



DEMYSTIFYING BRIDGE LAND STABILITY

I : Why care ?

① Algebra

$D^b(\text{mod } A)$

Triangulated
Category

$\text{Stab}(-)$

Geometry

$D^b(\text{coh } X)$

compar?

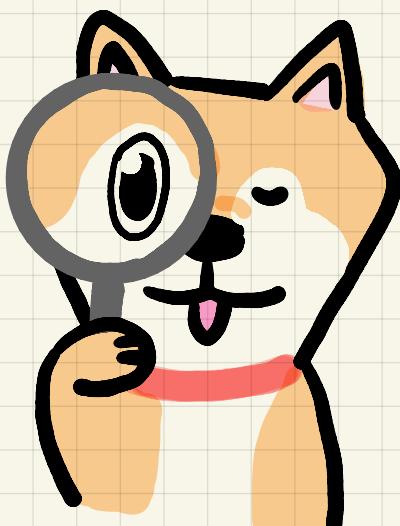
Complex Manifold

Bridgeland:
"Spaces of
stability conditions"

"moduli of
mirrors"

[Bridgeland
-Smith]
Teichmüller
spaces

[Bayer
-BridgeLand]
Autobiographies



2 Classification problems

vector
bundles

moduli over
f.d. algebras
(e.g. quiver rep.s)

GOAL: classify all objects of some kind



too many!



all objects built from "stable" ones

Fix discrete invariants \underline{v} (e.g. rank, degree)

$\rightarrow M^{\text{st}}(\underline{v})$: moduli of stable of class \underline{v} .

e.g. smooth & projective:

slope ✓ Gieseker ✓

Bridgeland: some examples



could have different ways to be "stable"

BRIDGELAND
STABILITY
BONUS

deformation + wall crossing properties

$\sigma_1 \dashdots \sigma_2$

$M_{\sigma_1}^{\text{st}}(\underline{v}) \dashrightarrow M_{\sigma_2}^{\text{st}}(\underline{v})$

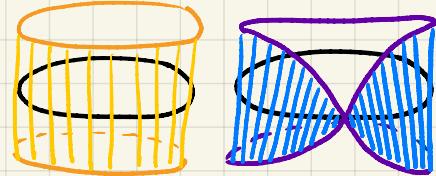
\rightarrow MMP / birationality
[Bayer-Maier '14]

\rightarrow Brill-Noether loci
e.g. [Feyzbaksh '20]

stab(D)



II Axiomatising Stability



Slope stability

C : sm. proj. curve / \mathbb{C}

~~E : vector bundle~~ $\in \text{coh}(C)$

Defⁿ E is μ -stable if ^{semi}

$$\mu(E) = \begin{cases} +\infty & \text{rk}(E) = 0 \\ \frac{\deg(E)}{\text{rk}(E)} & \text{otherwise} \end{cases}$$

$$0 \neq F \subset E$$

$$\mu(F) < \mu(E)$$

e.g. \mathcal{L} : line bundle $\mu(\mathcal{L}) = \deg(\mathcal{L})$

- $0 \neq F \subseteq \mathcal{L}$ must have $\text{rank } 1$

but $\deg(F) < \deg(\mathcal{L})$ $\therefore \mathcal{L}$ is stable

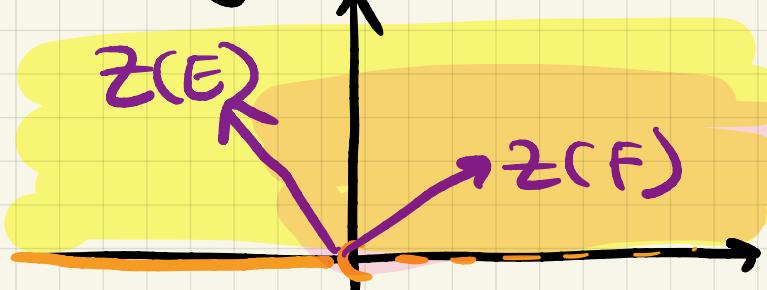
Thm E has a unique Harder-Narasimhan filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \dots \subsetneq E_{n-1} \subsetneq E_n = E \quad \text{s.t.}$$

(i) E_i/E_{i-1} μ -semistable (ii) $\mu(E_1/E_0) > \dots > \mu(E_n/E_{n-1})$

$\diamond \quad \sharp \quad Z_\mu(E) := -\deg(E) + i \cdot \text{rank}(E) \in \mathbb{H}$

E stable,
 $F \subset E$



$$Z(E) = m e^{i \pi \phi}$$

C

CHEAT SHEET: SLOPE STABILITY

① $E, F \in \text{coh}(C)$

② \bar{E} is semistable of phase ϕ ,

$$F \hookrightarrow E \Rightarrow \phi(F) \leq \phi(E)$$

i.e. \nexists map $F' \rightarrow E$ s.t. $\phi(F') > \phi(E)$

③ E is stable of phase ϕ ,

$$F \hookrightarrow E \Rightarrow \phi(F) < \phi(E)$$

i.e. E has no nontrivial subobjects of the same phase

④ E has a unique HN filtration by stables

$$\begin{array}{ccccccc} 0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_{n-1} \hookrightarrow E_n = E \\ \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \\ E/E_0 \quad \dots \quad \quad \quad E_n/E_{n-1} \\ e_1 > \dots > e_n \end{array}$$

⑤ $\chi_\mu(E) = -\deg(E) + i \operatorname{rank}(E) \in \mathbb{H}$

$$E \neq 0 \Rightarrow \chi_\mu(E) = \underbrace{m(E)}_{\operatorname{ER}_{>0}} \cdot e^{i\pi\phi} \quad \phi \in (0, 1]$$

CHEAT SHEET: BRIDGELAND STABILITY

① $E, F \in \mathcal{D}$: triangulated category
 ↴ "semistable"

② $\text{EFP}(\phi)$: subcat. of \mathcal{D} $\forall \phi \in \mathbb{R}$

$\phi_1 > \phi_2, \text{FEP}(\phi_1), \text{EFP}(\phi_2)$
 $\Rightarrow \text{Hom}(F_1, E) = 0$

③ $\text{EFP}(\phi)$ is stable if E is a simple object of $P(\phi)$

④ $P(\phi)[1] = P(\phi+1)$

- $[1] : \mathcal{D} \rightarrow \mathcal{D}$ "shift"
- $A \rightarrow B \rightarrow C \rightarrow A[1]$ exact triangle

④ E has a unique HN filtration by stables

$$0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_{n-1} \rightarrow E_n = E$$

$$\downarrow A_1 \text{EP}(\phi) \quad \dots \quad \downarrow A_n \text{EP}(\phi)$$

$$e_1 > \dots > e_n$$

⑤ $\chi : K(\mathcal{D}) \rightarrow \mathbb{C}$ group homomorphism

Grothendieck group = free group generated by objects
 $[B] = [A] + [C] \Leftrightarrow A \rightarrow B \rightarrow C \rightarrow A[1]$

$0 \neq E \in P(\phi) \Rightarrow \frac{\chi([E])}{\pi} = \frac{m(E)}{\epsilon R > 0} \cdot e^{i\pi\phi} \in \mathbb{C}$

PLUS

- \mathbb{Z} factors via $\mathcal{L} \cong \mathbb{Z}^r$
- support property



Def¹ $\sigma = (P, \mathbb{Z})$ is a Bridgeland and stability condition on D

$\text{Stab}_\perp(D) = \text{all stability conditions}$
on D wrt \perp



Th^m [Bridgeland '07] $\text{Stab}_\perp(D)$ is a \mathbb{C} -mfld.

- $\text{stab}(D) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Q}) \cong \mathbb{C}^r$
- $\sigma = (A, \mathbb{Z}) \mapsto \mathbb{Z}$
- ↑ local homeomorphism

Th^m [B '07] $\sigma = (P, \mathbb{Z})$ is equivalent to the data:

- A is a \heartsuit of a bdd t-structure on D
- $\mathbb{Z}_A : K(A) \setminus \{0\} \rightarrow \mathbb{H}$ w/ $M_{\mathbb{Z}_A} = \frac{-\text{Re } \mathbb{Z}_A}{\text{Im } \mathbb{Z}_A}$

+ MN property



$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_n = E$$

$$M_{\mathbb{Z}_A}(E/E_0) > \dots$$

10 SA: • $(P, \mathbb{Z}) \rightsquigarrow A = P(0, 1) := \langle P(\phi) \rangle_{\text{ext}}^{\Phi \in (0, 1]}$

- $(A, \mathbb{Z}_A) \cdot E \neq 0, \mathbb{Z}_A(E) = m e^{i \pi \phi} \Rightarrow E \in P(\phi) \quad \phi \in (0, 1]$
- $P(\phi') = P(n + \phi) = P(\phi)[n] \quad \phi \in (0, 1]$

EXAMPLES:

① $\mathcal{D} = \mathcal{D}^b X$

$\dim X = 1, g(X) \geq 1: \text{stab}(X) = (\text{coh } X, \mathbb{Z}_\mu) \cdot \widetilde{\mathcal{GL}_2^+(\mathbb{R})}$
 $\cong \mathbb{C} \times H_1$

$X = \mathbb{P}^1: \mathcal{D}^b \mathbb{P}^1 \xrightarrow{\sim} \mathcal{D}^b \underline{\text{Rep}}(\cdot \xrightarrow{\beta} \cdot)^{K_2}$

$E^\bullet \mapsto \text{Hom}(\mathcal{O} \oplus \mathcal{O}(1), E^\bullet)$

$\rightsquigarrow \sigma = (\text{Rep}^G, \mathbb{Z}')$

$\rightsquigarrow \text{stab}(X) \cong \mathbb{C}^2$

$\dim X = 2 (\text{coh } X, \mathbb{Z})$ never works

IDEA torsion pair $(\mathcal{T}_{n,\beta}, \mathcal{F}_{n,\beta}) \xrightarrow{\text{tilt}} \text{coh}^{n,\beta}(X)$
 $n\text{-slope } < \beta \uparrow \quad n\text{-slope } > \beta \quad \langle \mathcal{F}_{[-1]}, \mathcal{T} \rangle_{\text{ext}}$

$\sigma = (\text{coh}^{n,\beta}(X), \dots)$

$\dim X = 3$ "double tilt something works"

EXAMPLES:

(2) Q : finite quiver , $Q_0 = \{0, \dots, n\}$

Consider $\text{Rep}_C Q$

(A) $K_2: 0 \xrightarrow{\begin{matrix} x \\ y \end{matrix}} 1$

$\underline{V} \in \text{Rep}_C K_2 : (V_0, V_1, \phi_x, \phi_y: V_0 \rightarrow V_1)$

e.g. $\mathbb{C} \xrightarrow{\begin{matrix} \lambda \in \mathbb{C} \\ M \in \mathbb{C} \end{matrix}} \mathbb{C}$

$$\mathbb{C}^n \xrightarrow{\begin{matrix} \phi_x \\ \phi_y \end{matrix}} \mathbb{C}^m$$

$\triangle d(\underline{V}) = (\dim V_0, \dim V_1)$

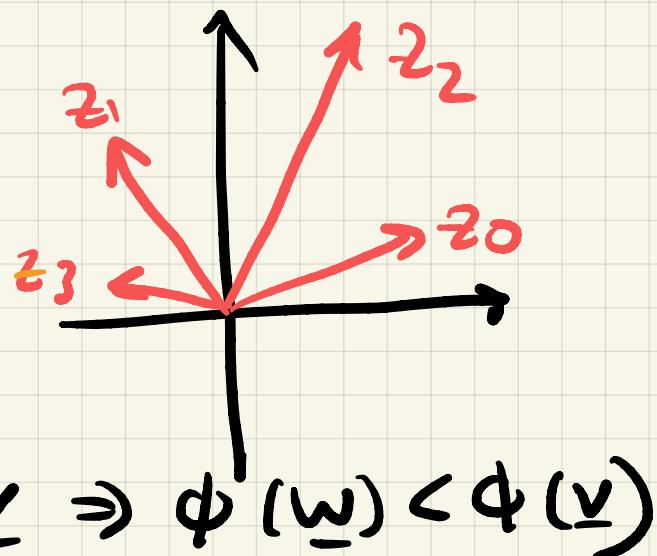
$(1,1): (\lambda, \mu) + (0, 0)$ - \mathbb{CP}^1 family of modules
w/modules of class $(1,1)$

$\mathbb{C} \xrightarrow{0} \mathbb{C} \rightsquigarrow M(1,1)$ is 1-dim variety
+ 0-dim.

want better behaved moduli \rightsquigarrow use stability conditions

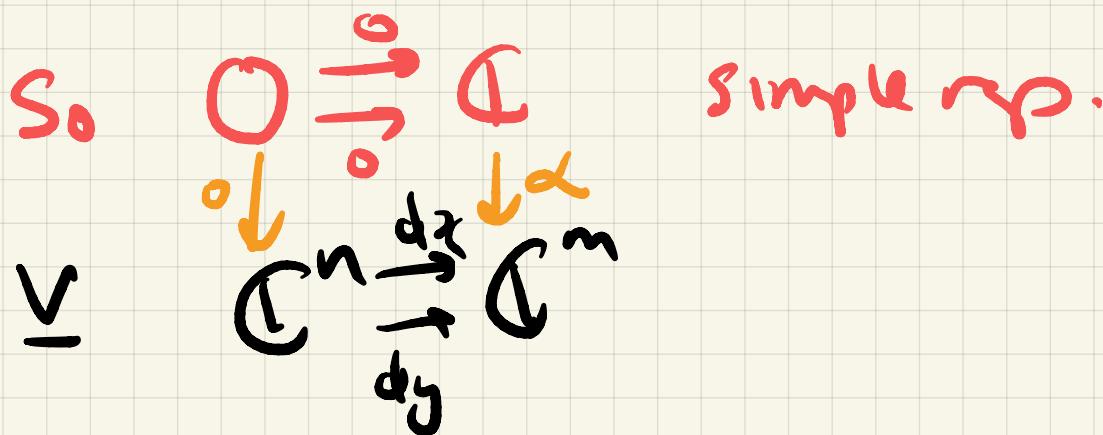
Pick $z_0, \dots, z_n \in \mathbb{H}$

$$z(\underline{v}) = \sum_{i=0}^n \dim V_i \cdot z_i \\ = r e^{i\pi\phi}$$



\underline{v} is stable if $\underline{w} \subset \underline{v} \Rightarrow \phi(w) < \phi(v)$

(A) : Pick $z_0, z_1 \in \mathbb{H}$



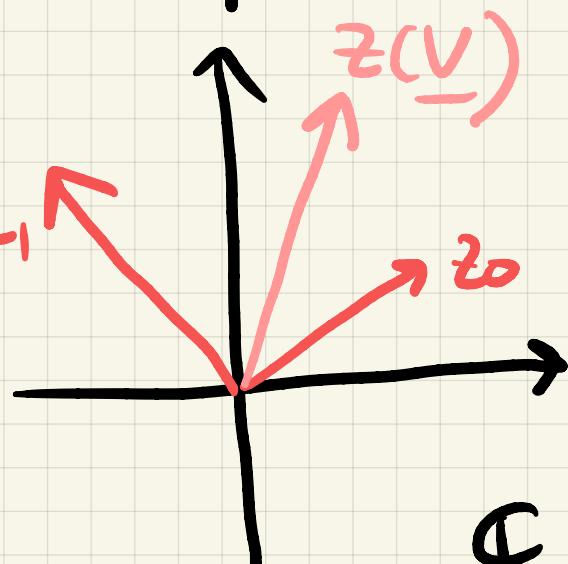
If $m > 0$, S_0 is always a subrep.

Assume $\phi(z_1) > \phi(z_0)$

$n > 0 \Rightarrow \phi(S_0) > \phi(V)$

$\Rightarrow V$ not stable

Sim. argument for $S_1 : \mathbb{C} \xrightarrow{\phi} O$



$\rightsquigarrow S_0, S_1$ only stable reps.

ASSUME $\phi(z_1) < \phi(z_0)$

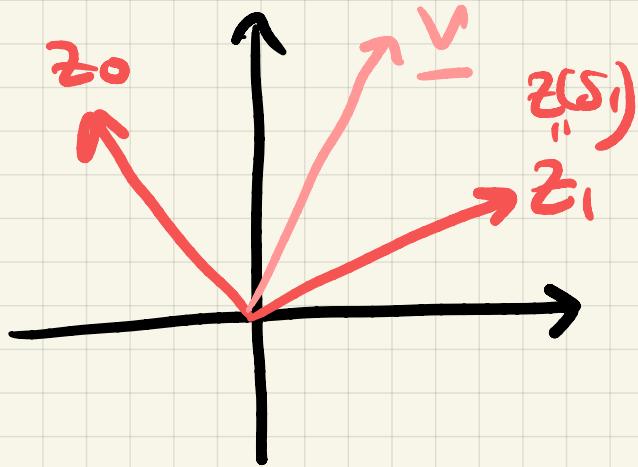
consider

$$\underline{S}_1 \quad 0 \xrightarrow{\sigma} \mathbb{C}$$

$\downarrow \alpha$

$$\underline{V} : \mathbb{C} \xrightarrow[\mu]{} \mathbb{C}$$

$$\phi(\underline{S}_1) < \phi(\underline{V}) \vee (\underline{S}_1 \subseteq \underline{V})$$



AND

$$\underline{S}_0 \quad \mathbb{C} \xrightarrow{\sigma} 0$$

$\downarrow \circ$

$$\underline{V} \quad \mathbb{C} \xrightarrow[\mu]{} \mathbb{C}$$

$$\underline{S}_0 \subseteq \underline{V}$$

(\Rightarrow) this commutes

$$(\lambda, \mu) = (0, 0)$$

$$\phi(\underline{S}_0) > \phi(\underline{V})$$

$$(\lambda, \mu) \neq (0, 0) \Rightarrow \underline{V} \text{ stable}$$

These correspond to $GL_2^+(\mathbb{R}) \cdot \sigma_M$.