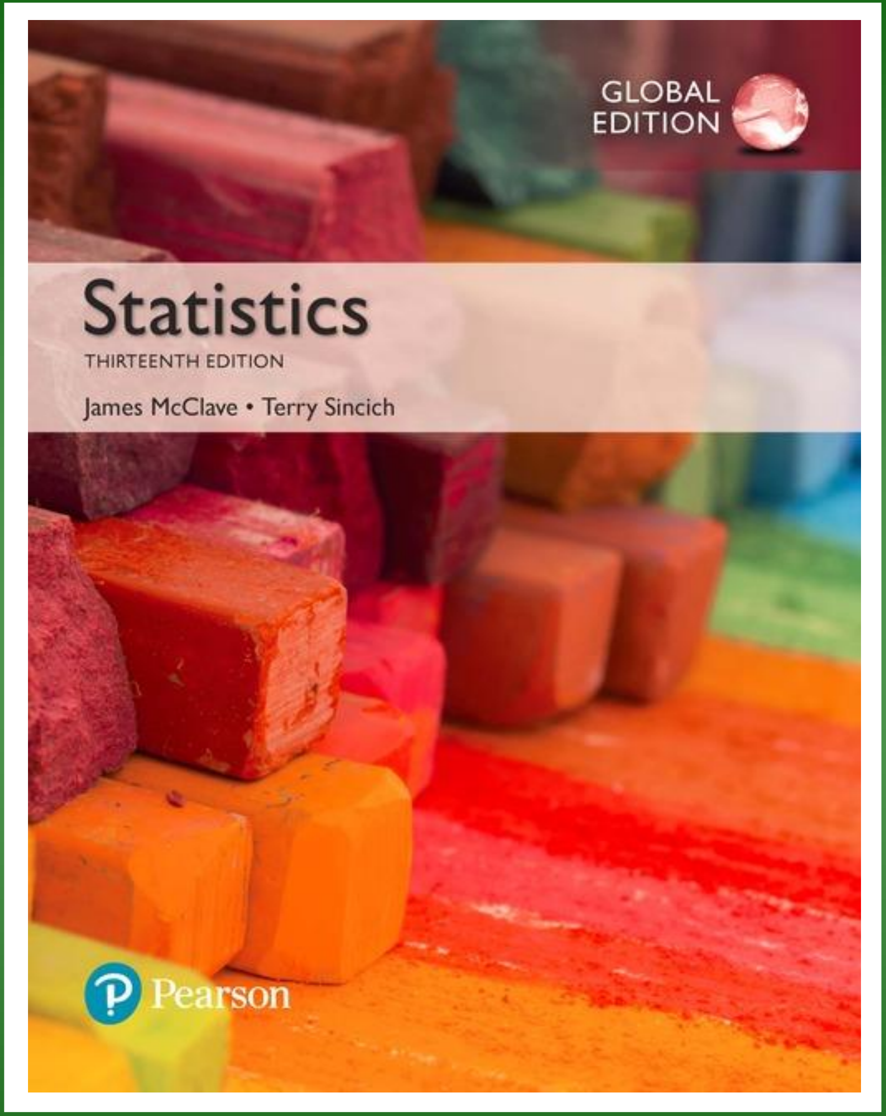



# Chapter 3

## Probability

# 3.1

## Events, Sample Spaces, and Probability

# Definition

- Suppose a coin is tossed once and the up face is recorded.
  - Result is an **observation**, or **measurement**,
  - Process of making an observation is **experiment.**
- Experiment examples:
  - recording an Internet user’s preference for a Web browser
  - recording a voter’s opinion on an important political issue
  - observing the level of anxiety of a test taker
  - counting the number of errors in an inventory
- ✓ The point is that a statistical experiment can be almost any act of observation, **as long as the outcome is uncertain**.

> An **experiment** is an act or process of observation that leads to a single outcome that cannot be predicted with certainty.

# Definition

- Consider tossing a die with six posible outcomes
  - Observe **1**, **2**, **3**, **4**, **5**, or **6**.
- If this experiment is conducted once, you can observe one and only one of these six basic outcomes, and the outcome cannot be predicted with certainty.
- Outcomes cannot be decomposed into more basic ones.
- Because observing the outcome of an experiment is similar to selecting a sample from a population, the basic possible outcomes of an experiment are called **sample points** (**simple event**)

A **sample point** is the most basic outcome of an experiment.

# Problem

- Two coins are tossed, and their up faces are recorded. List all the sample points for this experiment.
- Solution Even for a seemingly trivial experiment, we must be careful when listing the sample points. At first glance, we might expect one of three basic outcomes:
  - Observe two heads
  - Observe two tails
  - Observe one head and one tail.

# Solution

- However, further reflection reveals that the last of these, Observe one head and one tail, can be decomposed into two outcomes:
  - Head on coin 1, Tail on coin 2
  - Tail on coin 1, Head on coin 2

# **Figure 3.1** Tree diagram for the coin-tossing experiment

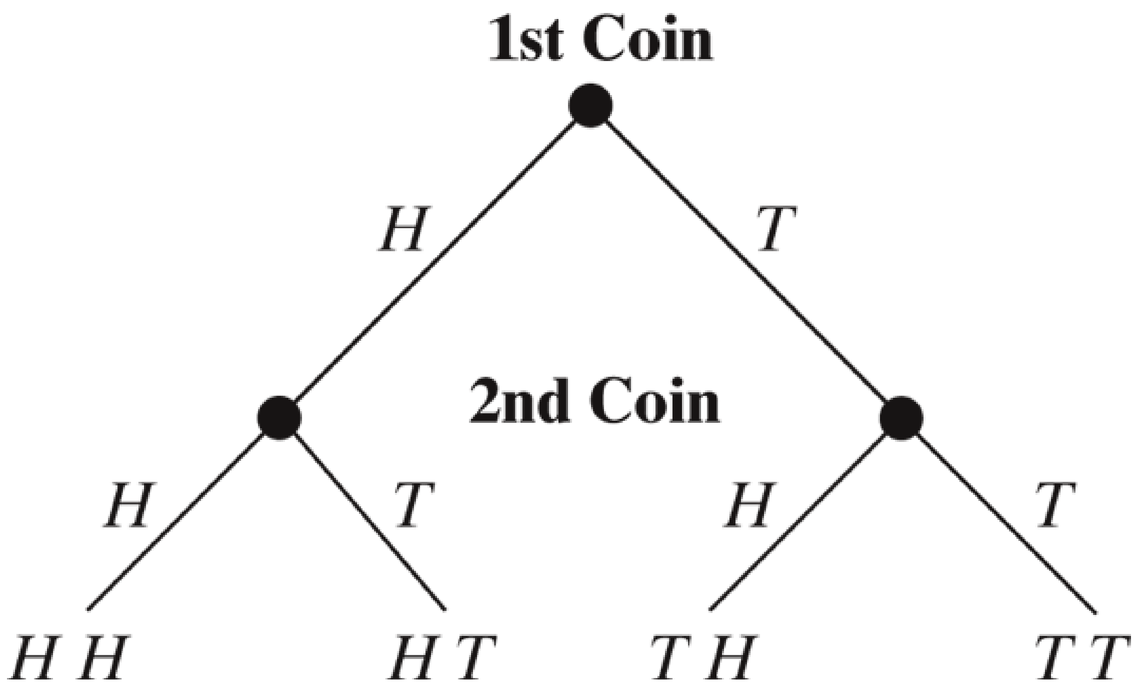
We have four sample points:

**1.** Observe ***HH***

**2.** Observe ***HT***

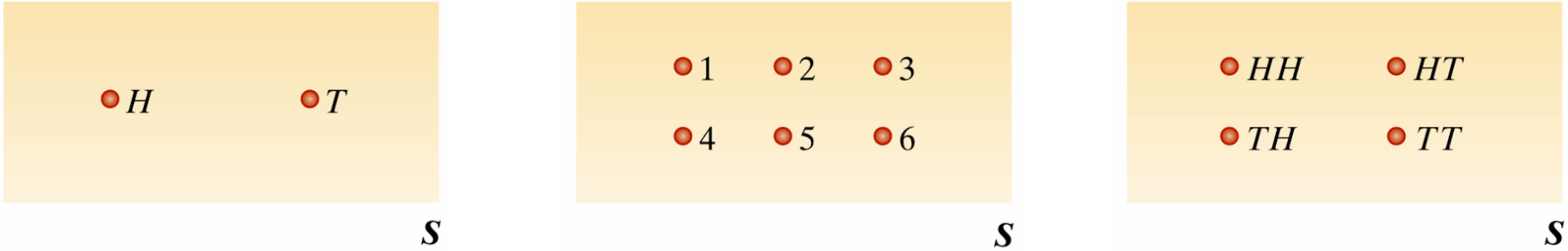
**3.** Observe ***TH***

**4.** Observe ***TT***



# **Figure 3.2** Venn diagrams for the three experiments from Table 3.1

- We often wish to refer to the collection of **all the sample points** of an experiment.
- This collection is called the **sample space** of the experiment.
- The sample spaces for the experiments discussed thus far are shown below.



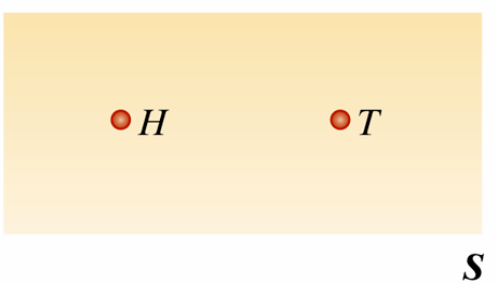

a. Experiment: Observe the up face on a coin

b. Experiment: Observe the up face on a die

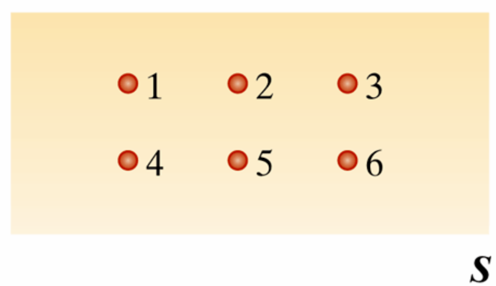
c. Experiment: Observe the up faces on two coins

# Definition

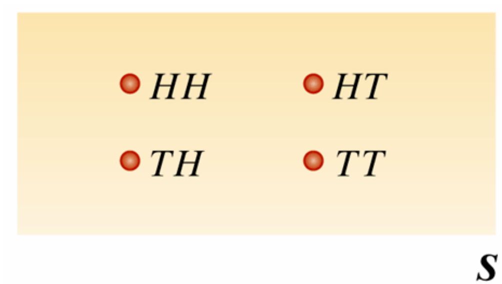
The **sample space** of an experiment is the collection of all its sample points.




a. Experiment: Observe the up face on a coin




b. Experiment: Observe the up face on a die




c. Experiment: Observe the up faces on two coins

# Table 3.1

The ***probability*** of a sample point is a number between **0 and 1** that measures the likelihood that the outcome will occur when the experiment is performed.

**Table 3.1 Experiments and Their Sample Spaces**

| | |
|---|---|
| *Experiment:* | Observe the up face on a coin. |
| *Sample Space:* | 1. Observe a head.<br>2. Observe a tail. |

This sample space can be represented in set notation as a set containing two sample points:

$$S: \{H, T\}$$

Here, $H$ represents the sample point Observe a head and $T$ represents the sample point Observe a tail.

| | |
|---|---|
| *Experiment:* | Observe the up face on a die. |
| *Sample Space:* | 1. Observe a 1.<br>2. Observe a 2.<br>3. Observe a 3.<br>4. Observe a 4.<br>5. Observe a 5.<br>6. Observe a 6. |

This sample space can be represented in set notation as a set of six sample points:

$$S: \{1, 2, 3, 4, 5, 6\}$$

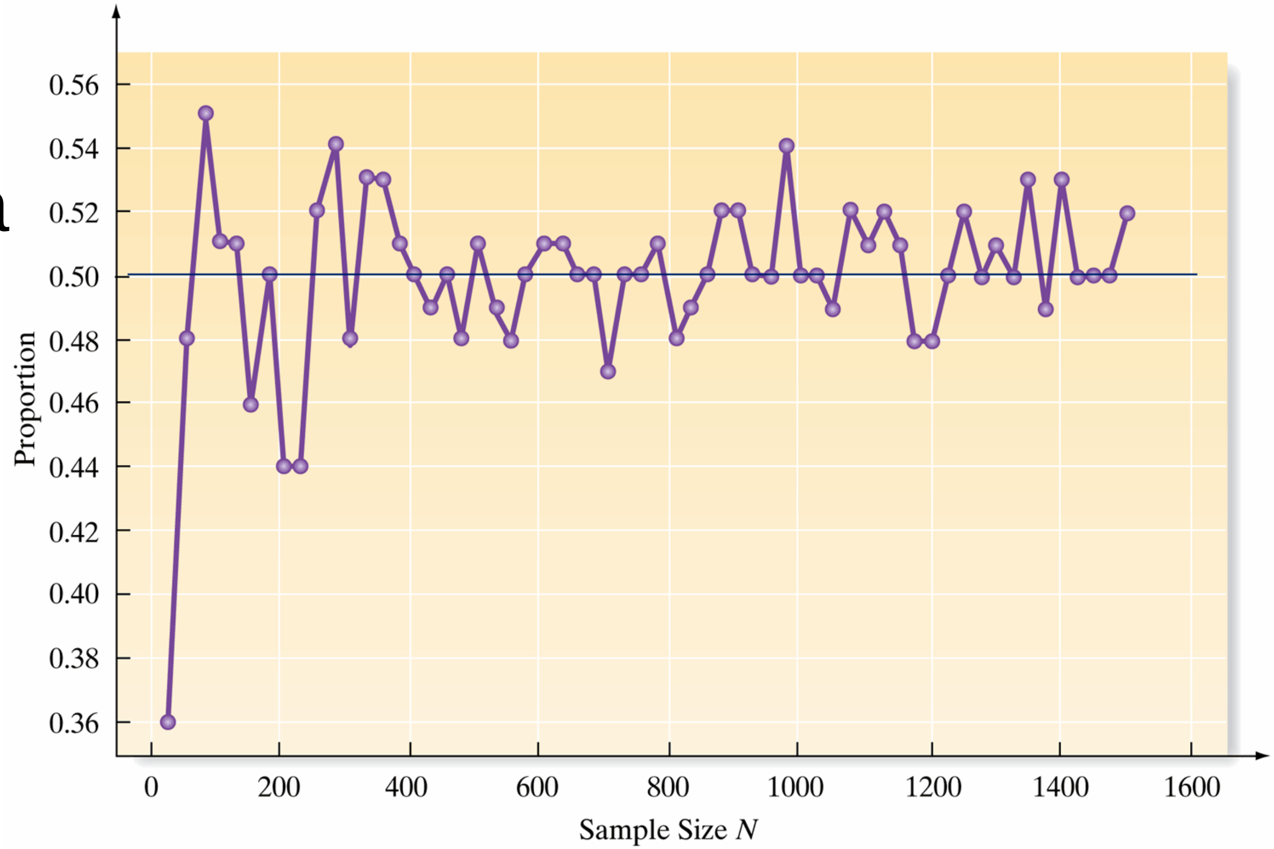
| | |
|---|---|
| *Experiment:* | Observe the up faces on two coins. |
| *Sample Space:* | 1. Observe *HH*.<br>2. Observe *HT*.<br>3. Observe *TH*.<br>4. Observe *TT*. |

This sample space can be represented in set notation as a set of four sample points:

$$S: \{HH, HT, TH, TT\}$$

# **Figure 3.3** Proportion of heads in *N* tosses of a coin

This probability is usually taken to be the *relative frequency* of the occurrence of a sample point in a *very long* series of repetitions of an experiment.



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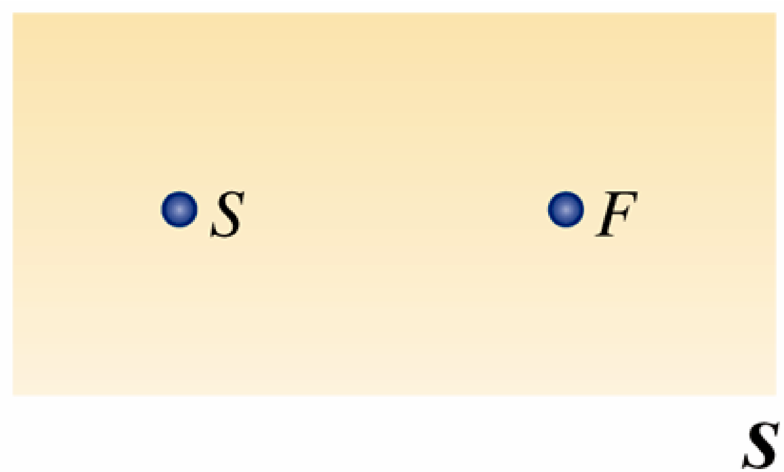
# **Figure 3.4** Experiment: invest in a business venture and observe whether it succeeds (S) or fails (F)

For some experiments, we may have little or no information on the relative frequency of occurrence of the sample points; consequently, we must assign probabilities to the sample points on the basis of general information about the experiment.

This probability can be interpreted as a measure of our degree of belief in the outcome of the business venture; it is a **subjective probability**.



# Procedure

No matter how you assign the probabilities to sample points, the probabilities assigned must obey two rules:

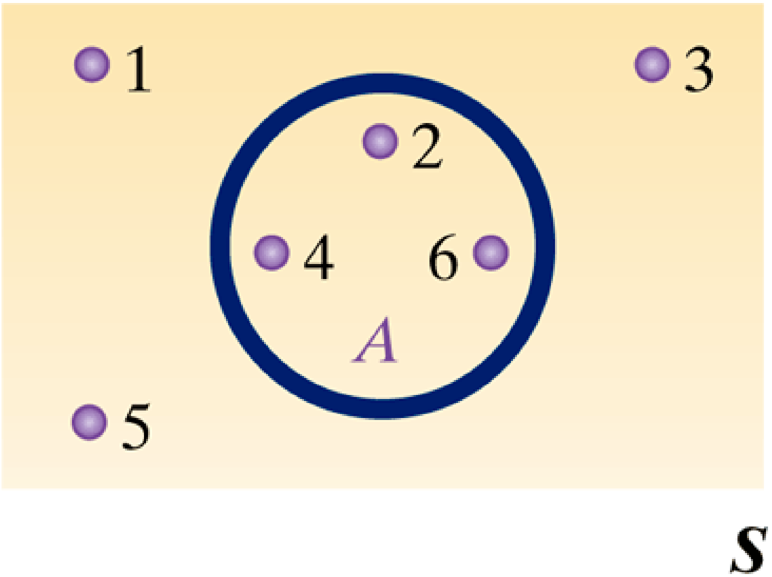
**Probability Rules for Sample Points**

Let $p_i$ represent the probability of sample point $i$. Then

1. All sample point probabilities *must* lie between 0 and 1 (i.e., $0 \leq p_i \leq 1$).
2. The probabilities of all the sample points within a sample space *must* sum to 1 (i.e., $\Sigma p_i = 1$).

## **Figure 3.5** Die-toss experiment with event A, observe an even number

Problem A fair die is tossed, and the up face is observed. If the face is even, you win $1. Otherwise, you lose $1. What is the probability that you win?



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# Solution

- Recall that the sample space for this experiment contains six sample points: $S = \{1, 2, 3, 4, 5, 6\}$
- Since the die is balanced, we assign a probability of $1/6$ to each of the sample points in this sample space. An even number will occur if one of the sample points Observe a 2, Observe a 4, or Observe a 6 occurs.
- A collection of sample points such as this is called **an event**, which we denote by the letter A. Since the event A contains three sample points—all with probability $1/6$—and since no sample points can occur simultaneously, we reason that the probability of A is the sum of the probabilities of the sample points in A. Thus, the probability of A is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1/2$

# Definition

- To summarize, we have demonstrated that an event can be defined in words, or it can be defined as a specific set of sample points.
- This leads us to the following general definition of an event:

An **event** is a specific collection of sample points.

# Problem

- Consider the experiment of tossing two unbalanced coins. Because the coins are not balanced, their outcomes (H or T) are not equiprobable. Suppose the correct probabilities associated with the sample points are given in the accompanying table. [Note: The necessary properties for assigning probabilities to sample points are satisfied.]
- Consider the events
  - A: {Observe exactly one head.}
  - B: {Observe at least one head.}
- Calculate the probability of A and the probability of B.

| Sample Point | Probability |
|---|---|
| $HH$ | $\frac{4}{9}$ |
| $HT$ | $\frac{2}{9}$ |
| $TH$ | $\frac{2}{9}$ |
| $TT$ | $\frac{1}{9}$ |

# Solution

- Event *A* contains the sample points *HT* and *TH*. Since two or more sample points cannot occur at the same time, we can easily calculate the probability of event *A* by summing the probabilities of the two sample points. Thus, the probability of observing exactly one head (event *A*), denoted by the symbol *P*(*A*), is

$$P(A) = P(\text{Observe } HT.) + P(\text{Observe } TH.) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

- Similarly, since B contains the sample points HH, HT, and TH, it follows that

$$P(B) = \frac{4}{9} + \frac{2}{9} + \frac{2}{9} = \frac{8}{9}$$

# Definition

**Probability of an Event**

The probability of an event $A$ is calculated by summing the probabilities of the sample points in the sample space for $A$.

# Procedure

- Thus, we can summarize the steps for calculating the probability of any event, as indicated in the next box.

**Steps for Calculating Probabilities of Events**

1. Define the experiment; that is, describe the process used to make an observation and the type of observation that will be recorded.
2. List the sample points.
3. Assign probabilities to the sample points.
4. Determine the collection of sample points contained in the event of interest.
5. Sum the sample point probabilities to get the probability of the event.

# Problem

The American Association for Marriage and Family Therapy (AAMFT) is a group of professional therapists and family practitioners that treats many of the nation's couples and families. The AAMFT released the findings of a study that tracked the postdivorce history of 100 pairs of former spouses with children. Each divorced couple was classified into one of four groups, nicknamed "**perfect pals (PP)**," "**cooperative colleagues (CC)**," "**angry associates (AA)**," and "**fiery foes (FF)**." The proportions classified into each group are shown in Table 3.2.

# Table 3.2

Suppose one of the 100 couples is selected at random.

a) Define the experiment that generated the data in Table 3.2 and list the sample points.

b) Assign probabilities to the sample points.

c) What is the probability that the former spouses are "fiery foes"?

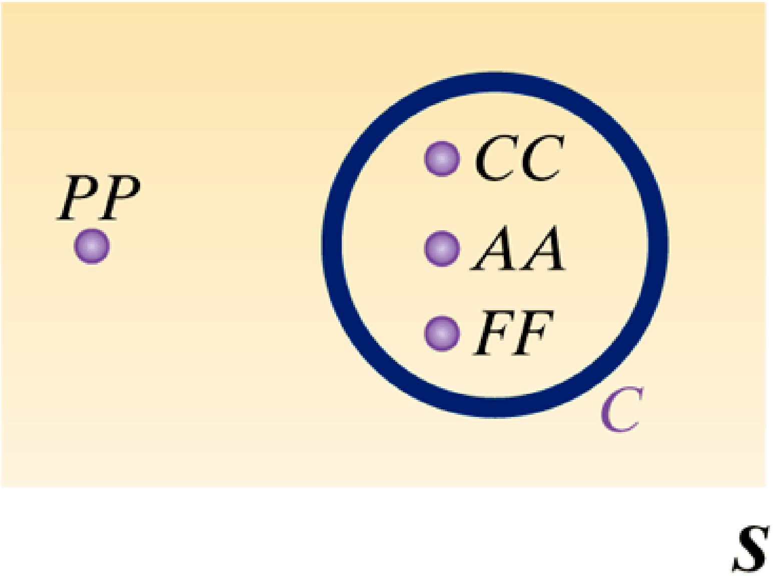
d) What is the probability that the former spouses have at least some conflict in their relationship?

**Table 3.2 Results of AAMFT Study of Divorced Couples**

| Group | Proportion |
|---|---|
| *Perfect Pals (PP)*<br>(Joint-custody parents who get along well) | .12 |
| *Cooperative Colleagues (CC)*<br>(Occasional conflict, likely to be remarried) | .38 |
| *Angry Associates (AA)*<br>(Cooperate on children-related issues only, conflicting otherwise) | .25 |
| *Fiery Foes (FF)*<br>(Communicate only through children, hostile toward each other) | .25 |

# Solution

a) **Define the experiment.** The experiment is the act of classifying the randomly selected couple. The sample points—the simplest outcomes of the experiment—are the four groups (categories) listed in Table 3.2.
They are shown in the Venn diagram.



# Solution

b) **Assign probabilities to the sample points.** If, as in Example 3.1, we were to assign equal probabilities in this case, each of the response categories would have a probability of one-fourth (1/4), or .25. But by examining Table 3.2, you can see that equal probabilities are not reasonable in this case because the response percentages are not all the same in the four categories. It is more reasonable to assign a probability equal to the response proportion in each class, as shown in Table 3.3.

# Table 3.3

**Table 3.3 Sample Point Probabilities for AAMFT Survey**

| Sample Point | Probability |
|---|---|
| PP | .12 |
| CC | .38 |
| AA | .25 |
| FF | .25 |

# Solution

c) **What is the probability that the former spouses are "fiery foes"?** The event that the former spouses are "fiery foes" corresponds to the sample point FF. Consequently, the probability of the event is the probability of the sample point. From Table 3.3, we find that P(FF) = .25. Therefore, there is a .25 probability (or one-fourth chance) that the couple we select are "fiery foes."

# Solution

c) **What is the probability that the former spouses have at least some conflict in their relationship?** The event that the former spouses have at least some conflict in their relationship, call it event C, is not a sample point because it consists of more than one of the response classifications (the sample points). In fact, as shown in Figure 3.6, the event C consists of three sample points: CC, AA, and FF. The probability of C is defined to be the sum of the probabilities of the sample points in C:

$$P(C) = P(\text{CC}) + P(\text{AA}) + P(\text{FF}) = .38 + .25 + .25 = .88$$

# What if the sample points run into thousands or millions?

- Suppose you wish to select 5 marines for a dangerous mission from a division of 1,000.
- Let's develop a system for counting the number of ways to select 2 marines from a total of 4.
- If the marines are represented by the symbols $M_1$, $M_2$, $M_3$, and $M_4$:

| | | |
|---|---|---|
| $(M_1, M_2)$ | $(M_2, M_3)$ | $(M_3, M_4)$ |
| $(M_1, M_3)$ | $(M_2, M_4)$ | |
| $(M_1, M_4)$ | | |

# Procedure

**Combinations Rule**

Suppose a sample of $n$ elements is to be drawn without replacement from a set of $N$ elements. Then the number of different samples possible is denoted by $\binom{N}{n}$ and is equal to

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

and similarly for $N!$ and $(N-n)!$ For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. [*Note:* The quantity $0!$ is defined to be equal to 1.]

# Problem

- Suppose a movie reviewer for a newspaper reviews 5 movies each month. This month, the reviewer has 20 new movies from which to make the selection. How many different samples of 5 movies can be selected from the 20?
- Solution For this example, $N = 20$ and $n = 5$.

$$\binom{20}{5} = \frac{20!}{5!(20-5)!} = \frac{20!}{5!15!}$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(15 \cdot 14 \cdot 13 \cdot \cdots \cdot 3 \cdot 2 \cdot 1)} = 15{,}504$$

# 3.2

## Unions and Intersections

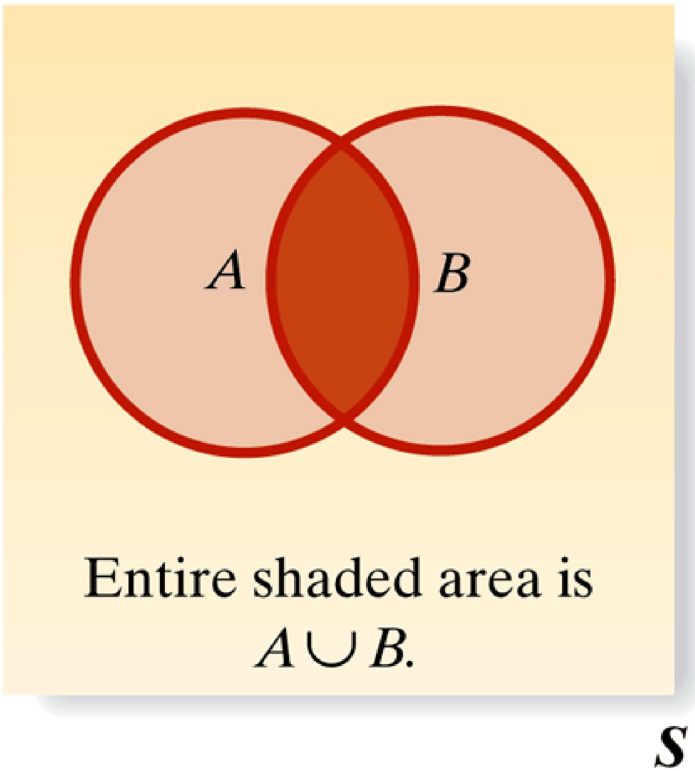
# Definition

- An event can often be viewed as a composition of two or more other events.
- Such events, which are called **compound events**, can be formed (composed) in two ways.

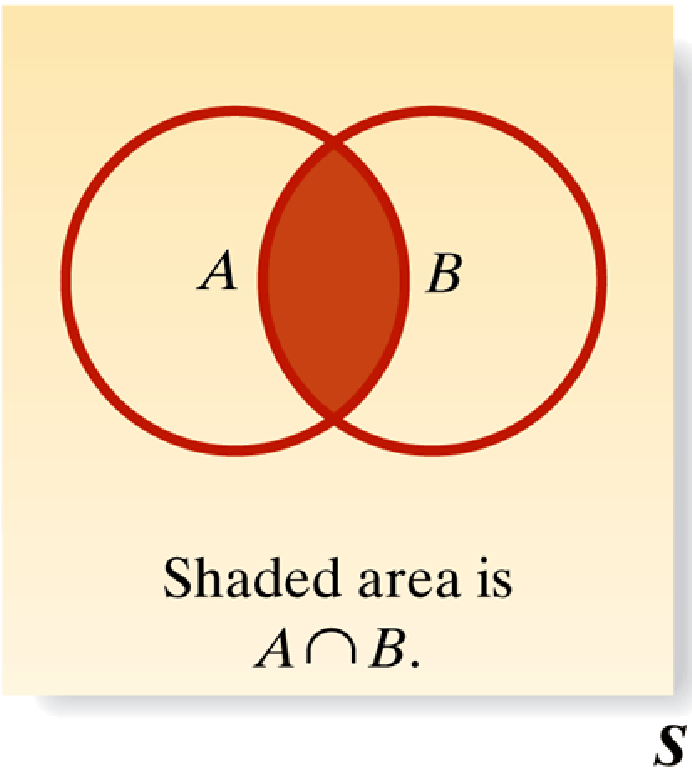
The **union** of two events $A$ and $B$ is the event that occurs if either $A$ or $B$ (or both) occurs on a single performance of the experiment. We denote the union of events $A$ and $B$ by the symbol $A \cup B$. $A \cup B$ consists of all the sample points that belong to *$A$ or $B$ or both*. (See Figure 3.7a.)

The **intersection** of two events $A$ and $B$ is the event that occurs if both $A$ and $B$ occur on a single performance of the experiment. We write $A \cap B$ for the intersection of $A$ and $B$. $A \cap B$ consists of all the sample points belonging to *both $A$ and $B$*. (See Figure 3.7b.)

# Figure 3.7 Venn diagrams for union and intersection




a. Union




b. Intersection

# Problem

- Consider a die-toss experiment in which the following events are defined:
  - $A$: {Toss an even number.}
  - $B$: {Toss a number less than or equal to 3.}

a) Describe $A \cup B$ for this experiment.

b) Describe $A \cap B$ for this experiment.

c) Calculate $\mathrm{P}(A \cup B)$ and $\mathrm{P}(A \cap B)$, assuming that the die is fair.

# **Solution - Figure 3.8** Venn diagram for die toss

Draw the Venn diagram

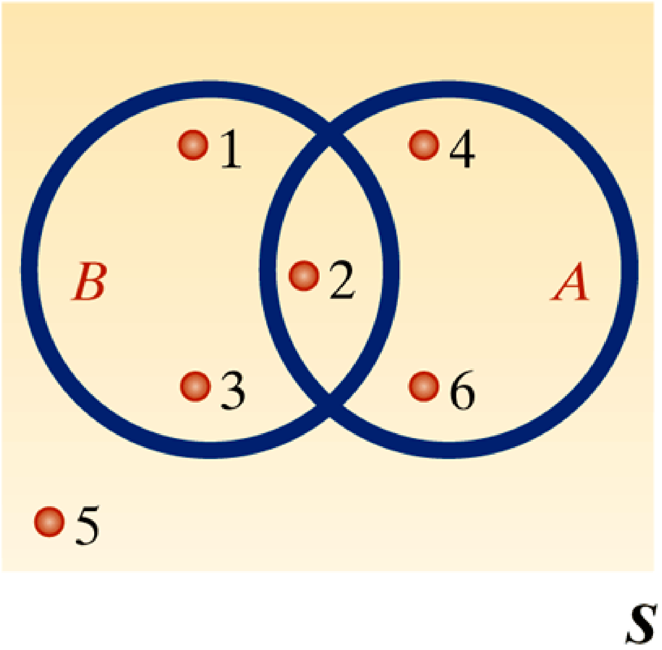
*a)* $A \cup B = \{1, 2, 3, 4, 6\}$

*b)* $A \cap B = \{2\}$

*c)* $P(A \cup B) = P(1) + P(2) + P(3) + P(4) + P(6) =$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$$

- $P(A \cap B) = P(2) = 1/6$



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# Problem

*Family Planning Perspectives* reported on a study of over 200,000 births in New Jersey over a recent two-year period. The study investigated the link between the mother's race and the age at which she gave birth (called maternal age). The percentages of the total number of births in New Jersey, by the maternal age and race classifications, are given in Table 3.4.

# Table 3.4

**Table 3.4 Percentage of New Jersey Birth Mothers, by Age and Race**

| | Race | |
|---|---|---|
| Maternal Age (years) | White | Black |
| $\leq 17$ | 2% | 2% |
| 18–19 | 3% | 2% |
| 20–29 | 41% | 12% |
| $\geq 30$ | 33% | 5% |

- $A$: {A New Jersey birth mother is white.}
- $B$: {A New Jersey mother was a teenager when giving birth.}

a) Find $P(A)$ and $P(B)$.

b) Find $P(A \cup B)$.

c) Find $P(A \cap B)$.



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# Solution

- The sample points are the eight different age–race classifications:

$E_1$: {≤17 yrs., white}
$E_2$: {18–19 yrs., white}
$E_3$: {20–29 yrs., white}
$E_4$: {≥30 yrs., white}
$E_5$: {≤17 yrs., black}
$E_6$: {18–19 yrs., black}
$E_7$: {20–29 yrs., black}
$E_8$: {≥30 yrs., black}

- Next, assign probabilities.

$$P(E_1) = .02 \qquad P(E_5) = .02$$
$$P(E_2) = .03 \qquad P(E_6) = .02$$
$$P(E_3) = .41 \qquad P(E_7) = .12$$
$$P(E_4) = .33 \qquad P(E_8) = .05$$

# Solution

a)

$$P(A) = P(E_1) + P(E_2) + P(E_3) + P(E_4) = .02 + .03 + .41 + .33 = .79$$
$$P(B) = P(E_1) + P(E_2) + P(E_5) + P(E_6) = .02 + .03 + .02 + .02 = .09$$

b) $P(A \cup B) = .02 + .03 + .41 + .33 + .02 + .02 = .83$

c) $P(A \cap B) = .02 + .03 = .05$

# 3.3

## Complementary Events

# Definition

- A very useful concept in the calculation of event probabilities is the notion of **complementary events**:

The **complement** of an event $A$ is the event that $A$ does *not* occur—that is, the event consisting of all sample points that are not in event $A$. We denote the complement of $A$ by $A^c$.



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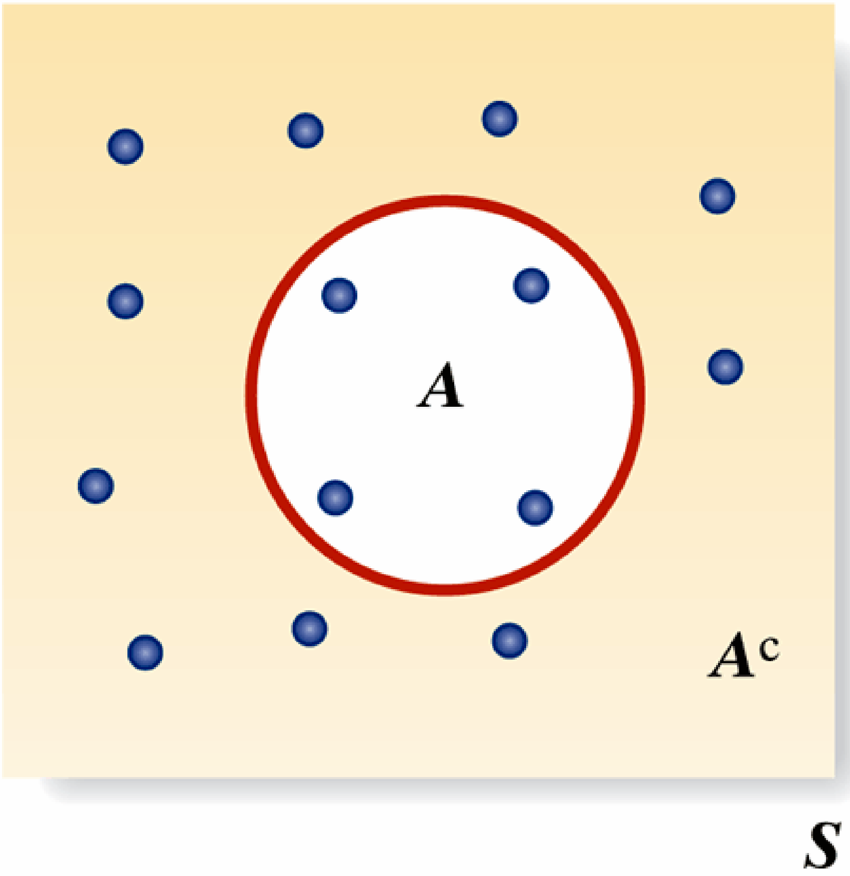
# **Figure 3.9** Venn diagram of complementary events

- An event $A$ is a collection of sample points, and the sample points included in $A^C$ are those not in $A$.
- Note from the figure that all sample points in $S$ are included in either $A$ or $A^C$ and that no sample point is in both $A$ and $A^C$.



# Procedure

- This leads us to conclude that the probabilities of an event and its complement *must sum to* 1:

**Rule of Complements**

The sum of the probabilities of complementary events equals 1; that is,

$$P(A) + P(A^c) = 1.$$

- In many probability problems, calculating the probability of the complement of the event of interest is easier than calculating the event itself.

# Problem

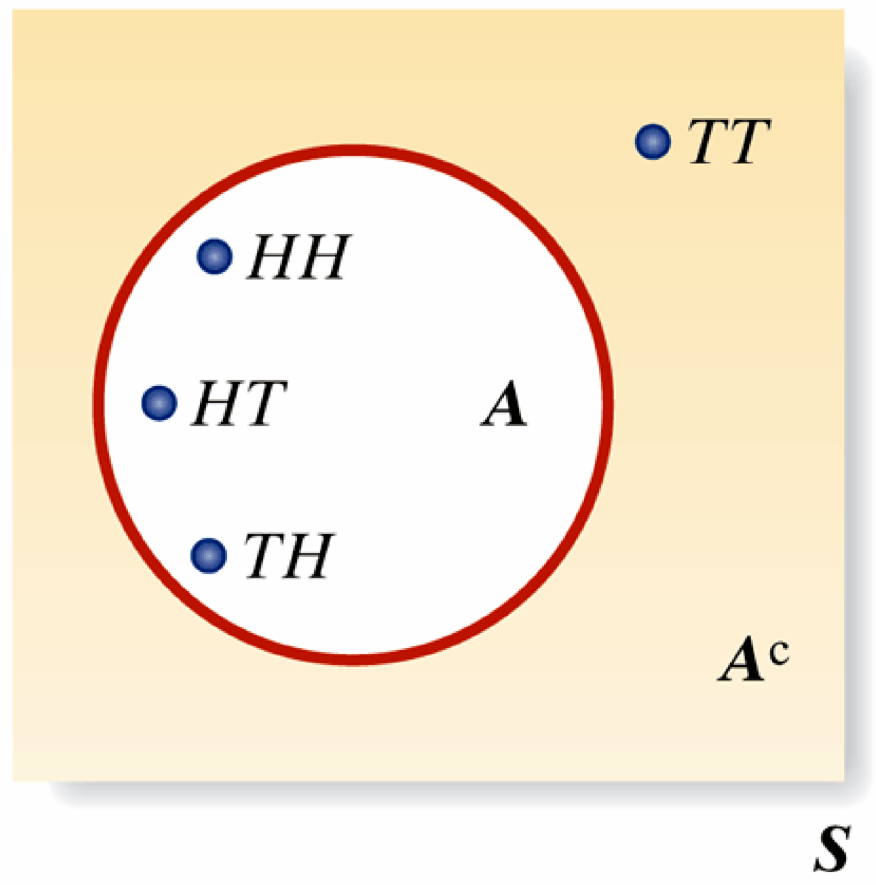
Consider the experiment of tossing fair coins. Define the following event:

- $A$: {Observing at least one head}.

a) Find $P(A)$ if 2 coins are tossed.

b) Find $P(A)$ if 10 coins are tossed.

# **Solution - Figure 3.10** Complementary events in the toss of two coins





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# Solution

a) When 2 coins are tossed, we know that the event $A$: {Observe at least one head.} consists of the sample points.

- $A: \{HH, HT, TH\}$ then $A^C: \{TT\}$
- $P(A^C) = P(TT) = \frac{1}{4}$ then $P(A) = 1 - P(A^C) = \frac{3}{4}$

*b)* $2^{10} = 1024$ sample points

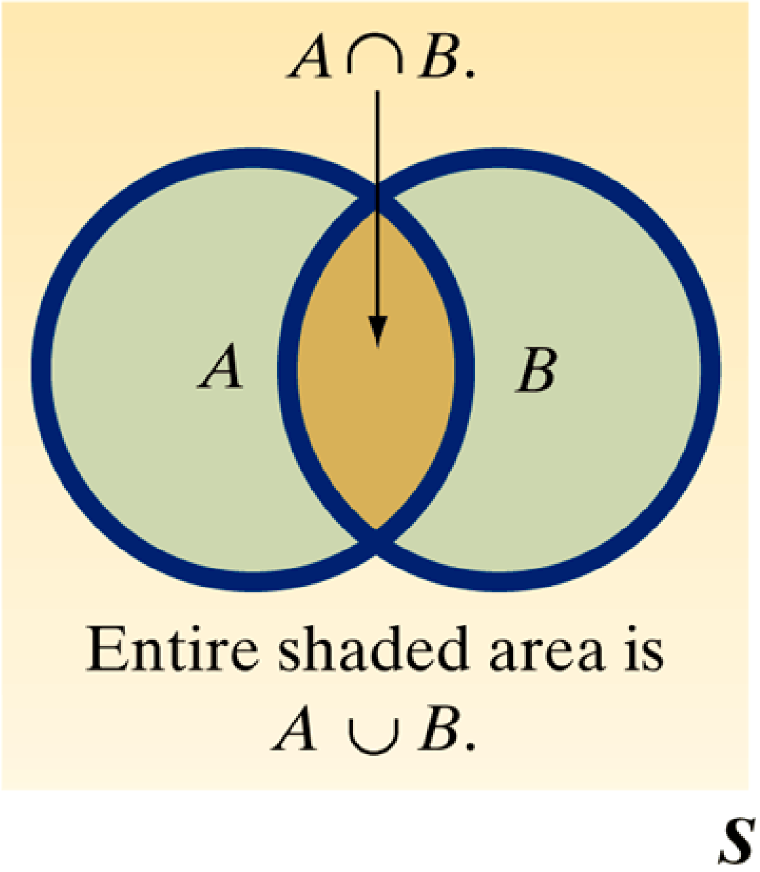
- $P(each\ sample) = 1/1024$
- $A^C = \{TTTTTTTTTT\}$, then $P(A^C) = 1/1024$
- $P(A) = 1 - \frac{1}{1024} = .999$

# 3.4

## The Additive Rule and Mutually Exclusive Events

# **Figure 3.11** Venn diagram of union

- It is also possible to obtain the probability of the union of two events by using the **additive rule of probability**.



# Procedure

- The formula for calculating the probability of the union of two events is given in the next box.

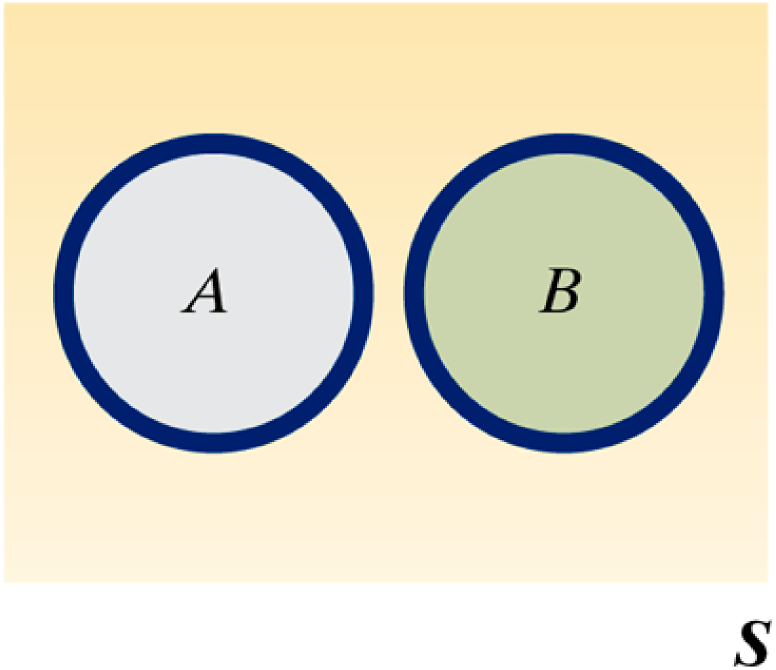
**Additive Rule of Probability**

The probability of the union of events $A$ and $B$ is the sum of the probability of event $A$ and the probability of event $B$, minus the probability of the intersection of events $A$ and $B$; that is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

# **Figure 3.12** Venn diagram of Mutually exclusive events

- A very special relationship exists between events $A$ and $B$ when $A \cap B$ contains no sample points. In this case, we call the events $A$ and $B$ ***mutually exclusive*** events.



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# Definition and Procedure

Events $A$ and $B$ are **mutually exclusive** if $A \cap B$ contains no sample points—that is, if $A$ and $B$ have no sample points in common. For mutually exclusive events,

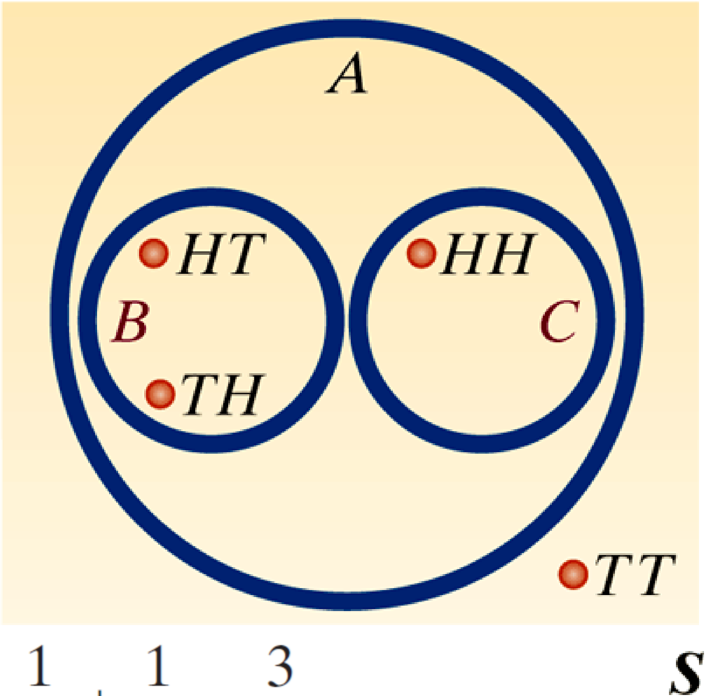
$$P(A \cap B) = 0$$

**Probability of Union of Two Mutually Exclusive Events**

If two events $A$ and $B$ are *mutually exclusive*, the probability of the union of $A$ and $B$ equals the sum of the probability of $A$ and the probability of $B$; that is, $P(A \cup B) = P(A) + P(B)$.

# Problem

- Consider the experiment of tossing two balanced coins. Find the probability of observing *at least* one head.
- Solution Define the events
  - $A$: {Observe at least one head.}
  - $B$: {Observe exactly one head.}
  - $C$: {Observe exactly two heads.}
- Note that $\boldsymbol{A = B \cup C}$ and $B$ and $C$ are mutually exclusive




$$P(A) = P(B \cup C) = P(B) + P(C) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$



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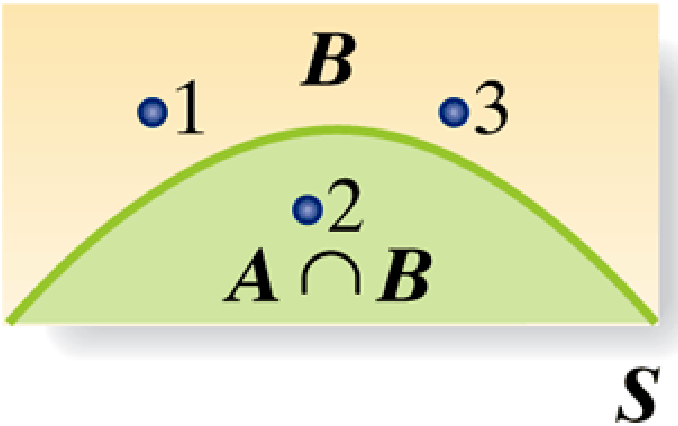
# 3.5

## Conditional Probability

# Figure 3.14 Reduced sample space for the die-toss experiment: given that event B has occurred

- We often have additional knowledge that might affect the outcome of an experiment, so we may need to alter the probability of an event of interest.
- A probability that reflects such additional knowledge is called the **conditional probability** of the event.
- Because the sample points for the die-toss experiment are equally likely, each of the three sample points in the reduced sample space is assigned an equal conditional probability of 1/3.

$$P(A|B) = \frac{1}{3}$$



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# Formula

**Conditional Probability Formula**

To find the *conditional probability that event A occurs given that event B occurs*, divide the probability that *both* $A$ and $B$ occur by the probability that $B$ occurs; that is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

[We assume that $P(B) \neq 0$.]

- For the coin toss example

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(2)}{P(1) + P(2) + P(3)} = \frac{1/6}{3/6} = \frac{1}{3}$$



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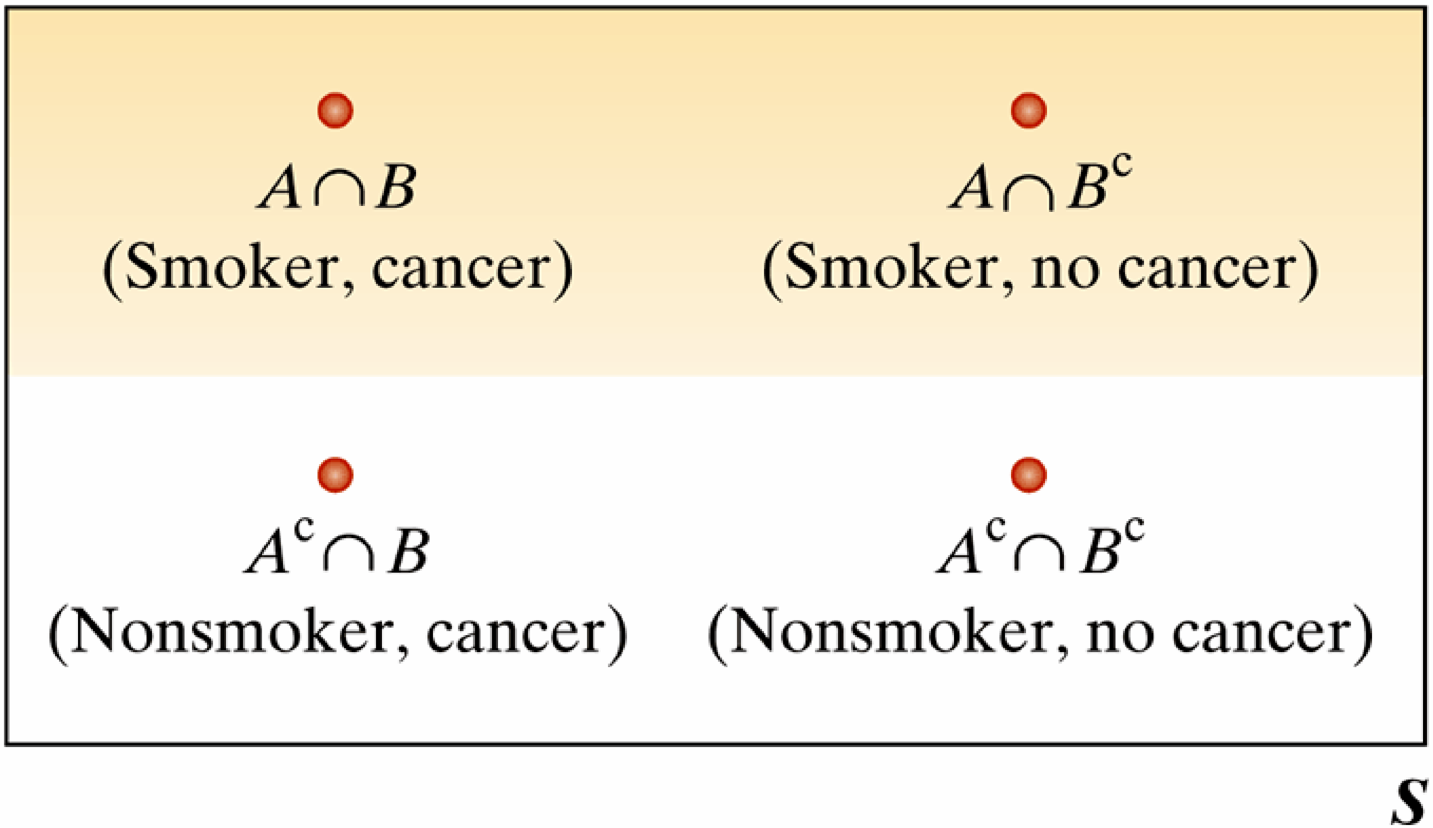
# Problem

Many medical researchers have conducted experiments to examine the relationship between cigarette smoking and cancer. Consider an individual randomly selected from the adult male population. Let $A$ represent the event that the individual smokes, and let $A^C$ denote the complement of $A$ (the event that the individual does not smoke). Similarly, let $B$ represent the event that the individual develops cancer, and let $B^C$ be the complement of that event. Then the four sample points associated with the experiment are shown in Figure 3.15, and their probabilities for a certain section of the United States are given in Table 3.5. Use these sample point probabilities to examine the relationship between smoking and cancer.

# **Figure 3.15** Sample space for Example 3.15



# Table 3.5

**Table 3.5 Probabilities of Smoking and Developing Cancer**

| | Develops Cancer | |
|---|---|---|
| Smoker | Yes, $B$ | No, $B^c$ |
| Yes, $A$ | .05 | .20 |
| No, $A^c$ | .03 | .72 |

# Solution

- We need to compare $P(B|A)$ and $P(B|A^C)$
- $P(B|A) = \frac{P(A \cap B)}{P(A)}$ and $P(B^C|A) = \frac{P(A \cap B^C)}{P(A)}$
- $P(A) = P(A \cap B) + P(A \cap B^C) = .05 + .2 = .25$
- $P(B|A) = \frac{.05}{.25} = .20$ and $P(B^C|A) = \frac{.20}{.25} = .80$
- $P(B|A^C) = \frac{P(A^C \cap B)}{P(A^C)} = \frac{.03}{.75} = .04$
- $P(B^C|A^C) = \frac{P(A^C \cap B^C)}{P(A^C)} = \frac{.72}{.75} = .96$

**Table 3.5 Probabilities of Smoking and Developing Cancer**

| | Develops Cancer | |
|---|---|---|
| Smoker | Yes, $B$ | No, $B^c$ |
| Yes, $A$ | .05 | .20 |
| No, $A^c$ | .03 | .72 |

# Problem

- If a consumer complaint is received, what is the probability that the cause of the complaint was the appearance of the product, given that the complaint originated during the guarantee period?

**Table 3.6 Distribution of Product Complaints**

| | Reason for Complaint | | | |
|---|---|---|---|---|
| | Electrical | Mechanical | Appearance | Totals |
| **During Guarantee Period** | 18% | 13% | 32% | 63% |
| **After Guarantee Period** | 12% | 22% | 3% | 37% |
| Totals | 30% | 35% | 35% | 100% |

# Solution

- Let $A$ represent the event that the cause of a particular complaint is the appearance of the product, and let $B$ represent the event that the complaint occurred during the guarantee period.
- $P(B) = (1 + 13 + 32)\% = 63\%$
- $P(A \cap B) = .32$
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.32}{.63} = .51$

# 3.6

## The Multiplicative Rule and Independent Events

# Procedure

- The probability of an intersection of two events can be calculated with the multiplicative rule, which employs the conditional probabilities we defined in the previous section.

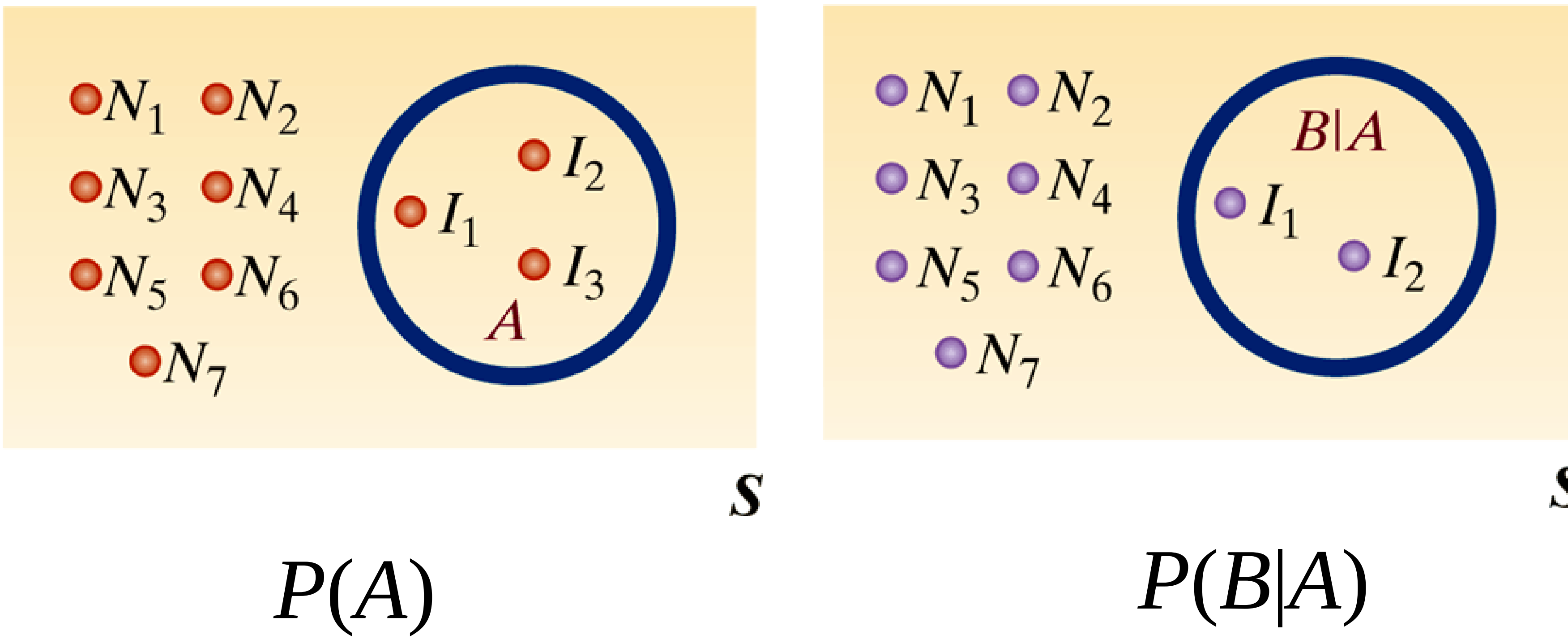
**Multiplicative Rule of Probability**

$P(A \cap B) = P(A)P(B|A)$ or, equivalently, $P(A \cap B) = P(B)P(A|B)$

# Problem

- A county welfare agency employs 10 welfare workers who interview prospective food stamp recipients. Periodically, the supervisor selects, at random, the forms completed by two workers and subsequently audits them for illegal deductions. Unknown to the supervisor, three of the workers have regularly been giving illegal deductions to applicants. What is the probability that both of the workers chosen have been giving illegal deductions?

# **Figure 3.16** Venn diagram for finding $P(A)$ and $P(B|A)$



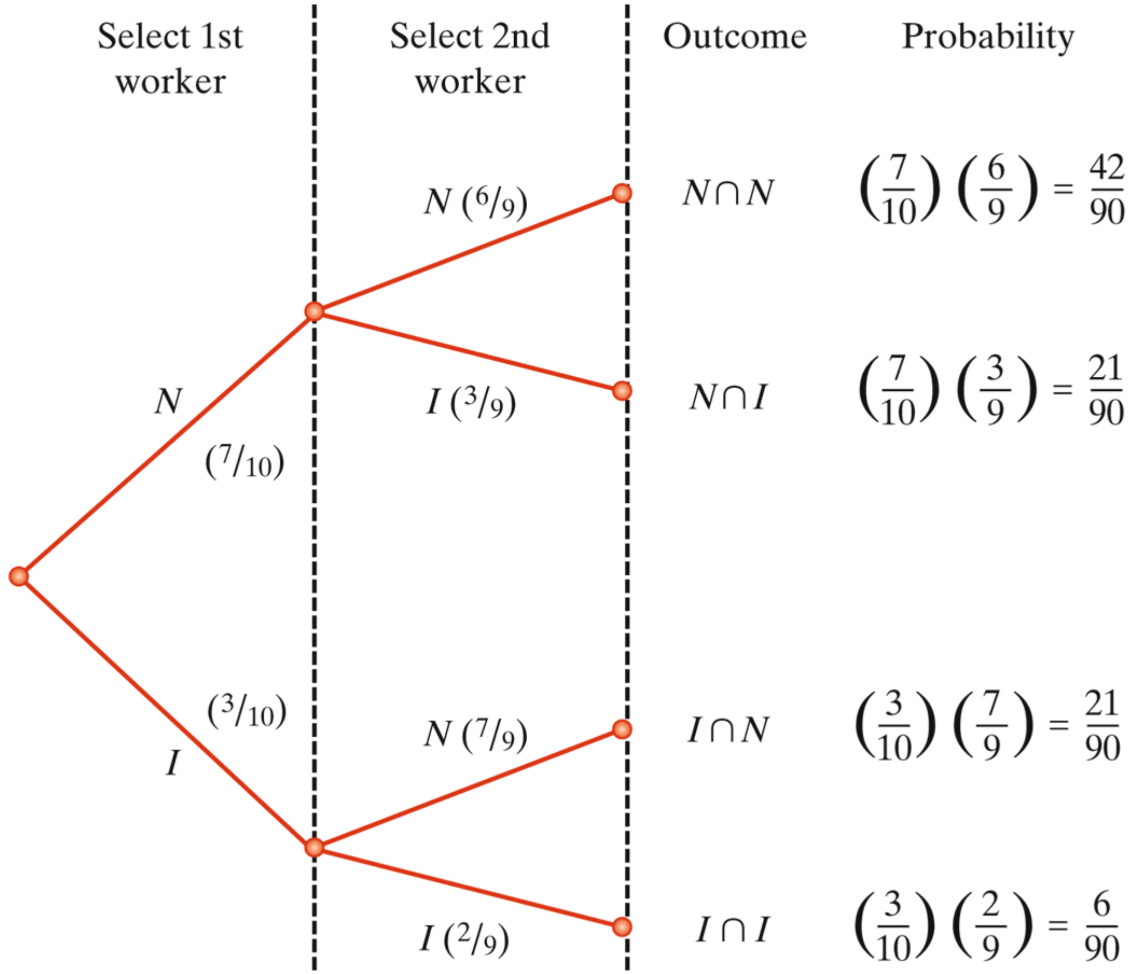
# Solution

- Define the following two events:
  - $A$: {First worker selected gives illegal deductions.}
  - $B$: {Second worker selected gives illegal deductions.}
- We want to find the probability that both workers selected have been giving illegal deductions.
  - {First worker gives illegal deductions and second worker gives illegal deductions.}.
- Thus, we want to find $P(A \cap B)$.

# Solution

- $P(A \cap B) = P(A)P(B|A)$
- $P(A) = P(I_1) + P(I_2) + P(I_3) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$
- $P(B|A) = P(I_1) + P(I_2) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$
- $P(A \cap B) = P(A)P(B|A) = \left(\frac{3}{10}\right)\left(\frac{2}{9}\right) = \frac{1}{15}$

# **Figure 3.18** Tree diagram for Example 3.17



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# Definition

- In some instances, the assumption that event B has occurred will not alter the probability of event A at all. When this occurs, we say that the two events A and B are **independent events**.

Events $A$ and $B$ are **independent events** if the occurrence of $B$ does not alter the probability that $A$ has occurred; that is, events $A$ and $B$ are independent if

$$P(A|B) = P(A)$$

When events $A$ and $B$ are independent, it is also true that

$$P(B|A) = P(B)$$

Events that are not independent are said to be **dependent.**



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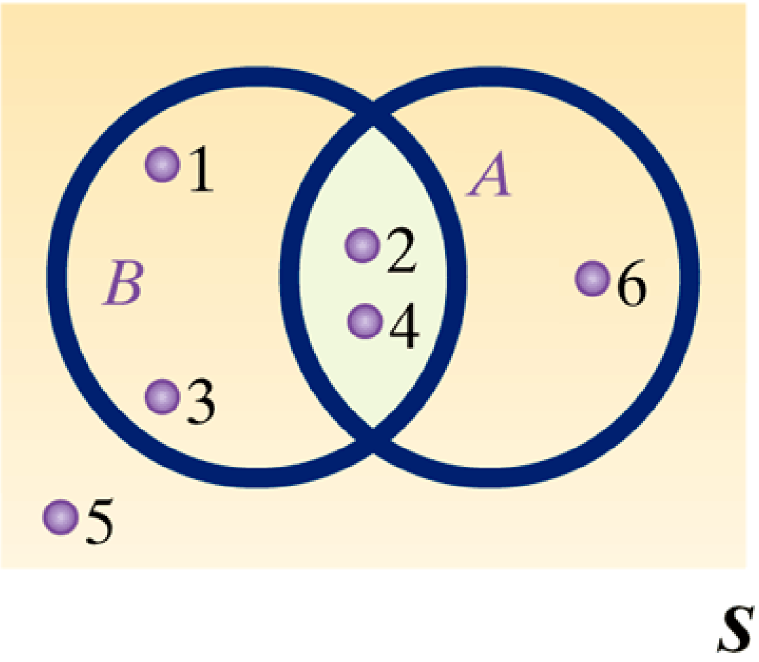
# Problem

- Consider the experiment of tossing a fair die, and let
  - $A$: {Observe an even number.}
  - $B$: {Observe a number less than or equal to 4.}
- Are $A$ and $B$ independent events?

# **Figure 3.19** Venn diagram for die-toss experiment

- $P(A) = P(2) + P(4) + P(6) = 1/2$
- $P(B) = P(1) + P(2) + P(3) + P(4) = 2/3$
- $P(A \cap B) = P(2) + P(4) = 1/3$
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2} = P(A)$



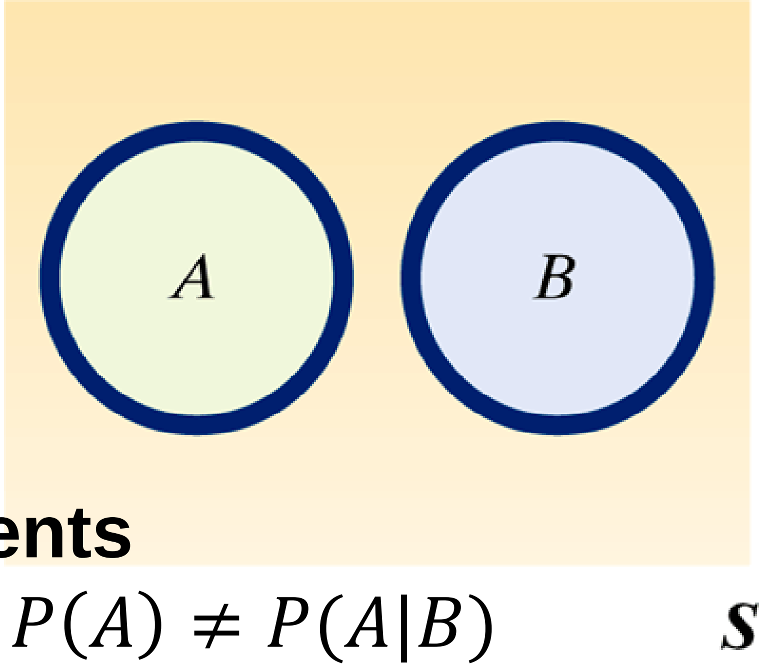
# Final points about independence

- **Point 1:** The property of independence, unlike the property of mutual exclusivity, generally **cannot be shown on, or gleaned from, a Venn diagram**. This means that you can't trust your intuition. In general, the only way to check for independence is by performing the calculations of the probabilities in the definition.

# Final points about independence

**Figure 3.20** Mutually exclusive events are dependent events

- **Point 2:** Suppose that events A and B are mutually exclusive and that both events have nonzero probabilities. Are these events independent or dependent?
- If we assume that B has occurred, it is impossible for A to have occurred simultaneously.
- Thus, **mutually exclusive events are dependent events**, since $P(A) \neq P(A|B)$



# Final points about independence

- **Point 3:** The probability of the intersection of independent events is very easy to calculate. Referring to the formula for calculating the probability of an intersection, we find that

$$P(A \cap B) = P(A)P(B|A)$$

- Thus, since $P(B|A) = P(B)$ when A and B are independent, we have the following useful rule:



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# Procedure

**Probability of Intersection of Two Independent Events**

*If events A and B are independent*, then the probability of the intersection of $A$ and $B$ equals the product of the probabilities of $A$ and $B$; that is,

$$P(A \cap B) = P(A)P(B)$$

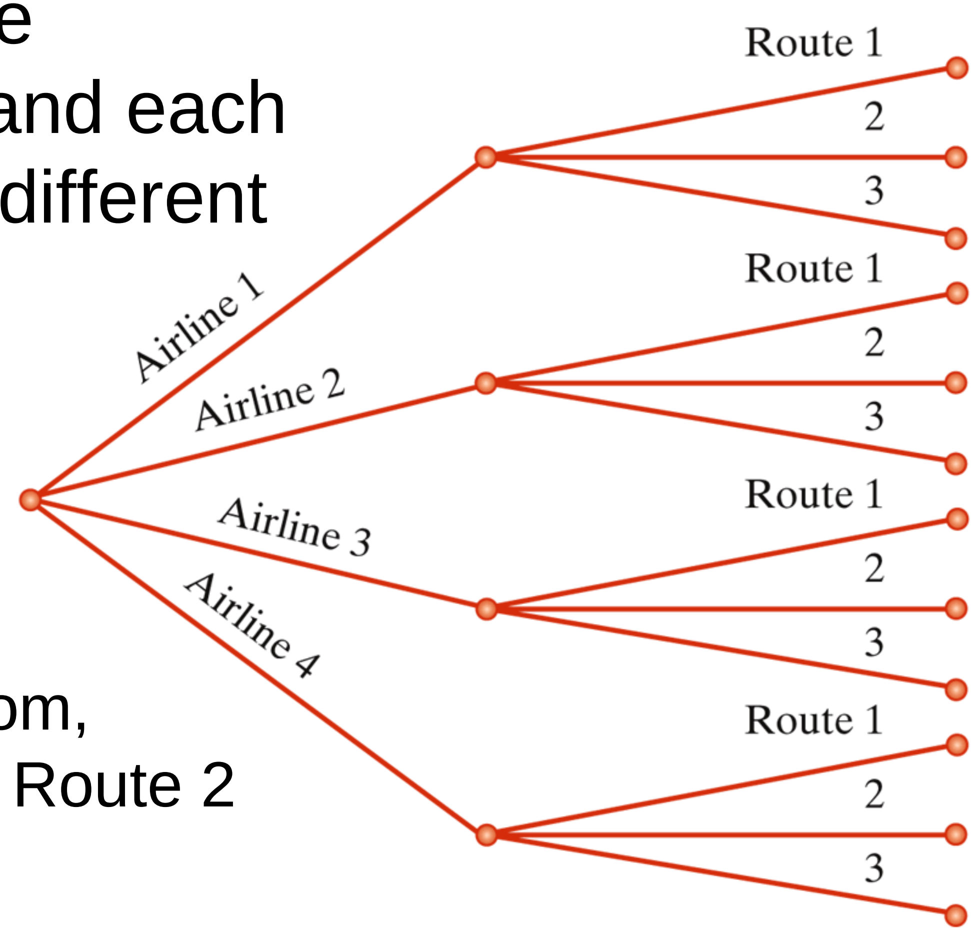
The converse is also true: If $P(A \cap B) = P(A)P(B)$, then events $A$ and $B$ are independent.

# 3.7

## Some Additional Counting Rules (Optional)

# Figure 3.23 Tree diagram for shipping problem

- Problem A product can be shipped by four airlines, and each airline can ship via three different routes.
  - How many distinct ways exist to ship the product?
    - $4 \times 3 = 12$
  - Assuming the airline and route are selected at random, what is the probability that Route 2 is used?
    - $4 \div 12 = 1/3$



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# Procedure

**The Multiplicative Rule**

You have $k$ sets of elements, $n_1$ in the first set, $n_2$ in the second set, …, and $n_k$ in the $k$th set. Suppose you wish to form a sample of $k$ elements by *taking one element from each* of the $k$ sets. Then the number of different samples that can be formed is the product

$$n_1 n_2 n_3 \ldots n_k$$

# Procedure

**Permutations Rule**

Given a *single set* of $N$ different elements, you wish to select $n$ elements from the $N$ and *arrange* them within $n$ positions. The number of different **permutations** of the $N$ elements taken $n$ at a time is denoted by $P_n^N$ and is equal to

$$P_n^N = N(N-1)(N-2)\cdot \cdots \cdot(N-n+1) = \frac{N!}{(N-n)!}$$

where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ and is called *n factorial*. (For example, $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$.) The quantity $0!$ is defined to be 1.

# Problem

- Suppose you are supervising four construction workers. You must assign three of them to job 1 and one to job 2. In how many different ways can you make this assignment?

# Solution

- To begin, suppose each worker is to be assigned to a distinct job. Then, using the multiplicative rule, you obtain $4 \times 3 \times 2 \times 1 = 24$ ways of assigning the workers to four distinct jobs.

**Table 3.7 Workers Assigned to Four Jobs, Example 3.26**

| Group 1 | Group 2 | Group 3 | Group 4 |
|---|---|---|---|
| *ABCD* | *ABDC* | *ACDB* | *BCDA* |
| *ACBD* | *ADBC* | *ADCB* | *BDCA* |
| *BACD* | *BADC* | *CADB* | *CBDA* |
| *BCAD* | *BDAC* | *CDAB* | *CDBA* |
| *CABD* | *DABC* | *DACB* | *DBCA* |
| *CBAD* | *DBAC* | *DCAB* | *DCBA* |

# Table 3.8

- Now suppose the first three positions represent job 1 and the last position represents job 2.
- You can see that all the listings in group 1 represent the same outcome of the experiment of interest.
- Similarly, group 2 listings are equivalent, as are group 3 and group 4 listings.
- Thus, there are only four different assignments of four workers to the two jobs.

**Table 3.8 Workers Assigned to Two Jobs, Example 3.26**

| Job 1 | Job 2 |
|---|---|
| *ABC* | *D* |
| *ABD* | *C* |
| *ACD* | *B* |
| *BCD* | *A* |

# Procedure

**Partitions Rule**

Suppose you wish to partition a *single* set of $N$ different elements into $k$ sets, with the first set containing $n_1$ elements, the second containing $n_2$ elements, …, and the $k$th set containing $n_k$ elements. Then the number of different partitions is

$$\frac{N!}{n_1!n_2!\ldots n_k!}, \text{ where } n_1 + n_2 + n_3 + \cdots + n_k = N$$



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# Definition

**Summary of Counting Rules**

1. *Multiplicative rule*. If you are drawing *one element from each of k sets of elements*, where the sizes of the sets are $n_1, n_2, \cdot \ldots \cdot, n_k$, then the number of different results is

$$n_1 n_2 n_3 \ldots n_k$$

2. *Permutations rule*. If you are drawing *n elements from a set of N elements and arranging the n elements in a distinct order*, then the number of different results is

$$P_n^N = \frac{N!}{(N-n)!}$$

3. *Partitions rule*. If you are partitioning the *elements of a set of N elements into k groups consisting* of $n_1, n_2, \ldots, n_k$ elements $(n_1 + \cdots + n_k = N)$, then the number of different results is

$$\frac{N!}{n_1! n_2! \cdot \ldots \cdot n_k!}$$

4. *Combinations rule*. If you are drawing *n elements from a set of N elements without regard to the order of the n elements*, then the number of different results is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

[*Note:* The combinations rule is a special case of the partitions rule when $k = 2$.]

# 3.9

## Bayes's Rule (Optional)

# Bayesian statistical methods

- The logic employed by the English philosopher Thomas Bayes in the mid-1700s involves converting an unknown conditional probability, say, $P(B|A)$, to one involving a known conditional probability, say, $P(A|B)$.

# Problem

- An unmanned monitoring system uses high-tech video equipment and microprocessors to detect intruders. A prototype system has been developed and is in use outdoors at a weapons munitions plant. The system is designed to detect intruders with a probability of **.90**. However, the design engineers expect this probability to vary with the weather. The system automatically records the weather condition each time an intruder is detected. On the basis of a series of controlled tests in which an intruder was released at the plant under various weather conditions, the following information is available: Given that the intruder was, in fact, **detected** by the system, the weather was **clear 75%** of the time, **cloudy 20%** of the time, and **raining 5%** of the time. When the system **failed to detect** the intruder, **60%** of the days were **clear**, **30% cloudy**, and **10% rainy**. Use this information to find the probability of detecting an intruder, given rainy weather. (Assume that an intruder has been released at the plant.)

# Solution

- Define $D$ to be the event that the intruder is detected by the system. Then $D^C$ is the event that the system failed to detect the intruder. Our goal is to calculate the conditional probability $P(D|Rainy)$. From the statement of the problem, the following information is available:

$$P(D) = .90 \qquad PD^c = .10$$

$$P(\text{Clear}|D) = .75 \qquad P(\text{Clear}|D^c) = .60$$

$$P(\text{Cloudy}|D) = .20 \qquad P(\text{Cloudy}|D^c) = .30$$

$$P(\text{Rainy}|D) = .05 \qquad P(\text{Rainy}|D^c) = .10$$

- Note that $P(D|Rainy)$ is not one of the conditional probabilities that is known.



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# **Figure 3.22** Tree diagram for Example 3.31

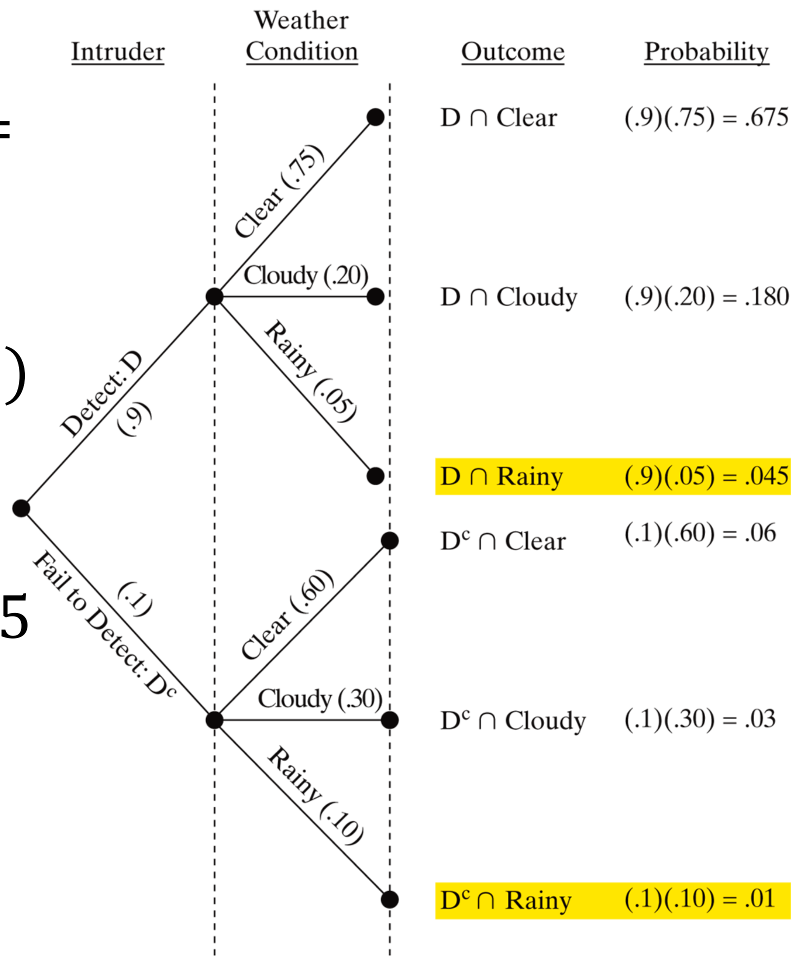
- However, we can find

$$P(Rainy \cap D) = P(D)P(Rainy|D) = (.90)(.05) = .045$$

and

$$P(Rainy \cap D^C) = P(D^C)P(Rainy|D^C) = (.10)(.10) = .01$$

- $P(Rainy) = P(Rainy \cap D) + P(Rainy \cap D^C) = .045 + .01 = .055$
- $P(D|Rainy) = \frac{P(Rainy \cap D)}{P(Rainy)} = \frac{.045}{.055} =$ 0.818



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# Procedure

**Bayes's Rule**

Given $k$ mutually exclusive and exhaustive events, $B_1, B_2, \ldots, B_k$ such that $P(B_1) + P(B_2) + \cdots + P(B_k) = 1$, and given an observed event $A$, it follows that

$$P(B_i|A) = P(B_i \cap A)/P(A)$$

$$= \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots + P(B_k)P(A|B_k)}$$

$$P(D|\text{Rainy}) = \frac{P(D)P(\text{Rainy}|D)}{P(D)P(\text{Rainy}|D) + P(D^c)P(\text{Rainy}|D^c)}$$

$$= \frac{(.90)(.05)}{(.90)(.05) + (.10)(.10)} = .818$$