Example: Elliptic curve

Let  $E : \{y^2z - x^3 - xz^2 = 0\}$ , so  $D^b(E) \simeq \mathrm{MF}(\mathcal{O}_{\mathbb{P}^2}(-3), fp) \simeq \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_3], f)$ .

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So  $\mathcal{O}_p \mapsto \mathcal{O}/(x, y) \in \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_3], y^2z - x^3 - xz^2)$ .

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 f \left( \begin{array}{c} \nearrow p \\ \searrow \end{array} \right) & & \nearrow & & & & \\
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Then restrict this to  $\{p \neq 0\}$ .

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$$\mathrm{RHom}(\mathcal{O}_0, \mathcal{O}_0) = (\mathbb{C}\langle \partial_x, \partial_y, \partial_z \rangle)^{\mathbb{Z}_3} = \mathbb{C}\{1, \partial_x \wedge \partial_y \wedge \partial_z\}.$$

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As expected:  $\mathrm{R}\Gamma(\mathcal{O}_E) = \mathbb{C} \oplus \mathbb{C}[-1]$ .



Thanks for listening!