30/11/2023 Midterm Duration: 90 minutes

Name: ____________ Student No: ____________

P1 [30 pts] In a software company, 21 software engineers will be assigned to 3 distinguishable projects. Every project needs 1 manager, 2 coordinators and 4 workers. Among the 21 engineers, only 6 of them are eligible to be a manager. Everyone will be assigned to exactly one role. In how many ways can these people be assigned to the roles?

(a) with no restrictions

$$\underbrace{6\cdot5\cdot4}_{\text{managers}}\cdot\underbrace{\binom{18}{2}\binom{16}{2}\binom{14}{2}}_{\text{coord.}}\underbrace{\binom{12}{4}\binom{8}{4}\binom{4}{4}}_{\text{workers.}}$$

(b) if manager candidates cannot be workers

$$\underbrace{6\cdot5\cdot4}_{\text{managers}}\cdot\underbrace{\binom{15}{4}\binom{11}{4}\binom{7}{4}}_{\text{workers}}\underbrace{\binom{6}{2}\binom{4}{2}\binom{2}{2}}_{\text{coord.}}$$

(c) if both coordinators of a project cannot be manager candidates (only one can be)

$$\underbrace{6\cdot5\cdot4}_{\text{man.}}\cdot\underbrace{\binom{15}{4}\binom{11}{4}\binom{7}{4}}_{\text{workers}}\cdot\underbrace{3!}_{\text{distrib. man. cand. coord.}}\cdot\underbrace{3!}_{\text{distrib. workers coord.}}$$

P2 [20 points]

(a) How many 6-digit numbers have 3 increasing odd digits followed by 3 decreasing even digits (as in 159820 or 137864)?

$$\binom{5}{3}\binom{5}{3}$$

(b) How many 10-digit numbers have 5 non-decreasing odd digits followed by 5 non-increasing even digits (as in 1337788840 or 5555964222)?

$x_1+\dots+x_5=5 \rightarrow \binom{9}{5}$ So, $\binom{9}{5}\binom{9}{5}$.

(c)
```
int a,b,c,d,count = 0;
for(a = 0; a<40; a++)
 for(b=a+5; b<40; b++)
  for(c=b+5; c<40; c++)
    for(d=c+5; d<40; d++){
     printf("%d %d %d %d\n",a,b,c,d);
     count += 1;
    }
printf("count=%d\n",count);
```
What will be the count above?

$0 \le a < b < c < d \le 39$

$x_1 = a-0$
$x_2 = b-a$
$x_3 = c-b$
$x_4 = d-c$
$+\, x_5 = 39-d$

$\sum x = 39$

$x_1+\dots+x_5=39$

$x_2, x_3, x_4 \geq 5$

$y_2 = x_2-5$
$y_3 = x_3-5$
$y_4 = x_4-5$
$y_1 = x_1$
$y_5 = x_5$

$\sum y = 24 \nearrow \binom{28}{4}$

P3 [10 points]

Prove the following argument:

$\forall x[p(x) \vee q(x)]$
$\exists x[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$
$\therefore \exists x[\neg r(x) \rightarrow p(x)]$

Steps	Reasons
1. $\forall x[p(x) \vee q(x)]$	Premise.
2. $\exists x[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$	Premise
3. $[\neg p(c) \wedge q(c)] \rightarrow r(c)$	R. Ex. Sp, 2 part. c
4. $p(c) \vee q(c)$	R. Un. Sp. 1
5. $\forall x[\neg p(x) \wedge \neg r(x)]$	Assumption
6. $\neg p(c) \wedge \neg r(c)$	R. Un. Sp. 5

Steps	Reasons
7. $\neg p(c)$	R. Conj Simp. 6
8. $q(c)$	R. Dis. Syl. 7, 4
9. $\neg p(c) \wedge q(c)$	R. Conj 7, 8
10. $r(c)$	M.P 3, 9.
11. $\neg r(c)$	R. Conj. S; 6
12. $r(c) \wedge \neg r(c) = \bot$	R. Conj 10, 11.
13. $\exists x(\neg r(x) \rightarrow p(x))$	P. by Contr. 5-13

P4 [20 points] Prove the following statements **by using mathematical induction**:

(a) $\forall n \in \mathbb{Z}^+$
$$[1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1]$$

B.S. for $n=1$, $1 \cdot 1! = 2! - 1$ ✓

I. H. for $n=k$, Assume $1.1! + 2.2! + \ldots + k k! = (k+1)! - 1$

I.S. for $n=k+1$.

$\underbrace{1.1! + 2.2! + \ldots + k k!}_{(k+1)!-1} + (k+1)(k+1)! \stackrel{?}{=} (k+2)! - 1$

$(k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(1 + k + 1) - 1 = (k+2)! - 1$ ✓

QED.

(b) $\forall n \in \mathbb{Z}^+, n \geq 90$:
$$\exists i, j, k \in \mathbb{N}\ [n = 10i + 15j + 21k]$$
(For all integers $n \geq 90$, there exists natural numbers i, j and k such that n can be written as $10i + 15j + 21k$. $\mathbb{N} = \{0, 1, 2, \ldots\}$)

B.S. for $n = 90$, $90 = 10 \cdot 9 + 15 \cdot 0 + 21 \cdot 0$

I.H. " $n = k$, assume $k = 10a + 15b + 21c$.

I.S. " $n = k+1$ we can always use at least one of these formulas:

$k+1 = 10(a+7) + 15(b+1) + 21(c-4)$

$k+1 = 10(a+1) + 15(b-2) + 21(c+1)$

$k+1 = 10(a-2) + 15b + 21(c+1)$

because: For us to be unable to use all, $c<4$, $b<2$, $a<2 \Rightarrow$ The max we can get is $10 \cdot 1 + 15 \cdot 1 + 21 \cdot 3 = 88$ which is less than 90. So, for $k \geq 90$, we can always use one of the formulas. QED.

P5 [20 points] (In this question, define your pigeons and pigeonholes clearly.)

(a) The parking area of a building has 12 parking lots which is filled by the same 12 cars randomly everyday. A kid takes the photo of the parking area every evening when all the lots are filled. How many days does the kid need to guarantee to take the photo of the same parking arrangement in two different days?

The kid needs $12! + 1$ days. Then, At least $\left\lfloor \frac{12! + 1 - 1}{12!} \right\rfloor + 1 = 2$ days will h the same parking arrangement.

Pigeons: days $(12! + 1)$

p.holes: possible arrangements $(12!)$

(b) Show that if we choose nine points having integer coordinates from the 3D Cartesian space, the midpoint of at least one pair of these points has to have integer coordinates in all axes. [For example, midpoint of (4,3,7) and (6,12,15) would be (5,7.5,11) which happens to have a non-integer coordinate.]

Observation: for a coordinate of ~~a~~ midpoint to be integer, the corresponding coordinates of the points must both be even or both be odd. Then, there are only 8 evenness-oddness for each point: $(e,e,e), (e,e,o), \ldots, (o,o,o)$.

Then, with 9 points, at least

$\left\lfloor \frac{9-1}{8} \right\rfloor + 1 = 2$ of them will have the

Same even-odd combination. Then, their midpoint have all integer coordinates

QED.

Pigeons: points

pholes: possible even-odd combinations