$S = [0, 1]$ (0.2, 0.8) 0 smallest

$S = (0, 1]$ smallest? ?? ← this set does not have well-ordering property.

$\mathbb{R} = (-\infty, +\infty)$ doesn't have w-o principle

$\mathbb{R} = [0, 5] \to (3, 4] \subset [0, 5]$ no well-ordering!

set

subset

- A set S is said to have well-ordering if $\forall C \subseteq S$ C has a smallest element.

Let's think about integers $\mathbb{Z}$

Every subset of an integer set has a smallest element.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \leftarrow \boxed{\forall n \geq 0}$$

$n=0$
$n=1$
$n=2$
$\vdots$
∞

Ex.

Mathematical induction

$S(3) = 1+2+3 = \frac{3 \cdot 4}{2}$

$S(2) = 1+2 = \frac{2 \cdot 3}{2}$

$S(1): 1 = \frac{1 \cdot 2}{2}$ ✓

M. Ponens.

P

$P \to q$

$\therefore q$

$S(1), S(2), \dots$

open st.

$S(x): \sum_{i=1}^{x} i = \frac{x(x+1)}{2} \quad x \in \mathbb{Z}^+$

smallest element in the universal set of x.

$S(1) = \frac{1 \cdot 2}{2} = 1$

1. $S(1)$ ✓

2. $S(k) \to S(k+1)$ $\Big|_{k=1}$ $\Rightarrow S(1) \to S(2)$ ✓

$k=2$ $\quad S(2) \to S(3)$

...

M Ponens. $S(2)$

M Pon $S(3)$

Q. Prove that $\forall n \in \mathbb{Z}^+, \sum_{i=1}^{n} i \overset{?}{=} \frac{n(n+1)}{2}$

Proof. We make math. induction on n as follows. Let $S(k)$ means $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Now, we need to show

1. $S(1)$
2. $\forall k \in \mathbb{Z}^+ \; S(k) \to S(k+1)$

1. $S(1)$: $\sum_{i=1}^{1} i = 1 \overset{\checkmark}{=} \frac{1 \cdot 2}{2} = 1$

comes from defn

comes from the theorem

Logic

$\forall x \; p(x) \to q(x)$

1. $p(x)$ arb. x Assume
2. $\vdots$
3. $q(x)$
4. $\forall x \; p(x) \to q(x)$

2. Assume that $S(k)$ is true i.e.

$\sum_{i=1}^{k} i = 1+2+\cdots+k = \frac{k(k+1)}{2}$ (Assumed)

$\frac{k(k+1)}{2} + k+1 \overset{?}{=}$

$\sum_{i=1}^{k+1} i = 1+2+\cdots+k + k+1 = \frac{(k+1)(k+2)}{2}$

$\frac{\cancel{k^2+k}}{2} + \frac{\cancel{2k+2}}{2} \overset{?}{=} \frac{\cancel{k^2+3k+2}}{2}$ $\square$

Therefore our ind. proof is complete. QED.

Cleaner Proof.

We make math. induction on n

from defn.

from thm.

show • Base step $S(1)$: $\sum_{i=1}^{1} i = 1 = \frac{1 \cdot 2}{2}$ ✓

$S(n)\big|_{n=k}$ Assume

assume • Inductive hypothesis $S(k)$: Assume that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ ✓

show • Inductive step: $S(k+1)$. $\sum_{i=1}^{k+1} i = 1+2+\cdots k-1 + k + k+1 = \frac{k(k+1)}{2} + k+1$

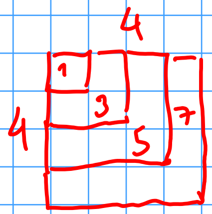
$S(n)\big|_{n=k+1}$ proof

comes from the theorem

$= \frac{k^2+k+2k+2}{2}$

$= \frac{(k+1)(k+2)}{2} = \frac{n(n+1)}{2}\Big|_{n=k+1}$ ✓ $\square$

calculated from the formula & inductive hypothesis



$\sum_{i=1}^{4}(2i-1)=16$

$1+3+5+7=16$

$\forall n \in \mathbb{Z}^+$ $\quad 1+3+5+\cdots+(2n-1)=n^2$

$\sum_{i=1}^{n}(2i-1)=n^2$

form $\quad$ then

- Base step • $1 = 1^2$ ✓
- Ind Hyp • Assume $\sum_{i=1}^{k}(2i-1)=k^2$
- Ind Step • when $n=k+1$,

$2(k+1)-1=2k+1$

$$\sum_{i=1}^{k+1}(2i-1) = \boxed{\underbrace{1+3+5+\cdots+2k-1}_{k^2}} + 2k+1 \stackrel{?}{=} (k+1)^2$$

then

$k^2+2k+1=(k+1)^2$ ✓

Thm. $\forall n \in \mathbb{Z}^+, n \geq 4$: $2^n < n!$ $2^4 < 4!$ ($16 < 24$) $2^5 < 5!$ ($32 < 120$) . . . .

Proof. We make ind. on n:

here we assumed our thm is correct for $n=k$

B.S: For $n=4$: $2^4 = 16 < 4! = 24$

I.H: Assume that $\forall k \geq 4$ $2^k < k!$

uses

So we showed that our thm also holds for $n = k+1$

$= k!(k+1)$

I.S. $2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < (k+1)! \Rightarrow 2^{k+1} < (k+1)! \Leftrightarrow 2^n < n! \Big|_{n=k+1}$ ✓

by defn. / from assumption / Since $k+1 > 2$

$S(k) \longrightarrow S(k+1)$

$\forall n \in \mathbb{Z}^+ \; n \geq 14: \quad \exists i, j \in \mathbb{N} \quad n = 3i + 8j$

$0, 1, 2, \ldots$

$14 = 3.2 + 8.1$

This is what our theorem tells us

$i=3 \quad j=1$

$17 = 3.\underline{3} + 8.\underline{1}$ ✓

$i=5 \quad j=0$

$15 = 3.5 + 8.0$ ✓

$218 = 3.\_\_ + 8.\_\_$ ✓

$15 = 3.5 + 8.0$

$16 = 3.8 + 8(-1)$

Proof: We make ind on n:

B.S. for $n = 14$ we can set $i=2$ $j=1$ and obtain $3.2 + 8.1 = 14$ ✓

I.H. Assume that $k \geq 14$ can be written as $3.a + 8.b$ where $a, b \in \mathbb{N}$.

[Imp. point: a, b are particular elements of $\mathbb{N}$ depending on k]

I.S.: Now we can write $k+1 = 3(a+3) + 8(b-1)$. (1)

or $k+1 = 3(a-5) + 8(b+2)$. (2)

To be unable to use both $a < 5$ and $b < 1$ but in this case $3.4 + 8.0 = 12$ is the largest k we can get, however we know that $k \geq 14$ so, it is guaranteed that we can always use either (1) or (2). Hence we're done QED.

$14 = 3.2 + 8.1$

$+3 \quad -1$

$15 = 3.5 + 8.0$

$-5 \quad +2$

$16 = 3.0 + 8.2$

$+3$

$17 = 3.3 + 8.1$

$18 = 3.6 + 8.0$

$19 = 3.1 + 8.2$

$20 = 3.4 + 8.1$

$21 = 3.7 + 8.0$

$22 = 3.2 + 8.2$

$\vdots$

$110 = 3.10 + 8.10$