MAT222
LINEAR ALGEBRA
HOMEWORK ASSIGNMENT 4
SOLUTIONS

(1) If A is a 2×2 matrix with $\mathrm{tr}(A) = 5$ and $\det(A) = -14$, find the eigenvalues of A.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Since $\mathrm{tr}(A) = a + d = 5$ and $\det(A) = ad - bc = -14$, we obtain

$$\lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2) = 0,$$

which gives the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 7$.■

(2) Find the values of k for which the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k & 3 & 0 \end{bmatrix}$$

has three distinct real eigenvalues.

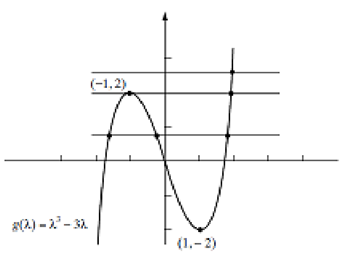
Solution: We have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -k & -3 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda - k,$$

so the eigenvalues of A are the solutions of the equation $\lambda^3 - 3\lambda - k = 0$, or $\lambda^3 - 3\lambda = k$. Let $g(\lambda) = \lambda^3 - 3\lambda$. Since

$$g'(\lambda) = 3\lambda^2 - 3 = 3(\lambda - 1)(\lambda + 1) = 0,$$

critical points are $(1, -2)$ and $(-1, 2)$. Since $g''(\lambda) = 6\lambda$ and $g''(1) > 0$, the point $(1, -2)$ is a local minimum, and since $g''(-1) < 0$, the point $(-1, 2)$ is a local maximum. In fact we have the following graph.



From the figure it is evident that the equation $\lambda^3 - 3\lambda = k$ has three solutions if $|k| < 2$, two solutions if $k = -2$ and $k = 2$, and one solution if $|k| > 2$. Thus, for three real eigenvalues, we must have $|k| < 2$.■

(3) Diagonalize the matrix

$$A = \begin{bmatrix} 0 & -3 & -3 \\ -3 & 0 & -3 \\ -3 & -3 & 0 \end{bmatrix}$$

if possible.

Solution: Since

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 3 & 3 \\ 3 & \lambda & 3 \\ 3 & 3 & \lambda \end{vmatrix} = (\lambda - 3)^2 (\lambda + 6),$$

the eigenvalues are $\lambda = 3$ and $\lambda = -6$.

For $\lambda = 3$, we have

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \underset{R_3 - R_1}{\overset{R_2 - R_1}{\longrightarrow}} \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{3}R_1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are the corresponding eigenvectors.

For $\lambda = -6$, we have

$$\begin{bmatrix} -6 & 3 & 3 \\ 3 & -6 & 3 \\ 3 & 3 & -6 \end{bmatrix} \underset{R_3 + \frac{1}{2}R_1}{\overset{R_2 + \frac{1}{2}R_1}{\longrightarrow}} \begin{bmatrix} -6 & 3 & 3 \\ 0 & -\frac{9}{2} & \frac{9}{2} \\ 0 & \frac{9}{2} & -\frac{9}{2} \end{bmatrix} \overset{R_3 + R_2}{\longrightarrow} \begin{bmatrix} -6 & 3 & 3 \\ 0 & -\frac{9}{2} & \frac{9}{2} \\ 0 & 0 & 0 \end{bmatrix} \underset{(-\frac{2}{9})R_2}{\overset{\frac{1}{3}R_1}{\longrightarrow}} \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 + \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is the corresponding eigenvector.

Now, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent eigenvectors, so we set the matrix

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since

$$\left[\begin{array}{ccc|ccc}-1 & -1 & 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc}-1 & -1 & 1 & 1 & 0 & 0\\ 0 & -1 & 2 & 1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right] \underset{R_3+R_2}{\xrightarrow{R_1-R_2}} \left[\begin{array}{ccc|ccc}-1 & 0 & -1 & 0 & -1 & 0\\ 0 & -1 & 2 & 1 & 1 & 0\\ 0 & 0 & 3 & 1 & 1 & 1\end{array}\right]$$

$$\overset{(-1)R_1}{\underset{\frac{1}{3}R_3}{\xrightarrow{(-1)R_2}}} \left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & -2 & -1 & -1 & 0\\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{array}\right] \underset{R_2+2R_3}{\xrightarrow{R_1-R_3}} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{array}\right],$$

we have

$$P^{-1} = \frac{1}{3}\begin{bmatrix}-1 & 2 & -1\\ -1 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}.$$

Thus, letting

$$D = \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -6\end{bmatrix},$$

we diagonalize the matrix A by $D = PAP^{-1}$.■

(4) Determine the value of k for which the matrix

$$A = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & k\\ 0 & 1 & 0\end{bmatrix}$$

is diagonalizable.

Solution: We first note that if the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable. Since

$$\det(\lambda I - A) = \begin{vmatrix}\lambda & 0 & 0\\ -1 & \lambda & -k\\ 0 & -1 & \lambda\end{vmatrix} = \lambda\left(\lambda^2 - k\right),$$

the characteristic equation is $\lambda\left(\lambda^2 - k\right) = 0$. If k is positive, then we have three distinct real eigenvalues, namely, 0, $\sqrt{k}$, and $-\sqrt{k}$, so the matrix A is diagonalizable. If k is negative or 0, then 0 is the only eigenvalue, and hence the matrix A is not diagonalizable.■

(5) If

$$A = \begin{bmatrix}0.5 & 0.75\\ 0.5 & 0.25\end{bmatrix},$$

find $\lim\limits_{k\to\infty} A^k$, where A^k is the kth power of A.

Solution: Since

$$\det(\lambda I - A) = \begin{vmatrix}\lambda - \frac{1}{2} & -\frac{3}{4}\\ -\frac{1}{2} & \lambda - \frac{1}{4}\end{vmatrix} = \lambda^2 - \frac{3}{4}\lambda - \frac{1}{4} = (\lambda - 1)\left(\lambda + \frac{1}{4}\right) = 0,$$

the eigenvalues of A are $\lambda_1 = -\frac{1}{4}$ and $\lambda_2 = 1$.

For $\lambda = -\frac{1}{4}$, we have

$$\lambda I - A = \begin{bmatrix}-\frac{3}{4} & -\frac{3}{4}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}.$$

Elementary row operation give

$$\begin{bmatrix} -\frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow[(-2)R_2]{\left(-\frac{4}{3}\right)R_1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = -x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and the corresponding eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda = 1$, we have

$$\lambda I - A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

Elementary row operation give

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 \\ x_2 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

and the corresponding eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Therefore, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}.$$

Since

$$P^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix},$$

we find, for any positive integer k, that

$$A^k = PD^kP^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{4}\right)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3+2\left(-\frac{1}{4}\right)^k & 3-3\left(-\frac{1}{4}\right)^k \\ 2-2\left(-\frac{1}{4}\right)^k & 2+3\left(-\frac{1}{4}\right)^k \end{bmatrix}.$$

Thus, we have $\lim\limits_{k\to\infty} A^k = \frac{1}{5} \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$. ■