MAT222
LINEAR ALGEBRA
HOMEWORK ASSIGNMENT 5
SOLUTIONS

(1) Find nonzero eigenvalues of the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y-z \\ -x+y-z \\ -x-y+z \end{bmatrix}$$

if any possible.

Solution: The eigenvalues of T are scalar satisfying $T(\mathbf{v}) = \lambda \mathbf{v}$ for some nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$. Therefore,

$$T(\mathbf{v}) = \lambda \mathbf{v} \Rightarrow T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x-y-z \\ -x+y-z \\ -x-y+z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

implies that

$$\left.\begin{array}{r} x-y-z = \lambda x \\ -x+y-z = \lambda y \\ -x-y+z = \lambda z \end{array}\right\} \Rightarrow \begin{array}{r} (1-\lambda)\,x - y - z = 0 \\ -x + (1-\lambda)\,y - z = 0 \\ -x - y + (1-\lambda)\,z = 0 \end{array}$$

Since we are asked to find nonzero λ values, the last homogeneous system must have nontrivial solutions. This means that the determinant of its coefficients matrix should be zero. Since

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} &= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1-\lambda \\ -1 & -1 \end{vmatrix} \\ &= (1-\lambda)\left(\lambda^2 - \lambda\right) + (\lambda - 2) + (\lambda - 2) = -(\lambda - 2)^2(\lambda + 1), \end{aligned}$$

we find the desired eigenvalues to be -1 and 2.■

(2) Let the sequence of numbers $\{G_k\}, k \geq 0$ be defined as $G_0 = 0$, $G_1 = 1$, and

$$G_{k+2} = \frac{G_{k+1} + G_k}{2}, \quad k \geq 0.$$

Using matrix diagonalization, find $\lim\limits_{k\to\infty} G_k$.

Solution: Since

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

we may consider the matrix equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$\mathbf{x}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

By iteration, we conclude that $\mathbf{x}_k = A^k \mathbf{x}_0$ with

$$\mathbf{x}_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{2} \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \frac{\lambda}{2} - \frac{1}{2},$$

so $\lambda = 1$ and $\lambda = -\frac{1}{2}$ are eigenvalues. If $\lambda = 1$, then

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{E.R.O.}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so a corresponding eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -\frac{1}{2}$, we have

$$\begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{\text{E.R.O.}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

so a corresponding eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Now, we set

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

so that

$$A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}.$$

Since

$$D^k = \begin{bmatrix} 1 & 0 \\ 0 & \frac{(-1)^k}{2^k} \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{-3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix},$$

we find that

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{(-1)^k}{2^k} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{(-1)^k}{2^k 3} & \frac{(-1)^{k+1}}{2^k 3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} + \frac{(-1)^k}{2^k 3} & \frac{1}{3} + \frac{(-1)^{k+1}}{2^k 3} \\ \frac{2}{3} + \frac{(-1)^{k+1}}{2^{k-1} 3} & \frac{1}{3} + \frac{(-1)^k}{2^{k-1} 3} \end{bmatrix}. \end{aligned}$$

We therefore conclude that

$$\begin{aligned} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \mathbf{x}_k = A^k \mathbf{x}_0 &= \begin{bmatrix} \frac{2}{3} + \frac{(-1)^k}{2^k 3} & \frac{1}{3} + \frac{(-1)^{k+1}}{2^k 3} \\ \frac{2}{3} + \frac{(-1)^{k+1}}{2^{k-1} 3} & \frac{1}{3} + \frac{(-1)^k}{2^{k-1} 3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} + \frac{(-1)^k}{2^k 3} \\ \frac{2}{3} + \frac{(-1)^{k+1}}{2^{k-1} 3} \end{bmatrix}, \end{aligned}$$

that is,

$$G_k = \frac{2}{3} + \frac{(-1)^{k+1}}{2^{k-1} 3},$$

from which we deduce that

$$\lim_{k\to\infty} G_k = \frac{2}{3} + \lim_{k\to\infty} \frac{(-1)^{k+1}}{2^{k-1}3} = \frac{2}{3}. \blacksquare$$

(3) Orthogonally diagonalize the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if possible.

Solution: Let A be the given matrix. We have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda - 1 & 0 & 0 \\ 0 & -1 & 0 & \lambda & 0 \\ -1 & 0 & 0 & 0 & \lambda \end{vmatrix} \overset{R_5 + \frac{1}{\lambda}R_1}{\underset{R_4 + \frac{1}{\lambda}R_2}{=}} \begin{vmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda^2-1}{\lambda} & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda^2-1}{\lambda} \end{vmatrix} = (\lambda-1)^3(\lambda+1)^2,$$

so eigenvalues are $\lambda = -1$ and $\lambda = 1$. If $\lambda = 1$, then

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\text{E.R.O.}}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

If $\lambda = -1$, then

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} \overset{\text{E.R.O.}}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so corresponding eigenvectors are

$$\mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore, the desired matrix S is found to be

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} . \blacksquare$$

(4) Let Q be a quadratic form in $\mathbb{R}^n$. For any $\mathbf{v} \in \mathbb{R}^n$ let $T : \mathbb{R}^n \to \mathbb{R}$ be a transformation defined by

$$T(\mathbf{x}) = Q(\mathbf{x} + \mathbf{v}) - Q(\mathbf{x}) - Q(\mathbf{v}).$$

Show that the transformation T is linear.

Solution: If T is a matrix transformation, that is, if there exists an $n \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, then T is a linear transformation.

Let B be the matrix for the quadratic form Q. Then B is symmetric. We have

$$\begin{aligned} T(\mathbf{x}) &= Q(\mathbf{x} + \mathbf{v}) - Q(\mathbf{x}) - Q(\mathbf{v}) = (\mathbf{x} + \mathbf{v})^T B (\mathbf{x} + \mathbf{v}) - \mathbf{x}^T B\mathbf{x} - \mathbf{v}^T B\mathbf{v} \\ &= \mathbf{x}^T B\mathbf{x} + \mathbf{x}^T B\mathbf{v} + \mathbf{v}^T B\mathbf{x} + \mathbf{v}^T B\mathbf{v} - \mathbf{x}^T B\mathbf{x} - \mathbf{v}^T B\mathbf{v} = \mathbf{x}^T B\mathbf{v} + \mathbf{v}^T B\mathbf{x}. \end{aligned}$$

Since $\mathbf{x}^T B\mathbf{v}$ is of size 1×1 and B is symmetric, we have

$$\mathbf{x}^T B\mathbf{v} = \left(\mathbf{x}^T B\mathbf{v}\right)^T = \mathbf{v}^T B^T\mathbf{x} = \mathbf{v}^T B\mathbf{x},$$

so

$$T(\mathbf{x}) = \mathbf{x}^T B\mathbf{v} + \mathbf{v}^T B\mathbf{x} = \mathbf{v}^T B\mathbf{x} + \mathbf{v}^T B\mathbf{x} = \left(2\mathbf{v}^T B\right)\mathbf{x}.$$

Therefore, T is a matrix transformation associated to the matrix $2\mathbf{v}^T B$.$\blacksquare$

(5) Show that the diagonal entries of a positive definite matrix are positive.

Solution: Let Q be a quadratic form with the matrix $A = [a_{ij}]$. Then, by definition, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for all nonzero $\mathbf{x}$ in $\mathbb{R}^n$. In particular, let $\mathbf{x} = \mathbf{e}_k$, the kth unit vector in $\mathbb{R}^n$. Then

$$Q(\mathbf{e}_k) = \mathbf{e}_k{}^T A\mathbf{e}_k = \mathbf{e}_k \cdot (A\mathbf{e}_k) = a_{kk}.$$

Since Q is positive definite, $Q(\mathbf{e}_k) > 0$ for all k, implying that $a_{kk} > 0$ for all k.$\blacksquare$