SPRING 2024
DEPARTMENT OF COMPUTER ENGINEERING
MAT222
LINEAR ALGEBRA
FINAL EXAM SOLUTIONS

June 6, 2024

1) A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins one buys 100 of them. Using Gauss-Jordan elimination, find all possible numbers of roosters, hens, and chicks that can be bought. **(20 p.)**

Solution: Let x, y, and z be the number of roosters, hens, and chicks, respectively. We have the system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & 100 \\ 5x & + & 3y & + & \frac{z}{3} & = & 100 \end{array}$$

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 100 \\ 5 & 3 & \frac{1}{3} & 100 \end{bmatrix}$$

We have

$$\begin{bmatrix} 1 & 1 & 1 & 100 \\ 5 & 3 & \frac{1}{3} & 100 \end{bmatrix} \xrightarrow{R_2-5R_1} \begin{bmatrix} 1 & 1 & 1 & 100 \\ 0 & -2 & -\frac{14}{3} & -400 \end{bmatrix} \xrightarrow{\left(-\frac{1}{2}\right)R_2} \begin{bmatrix} 1 & 1 & 1 & 100 \\ 0 & 1 & \frac{7}{3} & 200 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -100 \\ 0 & 1 & \frac{7}{3} & 200 \end{bmatrix}$$

The corresponding system is

$$\begin{array}{ccccc} x & = & \frac{4}{3}z & - & 100 \\ y & = & -\frac{7}{3}z & + & 200 \end{array}$$

Since $x \geq 0$ and $y \geq 0$, we must have

$$75 \leq z \leq \frac{600}{7}.$$

z must be an integer, so we find that $75 \leq z \leq 85$. But x and y are integers when z is a multiple of 3. 75, 78, 81, and 84 are the possible values of z within its range. Therefore, the possible numbers of the animals that can be bought are

$$(x, y, z) = \{(0, 25, 75), (4, 18, 78), (8, 11, 81), (12, 4, 84)\}. \blacksquare$$

(2) For the matrix

$$A = \begin{bmatrix} 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 4 & 8 & 1 & 1 & 6 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$$

find a basis for the kernel of A. **(20 p.)**

Solution: We must find the solution set of the system $A\mathbf{x} = \mathbf{0}$. We have

$$\begin{bmatrix} 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 4 & 8 & 1 & 1 & 6 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_4} \begin{bmatrix} 1 & 2 & 3 & 2 & 0 \\ 2 & 4 & 1 & 9 & 10 \\ 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 - 4R_1 \\ R_4 - 3R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 2 & 0 \\ 0 & 0 & -5 & 5 & 10 \\ 0 & 0 & -11 & -7 & 6 \\ 0 & 0 & -8 & -4 & 5 \end{bmatrix}$$

$$\xrightarrow{\left(-\frac{1}{5}\right) R_2} \begin{bmatrix} 1 & 2 & 3 & 2 & 0 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & -11 & -7 & 6 \\ 0 & 0 & -8 & -4 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 + 11R_2 \\ R_4 + 8R_2}]{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & 5 & 6 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -18 & -16 \\ 0 & 0 & 0 & -12 & -11 \end{bmatrix} \xrightarrow{\left(-\frac{1}{18}\right) R_3} \begin{bmatrix} 1 & 2 & 0 & 5 & 6 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & -12 & -11 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 + R_3 \\ R_4 + 12R_3}]{R_1 - 5R_3} \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{14}{9} \\ 0 & 0 & 1 & 0 & -\frac{10}{9} \\ 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} \end{bmatrix} \xrightarrow{(-3) R_4} \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{14}{9} \\ 0 & 0 & 1 & 0 & -\frac{10}{9} \\ 0 & 0 & 0 & 1 & \frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 + \frac{10}{9} R_4 \\ R_3 - \frac{8}{9} R_4}]{R_1 - \frac{14}{9} R_4} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the solution set for $A\mathbf{x} = \mathbf{0}$ is $\{(-2t, t, 0, 0, 0) : t \in \mathbb{R}\}$. This implies that $\ker(A)$ is spanned by only one vector, namely,

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so a basis for the kernel of A is $\mathcal{B} = \{\mathbf{v}\}$.■

(3) Let P_2 be the vector space of all polynomials of degree at most 2. Find the determinant of the matrix associated with the operator $T : P_2 \to P_2$ defined by $T(p(t)) = p(3t - 2)$. **(20 p.)**

Solution: Let $p(t) = a + bt + ct^2$, where a, b, and c are constants. Then

$$T(p(t)) = p(3t - 2) = a + b(3t - 2) + c(3t - 2)^2 = a - 2b + 4c + (3b - 12c)t + 9ct^2.$$

The associated matrix is then found to be

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}.$$

Thus, $\det(A) = 27$.■

(4) If V is a vector space of all 2×2 matrices for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector, find $\dim(V)$. **(20 p.)**

Solution: Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We need to find all matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $A\mathbf{v} = \lambda\mathbf{v}$, for some constant λ. We have

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a + 2b \\ c + 2d \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \end{bmatrix} \Rightarrow \begin{cases} a + 2b = \lambda \\ c + 2d = 2\lambda \end{cases}$$

This gives that

$$c + 2d = 2a + 4b \Rightarrow c = 2a + 4b - 2d.$$

Thus, A must be of the form

$$\begin{bmatrix} a & b \\ 2a + 4b - 2d & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

The matrices

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

are linearly independent and span the vector space V, so a basis of V is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\}.$$

Therefore, $\dim (V) = 3.\blacksquare$

(5) Consider the quadratic form

$$Q (x_1, x_2) = 8x_1^2 - 4x_1 x_2 + 5x_2^2.$$

Show that $(0, 0)$ is an absolute minimum point of Q. **(20 p.)**

Solution: For $\mathbf{x} = (x_1, x_2)$, we can write Q as $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where

$$A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}.$$

Since

$$\det (\lambda I - A) = \begin{vmatrix} \lambda - 8 & 2 \\ 2 & \lambda - 5 \end{vmatrix} = (\lambda - 9) (\lambda - 4),$$

the eigenvalues of A are 9 and 4. Since these eigenvalues are positive, the quadratic form Q is positive definite, that is, $Q (x_1, x_2) > 0$ for all $(x_1, x_2) \neq (0, 0)$. Since $Q (0, 0) = 0$, we observe that $Q (x_1, x_2) > Q (0, 0)$ for all (x_1, x_2), meaning that $(0, 0)$ is an absolute minimum point of Q with the absolute minimum value $0.\blacksquare$