CHAPTER 4

GENERAL VECTOR SPACES

In this chapter we shall generalize the concept of vector. The generalization will come in abstracting the vector idea to entirely different kinds of objects. This generalization is provided by the concept of vector space, which finds applications in mathematics, engineering, physics, chemistry, biology, and the social sciences. Vector spaces also arise in the studies of networks, functions as systems of vectors, and new perspectives on the linear system $Ax = b$.

4.1. The Concept of Vector Space

- **Vector space**

  A (real) ***vector space*** *is a nonempty set V of elements, called* ***vectors****, together with operations of* ***vector addition*** *and* ***scalar multiplication****, such that the following laws hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in \mathbb{R}$.*

  (1) $\mathbf{u} + \mathbf{v} \in V$ *(closure of vector addition)*

  (2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(commutativity of addition)*

  (3) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *(associativity of addition)*

  (4) *There exists an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ (additive identity)*

  (5) *There exists an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)*

  (6) $a\mathbf{u} \in V$ *(closure of scalar multiplication)*

  (7) $a\,(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ *(distributive law)*

  (8) $(a + b)\,\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ *(distributive law)*

  (9) $(ab)\,\mathbf{u} = a\,(b\mathbf{u})$ *(associative law)*

  (10) $1\mathbf{u} = \mathbf{u}$ *(monoidal law)*

In the following we give examples which illustrate the variety of vector spaces.

Example (The vector space $\mathbb{R}^n$) With the vector addition

$$\mathbf{x} + \mathbf{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and scalar multiplication

$$c\mathbf{x} = c(x_1, x_2, \dots, x_n) = (cx_1, cx_2, \dots, cx_n),$$

the Euclidean space $\mathbb{R}^n$ is a vector space. Indeed, we have

(1) Since $x_i + y_i$ is a real number for $1 \le i \le n$, we have

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n.$$

(2) We have

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) = \mathbf{y} + \mathbf{x}.$$

(3) We have

$$\begin{aligned} \mathbf{x} + (\mathbf{y} + \mathbf{z}) &= (x_1, x_2, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \dots, x_n + y_n + z_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n) = (\mathbf{x} + \mathbf{y}) + \mathbf{w}. \end{aligned}$$

(4) We have

$$\mathbf{x} + \mathbf{0} = (x_1, x_2, \dots, x_n) + (0, 0, \dots, 0) = (x_1 + 0, x_2 + 0, \dots, x_n + 0) = (x_1, x_2, \dots, x_n) = \mathbf{x}.$$

(5) We have

$$\begin{aligned} \mathbf{x} + (-\mathbf{x}) &= (x_1, x_2, \dots, x_n) + (-x_1, -x_2, \dots, -x_n) \\ &= (x_1 - x_1, x_2 - x_2, \dots, x_n - x_n) = (0, 0, \dots, 0) = \mathbf{0}, \end{aligned}$$

so the additive inverse of $\mathbf{x}$ is $-\mathbf{x} = (-x_1, -x_2, \dots, -x_n)$.

(6) Since ax_i is a real number for $1 \le i \le n$, we have

$$a\mathbf{x} = a(x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n) \in \mathbb{R}^n.$$

(7) We have

$$\begin{aligned} a(\mathbf{x} + \mathbf{y}) &= a(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (ax_1 + ay_1, ax_2 + ay_2, \dots, ax_n + ay_n) \\ &= a(x_1, x_2, \dots, x_n) + a(y_1, y_2, \dots, y_n) = a\mathbf{x} + a\mathbf{y}. \end{aligned}$$

(8) We have

$$\begin{aligned} (a + b)\mathbf{x} &= (a + b)(x_1, x_2, \dots, x_n) = ((a + b)x_1, (a + b)x_2, \dots, (a + b)x_n) \\ &= (ax_1 + bx_1, ax_2 + bx_2, \dots, ax_n + bx_n) \\ &= a(x_1, x_2, \dots, x_n) + b(x_1, x_2, \dots, x_n) = a\mathbf{x} + b\mathbf{x}. \end{aligned}$$

(9) We have

$$(ab)\,\mathbf{x} = (ab)\,(x_1, x_2, \ldots, x_n) = (abx_1, abx_2, \ldots, abx_n) = a\,(bx_1, bx_2, \ldots, bx_n) = a\,(b\mathbf{x})\,.$$

(10) We have

$$1\mathbf{x} = 1\,(x_1, x_2, \ldots, x_n) = (1x_1, 1x_2, \ldots, 1x_n) = (x_1, x_2, \ldots, x_n) = \mathbf{x}.\blacksquare$$

Example (Vector spaces of matrices) The set M_{mn} of all $m\times n$ matrices together with the matrix addition and scalar multiplication is a vector space since matrix addition and scalar multiplication satisfy vector space laws (see page 27).■

Example (Function space) Let $F\,(-\infty,\infty)$ be the set of all real-valued functions defined on the entire real line $(-\infty,\infty)$. If $\mathbf{f} = f\,(x)$ and $\mathbf{g} = g\,(x)$, and if c is any real number, define the standard function addition and scalar multiplication

$$\mathbf{f}+\mathbf{g} = (f+g)\,(x) = f\,(x) + g\,(x) \quad \text{and} \quad c\mathbf{f} = (cf)\,(x) = cf\,(x)\,.$$

Then $F\,(-\infty,\infty)$ is a vector space since defined addition and scalar multiplication satisfy vector space laws.■

Example (Function space) Let $C\,[0,1]$ be the set of all real-valued functions that are continuous on the interval $[0,1]$. With the standard function addition and scalar multiplication $C\,[0,1]$ is a vector space.■

In the last example, the vector space can be $C\,[a,b]$, the set of all continuous functions on the interval $[a,b]$. In general, $C\,(-\infty,\infty)$ is the vector space of all continuous functions defined on the real line $(-\infty,\infty)$.

Example (The zero vector space) Let V consist of a single object, which we denote it by $\mathbf{0}$, and define

$$\mathbf{0}+\mathbf{0} = \mathbf{0} \quad \text{and} \quad c\mathbf{0} = \mathbf{0}$$

for all scalars c. All the vector space laws are evidently satisfied. We call this space as the **zero vector space.**■

Example (A set that is not a vector space) Consider $\mathbb{R}^2$, the set of vectors in the plane, and define addition and scalar multiplication as follows: if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2) \quad \text{and} \quad c\mathbf{x} = (cx_1, 0)\,.$$

If $x_2 \neq 0$, then

$$1\mathbf{x} = 1\,(x_1, x_2) = (1x_1, 0) \neq \mathbf{x},$$

so the monodial law is not satisfied. This implies that $\mathbb{R}^2$ is not a vector space with indicated operations.■

Example (A set that is not a vector space) Consider the set

$$V = \left\{ f\,(x) \in C\,[0,1] : f\left(\frac{1}{2}\right) = 1 \right\}$$

together with the standard function addition and scalar multiplication. For $f\,(x)\,, g\,(x) \in V$, we have

$$(f+g)\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right) = 1 + 1 = 2 \neq 1,$$

so V is not closed under addition. Thus, V is not a vector space with given operations.■

We conclude this section with a result that lists laws of vector arithmetic.

- **Laws of vector arithmetic**

  Let $\mathbf{v}$ *be a vector in some vector space* V *and* c *be any scalar.*

  (1) $\mathbf{0v} = \mathbf{0}$.

  (2) $c\mathbf{0} = \mathbf{0}$.

  (3) $(-c)\,\mathbf{v} = c\,(-\mathbf{v}) = -\,(c\mathbf{v})\,.$

  (4) *If* $c\mathbf{v} = \mathbf{0}$, *then* $c = 0$ *or* $\mathbf{v} = \mathbf{0}$.

  (5) *A vector space has only one zero element.*

  (6) *Every vector space has only one additive inverse.*

Proof: All the laws are easily verified. For example, for (5), suppose that both $\mathbf{0}$ and $\mathbf{0}^*$ act as zero elements in the vector space. By additive identity of $\mathbf{0}$, we have

$$\mathbf{0}^* + \mathbf{0} = \mathbf{0}^*,$$

while additive identity of $\mathbf{0}^*$ implies

$$\mathbf{0} + \mathbf{0}^* = \mathbf{0}.$$

By the commutative law,

$$\mathbf{0}^* = \mathbf{0}^* + \mathbf{0} = \mathbf{0} + \mathbf{0}^* = \mathbf{0},$$

so $\mathbf{0}^* = \mathbf{0}$, that is, there can be only one zero element.■

4.2. Subspaces

We now consider the concept of a subspace, a rich source for examples of vector spaces. It often happens that a certain vector space of interest is a subset of a larger, and possibly better understood, vector space, and that the vector operations are the same for both spaces. As an example, the vector space $C[0,1]$ is a subset of the larger vector space $F(-\infty,\infty)$ with both spaces sharing the same definitions of vector addition and scalar multiplication. Such vector spaces are given a special name.

- **Subspace**

  A subset W of a vector space V is called a ***subspace*** *of V if W is itself a vector space under the addition and scalar multiplication defined on V.*

In general, we must verify the vector space laws to show that a nonempty set W is a vector space with addition and scalar multiplication. However, if W is a subspace of a known vector space V, then certain axioms need not be verified for W because they are inherited from V. We precisely use the following test.

- **Subspace test**

  If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions hold:

  (1) *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W (closure of addition).*

  (2) *If c is a scalar and $\mathbf{u}$ is a vector in W, then $c\mathbf{u}$ is in W (closure of scalar multiplication).*

Example: Determine whether the set

$$W=\left\{f(x)\in C[0,1]: f\left(\frac{1}{2}\right)=0\right\}$$

is a vector space with standard function addition and scalar multiplication.

Solution: For $f(x), g(x)\in W$, we have

$$(f+g)\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)+g\left(\frac{1}{2}\right)=0+0=0$$

and for a constant c

$$(cf)\left(\frac{1}{2}\right)=cf\left(\frac{1}{2}\right)=c0=0,$$

so W is a subspace of $C[0,1]$, hence it is a vector space itself.■

Example: Show that the subset P_n of $C(-\infty, \infty)$ consisting of all polynomials of degree at most n is a subspace of $C(-\infty, \infty)$.

Solution: Let p and q be two polynomials, say,

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \text{and} \quad q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n,$$

where n is a positive integer. We see that

$$(p+q)(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

and for a constant c

$$(cp)(x) = cp(x) = ca_0 + ca_1x + ca_2x^2 + \cdots + ca_nx^n$$

are elements of P_n. Thus, P_n is a subspace of $C(-\infty, \infty)$.■

Example: A function with a continuous derivative is said to be continuously differentiable. From calculus we know that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable. Thus, the functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C^1(-\infty, \infty)$, where the superscript emphasizes that the first derivatives are continuous. To take this a step further, the set of functions with m continuous derivatives on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$ as is the set of functions with derivatives of all orders on $(-\infty, \infty)$. We will denote these subspaces by $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$, respectively.■

Example: Show that the set S_n of all $n \times n$ symmetric matrices with standard matrix addition and scalar multiplication form a vector space.

Solution: Since S_n is a subset of M_{nn}, the vector space of all $n \times n$ matrices, we only need to check that S_n passes the subspace test. Let $A, B \in S_n$. This means that $A^T = A$ and $B^T = B$. We have

$$(A+B)^T = A^T + B^T = A + B$$

and for a scalar c

$$(cA)^T = cA^T = cA,$$

so $A + B$ and cA belong to S_n. Thus, S_n is a subspace, and hence is a vector space itself.■

Example: Determine which of the following subsets of the Euclidean space $\mathbb{R}^3$ are its subspaces.

(a) $W_1 = \{(x, y, z) : x - 2y + z = 0\}$.

(b) $W_2 = \{(x, y, z) : x, y, z \text{ are positive}\}$.

(c) $W_3 = \{(0, 0, 0)\}$.

(d) $W_4 = \{(x, y, z) : x^2 - y = 0\}$.

Solution:

(a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be in W_1. Then

$$u_1 - 2u_2 + u_3 = 0 \quad \text{and} \quad v_1 - 2v_2 + v_3 = 0.$$

We first show that $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in W_1$. Since

$$\left.\begin{aligned} u_1 - 2u_2 + u_3 = 0 \\ v_1 - 2v_2 + v_3 = 0 \end{aligned}\right\} \Rightarrow (u_1 + v_1) - 2(u_2 + v_2) + (u_3 + v_3) = 0,$$

the coordinates of $\mathbf{u} + \mathbf{v}$ fit the condition for being an element of W_1, that is, $\mathbf{u} + \mathbf{v} \in W_1$. Next, for a constant c, we have

$$cu_1 - 2cu_2 + cu_3 = c(u_1 - 2u_2 + u_3) = c0 = 0,$$

so $c\mathbf{u} \in W_1$. Therefore, W_1 is a subspace of $\mathbb{R}^3$.

(b) Let $\mathbf{u} = (u_1, u_2, u_3) \in W_2$. Since the coordinates of $-\mathbf{u} = (-1)\mathbf{u} = (-u_1, -u_2, -u_3)$ do not fit the requirement for being an element of W_2, W_2 is not a subspace of $\mathbb{R}^3$.

(c) The only possible choice for arbitrary elements $\mathbf{u}$ and $\mathbf{v}$ in the set W_3 is $\mathbf{u} = \mathbf{v} = (0, 0, 0)$. Then since

$$\mathbf{u} + \mathbf{v} = (0, 0, 0) + (0, 0, 0) = (0, 0, 0)$$

and for a scalar c

$$c\mathbf{u} = c(0, 0, 0) = (0, 0, 0),$$

we see that $\mathbf{u} + \mathbf{v} \in W_3$ and $c\mathbf{u} \in W_3$, so W_3 is a subspace of $\mathbb{R}^3$.

(d) By the definition of W_4, we see that $(1, 1, 0)$ and $(-1, 1, 0)$ are elements of W_4. But

$$(1, 1, 0) + (-1, 1, 0) = (0, 2, 0)$$

does not satisfy the condition $x^2 - y = 0$, so W_4 is not a subspace of $\mathbb{R}^3$.■

Part (c) of this example reveals a simple fact about vector spaces.

- **Trivial spaces**

  *Every vector space V has at least two subspaces, namely, $\{\mathbf{0}\}$, the set of zero vector in V, and V itself. These are called the **trivial subspaces**.*

There is a very useful type of subspace that requires the notion of a linear combination of vectors in a vector space.

- **Linear combination**

  A vector $\mathbf{w}$ *is called a* ***linear combination*** *of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *in a vector space* V *if it can be written in the form*

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

  where $c_1, c_2, \ldots, c_n$ *are scalars.*

Example: Every vector $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ is expressible as a linear combination of standard unit vectors

$$\mathbf{i} = (1,0,0)\,, \quad \mathbf{j} = (0,1,0)\,, \quad \mathbf{k} = (0,0,1)$$

since

$$\mathbf{x} = (x_1, x_2, x_3) = x_1\,(1,0,0) + x_2\,(0,1,0) + x_3\,(0,0,1) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.\blacksquare$$

Example: Consider the vectors $\mathbf{x} = (1, 2, -1)$ and $\mathbf{y} = (6, 4, 2)$ in $\mathbb{R}^3$. If $\mathbf{w} = (9, 2, 7)$, then

$$(9,2,7) = c_1\mathbf{x} + c_2\mathbf{y} = (c_1 + 6c_2, 2c_1 + 4c_2, -c_1 + 2c_2)\,,$$

which gives

$$\begin{array}{rcrcl} c_1 & + & 6c_2 & = & 9 \\ 2c_1 & + & 4c_2 & = & 2 \\ -c_1 & + & 2c_2 & = & 7 \end{array} \Rightarrow c_1 = -3, c_2 = 2$$

so $\mathbf{w} = -3\mathbf{x} + 2\mathbf{y}$, a linear combination of $\mathbf{x}$ and $\mathbf{y}$. On the other hand, if $\mathbf{w}' = (4, -1, 8)$, then the system

$$\begin{array}{rcrcr} c_1 & + & 6c_2 & = & 4 \\ 2c_1 & + & 4c_2 & = & -1 \\ -c_1 & + & 2c_2 & = & 8 \end{array}$$

is inconsistent since

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \underset{III+I}{\overset{II-2I}{\longrightarrow}} \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix} \overset{III+II}{\longrightarrow} \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 3 \end{bmatrix}$$

so no c_1 and c_2 exist such that $\mathbf{w}' = c_1\mathbf{x} + c_2\mathbf{y}$. Hence, $\mathbf{w}'$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$.$\blacksquare$

We can consider all possible linear combinations of a given list of vectors.

- **Spanning**

  Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in the vector space* V*. The* ***span*** *of these vectors, denoted by* span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$*, is the subset of* V *consisting of all possible linear combinations of these vectors, that is,*

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n : c_1, c_2, \ldots, c_n \text{ are scalars}\}.$$

Example: Determine whether the vectors $\mathbf{x} = (1,1,2)$, $\mathbf{y} = (1,0,1)$, and $\mathbf{z} = (2,1,3)$ span the vector space $\mathbb{R}^3$.

Solution: Let $\mathbf{w} = (w_1, w_2, w_3)$ be an arbitrary vector in $\mathbb{R}^3$. Then $\mathbf{w} = c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z}$ implies that

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & 2c_3 & = & w_1 \\ c_1 & & & + & c_3 & = & w_2 \\ 2c_1 & + & c_2 & + & 3c_3 & = & w_3 \end{array}$$

This linear system is consistent for all values of w_1, w_2, w_3 if the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

is invertible. But since

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 + 1 = 0,$$

the coefficient matrix is not invertible. Thus, the vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ do not span $\mathbb{R}^3$.■

Example: The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since every polynomial p in P_n can be written as

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. Thus, $P_n =$span$\{1, x, x^2, \ldots, x^n\}$.■

Spans are premier examples of subspaces.

- *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in the vector space* V*. Then* $W =$span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a subspace of* V*.*

Proof: Let $\mathbf{u}, \mathbf{w} \in W$ be any two elements of W, say,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \text{and} \quad \mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$

Adding these vectors gives

$$\mathbf{u} + \mathbf{w} = (c_1 + d_1)\,\mathbf{v}_1 + (c_2 + d_2)\,\mathbf{v}_2 + \cdots + (c_n + d_n)\,\mathbf{v}_n.$$

Thus, $\mathbf{u} + \mathbf{w}$ is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so W is closed under vector addition. For nay scalar c, we have

$$c\mathbf{u} = (cc_1)\,\mathbf{v}_1 + (cc_2)\,\mathbf{v}_2 + \cdots + (cc_n)\,\mathbf{v}_n,$$

which is again a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so W is closed under scalar multiplication. By the subspace test, W is a vector space of V.■

- **Spanning set**

  If $W =$span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, *then we say that* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a* ***spanning set*** *for the vector space* W, *and that* W *is spanned by the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The following fact is one of the most useful property of spans.

- *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in the vector space* V *and let* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ *be vectors in* span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. *Then*

$$\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

In particular, if $\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}\}$ is a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then

$$\operatorname{span}\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

This means that if we enlarge the list of spanning vectors, then we enlarge the spanning set. However, we may not obtain a strictly larger spanning set.

Example: Show that $\operatorname{span}\{(1,0), (1,1)\} = \operatorname{span}\{(1,0), (1,1), (1,2)\}$.

Solution: We let $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (1,1)$, and $\mathbf{v}_3 = (1,2)$. Every element of span$\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, since we can write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$. So we certainly have span$\{\mathbf{v}_1, \mathbf{v}_2\}$ $\subseteq$span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. On the other hand, since $(1,2) = -(1,0) + 2(1,1)$, we have $\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2$. Therefore, any linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(-\mathbf{v}_1 + 2\mathbf{v}_2) = (c_1 - c_3)\,\mathbf{v}_1 + (c_2 + 2c_3)\,\mathbf{v}_2.$$

Thus, span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq$span$\{\mathbf{v}_1, \mathbf{v}_2\}$. In conclusion, we have span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.■

As an application of vector spans, we consider the problem of determining all subspaces of the Euclidean plane $\mathbb{R}^2$ from a geometric point of view. First, we have the trivial subspaces $\{(0,0)\}$ and $\mathbb{R}^2$. Next, we consider the subspace V =span$\{\mathbf{v}\}$, where $\mathbf{v} \neq \mathbf{0}$. The set of all multiples of $\mathbf{v}$ forms a straight line through the origin. Finally, we consider the subspace V =span$\{\mathbf{v}, \mathbf{w}\}$, where $\mathbf{w} \notin$span$\{\mathbf{v}\}$. Any point in the plane can be a corner of a parallelogram with edges that are multiples of $\mathbf{v}$ and $\mathbf{w}$. Hence $V = \mathbb{R}^2$. Consequently, the only subspaces of $\mathbb{R}^2$ are $\{(0,0)\}$, $\mathbb{R}^2$, and lines through the origin. Similarly, the only subspaces of $\mathbb{R}^3$ are $\{(0,0,0)\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$.

4.3. Linear Independence

In the preceding section we learned that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans a given vector space V if every vector in V is expressible as a linear combination of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. In general, there may be more than one way to express a vector in V as a linear combination of vectors in the spanning set. In this section, we shall study conditions under which each vector in V is expressible as a linear combination of vectors in the spanning set uniquely. Spanning sets with this property play a fundamental role in the study of vector spaces.

- **Linear independence and linear dependence**

  If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a nonempty set of vectors, then the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

  has at least one solution, namely

$$c_1 = 0, c_2 = 0, \ldots, c_n = 0.$$

  If this is the only solution, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called ***linearly independent****. If there are other solutions, then these vectors are called* ***linearly dependent****.*

Examples:

(1) The vectors $\mathbf{x}_1 = (2, -1, 0, 3)$, $\mathbf{x}_2 = (1, 2, 5, -1)$, and $\mathbf{x}_3 = (7, -1, 5, 8)$ of $\mathbb{R}^4$ are linearly dependent since $3\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$.

(2) The polynomials $p_1 = 1 - x$, $p_2 = 5 + 3x - 2x^2$, and $p_3 = 1 + 3x - x^2$ are linearly dependent since $3p_1 - p_2 + 2p_3 = \mathbf{0}$.

(3) Any set of vectors that contains the zero vector is linearly dependent. For, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be such a set and suppose that for some index k, $\mathbf{v}_k = \mathbf{0}$. Since

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_k + \cdots + 0\mathbf{v}_n = \mathbf{0},$$

this set is linearly dependent by the definition of dependence.

(4) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$. For, if the set is $\{\mathbf{0}\}$, then $c_1\mathbf{0} + c_2\mathbf{0} = \mathbf{0}$ whenever c_1 and c_2 are not necessarily zero.

(5) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other. For, let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be such a set and suppose that for some nonzero constant k, $\mathbf{v}_2 = k\mathbf{v}_1$. Then choosing c_1 and c_2 such that $c_1 + kc_2 \neq 0$, we obtain that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ whenever c_1 and c_2 are not necessarily zero. So, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set.

(6) Consider the vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$ in $\mathbb{R}^3$. We have

$$c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = \mathbf{0} \Rightarrow (c_1, 0, 0) + (0, c_2, 0) + (0, 0, c_3) = \mathbf{0} \Rightarrow (c_1, c_2, c_3) = (0,0,0)\,,$$

which implies that $c_1 = c_2 = c_3 = 0$, so the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent. In general, the vectors

$$\mathbf{e}_1 = (1,0,0,\ldots,0,0)\,, \mathbf{e}_2 = (0,1,0,\ldots,0,0)\,, \ldots, \mathbf{e}_n = (0,0,0,\ldots,0,1)$$

are linearly independent in $\mathbb{R}^n$.

(7) Consider the vectors $\mathbf{x} = (1,-2,3)\,, \mathbf{y} = (5,6,-1)\,, \mathbf{z} = (3,2,1)$ in $\mathbb{R}^3$. The vector equation $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0}$ implies

$$\begin{array}{rcrcrcl} c_1 & + & 5c_2 & + & 3c_3 & = & 0 \\ -2c_1 & + & 6c_2 & + & 2c_3 & = & 0 \\ 3c_1 & - & c_2 & + & c_3 & = & 0 \end{array}$$

This system has only trivial solution if $\det(A) \neq 0$, where

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

Since

$$\det(A) = \begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} \underset{III-3I}{\overset{II+2I}{=}} \begin{vmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{vmatrix} \overset{III+II}{=} \begin{vmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.■

The concept of linear dependence suggests that the vectors depend on each other in some way.

- **Linear dependence and linear independence**

  A set with two or more vectors is

  (1) *linearly dependent if and only if at least one of the vectors in the set is expressible as a linear combination of the other vectors in the set*

  (2) *linearly independent if and only if no vector in the set is expressible as a linear combination of the other vectors in the set.*

The next result shows that a linearly independent set in $\mathbb{R}^n$ can contain at most n vectors.

- **Number of vectors in a linear independent set**

  Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *be a set of vectors in* $\mathbb{R}^n$*. If* $r > n$*, then the vectors in* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *are linearly dependent.*

Proof: Suppose that $\mathbf{v}_1 = (x_{11}, x_{12}, \ldots, x_{1n}), \mathbf{v}_2 = (x_{21}, x_{22}, \ldots, x_{2n}), \ldots, \mathbf{v}_r = (x_{r1}, x_{r2}, \ldots, x_{rn})$ and consider the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = \mathbf{0}$. If we express both sides of this equation in terms of components and then equate the corresponding components, we obtain a homogeneous system of n equations in the r unknowns $c_1, c_2, \ldots, c_r$. Since $r > n$, it follows that the system has nontrivial solutions. Therefore, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set.■

Suppose that $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are at least $(n-1)$ times differentiable functions on $(-\infty, \infty)$. There is a determinant which is useful for determining whether the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ form a linearly independent set of vectors in the vector space $C^{n-1}(-\infty, \infty)$.

- **The Wronskian of functions**

  If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ *are functions that are at least* $(n-1)$ *differentiable functions on* $(-\infty, \infty)$*, then the determinant*

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

  is is called the ***Wronskian*** *of* $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$.

Suppose that $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ are linearly dependent vectors in $C^{n-1}(-\infty, \infty)$. Then there exist scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + \cdots + c_n\mathbf{f}_n = \mathbf{0}$. Thus, for all x in $(-\infty, \infty)$, we have

$$\begin{array}{ccccccccc}
c_1 f_1(x) & + & c_2 f_2(x) & + & \cdots & + & c_n f_n(x) & = & 0 \\
c_1 f_1'(x) & + & c_2 f_2'(x) & + & \cdots & + & c_n f_n'(x) & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
c_1 f_1^{(n-1)}(x) & + & c_2 f_2^{(n-1)}(x) & + & \cdots & + & c_n f_n^{(n-1)}(x) & = & 0
\end{array}$$

Thus the linear dependence of $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ implies that the system

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a nontrivial solution for every x in $(-\infty, \infty)$. This implies in turn that for every x in $(-\infty, \infty)$ the coefficient matrix is not invertible, or equivalently, that its determinant (the Wronskian) is zero. Thus we have the following result.

- **The Wronskian and linear independence**

  If the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ have at least $(n-1)$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{n-1}(-\infty, \infty)$.

The converse of this result is not true in general. If the Wronskian of $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ is zero on $(-\infty, \infty)$, then $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ may form a linearly independent set.

Example: Consider the functions

$$f_1(x) = \begin{cases} 1 + x^3, & \text{if } x \le 0, \\ 1, & \text{if } x > 0, \end{cases} \qquad f_2(x) = \begin{cases} 1, & \text{if } x \le 0, \\ 1 + x^3, & \text{if } x > 0, \end{cases} \qquad f_3(x) = 3 + x^3.$$

The Wronskian is

$$\begin{vmatrix} 1 + x^3 & 1 & 3 + x^3 \\ 3x^2 & 0 & 3x^2 \\ 6x & 0 & 6x \end{vmatrix} = 0$$

if $x \le 0$, and

$$\begin{vmatrix} 1 & 1 + x^3 & 3 + x^3 \\ 0 & 3x^2 & 3x^2 \\ 0 & 6x & 6x \end{vmatrix} = 0$$

if $x > 0$. However, if $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x in $[-1, 1]$, then $c_1 = c_2 = c_3 = 0$. Indeed,

$$\left.\begin{array}{lll} x = -1 & \Rightarrow & c_2 + 2c_3 = 0 \\ x = 0 & \Rightarrow & c_1 + c_2 + 3c_3 = 0 \\ x = 1 & \Rightarrow & c_1 + 2c_2 + 4c_3 = 0 \end{array}\right. \Rightarrow c_1 = c_2 = c_3 = 0$$

So, $\{f_1(x), f_2(x), f_3(x)\}$ is a linearly independent set on $C^2[-1, 1]$.■

Example: The functions $f_1(x) = 1$, $f_2(x) = \mathrm{e}^x$, and $f_3(x) = \mathrm{e}^{2x}$ form a linearly independent set of vectors in $C^2(-\infty, \infty)$ since

$$W(x) = \begin{vmatrix} 1 & \mathrm{e}^x & \mathrm{e}^{2x} \\ 0 & \mathrm{e}^x & 2\mathrm{e}^{2x} \\ 0 & \mathrm{e}^x & 4\mathrm{e}^{2x} \end{vmatrix} = 2\mathrm{e}^{3x} \neq 0.■$$

4.4. Basis and Dimension

We usually think of a line as being one-dimensional, a plane as two-dimensional, and the space around us as three-dimensional because it takes two coordinate numbers to determine a vector in the plane $\mathbb{R}^2$ and three for a vector in the space $\mathbb{R}^3$. It is the primary purpose of this section to make this intuitive notion of dimension more precise.

The following key definition enables us to extend the concept of a coordinate system to general vector spaces.

- **Basis of vector space**

  *If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a **basis** for V if S is linearly independent set and S spans V.*

Example: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be any element of $\mathbb{R}^n$. Since

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, \ldots, 0) + \cdots + x_n(0, 0, \ldots, 1) \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n, \end{aligned}$$

where $\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0), \ldots, \mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$. Moreover, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a linearly independent set. Thus, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, called the **standard basis** for $\mathbb{R}^n$. In particular, the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, is the standard basis for $\mathbb{R}^3$.■

Example: Show that $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of degree at most n.

Solution: We denote the polynomials in S as

$$\mathbf{p}_0 = 1, \mathbf{p}_1 = x, \mathbf{p}_2 = x^2, \ldots, \mathbf{p}_n = x^n.$$

These vectors span P_n and they are linearly independent (which can be verified using the Wronskian). Thus, they form a basis for P_n which is called the **standard basis** for P_n.$\blacksquare$

Example: Show that the vectors $\mathbf{x}_1 = (1,1,0)$, $\mathbf{x}_2 = (0,2,2)$, and $\mathbf{x}_3 = (1,0,1)$ form a basis for $\mathbb{R}^3$.

Solution: We must show that these vectors are linearly independent and span $\mathbb{R}^3$. To prove linear independence we must have

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$$

has the only solution $c_1 = c_2 = c_3 = 0$, that is, the system

$$\begin{array}{ccccccc} c_1 & & & + & c_3 & = & 0 \\ c_1 & + & 2c_2 & & & = & 0 \\ & & 2c_2 & + & c_3 & = & 0 \end{array}$$

has only the trivial solution. On the other hand, to prove spanning we must have

$$\mathbf{b} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

or

$$\begin{array}{ccccccc} c_1 & & & + & c_3 & = & b_1 \\ c_1 & + & 2c_2 & & & = & b_2 \\ & & 2c_2 & + & c_3 & = & b_3 \end{array}$$

for any element $\mathbf{b} = (b_1, b_2, b_3)$ of $\mathbb{R}^3$. This system must have a solution for all choices of b_1, b_2, b_3.

For both conditions we must have $\det(A) \neq 0$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Since

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{vmatrix} \overset{II-I}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & 1 \end{vmatrix} \overset{III-II}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{vmatrix} = 4,$$

$\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ form a basis for $\mathbb{R}^3$.■

Example: Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Solution: We must show that the matrices are linearly independent and span M_{22}. To prove linear independence we must show that the equation

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \mathbf{0}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix. On the other hand, to prove that the matrices span M_{22} we must show that every 2×2 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

can be expressed as

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B.$$

These matrix equations respectively give

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

From the first of these equations, we have $c_1 = c_2 = c_3 = c_4 = 0$ and from the second one $c_1 = b_{11}, c_2 = b_{12}, c_3 = b_{21}, c_4 = b_{22}$. These prove that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}. More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the **standard basis** for M_{mn}.■

Example: The simplest of all vector spaces is the zero vector space $V = \{\mathbf{0}\}$. This space is spanned by the vector $\mathbf{0}$. However, it has no basis in the sense of the definition of a basis since $\{\mathbf{0}\}$ is not a linearly independent set. However, we will define the empty set $\varnothing$ to be a basis for this vector space.■

In the case of the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{R}^3$, we know that it is easy to write out any other vector $\mathbf{v} = (c_1, c_2, c_3)$ in terms of the standard basis as

$$\mathbf{v} = (c_1, c_2, c_3) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

We call the scalars c_1, c_2, c_3 the **coordinates** of the vector $\mathbf{v}$. We can see that these coordinates are strongly tied to the standard basis. However, $\mathbb{R}^3$ has many bases. Thanks to the following fact, we can give the notion of coordinates relative to other bases.

- **Uniqueness of the linear combination**

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Proof: Since S spans V, it follows that every vector in V is expressible as a linear combination of the vectors in S. To see there is only one way to express a vector as a linear combination of vectors in S, suppose that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$

Hence

$$\mathbf{0} = (c_1 - d_1)\,\mathbf{v}_1 + (c_2 - d_2)\,\mathbf{v}_2 + \cdots + (c_n - d_n)\,\mathbf{v}_n.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we must have $c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0$, that is, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$.$\blacksquare$

- **Vector coordinates and coordinate vector**

  *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of the vector space V and $\mathbf{v} \in V$ with $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. The **coordinate vector** of $\mathbf{v}$ with respect to $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is $[\mathbf{v}]_S = (c_1, c_2, \ldots, c_n)$.*

Coordinates with respect to other bases are fairly easy to calculate if there is enough information about the structure of the vector space.

Example: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$.

(a) Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ with respect to S.

(b) Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ whose coordinate vector with respect to the basis S is $[\mathbf{v}]_S = (-1, 3, 2)$.

Solution:

(a) We must find scalars c_1, c_2, c_3 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, that is, such that

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 5 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & -1 \\ c_1 & & & + & 4c_3 & = & 9 \end{array}$$

Solving the system, we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$, so $[\mathbf{v}]_S = (1, -1, 2)$.

(b) Using the definition of the coordinate vector, we obtain that

$$\mathbf{v} = (-1)\,\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = (11, 31, 7)\,.\blacksquare$$

Example: The set $S = \{1, x+1, x^2+x\}$ of vectors form a basis of P_2. Find the coordinate vector of $\mathbf{p} = p(x) = 2 - 2x - x^2$ with respect to this basis.

Solution: The coordinates are c_1, c_2, c_3, where

$$2 - 2x - x^2 = c_1(1) + c_2(1+x) + c_3\left(x^2+x\right) = (c_1+c_2) + (c_2+c_3)\,x + c_3x^2.$$

We note here that the order in which we list the basis elements matters for the coordinates. Equating the corresponding coefficients, we obtain that $c_1 + c_2 = 2$, $c_2 + c_3 = -2$, and $c_3 = -1$. It follows that $c_1 = 3, c_2 = -1, c_3 = -1$. Thus $[\mathbf{p}]_S = (3, -1, -1)$.$\blacksquare$

We know that the standard vector spaces always have a basis, namely, the standard basis. The same is true also for abstract vector spaces. The following concept turns out to be helpful.

- **Finite-dimensional vector space**

  *A vector space V is called **finite-dimensional** if it contains a finite set of vectors that forms a basis.*

Examples of finite-dimensional vector spaces are the standard vector spaces $\mathbb{R}^n$, P_n, and M_{mn}. However, some very important spaces are not finite-dimensional, and accordingly, we call them infinite-dimensional spaces.

- **Infinite-dimensional vector space**

  *A vector space V is called **infinite-dimensional** if it contains no finite set of vectors that forms a basis.*

For example, the vector spaces $F(-\infty,\infty)$, $C(-\infty,\infty)$, $C^m(-\infty,\infty)$ and $C^\infty(-\infty,\infty)$ are infinite-dimensional.

The following result provides the key to the concept of dimension.

- *Let V be an n-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.*

  (a) *If a set in V has more than n vectors, then it is linearly dependent.*

  (b) *If a set in V has fewer than n vectors, then it does not span V.*

This result says that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is any basis for V, then all the sets in V that simultaneously span V and are linearly independent must have precisely n vectors. Thus, all bases for V must have the same number of vectors as the arbitrary basis S. This yields the following result, which is one of the most important result in linear algebra.

- **Dimension theorem**

  All bases for a finite-dimensional space have the same number of vectors.

This result is related to the concept of dimension by the following definition.

- **Dimension**

  The ***dimension*** *of a finite-dimensional vector space V, denoted by* $\dim(V)$, *is defined to be the number of vectors in a basis for V. We define the zero vector space to have dimension zero.*

For example, we have

$$\dim(\mathbb{R}^n) = n, \quad \dim(P_n) = n+1, \quad \dim(M_{mn}) = mn.$$

The dimensions of a finite-dimensional vector space V and a subspace W of V relate to each other by the following result.

- *If W is a subspace of the finite-dimensional vector space V, then W is also finite-dimensional and* $\dim(W) \leq \dim(V)$, *with equality if and only if $V = W$.*

We conclude this section by stating a series of theorems which show the relationships among the concepts of spanning, linear independence and basis.

- *Let S be a nonempty set of vectors in a vector space V.*

  (a) *If S is a linearly independent set and if $\mathbf{v}$ is a vector space V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.*

  (b) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space, that is,* span(S) =span$(S - \{\mathbf{v}\})$.

In general, to show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, we must show that the vectors are linearly independent and span V. However, if we know that the dimension of V is n, then it suffices to check either linear independence or spanning. The remaining condition will hold automatically.

- *If V is an n-dimensional vector space, and if S is a subset of V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.*

For example, the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for $\mathbb{R}^3$ since $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the xz-plane and $\mathbf{v}_3$ is outside of the xz-plane. So the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Since $\mathbb{R}^3$ is 3-dimensional, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

- *Let S be a finite set of vectors in a finite-dimensional vector space V.*

  (a) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

  (b) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

4.5. Subspaces Associated with Matrices

Certain subspaces are a rich source of information about the behavior of a matrix. We define them and introduce their properties in this section.

- **Column space**

  The ***column space*** *of an $m \times n$ matrix A is the subspace $C(A)$ of $\mathbb{R}^m$ spanned by the columns of A.*

Example: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

We have $C(A) \subseteq \mathbb{R}^2$. Moreover, we have

$$C(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

But since

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we have

$$C(A) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

This shows that not all the columns of the matrix A are really needed to span the entire subspace $C(A)$. Since $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ are linearly independent, they form a basis for $C(A)$, and $\dim(C(A)) = 2$.$\blacksquare$

- **Row space**

  *The **row space** of an $m \times n$ matrix A is the subspace $R(A)$ of $\mathbb{R}^n$ spanned by the transposes of the rows of A.*

This definition simply suggests that $R(A) = C\left(A^T\right)$, so whatever valid for column spaces can be applied to row spaces.

Example: Consider again the matrix

$$A = \begin{bmatrix}1 & 1 & 1 & -1\\0 & 1 & 2 & 1\end{bmatrix}.$$

We have

$$R(A) = \text{span}\left\{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^4.$$

Neither of these vectors can be expressed as a multiple of the other, for if we would have

$$c\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix},$$

then c would be zero. Thus, these vectors form a basis for $R(A)$, and $\dim(R(A)) = 2$.$\blacksquare$

- **Null space**

  *The **null space** of the $m \times n$ matrix A is the subset $N(A)$ of $\mathbb{R}^n$ defined by*

$$N(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

We observe that $N(A)$ is the solution set to the homogeneous linear system $Ax = \mathbf{0}$. Let A be a square matrix. We know that A is invertible exactly when the system $Ax = \mathbf{0}$ has only the trivial solution Thus, we can add one more equivalent condition to the list of equivalences for invertibility.

- **Equivalent statements IV**

  If A is an $n \times n$ matrix, then the following statements are equivalent:

  (1) *A is invertible.*

  (2) *$Ax = \mathbf{0}$ has only the trivial solution.*

  (3) *The reduced row echelon form of A is I_n.*

  (4) *A is expressible as a product of elementary matrices.*

  (5) *$Ax = b$ is consistent for every $n \times 1$ matrix b.*

  (6) *$Ax = b$ has exactly one solution for every $n \times 1$ matrix b.*

  (7) $\det(A) \neq 0$.

  (8) $N(A) = \{\mathbf{0}\}$.

As column and row spaces, the null space of a matrix is a subspace of the Euclidean space.

- **Solution spaces of homogeneous systems**

  $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be arbitrary elements of $N(A)$. By definition, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Adding these two equations, we obtain

$$\mathbf{0} = \mathbf{0} + \mathbf{0} = A\mathbf{u} + A\mathbf{v} = A(\mathbf{u} + \mathbf{v}).$$

Thus, $\mathbf{u} + \mathbf{v} \in N(A)$. Multiplying the equation $A\mathbf{u} = \mathbf{0}$ by the scalar c, we see that

$$\mathbf{0} = c\mathbf{0} = c(A\mathbf{u}) = A(c\mathbf{u}).$$

Therefore, we have $c\mathbf{u} \in N(A)$. By the subspace test, $N(A)$ is a subspace of $\mathbb{R}^n$.$\blacksquare$

Example: Consider once the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{I-II} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - x_4 \end{aligned}$$

where x_3 and x_4 are free variables. From this we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

We then see that the general solution of the homogeneous system is a linear combination of the linearly independent vectors on the right, so

$$N(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4,$$

that is, $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $N(A)$, and $\dim(N(A)) = 2$.■

Example:

(a) Consider the system

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solutions are

$$x = 2t - s, \quad y = t, \quad z = s,$$

from which it follows that

$$x - 2y + z = 0.$$

The set of vectors $\mathbf{u} = (x, y, z)$ satisfying this equation is a subspace of $\mathbb{R}^3$.

(b) Consider the system

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $x = y = z = 0$, so the solution space is $\{\mathbf{0}\}$.

(c) Consider the system

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solutions are $x = t$, $y = s$, $z = r$, where t, s, r are arbitrary, so the solution space is $\mathbb{R}^3$.■