FACULTY OF ENGINEERING
DEPARTMENT OF COMPUTER ENGINEERING
MAT 222 LINEAR ALGEBRA AND NUMERICAL METHODS
MAKEUP EXAM
3rd July 2023

IMPORTANT: In all the problems consider your student ID as a-b-c-d-e-f-g-h-i-j-k.

1. Consider the matrix $A = \begin{bmatrix} j+1 & k+1 & 0 \\ k+1 & -(j+1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find three eigenvectors of A that form a basis for $\mathbb{R}^3$. Then express the vector $\begin{bmatrix} 1 \\ j \\ k \end{bmatrix}$ with respect to this basis. (Hint: A square matrix whose transpose is equal to itself is called **symmetric**. It is known that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal to each other.)

2. Suppose that there are three possibilities for the weather in a certain city, which are sunny, cloudy, rainy. The weather tomorrow is only affected by the weather today, with certain transition probabilities. Some of these probabilities are given below.

If it is rainy today, the probability that it is rainy tomorrow is $\dfrac{j+k+1}{j+k+10}$, while the probability that it is cloudy tomorrow is $\dfrac{5}{j+k+10}$.

If it is cloudy today, the probability that it is rainy tomorrow is $\dfrac{j+1}{j+k+5}$, while the probability that it is sunny tomorrow is $\dfrac{k}{j+k+5}$.

If it is sunny today, the probability that it is cloudy tomorrow is $\dfrac{2}{j+k+3}$, while the probability that it is sunny tomorrow is $\dfrac{j+k}{j+k+3}$.

According to this information, fill in the missing probabilities and answer the following: If it is NOT rainy today, what is the probability that it will be rainy three days later?

3. Assume $\mathbf{v}_1 = \begin{bmatrix} j+1 \\ k+1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2j+2 \\ 2k+3 \end{bmatrix}$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^2$. Find a matrix A such that $A\mathbf{x}$ gives the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$. (In other words, if $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $A\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.)

4. Let M be the 3×2 matrix given by $M = \begin{bmatrix} 1+j & 1+c+j+k \\ k & k-j-1 \\ 1 & 1-c-cj \end{bmatrix}$. Consider the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find the element of $C(M)$ which is closest to $\mathbf{b}$. (Hint: One way is to find an orthogonal basis for $C(M)$.)

Remarks: The duration of the exam (including upload file) is 90 minutes. Attempt all the questions. In order to get full marks, please show your work in sufficient detail.