

Guide to Orlov's theorem

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25 Nov 2025

Punchline

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- has $D^b(X)$ as a residual component if X is general type.

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- $[\mathbb{A}^n/\mathbb{Z}_d]$ has a d -torsion orbifold line bundle
 $\implies \exists T \in \text{Aut}(D^b(X))$ with $T^d = [2]$, for Calabi–Yau X .

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 $\implies \exists T \in \text{Aut}(D^b(X))$ with $T^d = [2]$, for Calabi–Yau X .
- The canonical bundle of $[\mathbb{A}^n/\mathbb{Z}_d]$ is d -torsion
 \implies Kuznetsov components are fractional Calabi–Yau
(for Fano hypersurfaces / complete intersections)

Proof structure

Hypersurface $X \subset \mathbb{P}^{n-1}$, $f \in H^0(\mathcal{O}(d))$.

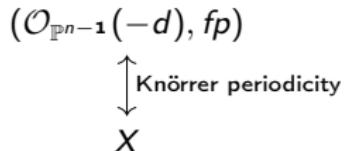
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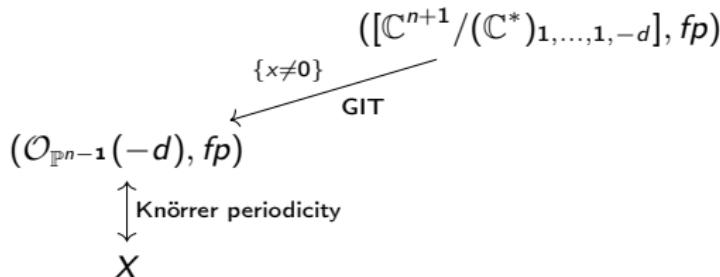
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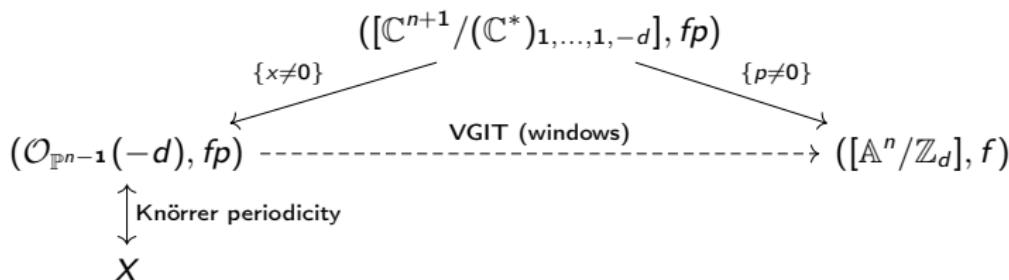
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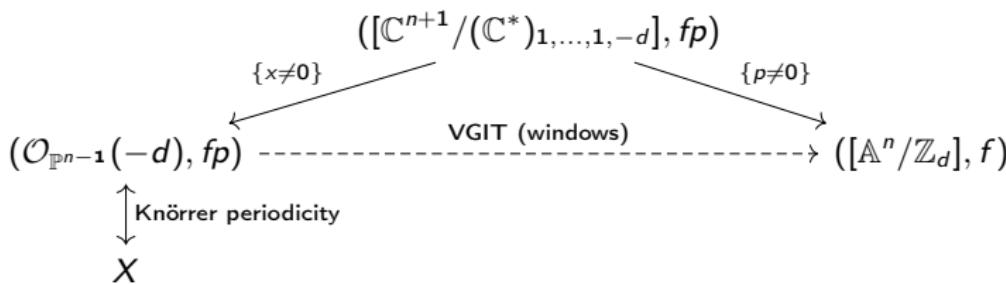
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Two steps:

- ① KP - easy geometric functor (push-pull).
- ② VGIT - mildly computationally annoying functor (iterate cones to move something into a window).

Tower of Babel

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- *Open sector* of B-model \rightsquigarrow dg-category of B-branes
= matrix factorization category
- *Closed sector* of B-model $\rightsquigarrow HH_\bullet$ of B-brane dg-category
(HKR isomorphism $\rightsquigarrow H^\bullet(\Omega_X^\bullet, dW) = \text{Jac}(W)$)

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Physics jargon:

- Landau–Ginzburg **A**-model $\rightsquigarrow (X, W)$; manifold + global function (symplectic Lefschetz fibration)
- Superpotential \rightsquigarrow global function W
- **A-branes** \rightsquigarrow thimbles / vanishing cycles
- Open sector of **A**-model $\rightsquigarrow A_\infty$ -category of **A**-branes = Fukaya–Seidel category
- Closed sector of **A**-model $\rightsquigarrow HH_\bullet$ of **A**-brane A_∞ -category (GPS, Abouzaid–Seidel, ... $\rightsquigarrow SH^\bullet(\text{Milnor fiber}) = \text{Jac}(W)$)

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Curved dg-algebra: (A, d, w) with $|w| = 2$, $d^2 = [w, -]$ and $dw = 0$.

Curved dg-module: (M, d) with $d^2(-) = w(-)$.

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Graded algebra A with $|w| = n$ central.

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Orlov: $X = \text{Spec } A$, $G = \mathbb{G}_m$, $L = \mathcal{O}(\chi_{\mathbb{G}_m}^n)$ [like $\text{Proj } A$ with $\mathcal{O}(n)$]

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Definition (Segal 2011)

Grading on \mathcal{O}_X : \mathbb{C}^* -action on X , $w \in \Gamma(\mathcal{O}_X)$ weight 2.

Sheaves of curved dg-modules over (\mathcal{O}_X, w) :

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Fact

This is a special case of BFK definition with $G = \mathbb{C}^*$, $L = \mathcal{O}(\chi_{\mathbb{C}^*}^2)$.

Conversely, BFK definition is equivalent to taking $X = [U(L)/G]$ here.

Derived categories: Objects

E.g. $D^b(\mathbb{P}^2)$, chain complexes, $d^2 = 0$:

$$\left(\begin{array}{ccc} & \begin{pmatrix} y \\ -x \end{pmatrix} & \\ \mathcal{O}(-2)[2] & \xrightarrow{\quad} & \mathcal{O}(-1)^2[1] \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O} \end{array} \right) \simeq \mathcal{O}/(x, y)$$

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Totalizations of complexes of complexes (e.g. cones):

$$\text{Tot} \left(\begin{array}{ccc} \mathcal{O}(-2)[2] & \xrightarrow{x} & \mathcal{O}(-1)[1] \\ \downarrow y & & \downarrow y \\ \mathcal{O}(-1)[1] & \xrightarrow{x} & \mathcal{O} \end{array} \right) \simeq (\mathcal{O}/y(-1)[1] \xrightarrow{x} \mathcal{O}/y) \simeq \mathcal{O}/(x, y)$$

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Definition (Orlov 2011 "... nonaffine LG-models")

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Example (Koszul resolutions)

Exact sequence:

$$\begin{array}{ccccc} \mathcal{O}(3) & \xrightarrow{P} & \mathcal{O} & \xrightarrow{1} & \mathcal{O}/p. \\ 1 \uparrow \downarrow xyzp & & p \uparrow \downarrow xyz & & \\ \mathcal{O}(3)[1] & \xrightarrow{1} & \mathcal{O}(3)[1] & & \end{array}$$

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Matrix factorizations: derived Homs

Fact (Koszul resolutions)

In $D^b(X)$, $Z \subset X$ complete intersection, $\mathrm{RHom}(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{R}\Gamma(\wedge^\bullet N_{Z/X}[-1])$.

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For $\text{MF}(\mathbb{A}^3, yz^2)$: $\partial_x \lrcorner dW = 0$ but $\partial_y \lrcorner dW = z^2$, so

$$R\text{Hom}(\mathcal{O}/(x, y), \mathcal{O}/(x, y)) \simeq (\mathbb{C}[z]/z^2)\langle \partial_x \rangle.$$

Knörrer periodicity

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Theorem ("Knörrer periodicity" Orlov '04, Shipman '11, Hirano '16, ...)

$D^b(X) \simeq \text{MF}(\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp)$ with graded fiber coordinate $|p| = 2$.

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Compute that $j_*\pi^* : D^b(X) \rightarrow \text{MF}(\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp)$ is an equivalence. □

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Recall:

$$\begin{array}{ccc} & \left([\mathbb{C}^{n+1}/\mathbb{C}_{1,\dots,1,-d}^*], fp\right) & \\ j_L^* \swarrow & \{x \neq 0\} & \searrow j_R^* \\ (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) & & ([\mathbb{A}^n/\mathbb{Z}_d], f) \end{array}$$

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Definition

The *window category* for the interval $[k : k + l - 1]$ is

$$\mathcal{W}_{k,l} = \langle \langle \mathcal{O}(k), \mathcal{O}(k+1), \dots, \mathcal{O}(k+l-1) \rangle \rangle \subset \text{MF}([\mathbb{C}^{n+1}/\mathbb{C}^*], fp).$$

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Corollary

- If $d = n$, $D^b(X) \simeq \text{MF}([\mathbb{A}^n/\mathbb{Z}_d])$.
- If $d < n$, $\text{MF}([\mathbb{A}^n/\mathbb{Z}_d]) \subset D^b(X)$.
- If $d > n$, $D^b(X) \subset \text{MF}([\mathbb{A}^n/\mathbb{Z}_d])$.

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The normal bundle of the unstable locus tells you the window size. □

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As expected: $\text{RHom}(\mathcal{O}_p, \mathcal{O}_p) = \mathbb{C} \oplus \mathbb{C}[-1]$.

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Example (Line bundle)

What about \mathcal{O}_E ?

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Then restrict this to $\{p \neq 0\}$.

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Sanity check

$$\text{RHom}(\mathcal{O}_0, \mathcal{O}_0) = (\mathbb{C}\langle \partial_x, \partial_y, \partial_z \rangle)^{\mathbb{Z}_3} = \mathbb{C}\{1, \partial_x \wedge \partial_y \wedge \partial_z\}.$$

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As expected: $\text{R}\Gamma(\mathcal{O}_E) = \mathbb{C} \oplus \mathbb{C}[-1]$.

Thanks for listening!