Discrete Mathematics

Lecture 1: Counting

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October 4, 2023

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The Rule of Sum

Suppose we have two different tasks to be done this weekend. But any task will take the whole weekend. If the first task can be performed in m ways, and the second can be performed in n ways, in how many ways can you perform a task on the weekend?

There are $m + n$ ways.

What will you do next Saturday?

1. Stay at home and...
   1. Watch TV all day
   2. Watch a few movies
   3. Do your homework
2. Do some recreation and...
   1. Go to picnic
   2. Go to the beach
   3. Cycling

Examples

A college library has 200 textbooks in physics and 350 books in economics. How many ways can a student learn about one or the other of these two subjects? Let's assume learning both is not possible.

The boss assigns 12 employees to two committees. Committee A consists of five members. Committee B consists of seven members. If the boss speak to just one member before making a decision, in how many ways can s/he do that?

Examples

Imagine you're a software engineer working on the user interface of an online store. You're responsible for developing a navigation menu that helps users quickly find products.

- Clothing:
  - Men's
  - Women's
  - Kids'
- Electronics:
  - Mobile Phones
  - Laptops
  - Audio Equipment
- Books:
  - Fiction
  - Non-fiction
  - Children's Books

We want to understand the number of primary choices a user has when navigating this menu.

The Rule of Product

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

Now reconsider your weekend:

1. Pick one task for Saturday:
    1. Watch TV all day
    2. Watch a few movies
    3. Do your homework
2. Pick another for Sunday:
    1. Go to picnic
    2. Go to the beach
    3. Cycling

Examples

- The drama club of a college is holding tryouts for a play. Eight men and five women are auditioning for the leading male and female roles. In how many ways can the director cast the leading couple?
- A restaurant sells six kinds of soups, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Each day a student buys a lunch consisting of a soup and a cold beverage, or a sandwich and a hot beverage. In how many ways can the student buy lunch?

Examples

- How many license plates can be produced if they have to consist of two letters followed by four digits?
  - With repetition of letters and digits allowed? (No constraints)
  - If no letter or digit can be repeated?
  - If repetitions are allowed, how many of the plates have only vowels and even digits?
- How about Turkish plate numbers?

Examples

Scenario: Combinatorial Testing of User Permissions Imagine you're developing a complex web application that has various types of user permissions. For instance:

- User Roles:
  - Admin
  - Moderator
  - Contributor
  - Viewer
- Resource Access:
  - Read
  - Write
  - Delete
  - Share
- Resource Types:
  - Documents
  - Images
  - Videos

We need to ensure each user role has the appropriate permissions for each resource. How many test cases are there?

Permuting Objects

- Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection / these objects.
- So, in how many ways, can you permute n objects?
- n factorial, denoted $n!$ is defined by:

$$0! = 1$$

$$n! = n(n-1)(n-2)\ldots(2)(1) \text{ for } n \geq 1$$

- As you can see, this is just a simple application of the rule of product.

Permutations

If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is defined as

$$\begin{aligned} P(n,r) &= n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned} \tag{1}$$

Examples

- In a class of 10 students, four are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?
- If five letters are to be chosen from COMPUTER, how many arrangements are there? What if the repetition is allowed?
- What is the number of linear arrangements of the four letters BALL?

Identical objects case

Table 1.1

(a)				(b)							
A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

- We can temporarily distinguish two Ls first, and then get rid of this distinction by dividing the total with permutations of these Ls themselves.
- 2 x Number of arrangements of B, A, L, L is the same as
- Number of permutations of the symbols B, A, L_1, L_2.

General formula for identical object case

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, ..., and n_r indistinguishable objects of a rth type, where

$$n_1 + n_2 + \ldots + n_r = n$$

Then, there are

$$\frac{n!}{(n_1)!(n_2)!\ldots(n_r)!}$$

(linear) arrangements of the given n objects.

Examples

- In how many ways can we arrange the letters of the word MASSASAUGA?
- In how many of these arrangements, do we have all four As together?
- How about MISSISSIPPI?

Examples

- Prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer.
- If six people are seated around a round table, how many different circular arrangements are possible, assuming arrangements are considered the same when one can be obtained from the other by rotation?
- Suppose that the above six people are three married couples. In how many of these circular arrangements, neither two men nor two women sits next to each other? (In other words, sexes should alternate in these arrangements.)

Combinations

Combination

If we want to select r of n distinct objects, we have

$$\frac{n!}{(n-r)!r!}$$

possible ways. This is called a combination and denoted as $C(n,r)$ or $\binom{n}{r}$ and read as "n choose r".

Note that $C(n,r) = P(n,r)/r!$. Why?

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Combinations

Example (1.22)

1. The gym teacher of example 1.21 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams?

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Binominal Theorem

Theorem (The Binomial Theorem)

If x and y are variables and n is a positive integer, then

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \dots \\ &\quad + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}\end{aligned} \tag{2}$$

Here, $\binom{n}{k}$ is referred as a binomial coefficient.

Binomial Theorem Application

Example 1.26

1. Find the coefficient of x^5y^2 in the expansion of $(x+y)^7$.
2. Find the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$.

Corollary (Corollary 1.1)

It follows from the Binomial Theorem that, for each integer $n > 0$,

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$, *and*
2. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$

Prove the statements of Corollary 1.1.

Binomial Coefficient Explained

Find the coefficient of x^5y^2 in the expansion of $(x+y)^7$.

$$(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$

When we expand this product, we get term x^5y^2 whenever we choose five xs and two ys from the above seven multiplicands.

$$(x+y)^7 = (\dot{x}+y)(x+\dot{y})(\dot{x}+y)(\dot{x}+y)(x+\dot{y})(\dot{x}+y)(\dot{x}+y)$$

The number of such possible multiplications (selections) is of course $\binom{7}{5}$ or $\binom{7}{2}$, so, we get that term $\binom{7}{5}$ times. That's why the coefficient is $\binom{7}{5}$.

Multinominal Theorem

Theorem (The Multinominal Theorem)

For positive integers n, t, the coefficient of ${x_1}^{n_1}{x_2}^{n_2}{x_3}^{n_3}\ldots{x_t}^{n_t}$ in the expansion of $(x_1 + x_2 + \ldots + x_t)^n$ is

$$\frac{n!}{n_1!n_2!\ldots n_t!}$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + \ldots + n_t = n$.

Combinations with Repetitions

When repetitions are allowed, we have seen that for n distinct objects an arrangement of size r of these objects can be obtained in n^r ways, for an integer $r \geq 0$.

Example 1.28

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible *from the viewpoint of the restaurant*?
Let c, h, t, and f represent cheeseburger, a hot dog, a taco, or a fish sandwich, respectively.
Here we are concerned with *how many of each item* are purchased, not who purchased what.

Combinations with Repetitions

Example 1.28 cont'd

In the table below we list some possible purchases in purchases column and the other means of representing each purchase in the right column.

#	Purchases	x-| Representation
1.	c,c,h,h,t,f,f	x x | x x | x | x x
2.	c,c,c,c,h,t,f	x x x x | x | x | x
3.	c,c,c,c,c,c,f	x x x x x x | | | x
4.	h,t,t,f,f,f,f	| x | x x | x x x x
5.	t,t,t,t,t,f,f	| | x x x x x | x x
6.	t,t,t,t,t,t,t	| | x x x x x x x |
7.	f,f,f,f,f,f,f	| | | x x x x x x x

Here, bars are used as separators for each type of food. For instance, an x to the left of all bars means a cheeseburger, and an x between 2nd and 3rd bars means a taco.

Combinations with Repetitions

Example 1.28 cont'd

Note that once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in the right column of the Table, we are enumerating all arrangements of 10 symbols consisting of seven xs and three $|$s, so by our correspondence the number of different purchases for column (a) is

$$\frac{10!}{7!3!} = \binom{10}{7}.$$

In this example we note that the seven xs (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the 3+1=4 possible food items that can be chosen.

Combinations with Repetitions

Combinations with repetition

When we wish to select, with repetition, r of n distinct objects, we find that we are considering all arrangement of r xs and $n-1$ $|$s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Consequently, the number of combinations of n objects taken r at time, with repetition, is $C(n+r-1, r)$.

Combinations with Repetitions

Example 1.29

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop. In how many ways can we select a dozen donuts?

Example 1.33

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7,$$

where $x_i \geq 0$ for all $1 \leq i \leq 4$.
One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.)

Number of solutions for an equation

Example 1.29

A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two child, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each non-negative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the identical pennies) from a collection of size 4 (the distinct children), so there are $C(4 + 7 - 1, 7) = 120$ solutions.

A crucial equivalence

At this point it is crucial that we recognize the equivalence of the following:

1. The number of integer solutions of the equation

$$x_1 + x_2 + \ldots + x_n = r,$$

   $x_i \geq 0, 1 \leq i \leq n$.
2. The number of selections, with repetition, of size r from a collection of size n.
3. The number of ways r identical objects can be distributed among n distinct containers.

Number of solutions for an equation

Example 1.34 and 1.35

1. Find the number of non-negative integer solutions to the equation
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10.$$
where $x_i \geq 0$ for all $1 \leq i \leq 6$.
2. How many non-negative integer solutions are there to the inequality
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10.$$
where $x_i \geq 0$ for all $1 \leq i \leq 6$.

Number of solutions for an equation

Example 1.35 cont'd

How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10.$$

It may seem very easy to solve this question by solving

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = k$$

for all $0 \leq k \leq 9$. But what if we have 100 or 1000 instead of 10? So, is there a better way? One that involves just a few calculations?

Number of solutions for an equation

Example 1.35 cont'd

Consider introducing an additional variable x_7 to get the following equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 9,$$

where $x_i \geq 0$ for all $1 \leq i \leq 7$. Note that the number of solutions to this equation is the same as number of solutions to our original inequality! So, we can directly write the result is $\binom{7+9-1}{9} = 5005$.

Number of solutions for an equation

Exercise 7

Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where $x_1, x_2 \geq 5$, $x_3, x_4 \geq 7$

Seemingly Irrelevant Problems

Example (Supplementary 29)

1. In how many ways can a particle move in the xy-plane from the origin to the point (7,4) if the moves that are allowed are:
    - (R): $(x, y) \rightarrow (x+1, y)$
    - (U): $(x, y) \rightarrow (x, y+1)$
2. How many of the paths above do not use the path from (2,2) to (3,2) to (4,2) to (4,3)?
3. Answer the first two questions when a third type of move is allowed:
    - (D): $(x, y) \rightarrow (x+1, y+1)$

Seemingly More Irrelevant Problems

Example

In the play room there are two identical boxes each having 10 balloons. A boy randomly (with equal probability) selects a box and bursts a balloon. He continues this process until the box he selected turns out to be empty. When he stops what is the probability that the other box has exactly 5 balloons?

Summary Table

The following table summarizes the number of choosing r objects from n distinct objects with different rules.

Order	Repetitions	Result	Formula
Relevant	Not allowed	Permutation	$\frac{n!}{(n-r)!}$
Relevant	Allowed	Arrangement	n^r
Irrelevant	Not allowed	Combination	$\binom{n}{r}$
Irrelevant	Allowed	Comb w/ rep	$\binom{n+r-1}{r}$

Catalan Numbers

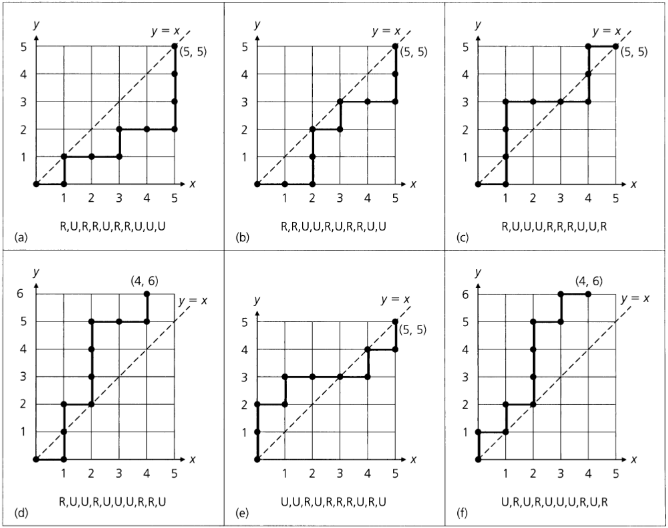
Moving Up(North) or Right(East)

- Consider the Cartesian plane
- We start at (0,0)
- Two types of moves allowed:
  - $R : (x, y) \longrightarrow (x + 1, y)$ (Right move)
  - $L : (x, y) \longrightarrow (x, y + 1)$ (Up move)

The question: Never cross the line!

In how many ways can we go to (5,5) **without ever rising above the line** $x = y$?

Catalan Numbers



Catalan Numbers

- If a path goes above the line, we immediately stop
- and transpose the rest of the path (Us and Rs switched).
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- So, we can easily find the number of paths we should exclude: $\binom{2n}{n-1}$
- So, the number of paths we are looking for is: $b_n = \binom{2n}{n} - \binom{2n}{n-1}$
- Or equivalently, $b_n = \binom{2n}{n}/(n+1)$

Catalan Numbers

Number of Possible Parenthesizations

What is the number of possible ways to parenthesize a string of length n?

Can you use the idea of Catalan Numbers to solve this question?