CHAPTER 2

DETERMINANTS

We are familiar with functions which associate a real number $f(x)$ with a real value of the variable x. In this chapter we shall study the determinant function, which is a real-valued function of a matrix variable in the sense that it associates a real number $f(A)$ with a matrix A. Determinant functions have important applications to the theory of linear systems and also lead us to an explicit formula for the inverse of an invertible matrix.

2.1. The Determinant Concept

To motivate the definition of determinant, we consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{a}I} \left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{II-cI} \left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$
$$\xrightarrow{\frac{a}{ad-bc}II} \left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{I-\frac{b}{a}II} \left[\begin{array}{cc:cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 1 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

we have

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This result gives a formula for the inverse of any 2×2 matrix and enables us to compute a single number $D = ad - bc$ which implies that whether A is invertible. The condition for invertibility is $D \neq 0$. The quantity $D = ad - bc$ is called the determinant of the 2×2 matrix A and is denoted by the symbol $\det(A)$ or $|A|$. With this notation, the formula for A^{-1} is

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

One of the goals of this chapter is to obtain analogs of this formula to square matrices of higher size. There are several definitions of determinants, but the following will suit our purposes. It is sometimes referred to as expansion down the first column.

- **Determinant**

  *The **determinant** of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar quantity $\det(A)$ defined recursively as follows:*

  ·If $n = 1$, then $\det(A) = a_{11}$.

  ·If $n > 1$, then

$$\begin{aligned}\det(A) &= \sum_{k=1}^{n} a_{k1}(-1)^{k+1} M_{k1}(A) \\ &= a_{11}M_{11}(A) - a_{21}M_{21}(A) + \cdots + (-1)^{n+1} a_{n1}M_{n1}(A)\end{aligned}$$

  where $M_{ij}(A)$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column of A.

We note that the determinant of a matrix A is a scalar number, it is not a matrix.

- **Minors and cofactors**

  *The number $M_{ij}(A)$ is called the (i,j)th **minor** of the matrix A. The (i,j)th **cofactor** of the matrix A is defined as $C_{ij} = (-1)^{i+j} M_{ij}(A)$. In the terminology of cofactors, we have*

$$\det(A) = \sum_{k=1}^{n} a_{k1}C_{k1}.$$

Example: For

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

we have

$$M_{11}(A) = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40 - 24 = 16, \quad C_{11} = (-1)^{1+1} M_{11}(A) = 16$$

and

$$M_{32}(A) = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 18 + 8 = 26, \quad C_{32} = (-1)^{3+2} M_{32}(A) = -26. \blacksquare$$

Example: Use the definition to compute the determinants of the following matrices.

(a) $A = [-4]$ **(b)** $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **(c)** $C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Solution:

(a) From the first part of the definition we see that $\det(A) = -4$.

(b) Using the formula in the definition, we find that

$$\det(B) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = aM_{11} - cM_{21} = a\det([d]) - c\det([b]) = ad - bc.$$

(c) We use the definition to obtain

$$\det(C) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 6 - 2 = 4. \blacksquare$$

2.2. Laws of Determinants

Our main goal in this section is to show that determinants have a significant property: a matrix is not invertible exactly when its determinant is 0. Along the way we will examine some useful properties of determinants.

Throughout this section we assume that the matrices are square, and that $A = [a_{ij}]$ is an $n \times n$ matrix, unless otherwise is stated.

- **Property 1**

  If A is an upper triangular matrix, then the determinant of A is the product of all the diagonal elements of A.

Proof: Since

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

we see that $a_{k1} = 0$ if $k > 1$. So

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \\ &= \cdots = a_{11}a_{22}\cdots a_{nn}. \blacksquare \end{aligned}$$

For example, we have

$$\begin{vmatrix} 4 & 4 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 4 \cdot (-1) \cdot 2 \cdot 2 = -16$$

and

$$\det(I_n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1.$$

- **Property 2**

  If B is obtained from A by multiplying one row of A by the scalar c, then $\det(B) = c\det(A)$.

Here is a simple illustration:

Example: Compute the determinant of

$$A = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: We have

$$\det(A) = \begin{vmatrix} 5 & 0 & 10 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 0 & 2 \\ 5 & 5 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 25 \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 25 \left(\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + 0 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \right) = 50. \blacksquare$$

- **Property 3**

  If B is obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$.

We can use this property to show the following useful fact.

Example: Show that if a determinant has a repeated row, then it must be 0.

Solution: Suppose that the ith and jth rows of the matrix A are identical, and B is obtained by switching these two rows of A. Clearly, $B = A$. According to above result, $\det(B) = -\det(A)$. It follows that

$$\det(A) = -\det(A) \Rightarrow 2\det(A) = 0,$$

so $\det(A) = 0.\blacksquare$

- **Property 4**

  If B is obtained from A by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

Example: Compute

$$\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

Solution: What above result says is that adding a multiple of one row to another row will not change the determinant. Hence

$$\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} \underset{IV-\frac{1}{2}III}{\overset{II-I}{=}} \begin{vmatrix} 1 & 4 & 1 & 1 \\ 0 & -5 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & \frac{1}{2} \end{vmatrix} = -5. \blacksquare$$

Example: Show that a matrix with a row of zeros has zero determinant.

Solution: Suppose that A has a row of zeros. If we add any other row of the matrix A to this zero row to obtain a matrix B with repeated rows, then $\det(B) = 0 = \det(A)$.$\blacksquare$

If we let $A = I_n$ in the last three results, then the matrix B is elementary, and we have the following conclusion.

- **Determinants of elementary matrices**

  Let E be an $n \times n$ elementary matrix.

  (1) *If E results from multiplying a row of I_n by a scalar c, then* $\det(E) = c$.

  (2) *If E results from I_n by interchanging two rows of I_n, then* $\det(E) = -1$.

  (3) *If E results from I_n by adding a multiple of one row of I_n to another row of I_n, then* $\det(E) = 1$.

For the next property, we apply a sequence of elementary row operations on the $n \times n$ matrix A to reduce it to its reduced row-echelon form R, or equivalently we multiply A on the left by elementary matrices $E_1, E_2, \ldots, E_k$ and obtain

$$R = E_1 E_2 \cdots E_k A.$$

We take the determinant of both sides to obtain

$$\det(R) = \det(E_1 E_2 \cdots E_k A) = \mp(\text{nonzero constant})\det(A).$$

Therefore, $\det(A) = 0$ precisely when $\det(R) = 0$. Now the reduced row-echelon form of A is certainly upper triangular. In fact, on the main diagonal it has zeros, and hence have zero determinant unless $R = I_n$. But if $R = I_n$, then A is invertible. Thus we have

- **Invertibility and determinant**

  The matrix A is invertible if and only if $\det(A) \neq 0$.

This result enables us to add one more statement to the list of equivalent stataments.

- **Equivalent statements III**

  If A is an $n \times n$ matrix, then the following statements are equivalent:

  (1) *A is invertible.*

  (2) *$Ax = \mathbf{0}$ has only the trivial solution.*

  (3) *The reduced row echelon form of A is I_n.*

  (4) *A is expressible as a product of elementary matrices.*

  (5) *$Ax = b$ is consistent for every $n \times 1$ matrix b.*

  (6) *$Ax = b$ has exactly one solution for every $n \times 1$ matrix b.*

  (7) $\det(A) \neq 0$.

It is a surprising fact that the determinants do not distribute over sums.

- **Determinant of an addition**

  In general, we have

$$\det(A + B) \neq \det(A) + \det(B).$$

For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

then we have

$$A + B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

and $5 = \det(A + B) \neq 1 + 2 = \det(A) + \det(B)$.

Although determinants do not preserve addition, they are additive in one row at a time.

- **Property 5**

  Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then

$$\det(C) = \det(A) + \det(B).$$

Example: Let

$$A = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7-1 \end{bmatrix}$$

Since

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} - 2 \begin{vmatrix} 7 & 5 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 7 & 5 \\ 0 & 3 \end{vmatrix} = -49 \\ \det(B) &= \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 7 & 5 \\ 1 & -1 \end{vmatrix} + 0 \begin{vmatrix} 7 & 5 \\ 0 & 3 \end{vmatrix} = 21 \\ \det(C) &= \begin{vmatrix} 0 & 3 \\ 5 & 6 \end{vmatrix} - 2 \begin{vmatrix} 7 & 5 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 7 & 5 \\ 0 & 3 \end{vmatrix} = -28, \end{aligned}$$

we have $-28 = \det(C) = -49 + 21 = \det(A) + \det(B)$.■

Considering the complexity of the formulas for determinants and matrix multiplication, a simple relation between them is unlikely to happen. However, we have something so surprising.

- **Determinant of a product**

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B).$$

Proof: We first show that if B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B).$$

We consider three cases each corresponding to row operations that produces the matrix E.

·If E results from multiplying a row of I_n by c, then EB results from B by multiplying the corresponding row by c, so

$$\det(EB) = c\det(B).$$

But $\det(E) = c$ in this case, so $\det(EB) = \det(E)\det(B)$.

·If E results from interchanging two rows of I_n, then EB results from B by interchanging corresponding two rows. Thus

$$\det(EB) = -\det(B).$$

Since $\det(E) = -1$ in this case, we have $\det(EB) = \det(E)\det(B)$.

·If E results from adding a multiple of one row of I_n to its another row, then EB results from B by adding a multiple of one row to its another row correspondingly, so

$$\det(EB) = \det(B).$$

But $\det(E) = 1$ in this case, so $\det(EB) = \det(E)\det(B)$.

It follows by repeated applications that if B is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1E_2\cdots E_rB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B).$$

Now, if A is not invertible, then AB is not invertible either. Thus, $\det(AB) = 0$ and $\det(A) = 0$, so it follows that

$$\det(AB) = \det(A)\det(B).$$

On the other hand, if A is invertible, then it can be expressed as a product of elementary matrices, say, $A = E_1E_2\cdots E_r$, so $AB = E_1E_2\cdots E_rB$. Then we have

$$\det(AB) = \det(E_1E_2\cdots E_rB) = \det(E_1E_2\cdots E_r)\det(B) = \det(A)\det(B),$$

which is the desired equality.■

A consequence of this result shows the relationship between determinant of an invertible matrix and determinant of its inverse.

- **Determinant of the inverse matrix**

  If A is an invertible matrix, then

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

Proof: If A is invertible, then $\det(A) \neq 0$. We then have

$$AA^{-1} = I \Rightarrow \det\left(AA^{-1}\right) = \det(I) \Rightarrow \det\left(A^{-1}\right)\det(A) = 1 \Rightarrow \det\left(A^{-1}\right) = \frac{1}{\det(A)},$$

the desired result.■

Some simple cases suggest that there is no difference between determinants of A and A^T. This is exactly what happens in general.

- **Determinant of transpose**

  For all square matrices A, we have $\det\left(A^T\right) = \det(A)$.

Proof: Since $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$, we note that A is invertible if and only if its transpose is invertible. First assume that $\det(A) = 0$. Then A is not invertible. A^{-1} is not invertible either, and hence $\det\left(A^T\right) = 0$, which implies that $\det(A) = \det\left(A^T\right)$. Now let $\det(A) \neq 0$. Then A is invertible, so there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $A = E_1 E_2 \cdots E_k$. If we knew for each elementary matrix E that $\det(E) = \det\left(E^T\right)$, then it would follow that

$$\begin{aligned}
\det\left(A^T\right) &= \det\left(\left(E_1 E_2 \cdots E_k\right)^T\right) = \det\left(E_k^T \cdots E_2^T E_1^T\right) \\
&= \det\left(E_k^T\right) \cdots \det\left(E_2^T\right) \det\left(E_1^T\right) = \det\left(E_k\right) \cdots \det\left(E_2\right) \det\left(E_1\right) \\
&= \det\left(E_1 E_2 \cdots E_k\right) = \det(A).
\end{aligned}$$

Now, if E is obtained by multiplying a row of I_n by a nonzero constant, then $E = E^T$, so $\det(E) = \det\left(E^T\right)$. If E is obtained by interchanging two rows of I_n, then $E = E^T$, so $\det(E) = \det\left(E^T\right)$. If E is obtained by adding a multiple of a row of I_n to its another row, than E^T is of this form. Since such an elementary matrix has determinant 1, we again have $\det(E) = \det\left(E^T\right)$. This completes the proof.■

Since transposing a matrix interchanges the rows and columns of the matrix, above result implies that everything said about rows of determinants applies equally well to the columns, including the definition of determinant itself. Therefore, the definition of determinant can be given in terms of expanding across the first row. Likewise, elementary column operations can be performed instead of row operations. By interchanging rows or columns, and then expanding by first row or column, we see that the same effect is obtained by simply expanding the determinant down any column or across any row.

All these properties can be put together and compute determinants no worse than Gaussian elimination in the amount of work has to be done. We simply reduce a matrix to triangular form by elementary operations, then take the product of the diagonal terms.

Example: Calculate $\det(A)$ when

$$A = \begin{bmatrix} 3 & 0 & 6 & 6 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}$$

Solution: We have

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 6 & 6 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix} &= 3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix} \overset{II-I}{\underset{\substack{III-2II \\ IV+I}}{=}} 3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -4 & -3 \\ 0 & 2 & 2 & 2 \end{vmatrix} \overset{II \leftrightarrow IV}{=} -3 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & -1 \end{vmatrix} \\ &= (-3)\,2\,(-4)\,(-1) = -24 \blacksquare \end{aligned}$$

Example: Calculate $\det(A)$ when

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Solution: We start by expanding the determinant down the second column since it has the most zeros. We then have

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

Now we again expand the determinant down the second column to obtain

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -6. \blacksquare$$

2.3. A Formula for the Inverse of a Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We can calculate $\det(A)$ by expanding the determinant down any column of A. These expansions lead to cofactor formulas for each column number j, namely,

$$\det(A) = \sum_{k=1}^{n} C_{kj} a_{kj}.$$

This formula looks like a matrix multiplication formula. We now consider the sum

$$\sum_{k=1}^{n} C_{ki}a_{kj} = C_{1i}a_{1j} + C_{2i}a_{2j} + \cdots + C_{ni}a_{nj}.$$

This formula implies that the ith column of the matrix A is replaced by its jth column and then the determinant of the resulting matrix is computed by expanding down the ith column. But such a matrix has two equal columns and hence has a zero determinant. So, this sum is 0 unless $i \neq j$. We can combine these two sums in the formula

$$\sum_{k=1}^{n} C_{ki}a_{kj} = \delta_{ij} \det(A),$$

where δ_{ij} is the Kronecker delta symbol (that is, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise). In order to use this formula we make the following definitions:

- **Minors, cofactors, and adjoint matrices**

  The ***matrix of minors*** *of the $n \times n$ matrix $A = [a_{ij}]$ is the matrix $M(A) = [M_{ij}(A)]$ of the same size. The* ***matrix of cofactors*** *of A is the matrix $C(A) = [C_{ij}]$ of the same size. The* ***adjoint matrix*** *of A is the matrix $adj(A) = C(A)^T = [C_{ij}]^T$.*

Example: Compute the determinant, minors, cofactors, and adjoint matrices for

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

and compute $A\text{adj}(A)$.

Solution: We have

$$\det(A) = 1 \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 2,$$

$$M(A) = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

$$C(A) = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & -2 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

and

$$A\operatorname{adj}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = [\det(A)]\, I_3. \blacksquare$$

Last part of the above example holds in general.

- **Adjoint formula**

  For a square matrix A, we have

$$A\operatorname{adj}(A) = \operatorname{adj}(A)\, A = [\det(A)]\, I.$$

Proof: Consider the product

$$A\operatorname{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

The entry in the ith row and jth column of the product $A\operatorname{adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

If $i = j$, then this is the cofactor expansion of $\det(A)$ across the ith row of A. If $i \neq j$, then this value is zero. Therefore, we have

$$A\operatorname{adj}(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = [\det(A)]\, I,$$

the desired result.■

We already know that A is invertible exactly when $\det(A) \neq 0$. This fact together with the above result gives an inverse formula.

- **Inverse formula**

  If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

For example, for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

we already know that $\det(A) = 2$ and

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

Thus,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 \end{bmatrix}$$

2.4. Cramer's Rule

Using inverse formula, we can find an explicit formula for solving linear systems with invertible coefficient matrices.

- **Cramer's rule**

  Let A be an invertible $n \times n$ matrix and b be an $n \times 1$ column matrix. If B_j is the matrix obtained from A by replacing the jth column of A by b, then the linear system $Ax = b$ has unique solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where

$$x_j = \frac{\det(B_j)}{\det(A)}, \quad j = 1, 2, \ldots, n.$$

Proof: We multiply both sides of $Ax = b$ on the left A^{-1} to obtain $x = A^{-1}b$. Now we use the inverse formula to obtain

$$x = A^{-1}b = \frac{1}{\det(A)}\operatorname{adj}(A)\,b.$$

Then the explicit formula for the jth coordinate of x is

$$x_j = \frac{1}{\det(A)}\sum_{i=1}^{n} C_{ij}b_i.$$

The sum on the right is the determinant of B_j, so we have

$$x_j = \frac{\det(B_j)}{\det(A)},$$

the desired result.■

Example: Use Cramer's rule to solve

$$\begin{array}{ccccccc} x & + & y & + & z & = & 4 \\ 2x & + & 2y & + & 5z & = & 11 \\ 4x & + & 6y & + & 8z & = & 24 \end{array}$$

Solution: We write the system in matrix equation $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 11 \\ 24 \end{bmatrix}$$

We have

$$\det(A) = \begin{vmatrix} 2 & 5 \\ 6 & 8 \end{vmatrix} - \begin{vmatrix} 2 & 5 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 4 & 6 \end{vmatrix} = -6$$

$$B_1 = \begin{bmatrix} 4 & 1 & 1 \\ 11 & 2 & 5 \\ 24 & 6 & 8 \end{bmatrix} \Rightarrow \det(B_1) = 4\begin{vmatrix} 2 & 5 \\ 6 & 8 \end{vmatrix} - \begin{vmatrix} 11 & 5 \\ 24 & 8 \end{vmatrix} + \begin{vmatrix} 11 & 2 \\ 24 & 6 \end{vmatrix} = -6$$

$$B_2 = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 11 & 5 \\ 4 & 24 & 8 \end{bmatrix} \Rightarrow \det(B_2) = \begin{vmatrix} 11 & 5 \\ 24 & 8 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 24 & 8 \end{vmatrix} + 4\begin{vmatrix} 4 & 1 \\ 11 & 6 \end{vmatrix} = -12$$

$$B_3 = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 11 \\ 4 & 6 & 24 \end{bmatrix} \Rightarrow \det(B_3) = \begin{vmatrix} 2 & 11 \\ 6 & 24 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ 6 & 24 \end{vmatrix} + 4\begin{vmatrix} 1 & 4 \\ 2 & 11 \end{vmatrix} = -6$$

Thus

$$x = \frac{\det(B_1)}{\det(A)} = \frac{-6}{-6} = 1, \quad y = \frac{\det(B_2)}{\det(A)} = \frac{-12}{-6} = 2, \quad z = \frac{\det(B_3)}{\det(A)} = \frac{-6}{-6} = 1.$$■