SPRING 2024
DEPARTMENT OF COMPUTER ENGINEERING
MAT222
LINEAR ALGEBRA
MAKEUP EXAM SOLUTIONS

June 27, 2024

1) Are there invertible 3×3 matrices S and A such that $S^T AS = -A$? Explain your answer. **(20 p.)**

Solution: Suppose that there are invertible 3×3 matrices S and A such that $S^T AS = -A$. Then

$$\det(-A) = \det\left(S^T AS\right) = \det\left(S^T\right)\det(A)\det(S).$$

We have $\det\left(S^T\right) = \det(S)$ and since A is of size 3,

$$\det(-A) = (-1)^3 \det(A) = -\det(A).$$

Thus, above equality implies that

$$-\det(A) = (\det(S))^2 \det(A) \Rightarrow (\det(S))^2 = -1$$

since A is invertible. But this is impossible. Therefore, there do not exist such matrices.$\blacksquare$

(2) Find a basis of the space of all matrices that commute with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

if possible. **(20 p.)**

Solution: We need to find all matrices

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

satisfying

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

This equality gives that

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}.$$

So, $d = g = h = 0$, $a = e = i$, and $f = b$. Let V be the space consists of all matrices of the form B. Since

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the linearly independent matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

span V. Thus, the set

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is a basis of the space V.■

(3) Let P_2 be the vector space of all polynomials of degree at most 2. Find the dimension of the kernel of the operator $T : P_2 \to P_2$ defined by $T\left(p\left(t\right)\right) = \int\limits_0^2 p\left(t\right) \mathrm{d}t$. **(20 p.)**

Solution: Let $p\left(t\right) = a + bt + ct^2$, where a, b, and c are constants. Then

$$T\left(p\left(t\right)\right) = \int\limits_0^2 p\left(t\right) \mathrm{d}t = \int\limits_0^2 \left(a + bt + ct^2\right) \mathrm{d}t = \left(at + b\frac{t^2}{2} + c\frac{t^3}{3} \Big|_0^2 \right) = 2a + 2b + \frac{8}{3}c.$$

The associated matrix is then found to be

$$A = \begin{bmatrix} 2 & 2 & \frac{8}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 2 & 2 & \frac{8}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and hence $\ker\left(T\right)$ consists of the matrices of the form

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b - \frac{4}{3}c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis of $\ker\left(T\right)$ is

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \right\},$$

so $\dim\left(\ker\left(T\right)\right) = 2$.■

(4) Find a for which the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$$

is diagonalizable. **(20 p.)**

Solution: Since

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -a & -b \\ 0 & \lambda - 1 & -c \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2 (\lambda - 2),$$

the eigenvalues are $\lambda = 1$ and $\lambda = 2$. If $\lambda = 1$, then

$$\begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{E.R.O.}} \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, if $a = 0$, then the dimension of the kernel of the matrix is 2, otherwise it is 1. On the other hand, if $\lambda = 2$, then

$$\begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{E.R.O.}} \begin{bmatrix} 1 & 0 & -ac - b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix},$$

so the dimension of the matrix is 1. Therefore, we must have $a = 0$ for which A is diagonalizable.■

(5) Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and A is an $n \times n$ symmetric matrix. If $\mathbf{u}$ is a unit eigenvector of A, find $Q(\mathbf{u})$. **(20 p.)**

Solution: Let λ be the eigenvalue corresponding to $\mathbf{u}$. Then

$$Q(\mathbf{u}) = \mathbf{u}^T A\mathbf{u} = \mathbf{u} \cdot (A\mathbf{u}) = \mathbf{u} \cdot (\lambda\mathbf{u}) = \lambda (\mathbf{u} \cdot \mathbf{u}).$$

Since $\mathbf{u}$ is a unit vector, we have

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = 1,$$

so $Q(\mathbf{u}) = \lambda$.■