Discrete Mathematics

Lecture 2: Logic

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First Created: October 10, 2016,
Last Modified: October 16, 2023

Table of contents

Basic Connectives and Truth Tables

Terminology: In the development of any mathematical theory, assertions are made in the form of sentences. Such verbal or written assertions, called **statements** (or **propositions**), are declarative sentences that are either **true** of **false** but not both. For example, the following are statements, and we use the lowercase letters of the alphabet (such as p, q and r) to present these statements.

- p : Combinatorics is a required course for sophomores.
- q : Margaret Mitchell wrote Gone with the Wind.
- r : 2+3=5

Basic Connectives and Truth Tables

On the other hand, we do not regard sentences such as the exclamation

"What a beautiful evening!"

or the command:

"Get up and do your exercises."

as statements since they do not have truth values (true or false).
The preceding statements represented by p, q, and r are considered to be primitive statements, for there is really no way to break them down into anything simpler.

Basic Connectives and Truth Tables

A statement can be obtained from existing ones in two ways:

1. Transform a given statement p into the statement $\neg p$, which denotes its negation and is read "Not p." For the statement p above, $\neg p$ is the statement "Combinatorics is not a required course for sophomores." (We do not consider the negation of a primitive statement to be a primitive statement.)
2. Combine two or more statements into a compound statement, using the following logical connectives.
   - $p \wedge q$: "p and q". (Conjunction)
   - $p \vee q$: "p or q". (Disjunction)
   - $p \to q$: "p implies q". (Implication) p is called the hypothesis and q the conclusion.
   - $p \leftrightarrow q$: "p if and only if q", "p iff q". (Biconditional)

Implication

$p \to q$ is read as:

- (i) "If p, then q."
- (ii) "p is sufficient for q."
- (iii) "p is a sufficient condition for q."
- (iv) "q is necessary for p."
- (v) "q is a necessary condition for p."
- (vi) "p only if q."

Truth of basic compound statements

Let p and q be primitive statements. Then,

- $p \wedge q$ is true only both are true.
- $p \vee q$ is true if either one is true. (It is false only if both are false.)
- $p \rightarrow q$ is true except when p is true but q is not.
- $p \leftrightarrow q$: is true when p and q have the same truth value. (i.e., when they are both true or both false.)

Truth tables

Truth tables are simple tables that enumerates all possible input values for the input statements and show the corresponding truth values of the compound statement. Here is the truth table for $\neg$ (**negation**), $\wedge$ (**logical and**), and $\vee$ (**logical or**):

p	$\neg p$
F	T
T	F

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truth tables

Especially in computer science and engineering, most of the time, we use 0 instead of False (F) and 1 instead of True (T). Hence our truth tables will look like:

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Arguments

Definition (Argument)

In general, an **argument** starts with a list of given statements called premises, say $p_1, p_2, p_3, \ldots, p_n$, and a statement called the **conclusion** of the argument. For example, the following is an argument:

$$p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n \rightarrow q$$

Truth of basic compound statements: Example

Example (2.1)

Let s, t, and u denote the following primitive statements.

- s: Phyllis goes out for a walk.
- t: The moon is out.
- u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

- If the moon is out and it is not snowing, then Phyllis goes out for a walk.
- If the moon is out, then if it is not snowing Phyllis goes out for a walk.
- It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Truth of basic compound statements: Example

Example (2.1) cont'd.

Let s, t, and u denote the following primitive statements.

- s: Phyllis goes out for a walk.
- t: The moon is out.
- u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

- $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.
- $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk.
- $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Truth of basic compound statements: Example

Example (2.1) cont'd.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences.

- "Phyllis will go out walking if and only if the moon is out."
- "If it is snowing and the moon is not out, then Phyllis will not go out for a walk."
- "It is snowing but Phyllis will still go out for a walk."

Truth of basic compound statements: Example

Example (2.1) cont'd.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences.

- "Phyllis will go out walking if and only if the moon is out." $s \leftrightarrow t$
- "If it is snowing and the moon is not out, then Phyllis will not go out for a walk." $(u \wedge \neg t) \rightarrow \neg s$
- "It is snowing but Phyllis will still go out for a walk." $u \wedge s$

Truth of basic compound statements: Example

Example 2.3 Decision (selection) structure

- In computer science, the if-then and if-then-else decision structures arise in high-level programming languages such as Java and C++.
- E.g., "if p then q else r": q is executed when p is true and r is executed when p is false.

Truth tables

Example 2.5

Construct a truth table for each of the following compound statements, where p, q, r denote primitive statements.

1. $p \vee (q \wedge r)$
2. $(p \vee q) \wedge r$

Remark: Because the truth values in $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$ differ, we must avoid writing a compound statement such as $p \vee q \wedge r$. Without parentheses to indicate which of the connectives $\vee$ and $\wedge$ should be applied first, we have no idea whether we are dealing with $p \vee (q \wedge r)$ or $(p \vee q) \wedge r$.

Truth tables

Example 2.5 cont'd.

1. $p \vee (q \wedge r)$
2. $(p \vee q) \wedge r$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Truth tables

Example 2.6

Construct a truth table for each of the following compound statements, where p and q denote primitive statements.

1. $p \to (p \vee q)$
2. $p \wedge (\neg p \wedge q)$

p	q	$p \vee q$	$p \to (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Truth tables

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Definition (2.1)

A compound statement is called a **tautology** if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a **contradiction**.

Logical Equivalence

Example 2.7

Construct a truth table for each of the following compound statements, where p and q denote primitive statements.

1. $\neg p \vee q$
2. $p \rightarrow q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Logical Equivalence

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Definition (2.2)

Two statement s_1, s_2 are said to be **logically equivalent**, and we write $s_1 \Leftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

Hence, from the table, we know that $\neg p \vee q \Leftrightarrow p \to q$.

Logical Equivalence

Example 2.8

Construct a truth table for each of the following statements, where p and q are primitive statements.

1. $\neg(p \wedge q)$
2. $\neg p \vee \neg q$
3. $\neg(p \vee q)$
4. $\neg p \wedge \neg q$

Here, a crucial difference emerges: The negation of the conjunction of two primitive statement p, q results in the disjunction of their negations $\neg p$, $\neg q$, whereas the negation of the disjunction of these same statements p, q is logically equivalent to the conjunction of their negations $\neg p$, $\neg q$.

The Laws of Logic

For any primitive statements p, q, r , any tautology T_0; and any contradiction F_0,

1	$\neg\neg p \Leftrightarrow p$	Double Negation Law
2	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Laws
3	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
4	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative Laws
5	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws

The Laws of Logic

For any primitive statements p, q, r , any tautology T_0; and any contradiction F_0,

6	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	Idempotent Laws
7	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	Identity Laws
8	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	Inverse Laws
9	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	Domination Laws
10	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption Laws

The Laws of Logic

Definition (2.3 Dual)

Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the dual of s denoted s^d , is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

Theorem (2.1 The Principle of Duality)

Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Substitution Rules

Substitution rules

1. Suppose that the compound statement P is a tautology. If p is a primitive statement that appears in P and we replace each occurrence of p by the same statement q, then the resulting compound statement P_1 is also a tautology.
2. Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Leftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Leftrightarrow P$.

Substitution Rules

Example (2.12)

Negate and simplify the compound statement $(p \vee q) \rightarrow r$.

Example (2.15)

Verify the following compound statements are logically equivalent:

- $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
- $(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$

The statement $\neg q \rightarrow \neg p$ is called the **contrapositive** of the implication $p \rightarrow q$. The statement $q \rightarrow p$ is called the **converse** of $p \rightarrow q$. And $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

Substitution Rules

Example (2.16,2.17,2.18)

Let p, q, r, and t denote primitive statements. Simplify each of the following compound statements:

- $(p \vee q) \wedge \neg(\neg p \wedge q)$
- $\neg(\neg((p \vee q) \wedge r) \vee \neg q)$
- $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$

Rules of Inference

Example (2.19)

Let p, q, r denote primitive statements given as

- p: Roger studies.
- q: Roger plays basketball.
- r: Roger passes discrete mathematics.

Now let p_1, p_2, p_3 denote the premises. p_1: If Roger studies, then he will be pass discrete mathematics. p_2: If Roger doesn't play basketball, then he'll study. p_3: Roger failed discrete mathematics.

Rules of Inference

Example (2.19 cont'd)

We want to determine whether the argument

$$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$$

is valid. To do so, we rewrite p_1, p_2, p_3 as

$$p_1 : p \rightarrow r, p_2 : \neg q \rightarrow p, p_3 : \neg r$$

and examine the truth table for the implication

$$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$$

given in Table 2.14.

Rules of Inference

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \to q$
p	q	r	$p \to r$	$\neg q \to p$	$\neg r$	$((p \to r) \wedge (\neg q \to p) \wedge \neg r) \to q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

Because the final column in Table 2.14 contains all ones, the implication is a tautology. Hence we can say that $(p_1 \wedge p_2 \wedge p_3) \to q$ is a valid argument.

Rules of Inference

Definition (2.4)

If p, q are arbitrary statements such that $p \to q$ is a tautology, then we say that p logically implies q and we write $p \Rightarrow q$ to denote this situation.

When p, q are statements and $p \Rightarrow q$, the implication $p \to q$ is a tautology and we refer to $p \to q$ as a logical implication.

Modus Ponens

Example (2.22)

We consider the rule of inference called Modus Ponens, or the Rule of Detachment. In symbolic form this rule is expressed by the logical implication

$$(p \wedge (p \to q)) \to q$$

which is verified in Table 2.16, where we find that the fourth row is the only one where both of the premises p and $p \to q$ (and the conclusion q) are true.

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Rules of Inference

Example (2.22)

The actual rule will be written in the tabular form

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

where the three dots ($\therefore$) stand for the word "therefore" indicating that q is the conclusion for the premises p and $p \to q$, which appear above the horizontal line. This rule arises when we argue that if (1) p is true, and (2) $p \to q$ is true (or $p \Rightarrow q$), then the conclusion q must be true. (After all, if q were false and p were true, then we could not have $p \to q$ is true.)

The Law of the Syllogism

Example (2.23)

A second rule of inference is given by the logical implication

$$((p \to q) \land (q \to r)) \to (p \to r),$$

where p, q, and r are arbitrary statements. In tabular form it is written

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

This rule, which is referred to as the **Law of the Syllogism**, arises in many arguments.

The Law of the Syllogism: Example

Example (2.23 cont'd)

For example, we may use it as follows:

1. If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $p \to q$
2. If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $q \to r$
3. Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\therefore p \to r$.

Rules of inference

Rule of Inference	Name of Rule
1) $\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	Rule of Detachment (Modus Ponens)
2) $\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$	Law of Syllogism
3) $\begin{array}{l} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$	Modus Tollens
4) $\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$	Rule of Conjunction

Rules of inference cont'd

Rule of Inference	Name of Rule
5) $\dfrac{\begin{array}{l} p \vee q \\ \neg p \end{array}}{\therefore q}$	Rule of Disjunctive Syllogism
6) $\dfrac{\neg p \rightarrow F_0}{\therefore p}$	Rule of Contradiction
7) $\dfrac{p \wedge q}{\therefore p}$	Rule of Conjunctive Simplification
8) $\dfrac{p}{\therefore p \vee q}$	Rule of Disjunctive Amplification
9) $\dfrac{\begin{array}{l} p \wedge q \\ p \rightarrow (q \rightarrow r) \end{array}}{\therefore r}$	Rule of Conditional Proof

Rules of inference cont'd

Rule of Inference	Name of Rule
10) $\begin{array}{l} p \to r \\ q \to r \\ \hline \therefore (p \lor q) \to r \end{array}$	Rule of Proof by Cases
11) $\begin{array}{l} p \to q \\ r \to s \\ p \lor r \\ \hline \therefore q \lor s \end{array}$	Rule of the Constructive Dilemma
12) $\begin{array}{l} p \to q \\ r \to s \\ \neg q \lor \neg s \\ \hline \therefore \neg p \lor \neg r \end{array}$	Rule of the Destructive Dilemma

Rule of Inference

Example (2.31)

Establish the validity of the argument

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

Rule of Inference

Example (2.31 Solution)

We are interested in proving that p is true.

	Steps	**Reasons**
1)	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
2)	$r \rightarrow t$	Premise
3)	$\neg t$	Premise
4)	$\neg r$	Modus Tollens (Steps 2,3)
5)	$\neg(r \wedge s) \rightarrow \neg(\neg p \vee \neg q)$	Contrapositive (Step 1)
6)	$(\neg r \vee \neg s) \rightarrow \neg(\neg p \vee \neg q)$	De Morgan (Step 5)
7)	$(\neg r \vee \neg s) \rightarrow (p \wedge q)$	De Morgan (Step 6)
8)	$\neg r \vee \neg s$	Disjunctive Amplification (Step 4)
9)	$p \wedge q$	Modus Ponens (Steps 7,8)
10)	p	Conjunctive Simplification (Step 9)

Rule of Inference

Example (2.32)

In this instance we shall use the method of Proof by Contradiction. Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array}$$

To establish the validity for this argument, we assume the negation $\neg p$ of the conclusion p as another premise. The objective now is to use these four premises to derive a contradiction F_0. Our derivation follows.

The Use of Quantifiers

"The number $x + 2$ is an even integer" is not a statement. When x is replaced by 1, 3 or 5 the resulting statement is false. However, when x is replaced by 2, 4 or 6 the resulting statement is true. We refer to the sentence "The number $x + 2$ is an even integer" as an **open statement**.

Definition (2.5)

A declarative sentence is an open statement if

1. it contains one or more variables, and
2. it is not a statement, but
3. it becomes a statement when the variables in it are replaced by certain allowable choices.

The Use of Quantifiers

Let $p(x)$, $q(x, y)$ denote the following open statements

- $p(x)$: $x + 2$ is an even integer.
- $q(x, y)$: $x - y$ is an even integer.

For example, $p(2)$ (true), $p(5)$ (false), $q(4, 2)$ (true) and $q(4, 7)$ (false). Hence, we can make the following true statements.

- (1) For some x, $p(x)$.
- (2) For some x, y, $q(x, y)$.

- Statement (1) uses the existential quantifiers "For some x", which can also be expressed as "For as least one x" or "There exists an x such that". This quantifier is written in symbolic form as $\exists x$.
- Hence the statement "For some x, $p(x)$" becomes $\exists x \ p(x)$, in symbolic form.

The Use of Quantifiers

- The phrases "For some x" and "For some x, y" are said to quantify the open statements $p(x)$ and $q(x, y)$, respectively.
- Many postulates, definitions, and theorems in mathematics involve statements that are quantified open statements.
- These results from the two types of quantifiers, which are called the existential and the universal quantifiers.

The Use of Quantifiers

Statement (2) becomes $\exists x\ \exists y\ q(x,y)$ in symbolic form. The notation $\exists x, y$ can be used to abbreviate $\exists x, y\ q(x,y)$ to $\exists x\ \exists y\ q(x,y)$.
The universal quantifier is denoted by $\forall x$ and is read "For all x", "For any x", "For each x", or "For every x". "For all x, y", "For any x, y", "For every x, y", or "For all x and y" is denoted by $\forall x\ \forall y$, which can be abbreviated to $\forall x, y$.

The Use of Quantifiers

Stmnt	When is it true?	When is it false?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every a in the universe, $p(a)$ is true	There is at least one a in the universe, for which $p(a)$ is false.

The Use of Quantifiers

Definition (2.6)

Let $p(x), q(x)$ be open statements defined for a given universe. The open statements $p(x)$ and $q(x)$ are called **(logically) equivalent**, and we write $\forall x\ (p(x) \Leftrightarrow q(x))$ when the bi-conditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe (that is, $p(a)$, $q(a)$ for each a in the universe). If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe (that is, $p(x) \Rightarrow q(x)$ for each x in the universe), then we write $\exists x\ (p(x) \Rightarrow q(x))$ and say that $p(x)$ **logically implies** $q(x)$.

The Use of Quantifiers

Definition (2.7)

For open statements $p(x)$ and $q(x)$–defined for a prescribed universe–and the universally quantified statement $\forall x\ [p(x) \to q(x)]$, we define:

1. The **contrapositive** of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [\neg q(x) \to \neg p(x)]$.
2. The **converse** of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [q(x) \to p(x)]$.
3. The **inverse** of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [\neg p(x) \to \neg q(x)]$.

The Use of Quantifiers

Logical equivalences and implications for quantified statements in one variable

For a prescribed universe and any open statements $p(x), q(x)$ in the variable x:

- $\exists x\ [p(x) \wedge q(x)] \Rightarrow [\exists x\ p(x) \wedge \exists x\ q(x)]$
- $\exists x\ [p(x) \vee q(x)] \Rightarrow [\exists x\ p(x) \vee \exists x\ q(x)]$
- $\forall x\ [p(x) \wedge q(x)] \Rightarrow [\forall x\ p(x) \wedge \forall x\ q(x)]$
- $[\forall x\ p(x) \vee \forall x\ q(x)] \Rightarrow \forall x\ [p(x) \vee q(x)]$

The Use of Quantifiers

Rules for negating statements with one quantifier:

- $\neg[\forall x\ p(x)] \Leftrightarrow \exists x\ \neg p(x)$
- $\neg[\exists x\ p(x)] \Leftrightarrow \forall x\ \neg p(x)$
- $\neg[\forall x\ \neg p(x)] \Leftrightarrow \exists x\ \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$
- $\neg[\exists x\ \neg p(x)] \Leftrightarrow \forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$

The Use of Quantifiers

Example (2.47)

For the universe of all integers, consider the true statement "There exist integers x, y such that $x + y = 6$." We may represent this in symbolic form by

$$\exists x\ \exists y\ (x + y = 6).$$

If we let $p(x, y)$ denote the open statement "$x + y = 6$," then an equivalent statement can be given by $\exists y\ \exists x\ p(x, y)$. In general, for any open statement $p(x, y)$ and universe(s) prescribed for the variables x, y,

$$\exists x\ \exists y\ p(x, y) \Leftrightarrow \exists y\ \exists x\ p(x, y).$$

Similar results follow for statements involving three or more quantifiers.

The Use of Quantifiers

Example (2.49)

Let $p(x, y)$, $q(x, y)$, and $r(x, y)$ represent three open statements, with replacements for the variables x, y chosen from some prescribed universe(s). What is the negation of the following statement?

$$\forall x \; \exists y \; [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

Example (2.50)

$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; [(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)]$.

Then what is the symbolic form for the limit of $f(x) \neq L$, as x approaches a?

The Use of Quantifiers

Example (Q14)

Let $p(n), q(n)$ represent the open statements "n is odd" and "n^2 is odd" for the universe of all integers, respectively. Which of the following statements are logically equivalent to each other?

1. If the square of an integer is odd, then the integer is odd.
2. $\forall n\ [p(n)$ is necessary for $q(n)]$
3. The square of an odd integer is odd.
4. There are some integers whose squares are odd.
5. Given an integer whose square is odd, that integer is likewise odd.
6. $\forall n\ [\neg p(n) \rightarrow \neg q(n)]$
7. $\forall n\ [p(n)$ is sufficient for $q(n)]$

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Example (Q14)

Let $p(n), q(n)$ represent the open statements "n is odd" and "n^2 is odd" for the universe of all integers, respectively. Which of the following statements are logically equivalent to each other?

1. If the square of an integer is odd, then the integer is odd.
   $\forall n\ [q(n) \rightarrow p(n)]$
2. $\forall n\ [p(n)$ is necessary for $q(n)]$ $\quad \forall n\ [q(n) \rightarrow p(n)]$
3. The square of an odd integer is odd. $\quad \forall n\ [p(n) \rightarrow q(n)]$
4. There are some integers whose squares are odd. $\quad \exists n\ q(n)$
5. Given an integer whose square is odd, that integer is likewise odd.
   $\forall n\ [q(n) \rightarrow p(n)]$
6. $\forall n\ [\neg p(n) \rightarrow \neg q(n)]$ $\quad \forall n\ [q(n) \rightarrow p(n)]$
7. $\forall n\ [p(n)$ is sufficient for $q(n)]$ $\quad \forall n\ [p(n) \rightarrow q(n)]$

The Use of Quantifiers

Example (2.53.b)

Consider the universe of all triangles in the plane in conjunction with the open statements

- $p(t)$: t has two sides of equal length.
- $q(t)$: t is an isosceles triangle.
- $r(t)$: t has two angles of equal measure.

Consider a particular triangle c. Here is an argument:

1. In triangle c there is no pair of angles of equal measure. $\neg r(c)$
2. If a triangle has two sides of equal length, it is isosceles. $\forall t\ [p(t) \to q(t)]$
3. If a triangle is isosceles, then it has two angles of equal measure. $\forall t\ [q(t) \to r(t)]$
4. Therefore c has no two sides of equal length.

The Use of Quantifiers

Example (2.53.b cont'd)

We are interested in proving that the argument above is valid, i.e., $\neg p(c)$, i.e., c has no two sides of equal length.

	Steps	**Reasons**
1)	$\forall t\ [p(t) \rightarrow q(t)]$	Premise
2)	$p(c) \rightarrow q(c)$	Rule of Univ. Specification (Step 1)
3)	$\forall t\ [q(t) \rightarrow r(t)]$	Premise
4)	$q(c) \rightarrow r(c)$	Rule of Univ. Specification (Step 3)
5)	$p(c) \rightarrow r(c)$	Law of Syllogism (Steps 2,4)
6)	$\neg r(c)$	Premise
7)	$\therefore \neg p(c)$	Modus Tollens (Steps 5,6)

The Use of Quantifiers

Example (Prove:)

$$
\begin{array}{l}
\forall x\ [p(x) \vee q(x)] \\
\exists x\ \neg p(x) \\
\forall x\ [\neg q(x) \vee r(x)] \\
\forall x\ [s(x) \rightarrow \neg r(x)] \\
\hline
\therefore \exists x\ \neg s(x)
\end{array}
$$

The Use of Quantifiers

Example (Proof)

	Steps	Reasons
1)	$\forall x\ [p(x) \vee q(x)]$	Premise
2)	$\exists x\ \neg p(x)$	Premise
3)	$\forall x\ [\neg q(x) \vee r(x)]$	Premise
4)	$\forall x\ [s(x) \rightarrow \neg r(x)]$	Premise
5)	$\ldots$	$\ldots$
6)	$\ldots$	$\ldots$
7)	$\therefore \exists x\ \neg s(x)$	$\ldots$