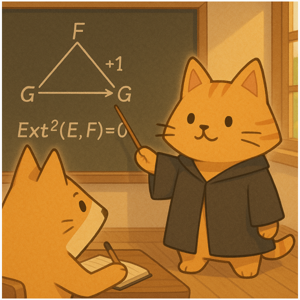

Derived Obsessed Graduate Students

Gepner type Bridgeland Stability conditions

Chunkai Xu (Thanks Peize Liu for providing this beamer template)

University of Warwick

11 November 2025

Introduction

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$$\begin{array}{ccc} \text{Slope stability} & \xrightarrow{\textit{generalise}} & \text{Bridegland stability} \\ \text{on curves} & & \text{on triangulated categroies} \\ E \in \mathrm{Coh}(C) & & E \in \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}\,X) \end{array}$$

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$E \in \mathsf{Coh}(C)$		$E \in \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$

Stability conditions have been proved to exist on D^{b} of varieties...

- Varieties whose derived category admits a full exceptional collection: projective spaces $\mathbb{P}^n$, quadrics $Q^n \subseteq \mathbb{P}^{n+1}$, Grassmannians $\mathrm{Gr}(k, n)$.
- Curves (= slope stability);
- Surfaces (tilt stability);
- Fano threefolds;
- Abelian threefolds;
- Quintic threefolds, and some other CY_3 with Picard number 1.

Stability conditions

Let $\mathcal{D}$ be a triangulated category

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Remark

With the Support property, Bridgeland shows that the set of stability conditions on $\mathcal{D}$ can be endowed with a manifold structure.

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In the rest of this talk, we will take

- $\mathcal{D} \coloneqq D^b(X)$ for some smooth projective variety of dimension n,
- $v : K(\mathcal{A}) \to \Lambda \coloneqq A(X)$ to be Chern characters

$$\mathrm{ch} : E \mapsto (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \cdots, \mathrm{ch}_n(E)).$$

Examples

- For a smooth projective curve C, $\sigma = (\mathrm{Coh}(X), Z)$ where

$$\begin{aligned} Z(E) &= -\deg(E) + i\,\mathrm{rk}(E) \\ &= -\mathrm{ch}_1(E) + i\,\mathrm{ch}_0(E) \end{aligned}$$

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  gives a stability condition.

- For a smooth projective surface X, the same construction does not give a stability condition. ($\mathrm{ch}(k_p) = (0, 0, 1)$ hence $Z(k_p) = 0$).

Tilting construction

For simplicity, assume $\mathrm{Pic}(X) = \mathbb{Z}H$. Let $\mu_H(E) := H\,\mathrm{ch}_1(E)/H^2\,\mathrm{ch}_0(E)$ be the slope of E with H. Denote by $\mu_H^{\pm}(E)$ the maximal (minimal) slope of its Harder–Narasimhan factors.

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For $\beta \in \mathbb{R}$, consider the torsion pair

$$\mathcal{T}_\beta := \{E \in \mathrm{Coh}(X) \mid \mu_H^-(E) > \beta\};$$
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Let $\mathrm{Coh}^\beta(X) = \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$ be the **tilt** of $\mathrm{Coh}(X)$ with this pair, i.e. for $B \in \mathrm{Coh}^\beta(X)$ iff

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$$\mathcal{H}^0(B) \in \mathcal{T}_\beta, \mathcal{H}^{-1}(B) \in \mathcal{F}_\beta, \text{ and } \mathcal{H}^i(B) = 0 \text{ for } i \neq 0, -1.$$

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This slope function has the advantage of actually being ‘slope’.

Visualize Stability conditions

We have the Bogomolov-Gieseker inequality:

Theorem

For any (β, α)*-tilt-semistable object* E *in* $\mathrm{Coh}^{\beta}(X)$*, we have*

$$Q^{tilt}(E) = \mathrm{ch}_1(E)^2 - 2\,\mathrm{ch}_0(E)\,\mathrm{ch}_2(E) \geqslant 0.$$

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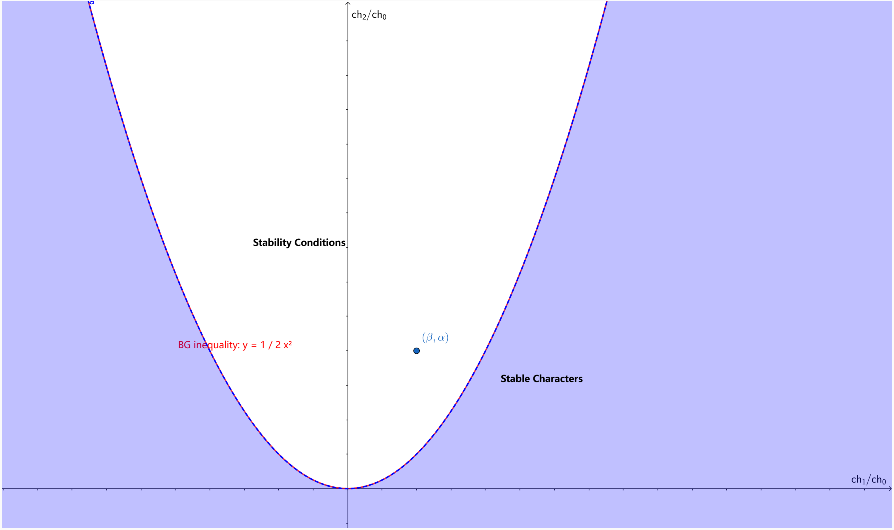
Equivalently,

$$\frac{1}{2}\left[\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)}\right]^2 \geqslant \frac{\mathrm{ch}_2(E)}{\mathrm{ch}_0(E)}.$$

Visualize Tilt Stability conditions



Stability condition on threefolds

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Denote by $\nu_{\alpha,\beta}(E) := \frac{\mathrm{ch}_2(E) - \alpha\,\mathrm{ch}_0(E)}{\mathrm{ch}_1^{\beta}(E)}$ the tilt slope. Consider the following torsion pair

$$\mathcal{T}_{\alpha,\beta} := \{E \in \mathsf{Coh}^{\beta}(X) | \nu^{-}_{\alpha,\beta}(E) > 0\}$$
$$\mathcal{F}_{\alpha,\beta} := \{F \in \mathsf{Coh}^{\beta}(X) | \nu^{+}_{\alpha,\beta}(F) \leqslant 0\}$$

Then we consider the **double tilt** $\mathsf{Coh}^{\alpha,\beta}(X) := \left\langle \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} \right\rangle$.

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Theorem (Bayer, Macrì, Toda)

The pair $\sigma_{\alpha,\beta,s} = (\mathrm{Coh}^{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ is a BSC on X if ***generalized Bogomolov–Gieseker inequality*** *holds on X.*

Conjecture (generalized BG inequality)

Let (X, H) be a threefold pair with $\mathrm{Pic}(X) = \mathbb{Z}H$. If $E \in \mathrm{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$-tilt-semistable, then

$$Q_{\alpha,\beta}(E) = \alpha Q^{\mathrm{tilt}}(E) + 4\operatorname{ch}_2^\beta(E)^2 - 6\operatorname{ch}_1^\beta(E)\operatorname{ch}_3^\beta(E) \geqslant 0.$$

Graded matrix factorization and Derived category of hypersurfaces

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A **graded matrix factorization** of W is the following data

$$P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d) \tag{1}$$

where P^i are graded free A-modules of finite rank, p^i are homomorphisms of graded A-modules, satisfying the following conditions:

$$p^1 \circ p^0 = \cdot W, \qquad p^0(d) \circ p^1 = \cdot W. \tag{2}$$

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$$\tau^{\times d} = [2].$$

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Conjecture (Toda)

There is a stability condition

$$\sigma_G = (Z_G, \{\mathcal{P}_G(\phi)\}_{\phi \in \mathbb{R}}) \in \mathrm{Stab}(\mathrm{HMF}^{\mathrm{gr}}(W)), \tag{3}$$

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Here the $e^{2\pi i/d}$-action on $P^\bullet = P^0 \oplus P^1$ is induced by the $\mathbb{Z}$-grading on each P^i, and "str" is the supertrace which respects the $\mathbb{Z}/2\mathbb{Z}$-grading on $P^\bullet$.

Upshot: This stability condition has some symmetry: $(\tau_*)^d\sigma_G = \sigma_G[2]$.

Relation with $D^b(\mathrm{Coh}(X))$

Orlov proved that $\mathrm{HMF}^{gr}(W)$ is related to the derived category of coherent sheaves on the stack

$$X := (W = 0) \subseteq \mathbb{P}(x_1, \cdots, x_n) \tag{4}$$

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Hence, we might expect there is a stability condition with nice symmertry on quintic threefolds.

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A stability condition σ on a triangulated category $\mathcal{D}$ is called **Gepner type with respect to** $(\Phi, \lambda) \in \mathrm{Aut}(\mathcal{D}) \times \mathbb{C}$ if the following condition holds:

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In this notion, Toda's conjectural stability condition on matrix factorization is Gepner type with respect to $(\tau, 2/d)$.

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From now on, We will call a stability condition σ on X of **Gepner type** if it is Gepner type with respect to $(\mathrm{ST}_{\mathcal{O}_X} \circ \otimes \mathcal{O}_X(1), \frac{2}{5})$.

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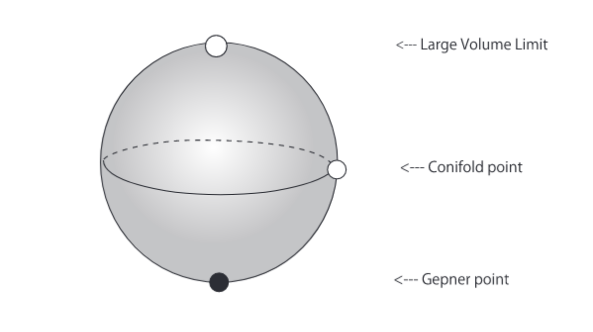

Figure 1: Stringy Kähler moduli space of a quintic threefold

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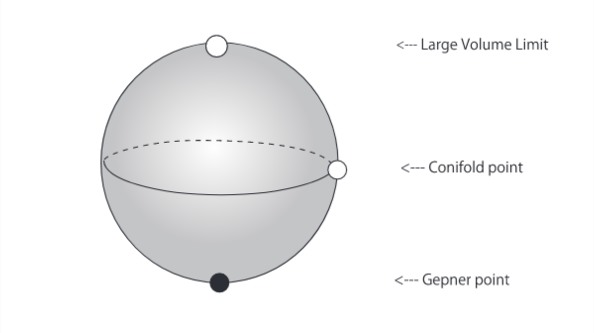

Figure 1: Stringy Kähler moduli space of a quintic threefold

Gepner type stability condition will correspond to Gepner point.

Conjectural construction

Recall a conjectural stability condition on $\mathrm{HMF}^{\mathrm{gr}}(W)$ has the central charge:

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where

$$\beta = -\frac{1}{2}, \qquad a = -0.8819\cdots$$
$$b = 0.68819, \qquad c = 0.104176\cdots$$

Comparison with BMT's construction

Recall that in Bayer, Macrí, Toda's construction, the central charge is

$$Z_{\alpha,\beta,s}(E) = -\operatorname{ch}_3^{\beta}(E) + (s+\frac{1}{6})(\alpha-\frac{1}{2}\beta^2)H^2\operatorname{ch}_1^{\beta}(E) + i\left[H\operatorname{ch}_2(E) - \alpha H^3\operatorname{ch}_0(E)\right],$$

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$$\alpha_0 := \frac{\beta^2}{2} - \frac{c}{b} = -0.02637\cdots < 0$$

Sign difference in α!

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With this inequality, at least the tilt-stability at the point $(\beta, \alpha) = (-\frac{1}{2}, -0.02637\cdots)$ makes sense.

Further conjectures

Conjecture (Toda)

Let X be a smooth quintic threefold. The pair

$$\sigma_G \coloneqq (\mathrm{Coh}^{\alpha_0, -\frac{1}{2}}(X), Z_G)$$

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• Use restriction lemma and Riemann-Roch, with some Clifford type bound on the curve $C_{2,2,5} \in |2H_{S_{2,5}}|$, get the inequality.

Comparison of bounds

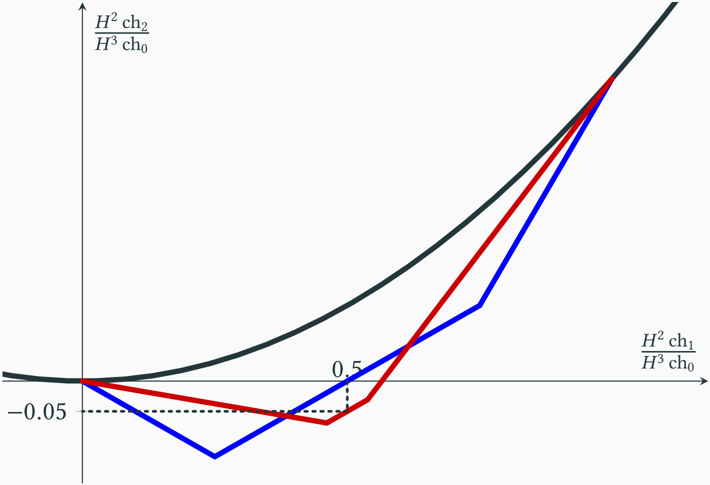

Figure 2: Blue:Li's bound   Red: my bound

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- Disadvantage: the bound is weaker near 0 and 1.

Thank you for your attention!

D*erived* ***O****bsessed* ***G****raduate* ***S****tudents*