MAT222
LINEAR ALGEBRA
HOMEWORK ASSIGNMENT 3
SOLUTIONS

(1) Find the dimension of the subspace spanned by the vectors $\mathbf{v}_1 = (1,0,1)$, $\mathbf{v}_2 = (2,1,1)$, $\mathbf{v}_3 = (1,1,0)$ and $\mathbf{v}_4 = (3,1,2)$.

Solution: By elementary row operations, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow[R_4-3R_1]{\substack{R_2-2R_1 \\ R_3-R_1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_4-R_2]{R_3-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the vectors $\mathbf{v}_1 = (1,0,1)$ and $\mathbf{v}_2 = (2,1,1)$ are linearly independent and span the subspace, so they form a basis. Therefore, the dimension of the subspace is 2.■

(2) Find the dimension of the subspace

$$W = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_{22} : x + t = 0 \right\}$$

Solution: For any element $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ of W, we have

$$\begin{aligned} \begin{bmatrix} x & y \\ z & t \end{bmatrix} &= \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} + \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix} \\ &= x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Thus, the set $\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

spans W. Moreover, since

$$\begin{aligned} &c_1A_1 + c_2A_2 + c_3A_3 = \mathbf{0} \Rightarrow c_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0, \end{aligned}$$

the set $\{A_1, A_2, A_3\}$ is linearly independent. Thus, the set $\{A_1, A_2, A_3\}$ is a basis for W, and the dimension of W is 3.■

(3) Find the image of the circle $x^2 + y^2 = 4$ under the transformation $T\left((x,y)\right) = (x, y + 2x)$.

Solution: The transformation T is a shearing transformation with

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Hence, the circle $x^2+y^2=4$ transforms into the ellipse determined by

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ \mp\sqrt{4-x^2}\end{bmatrix}=\begin{bmatrix}x\\ 2x\mp\sqrt{4-x^2}\end{bmatrix}$$
$$\Rightarrow y=2x\mp\sqrt{4-x^2}\Rightarrow 5x^2-4xy+y^2=4.\blacksquare$$

(4) If

$$A=\begin{bmatrix}1 & -1 & -1 & 1 & 1\\ -1 & 1 & 0 & -2 & 2\\ 1 & -1 & -2 & 0 & 3\\ 2 & -2 & -1 & 3 & 4\end{bmatrix}$$

find the dimension of the kernel of the matrix transformation $T_A:\mathbb{R}^4\to\mathbb{R}^5$.

Solution: We have

$$\begin{bmatrix}1 & -1 & -1 & 1 & 1\\ -1 & 1 & 0 & -2 & 2\\ 1 & -1 & -2 & 0 & 3\\ 2 & -2 & -1 & 3 & 4\end{bmatrix}\xrightarrow[\substack{R_3-R_1\\ R_4-2R_1}]{R_2+R_1}\begin{bmatrix}1 & -1 & -1 & 1 & 1\\ 0 & 0 & -1 & -1 & 3\\ 0 & 0 & -1 & -1 & 2\\ 0 & 0 & 1 & 1 & 2\end{bmatrix}\xrightarrow{(-1)R_2}\begin{bmatrix}1 & -1 & -1 & 1 & 1\\ 0 & 0 & 1 & 1 & -3\\ 0 & 0 & -1 & -1 & 2\\ 0 & 0 & 1 & 1 & 2\end{bmatrix}$$
$$\xrightarrow[\substack{R_3+R_2\\ R_4-R_2}]{R_1+R_2}\begin{bmatrix}1 & -1 & 0 & 2 & -2\\ 0 & 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 5\end{bmatrix}\xrightarrow{(-1)R_3}\begin{bmatrix}1 & -1 & 0 & 2 & -2\\ 0 & 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 5\end{bmatrix}\xrightarrow[\substack{R_2+3R_3\\ R_4-5R_3}]{R_1+2R_3}\begin{bmatrix}1 & -1 & 0 & 2 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Thus, the equations

$$\begin{aligned}x_1-x_2+2x_4&=0\\ x_3+x_4&=0\\ x_5&=0\end{aligned}$$

give

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}x_2-2x_4\\ x_2\\ -x_4\\ x_4\\ 0\end{bmatrix}=x_2\begin{bmatrix}1\\ 1\\ 0\\ 0\\ 0\end{bmatrix}+x_4\begin{bmatrix}-2\\ 0\\ -1\\ 0\\ 0\end{bmatrix}.$$

Since the vectors

$$\mathbf{v}_1=\begin{bmatrix}1\\ 1\\ 0\\ 0\\ 0\end{bmatrix}\quad\text{and}\quad\mathbf{v}_2=\begin{bmatrix}-2\\ 0\\ -1\\ 0\\ 0\end{bmatrix}$$

are linearly independent, a basis for $\ker(T_A)$ is $\{\mathbf{v}_1,\mathbf{v}_2\}$, so $\dim(\ker(T_A))=2.\blacksquare$

(5) For two nonparallel vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, consider the linear transformation $T(\mathbf{u})=\det\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}$. Find $\ker(T)$.

Solution: The kernel of T consists of all vectors $\mathbf{u}$ such that $\det\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix}=0$, that is, the matrix $\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix}$ fails to be invertible. This is the case if $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Thus, $\ker(T)=\text{span}\{\mathbf{v},\mathbf{w}\}.\blacksquare$