Fundamental thm. of arithmetic : Every pos. int. can be written as a product of primes.

$2 \ldots k \quad k+1$

Ex $47 = 47$

Ex $42 = 2 \cdot 21$

divide

subproblem — conquer     subproblem — conquer

$42 = 2 \cdot 3 \cdot 7$

$4 = u \cdot v \quad (u, v < 4)$

$= 2 \cdot 2$

$2 \checkmark \quad 2 \checkmark$

explicit formula??? $a_n = \frac{n^2 - n}{2}$ ??

$a_0 = 1 \quad a_1 = 2 \quad a_2 = 3 \qquad a_n = a_{n-1} + \cdots a_{n-3}$

1 2 3 6 11 20 37 68 ...

Divide & conquer example: mergesort

7 1 3 2 9 4 5 6

1 ... 2

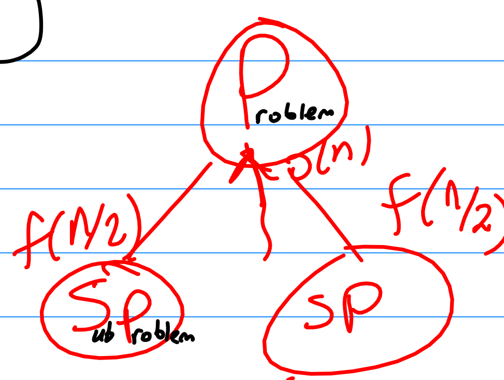
sort     sort

1 2 3 7     4 5 6 9

merge $O(n)$

1 2 3 4 5 6 7 9

$n = f(n)$



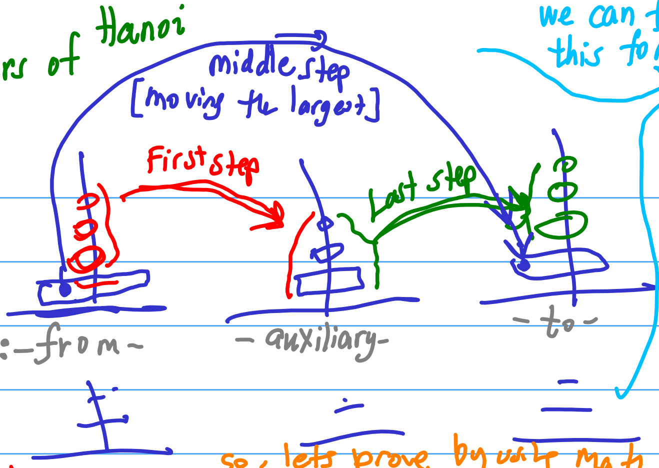

$f(\frac{n}{2})$ needs

$\Rightarrow = 2 f(\frac{n}{2}) + O(n)$

$\boxed{f(n) = f(\frac{n}{2}) + f(\frac{n}{2}) + O(1)}$

$f(n) = n \log(n)$

Towers of Hanoi




we can figure out this formula!

$T(n) = T(n-1) + 1 + T(n-1)$

but, it is not an explicit one !!

so let's find one...

$T(n) = 2T(n-1) + 1 \overset{\text{hyp.}}{=} 2^n - 1$

~~Proof by exam.~~ no such proof

$T(0) = 0$

$T(1) = 1$

$T(2) = 3$

$T(3) = 7$ looks like a good guess $2^3 - 1$

$T(4) = 15$ $\quad 2^4 - 1$

$T(5) = 31$ $\quad = 2^5 - 1$

$T(6) = 63$ $\quad = 2^6 - 1$

hypothesize

so, lets prove by using math. induction...

Math. Ind.

Thm.. Let $T(n)$ be defined as

Defn. of the formula: ( for $n=1$, $T(1) = 1$ and $T(n) = 2T(n-1) + 1$ for $n \geq 2$ ) Then $T(n) = 2^n - 1 \;\; \forall n \geq 1$ Thm

Proof: Base Step: $T(1) = 1 = 2^1 - 1$ Thm ✓

Ind. Hyp. Assume that $T(k) = 2^k - 1$

$n \to k+1$

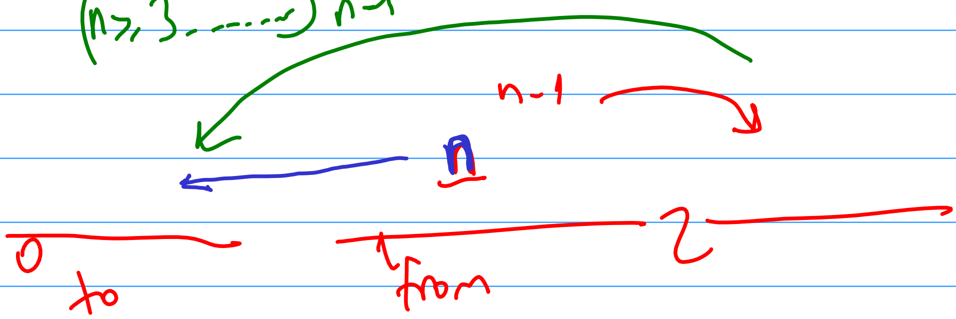
Ind. Step. $T(k+1) = 2T(k) + 1 = 2 \cdot (2^k - 1) + 1$

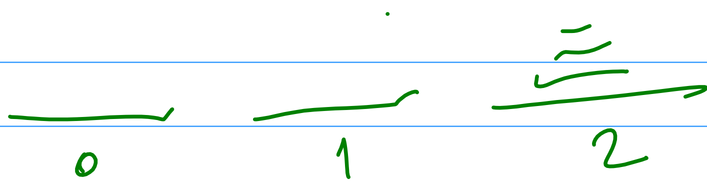
$= 2^{k+1} - 1 \overset{?✓}{=} 2^{k+1} - 1$

So we are done, our thm is correct. QED.

$T(n)$ :- 1 5 12 72 . — —
0 1 2 3

$(n \geq 3 \ldots\ldots)$ $n-1$



# Harmonic numbers

$H_1 = 1 \qquad H_2 = 1 + \frac{1}{2} \qquad H_3 = 1 + \frac{1}{2} + \frac{1}{3} \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

$$\forall n \in \mathbb{Z}^+ \quad \sum_{j=1}^{n} H_j = (n+1)H_n - n$$

Ex. when $n=3$: $1 + 1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} = 4\left(1 + \frac{1}{2} + \frac{1}{3}\right) - 3$

$4 + \frac{1}{3} \overset{?}{=} 4 + \cancel{2} + \frac{4}{\cancel{3}} - \cancel{3}$

$+ \cancel{1} + \frac{1}{3}$

$= 4 + \frac{1}{3}$

Proof. (By induction on $n$)

Base step for $n=1$: $\sum_{j=1}^{1} H_j = H_1 = 1 \overset{\checkmark}{=} 2.1 - 1$

Ind. hyp. Assume for $n=k$: $\boxed{\sum_{j=1}^{k} H_j = (k+1)H_k - k}$

Ind. step for $n=k+1$: $\sum_{j=1}^{k+1} H_j = \sum_{j=1}^{k} H_j + H_{k+1}$

fact by defn.

$H_{k+1} = H_k + \frac{1}{k+1}$

$= (k+1)H_k - k + H_{k+1} \overset{?}{=} (k+2)H_{k+1} - k - 1$

$= (k+1)H_k - k + H_k + \frac{1}{k+1} \overset{?}{=} (k+2)\left(H_k + \frac{1}{k+1}\right) - (k+1)$

$= \cancel{(k+2)H_k} - \cancel{k} + \frac{1}{\cancel{k+1}} \overset{?}{=} \cancel{(k+2)H_k} + \frac{\cancel{k+2}}{\cancel{k+1}} - \cancel{k} - \cancel{1}$

✓ proven. QED.

$\forall n \in \mathbb{N}$, prove that $1 + \frac{n}{2} \le H_{2^n}$

smallest

Proof. Base step: for $n = 0$ $\quad 1 + \frac{0}{2} \le H_1 = 1$ ✓

Ind hyp. for $n = k$, $\boxed{1 + \frac{k}{2} \le H_{2^k}}$ $\qquad H_{32} = 1 + \frac{1}{2} + \cdots \frac{1}{32}$

Ind step for $n = k+1$, $\quad 1 + \frac{k+1}{2} \overset{?}{\le} H_{2^{k+1}}$ $\qquad H_{64} = \left(1 + \frac{1}{2} + \cdots \frac{1}{32}\right) + \frac{1}{33} \cdots \frac{1}{64}$

$$1 + \frac{k}{2} + \frac{1}{2} \overset{?}{\le} H_{2^k} + \boxed{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}}$$

$$\frac{1}{2} = \frac{2^k}{2^{k+1}} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \cdots\cdots + \frac{1}{2^{k+1}}$$

We are done, and theorem is correct QED.

$k=3$ ex.

$$\frac{1}{2} \le \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}$$

$$8\left(\frac{1}{16} \quad \frac{1}{16} \quad \frac{1}{16}\right) = \frac{1}{2}$$

$$\frac{1}{2} = \underbrace{\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}}_{8} \le \underbrace{\boxed{\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}}}_{8}$$

$H_1 = 1 \qquad H_2 = 1 + \frac{1}{2}$

Base : $\sum_{j=1}^{1} j H_j = H_1 = \frac{1.2}{2} H_2 - \frac{1.2}{4}$

$H_1 = H_2 - \frac{1}{2}$

$1 = \frac{3}{2} - \frac{1}{2} = 1$

Ind. hyp. $\sum_{j=1}^{k} j H_j = \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4}$

Ind. Step $\sum_{j=1}^{k+1} j H_j = \sum_{j=1}^{k} j H_j + (k+1) H_{k+1} = \frac{(k+1)(k+2)}{2} H_{k+2} - \frac{(k+1)(k+2)}{4}$

$= \frac{(k+1)(k+2)}{2}\left(H_{k+1} + \frac{1}{k+2}\right) - \frac{(k+1)(k+2)}{4}$

$\left(\frac{k(k+1)}{2} + \frac{2(k+1)}{2}\right)\left(H_{k+1} + \frac{1}{k+2}\right) - \frac{k(k+1)}{4} - \frac{2(k+1)}{4}$

$\frac{k(k+1)}{2(k+2)} + (k+1) H_{k+1} + \frac{k+1}{k+2} - \frac{(k+1)}{2}$

QED

$2k + 2 - k^2 - 3k - 2$ $\qquad -k^2 - k$

$\frac{2(k+1) - (k+2)(k+1)}{2.(k+2)}$ $\qquad -k(k+1)$