MAT222
LINEAR ALGEBRA
HOMEWORK ASSIGNMENT 1
SOLUTIONS

(1) Suppose that the row echelon form of the augmented matrix of a given linear system is

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & m & m & \vdots & m^2-m \\ 0 & 0 & m^2-m & \vdots & m \end{bmatrix}$$

Determine (and explain your reasoning) whether
(a) the system has infinitely many solutions depending on one parameter if $m = 0$
(b) the system has infinitely many solutions depending on one parameter if $m = 1$
(c) the system is inconsistent for $m = 1$
(d) the system has infinitely many solutions for $m = 0$ and $m = 1$
(e) the system has exactly one solution for $m \neq 0$

Solution:
(a) If $m = 0$, then we have

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

which shows that the system has infinitely many solutions depending on two parameters. Hence the statement is false.
(b) If $m = 1$, then we have

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

which shows that the system has no solutions. Hence the statement is false.
(c) If $m = 1$, then we have

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

which shows that the system is inconsistent. Hence the statement is true.
(d) Because of parts (a) and (b), the statement is false.
(e) If $m \neq 0$, then we have

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & m & m & \vdots & m^2-m \\ 0 & 0 & m^2-m & \vdots & m \end{bmatrix} \xrightarrow[\frac{1}{m^2-m}III]{\frac{1}{m}II} \begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & m-1 \\ 0 & 0 & 1 & \vdots & \frac{1}{m-1} \end{bmatrix}$$

which shows that the system has no solutions if $m = 1$. Hence the statement is false.■

(2) Determine the values of k for which the system

$$\begin{aligned} y + 2kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2z &= 1 \end{aligned}$$

has no solution.

Solution: The augmented matrix corresponding to the system is

$$\begin{bmatrix} 0 & 1 & 2k & \vdots & 0 \\ 1 & 2 & 6 & \vdots & 2 \\ k & 0 & 2 & \vdots & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 0 & 1 & 2k & \vdots & 0 \\ 1 & 2 & 6 & \vdots & 2 \\ k & 0 & 2 & \vdots & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ k & 0 & 2 & \vdots & 1 \end{bmatrix} \xrightarrow{R_3 - kR_1} \begin{bmatrix} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & -2k & 2-6k & \vdots & 1-2k \end{bmatrix}$$

$$\xrightarrow[R_3 + 2kR_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 6-4k & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & 0 & 2(2k-1)(k-1) & \vdots & 1-2k \end{bmatrix}.$$

The system has no solution when

$$2(2k-1)(k-1) = 0 \quad \text{and} \quad 1 - 2k \neq 0.$$

This happens only when $k = 1$.■

(3) A conic section is a curve in $\mathbb{R}^2$ that can be described by an equation of the form

$$f(x, y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 = 0,$$

where at least one of the coefficients c_k is nonzero. Find all conics through the points $(1, 0)$, $(2, 0)$, $(2, 2)$, $(5, 2)$, and $(5, 6)$ using Gauss-Jordan elimination.

Solution: We have the system of equations

$$\begin{array}{ccccccccccccc} & & & & & & c_1 & + & c_2 & + & c_4 & = & 0 \\ & & & & & & c_1 & + & 2c_2 & + & 4c_4 & = & 0 \\ c_1 & + & 2c_2 & + & 2c_3 & + & 4c_4 & + & 4c_5 & + & 4c_6 & = & 0 \\ c_1 & + & 5c_2 & + & 2c_3 & + & 25c_4 & + & 10c_5 & + & 4c_6 & = & 0 \\ c_1 & + & 5c_2 & + & 6c_3 & + & 25c_4 & + & 30c_5 & + & 36c_6 & = & 0 \end{array}$$

with the corresponding augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 \\ 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 5 & 2 & 25 & 10 & 4 \\ 1 & 5 & 6 & 25 & 30 & 36 \end{bmatrix}.$$

We have

$$\begin{bmatrix}1&1&0&1&0&0\\1&2&0&4&0&0\\1&2&2&4&4&4\\1&5&2&25&10&4\\1&5&6&25&30&36\end{bmatrix}\xrightarrow[\substack{R_4-R_1\\R_5-R_1}]{\substack{R_2-R_1\\R_3-R_1}}\begin{bmatrix}1&1&0&1&0&0\\0&1&0&3&0&0\\0&1&2&3&4&4\\0&4&2&24&10&4\\0&4&6&24&30&36\end{bmatrix}\xrightarrow[\substack{R_4-4R_2\\R_5-4R_2}]{\substack{R_1-R_2\\R_3-R_2}}\begin{bmatrix}1&0&0&-2&0&0\\0&1&0&3&0&0\\0&0&2&0&4&4\\0&0&2&12&10&4\\0&0&6&12&30&36\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3}\begin{bmatrix}1&0&0&-2&0&0\\0&1&0&3&0&0\\0&0&1&0&2&2\\0&0&2&12&10&4\\0&0&6&12&30&36\end{bmatrix}\xrightarrow[R_5-6R_3]{R_4-2R_3}\begin{bmatrix}1&0&0&-2&0&0\\0&1&0&3&0&0\\0&0&1&0&2&2\\0&0&0&12&6&0\\0&0&0&12&18&24\end{bmatrix}\xrightarrow{\frac{1}{12}R_4}\begin{bmatrix}1&0&0&-2&0&0\\0&1&0&3&0&0\\0&0&1&0&2&2\\0&0&0&1&\frac{1}{2}&0\\0&0&0&12&18&24\end{bmatrix}$$

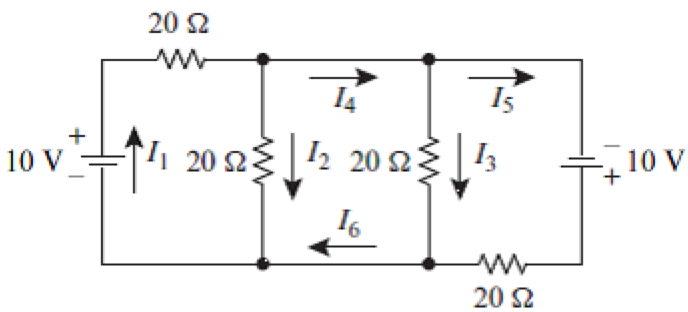
$$\xrightarrow[\substack{R_2-3R_4\\R_5-12R_4}]{R_1+2R_4}\begin{bmatrix}1&0&0&0&1&0\\0&1&0&0&-\frac{3}{2}&0\\0&0&1&0&2&2\\0&0&0&1&\frac{1}{2}&0\\0&0&0&0&12&24\end{bmatrix}\xrightarrow{\frac{1}{12}R_5}\begin{bmatrix}1&0&0&0&1&0\\0&1&0&0&-\frac{3}{2}&0\\0&0&1&0&2&2\\0&0&0&1&\frac{1}{2}&0\\0&0&0&0&1&2\end{bmatrix}\xrightarrow[\substack{R_3-2R_5\\R_4-\frac{1}{2}R_5}]{\substack{R_1-R_5\\R_2+\frac{3}{2}R_5}}\begin{bmatrix}1&0&0&0&0&-2\\0&1&0&0&0&3\\0&0&1&0&0&-2\\0&0&0&1&0&-1\\0&0&0&0&1&2\end{bmatrix}.$$

Thus, the system reduces to

$$\begin{array}{rcrcl}c_1&-&2c_6&=&0\\c_2&+&3c_6&=&0\\c_3&-&2c_6&=&0\\c_4&-&c_6&=&0\\c_5&+&2c_6&=&0\end{array}$$

The solution set is then found to be $\{(2c_6,-3c_6,2c_6,c_6,-2c_6,c_6):c_6\in\mathbb{R}\}$. Choosing, for example, $c_6=1$, we find the conic $2-3x+2y+x^2-2xy+y^2=0.$■

(4) Analyze the given electrical circuit by finding the unknown currents.



Solution: By Kirchhoff's current and voltage laws, we have

$$\begin{array}{rcrcrcrcl}&&I_1&=&I_2&+&I_4\\&&I_2&=&I_1&-&I_6\\&&I_4&=&I_3&+&I_5\\&&I_3&=&I_6&-&I_5\\20I_1&+&20I_2&=&10\\20I_3&-&20I_2&=&0\\-20I_3&+&20I_5&=&10\\20I_1&+&20I_5&=&20\end{array}\quad\Rightarrow\quad\begin{array}{rcrcrcl}I_1&-&I_2&-&I_4&=&0\\I_1&-&I_2&-&I_6&=&0\\I_3&-&I_4&+&I_5&=&0\\I_3&+&I_5&-&I_6&=&0\\&&2I_1&+&2I_2&=&1\\&&-2I_2&+&2I_3&=&0\\&&-2I_3&+&2I_5&=&1\\&&2I_1&+&2I_5&=&2\end{array}$$

In the augmented matrix form, we have

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[R_4-2R_1]{\substack{R_2-R_1 \\ R_3-R_1}} \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[interchangings]{Row} \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[R_4+R_2]{\substack{R_1+R_2 \\ R_3-4R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & -4 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[interchangings]{Row} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & -4 & 0 & -3 \end{bmatrix} \xrightarrow[R_6+2R_3]{\substack{R_4-R_3 \\ R_5-R_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & -4 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow[R_8+R_6]{\substack{R_3+R_5 \\ R_7+R_5}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the system reduces to

$$\begin{array}{rcrcrcl} & & & & -2I_6 & = & -1 \\ & & & & -2I_4 & = & -1 \\ & & I_1 & + & I_5 & = & 1 \\ & & 2I_4 & + & 4I_5 & = & 3 \\ I_2 & + & I_5 & + & I_6 & = & 1 \\ I_3 & + & I_5 & - & I_6 & = & 0 \end{array}$$

The solution is then found to be $I_1 = I_4 = I_5 = I_6 = 0.5$, $I_2 = I_3 = 0.$■

(5) Let A be the matrix of size 4×10 with entries $a_{ij} = \dfrac{j}{j+i}$ and B be the matrix of size 10×3 with entries $b_{ij} = \dfrac{j}{i^2+i}$. If $AB = C$ with entries c_{ij}, find c_{21}.

Solution: By the definition of matrix multiplication, we have

$$c_{21} = \sum_{k=1}^{10} a_{2k} b_{k1}.$$

We then find

$$\begin{aligned} c_{21} &= \sum_{k=1}^{10} a_{2k} b_{k1} = \sum_{k=1}^{10} \left(\frac{k}{k+2} \cdot \frac{1}{k^2+k} \right) = \sum_{k=1}^{10} \frac{1}{(k+2)(k+1)} \\ &= \sum_{k=1}^{10} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{10} - \frac{1}{11} + \frac{1}{11} - \frac{1}{12} \\ &= \frac{1}{2} - \frac{1}{12} = \frac{5}{12}, \end{aligned}$$

the desired entry.■