MAT222
LINEAR ALGEBRA
HOMEWORK ASSIGNMENT 2
SOLUTIONS

(1) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

if any exists.

Solution: We have

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3-R_1}{\overset{R_2-R_1}{\longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{(-1)R_2}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3-R_1}{\overset{R_2-R_1}{\longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_4-R_3}{\overset{R_1-R_3}{\longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 1 \end{array}\right] \overset{(-1)R_2}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & -1 \end{array}\right]$$

$$\underset{R_3-2R_4}{\overset{\substack{R_1+R_4 \\ R_2+R_4}}{\longrightarrow}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & -1 \end{array}\right]$$

so

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 2 & -1 & -1 & 2 \\ -2 & 1 & 1 & -1 \end{bmatrix}$$

is the inverse of A.■

(2) If A is an $n \times n$ matrix with integer entries such that $\det(A) = 1$, are the entries of A^{-1} necessarily integers? Explain your answer.

Solution: We have $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$, so if $\det(A) = 1$, then $A^{-1} = \mathrm{adj}(A)$. If A has integer entries, then $(-1)^{i+j}\det(A_{ij})$ is integer for all $1 \leq i, j \leq n$. Hence, $\mathrm{adj}(A)$ has integer entries. Therefore, A^{-1} has integer entries.■

(3) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & -3 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & x & 2 \end{bmatrix}.$$

If $\det(A) = 30$, find x.

Solution: Expanding down the second column, we have

$$30 = \det(A) = -2 \begin{vmatrix} 0 & 3 & 0 & 1 \\ 0 & -3 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & x & 2 \end{vmatrix}.$$

Since

$$\begin{vmatrix} 0 & 3 & 0 & 1 \\ 0 & -3 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & x & 2 \end{vmatrix} \underset{R_4+R_2}{\overset{R_1+R_2}{=}} \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & -3 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & x & 6 \end{vmatrix},$$

we have

$$-15 = \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & -3 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & x & 6 \end{vmatrix}.$$

Expanding across the first row, we have

$$-15 = -5 \begin{vmatrix} 0 & -3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & x \end{vmatrix} \Rightarrow 3 = \begin{vmatrix} 0 & -3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & x \end{vmatrix}.$$

Again, expanding across the first row, we find

$$3 = 3 \begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} = 3(x-1) \Rightarrow x - 1 = 1 \Rightarrow x = 2,$$

the desired value of x.■

(4) Show that the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

cannot be the adjoint of any invertible matrix with real entries.

Solution: Let A be an invertible 3×3 matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

implies that

$$\det[\mathrm{adj}(A)] = \det\left(\det(A) A^{-1}\right) = [\det(A)]^3 \det\left(A^{-1}\right) = [\det(A)]^3 \frac{1}{\det(A)} = [\det(A)]^2.$$

Now let

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Since

$$\det[\operatorname{adj}(A)] = 2\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = 2(-3-2) = -10,$$

we must have $-10 = [\det(A)]^2$, which is impossible. Thus, the given matrix cannot be the adjoint of any invertible matrix with real entries.■

(5) Show that the adjoint matrix of the transpose of a matrix is the transpose of adjoint of that matrix.

Solution: Since $\operatorname{adj}(A) = (\det A)\, A^{-1}$, $\det\left(A^T\right) = \det(A)$, and $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$, we have

$$[\operatorname{adj}(A)]^T = \left[(\det A)\, A^{-1}\right]^T = (\det A)\left(A^{-1}\right)^T = \left(\det A^T\right)\left(A^T\right)^{-1} = \operatorname{adj}\left(A^T\right),$$

the desired conclusion.■