5/11/2022 Saturday Midterm Duration: 90 minutes

$\binom{16}{8}\left(\binom{8}{4}-\binom{8}{3}\right)\cdot\frac{8!}{4!4!}$

Choose A&T pos. Catalan Place G&Cs.

Name: TTAACCGGGGCCAATT Student No:

TAG . . . . TA A T A T AT A T

P1 [30 pts] Suppose that in a ripped-apart DNA strand, there are 4 adenine(A), 4 cytosine(C), 4 guanine(G) and 4 thymine(T) nucleotides. (For example CATCAGGGCATTCGAT. Note that reverse of a strand is the same as the original one.)

(a) How many such DNA segments can there be? (b) Suppose that As and Ts must alternate. (If Gs and Cs are deleted, there must be no consecutive As or Ts. For example CATCAGGGCATTCGAT will not be counted because when we only take As and Ts, we get ATAATTAT where we can see an AA) (c) Suppose that in no prefix of this stand, the number of As cannot be greater than the number of Ts. (For example, CATCAGGGCATTCGAT will not be counted since the prefix CATCA contains more As then Ts)

(a) $\left(\frac{16!}{4!\,4!\,4!\,4!}\right)\cdot\frac{1}{2}$ $\propto$ incorrect. $P+\frac{Q}{2}$

(b) ACTGGACGTCATCGATC $\left(1\cdot\binom{16}{8}\right)\frac{8!}{4!4!}$ A T AT AT A T

(c) $\left(1\cdot\binom{16}{8}\right)\frac{8!}{4!4!}$ A↑ →T TATTAAA $\binom{8}{4}-\binom{8}{3}$

P2 [20 points]

(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

10000
10001
10002
⋮
99999

$\binom{9}{5}$

(b) How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

$X_i \geq 0$ $x_1+x_2+\dots+x_9=5$ $\binom{13}{5}$

$X_5=5$, $X_i=0$; $x_2=1$, $x_3=2$, $x_7=2$, $x_i=0$ 72337

(c)
```
int a,b,c,d,count = 0;
for(a = 0; a<30; a++)
 for(b=a+3; b<30; b++)
  for(c=b+6; c<30; c++)
   if(c==20)
    for(d=c+2; d<30; d++){
     printf("%d %d %d %d\n",a,b,c,d);
     count += 1;
    }
printf("count=%d\n",count);
```

What will be the count above?

6 14 20 27
$a<b<c<d$

$0\leq a < b < c=20 < d \leq 29$

$b-a\geq 3$ $c-b\geq 6$ $c=20$ $d-c\geq 2$

$20-b=x_1$, $b-a=x_2$, $a-0=x_3$ 20

$y_1+6+y_2+3+y_3=20$ $\sum y=11$

$x_1+x_2+x_3=20$

$x_1\geq 6$ $x_2\geq 3$ $x_3\geq 0$

$y_1=x_1-6$ $y_2=x_2-3$ $y_3=x_3$

$x_1+x_2=9$ $x_2\geq 2$ $x_1\geq 0$ $\sum y=7$

$29-d=x_1$ $d-20=x_2$ 9

$\binom{13}{2}\binom{8}{1}$

P3 [10 points]

Provide the reasons for the steps verifying the following argument. (In the proof, a denotes a specific but arbitrarily chosen element from the given universe.) Two reasons are already given, fill the rest.

$$\begin{array}{l} \forall x[p(x) \rightarrow ((q(x) \wedge r(x))] \\ \forall x[p(x) \wedge s(x)] \\ \hline \therefore \forall x[r(x) \wedge s(x)] \end{array}$$

Steps	Reasons
1. $\forall x[p(x) \rightarrow ((q(x) \wedge r(x))]$.	Premise
2. $\forall x[p(x) \wedge s(x)]$................	Premise
3. $p(a) \rightarrow ((q(a) \wedge r(a))$......	S1,Un.Sp.
4. $p(a) \wedge s(a)$.....................	S2, Un.Sp.
5. $p(a)$...............................	S4. Conj. Simp.
6. $q(a) \wedge r(a)$......................	S 3,5 MP.
7. $r(a)$...............................	S6 R. Conj Simp.
8. $s(a)$...............................	S4 R. Conj. Simp.
9. $r(a) \wedge s(a)$......................	S 7,8 R. Conj.
10. $\forall x[r(x) \wedge s(x)$................	S9,Un.Gn.

P4 [20 points] Prove the following statements **by using mathematical induction**:

(a) $\forall n \in \mathbb{Z}^+$

$$[1/2^1] + 1/2^2 + 1/2^3 + \ldots + 1/2^n = 1 - \frac{1}{2^n}$$

- Base step for $n = 1$: $\frac{1}{2^1} = 1 - \frac{1}{2^1}$ ✓
- Ind. hyp. for $n = k$. Assume that $\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$
- Ind. step for $n = k+1$

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \stackrel{?}{=} 1 - \frac{1}{2^{k+1}}$$

$$1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}} \cancel{- \frac{1}{2^{k+1}}} \cancel{+ \frac{1}{2^{k+1}}}$$

$$-\frac{1}{2^{k+1}} = \frac{1}{2^{k+1}}$$

QED.

(b) $\forall n \in \mathbb{Z}^+, n \geq 35$:

$$\exists i, j, k \in \mathbb{N}\ [n = 6i + 7j + 13k]$$

(For all integers $n \geq 35$, there exists natural numbers i, j and k such that n can be written as $6i + 7j + 13k$. $\mathbb{N} = \{0, 1, 2, \ldots\}$)

- Base step: for $n = 35$, $35 = 6.0 + 7.5 + 13.0$, $i=0$ $j=5$ $k=0$ ✓
- Ind. hyp. Assume that for $n = m$

$$m = 6a + 7b + 13c$$

- Ind. step. for $n = m+1$,

$$m+1 = 6(a-1) + 7(b+1) + 13c$$

$$m+1 = 6a + 7(b+2) + 13(c-1)$$

$$m+1 = 6(a+6) + 7(b-5) + 13c$$

For us to be unable to use all 3 formulas above it must be the case that $a<1$ and $b<5$ and $c<1$ but the largest int. we can get would be $6.0 + 7.4 + 13.0 = 28$ Since $m \geq 35$, this is impossible (i.e. we can always use at least one of the formulas) So, we are done. QED.)

P5 [20 points] (In this question, define your pigeons and pigeonholes clearly.)

(a) Suppose that there are 100 people in a party where everybody will shake everybody else's hand. All possible handshakes will be done in a random order. Show that after every handshake, we can find two people who had equal number of handshakes. (For example if there were 5 people A,B,C,D,E, after some random handshakes, say AB, BC, CD, we can find B and C who both had 2 handshakes)

	#
A_1	53
A_2	0
⋮	⋮
	99
A_{100}	⋮

At some point in time we can see at most 99 different values in this column (# handshakes people had) If we see 0, we can't see 99 obviously & vica versa)

People : Pigeon (100)

Possible handshake counts : pigeonholes (99)

$\left\lceil \frac{100-1}{99} \right\rceil + 1$ people must have the same handshake count.

(b) We know that any binary string of length n must contain the same 10-bit substring at least twice for sure. Then what is the smallest value for n? (For example $n = 15$ is not possible because all of the 10 bit substrings of 111110000011111 are distinct. So n must be larger than 15. Can it be 16? 17?... What is the smallest possible n that guarantees the given condition?)

$n = 2^{10} + 10$

In a bin. string of length $2^{10} + 10$ there are $2^{10} + 1$ 10-bit substrings.

1034

Therefore, by p.h.p. at least $\left\lfloor \frac{2^{10}+1-1}{2^{10}} \right\rfloor + 1$ of these substrings are the same.

pigeons = 10-bit substrings in any string of length $2^{10}+10$

p.holes = possible 10-bit substrings.

$2^2 + 2$

3 ⇒ 0010111000 $2^3 + 3$