

Defn. Harmonic numbers $H_1 = 1$ $H_2 = 1 + \frac{1}{2}$ $H_3 = 1 + \frac{1}{2} + \frac{1}{3} \dots$
 $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

Thm. $\forall n \in \mathbb{Z}^+$, $\sum_{j=1}^n j H_j = \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4}$

Proof. We'll make induction on n .

• Base step: Idea: Show that thm holds for the smallest element of the universal set that the thm is about.

For $n=1$ we have $\sum_{j=1}^1 j H_j = 1 = \frac{1(2)}{2} H_2 - \frac{1(2)}{4} = \frac{2}{2} \cdot \frac{3}{2} - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1 \checkmark$

• Ind. hypothesis: For $n=k$ we assume

$$\sum_{j=1}^k j H_j = \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4}$$

• Ind. step: For $n=k+1$ we'll show that

$$\sum_{j=1}^{k+1} j H_j \stackrel{?}{=} \frac{(k+1)(k+2)}{2} H_{k+2} - \frac{(k+1)(k+2)}{4}$$

$H_{k+2} = H_{k+1} + \frac{1}{k+2}$ by defn

$$= \sum_{j=1}^k j H_j + (k+1) H_{k+1}$$

$$= \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4} + (k+1) H_{k+1} \stackrel{?}{=} \frac{(k+1)(k+2)}{2} (H_{k+1} + \frac{1}{k+2}) - \frac{(k+1)(k+2)}{4}$$

$$= H_{k+1} \left(\frac{k(k+1)}{2} + \frac{2(k+1)}{2} \right) - \frac{k(k+1)}{4} = H_{k+1} \left(\frac{(k+1)(k+2)}{2} \right) + \frac{2(k+1)}{2 \cdot 2} - \frac{(k+1)(k+2)}{4}$$

$$= H_{k+1} \left(\frac{k^2 + k + 2k + 2}{2} \right) - \frac{k(k+1)}{4} = H_{k+1} \left(\frac{k^2 + 3k + 2}{2} \right) + \frac{2k + 2}{4} - \frac{(k+1)(k+2)}{4}$$

So we are done, our thm is correct QED

for $n=4$

2 5 6 7 11 13 14 27 33 46 51 53 54 62 65 72

Does 51 exist in this array

↓
of comparisons we have made is at most $n+1 = 5$

Proof of binary search tm.

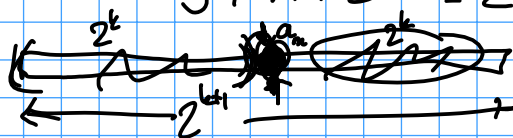
• Base step: For $n=0$ $|A_n| = 2^n = 2^0 = 1$

of comp we need is obv. 1 = $0+1=1$

• Ind. hyp: Assume that for $n=k$ $|A_n| = 2^k$ # of comp. we need (at most) is $k+1$

• Ind. step: for $n=k+1$, the size of the array $|A_n| = 2^{k+1} = 2 \cdot 2^k$

Search x



If $x = a_m$
 $x < a_m$
 $x > a_m$

of com

1
 $(k+1) + 1 = k+2$
 $(k+1) + 1 = k+2$

So, the number of comp. needed to find x in an array of size 2^{k+1}

is $k+2$

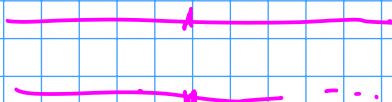
Our hyp also tells us it must be $k+1+1 = k+2$ com.

6:5
12:5

8:4

4:

1:2



Defn. Fibonacci numbers

$$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$$

0	1	1	2	3	5	8	13	21	34	55
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Thm. $\sum_{i=0}^n F_i^2 = F_n F_{n+1} \quad \forall n \geq 0$

Proof. Base step: For $n=0$

$$\sum_{i=0}^0 F_i^2 = 0^2 = F_0 F_1 = 0 \cdot 1 = 0$$

Ind. hyp. Assume true for $n=k$,

$$\sum_{i=0}^k F_i^2 = F_k F_{k+1}$$

Ind. step. We'll show that for $n=k+1$,

$$\sum_{i=0}^{k+1} F_i^2 = \left(\sum_{i=0}^k F_i^2 \right) + F_{k+1}^2 \stackrel{?}{=} F_{k+1} F_{k+2}$$

$$= F_k F_{k+1} + F_{k+1}^2 \stackrel{?}{=} F_{k+1} F_{k+2}$$

$$= F_{k+1} (F_k + F_{k+1})$$

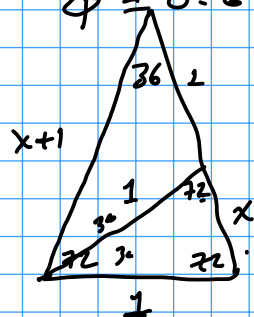
$$F_{k+1} \cdot F_{k+2} = \quad \checkmark \quad \text{We're done QED.}$$

n=0	0
n=1	1
n=2	1
n=3	2
n=4	3
n=5	5
n=6	8
n=7	13
n=8	21
n=9	34
n=10	55
n=11	89
n=12	144
n=13	233
n=14	377
n=15	610
n=16	987
n=17	1597
n=18	2584
n=19	4181
n=20	6765
n=21	10946
n=22	17711
n=23	28657
n=24	46368
n=25	75025
n=26	121393
n=27	196418
n=28	317811
n=29	514132
n=30	832040
n=31	1346269
n=32	2178309
n=33	3542248
n=34	5726057
n=35	9273682
n=36	14943581
n=37	24214301
n=38	39186601
n=39	63478312
n=40	102398763
n=41	165580141
n=42	268015401
n=43	433494461
n=44	701408861
n=45	1123752681
n=46	1825261781
n=47	2949016641
n=48	4774278421
n=49	7723295061
n=50	12538897831
n=51	20273359751
n=52	32816162581
n=53	53093795351
n=54	85910214761
n=55	138903014961
n=56	224816344641
n=57	363720369501
n=58	588536714141
n=59	952353183641
n=60	1541078907781
n=61	2493413601921
n=62	4034492509701
n=63	6527906111621
n=64	10561408621321
n=65	17089314733041
n=66	27650723354361
n=67	44730038087401
n=68	72380761441761
n=69	117110799529201
n=70	190491560970961
n=71	307602360500161
n=72	498093921471121
n=73	795796281971281
n=74	1253889783162401
n=75	1998686065133581
n=76	3124575848304801
n=77	4923265613438381
n=78	7621955461742981
n=79	11945641275181361
n=80	18567597236924341
n=81	28513238512105721
n=82	43780835749030061
n=83	67294074261135781
n=84	103074910000165841
n=85	158368984261301621
n=86	243663994261467461
n=87	372032978522769081
n=88	565696972784136541
n=89	868759951306905621
n=90	1334456924091042161
n=91	2053216875397947781
n=92	3127673799489089941
n=93	4810889674886032121
n=94	7338563474375122061
n=95	11149453169261154181
n=96	17188016643636276241
n=97	26337470812907430421
n=98	40125487456543706661
n=99	61062958269451137081
n=100	92736815726004843701

Hypoth $\frac{1}{0} \frac{1}{1} \frac{2}{1} \frac{3}{2} \frac{5}{3} \frac{8}{5} \frac{13}{8} \frac{21}{13} \dots$

$$\phi = 1.618033 \dots = \frac{\sqrt{5}+1}{2}$$

$$\phi' = 0.618033 \dots = \frac{\sqrt{5}-1}{2}$$



$$\frac{1}{x} = \frac{x+1}{1}$$

$$x(x+1) = 1$$

$$x^2 + x - 1 = 0$$

$$\frac{-1 \pm \sqrt{1^2 - 4(-1)(1)}}{2}$$

$$\frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{\sqrt{5}-1}{2} = \phi'$$

Defn. Lucas numbers: $L_0 = 2$ $L_1 = 1$ $L_n = L_{n-1} + L_{n-2}$

0	1	2	3	4	5	6	7	8
2	1	3	4	7	11	18	29	47 ...

Thm. $\forall n \in \mathbb{Z}^+$, $L_n = F_{n-1} + F_{n+1}$

ind.	0	1	2	3	4	5	6	7
F	0	1	1	2	3	5	8	13
L	2	1	3	4	7	11	18	29

} illustr.

Proof. We make induction on n . Ind. hyp. Assume that for $m \leq k$ from 0 to k m

Base step For $n=1$ $L_1 = F_0 + F_2$
 $1 = 0 + 1 \quad \checkmark$

$$L_m = F_{m-1} + F_{m+1}$$

Ind. step. for $n=k+1$, we need to show

$$\rightarrow L_{k+1} = F_k + F_{k+2}$$

$$L_{k-1} + L_k \stackrel{?}{=} F_{k-1} + F_{k-2} + F_k + F_{k+1}$$

$m=k$
 $m=k-1$

\rightarrow we're done QED.

$$\frac{a}{b} \mid \frac{b}{n}$$

$$\begin{array}{ccc} x & = & y + z \\ \downarrow & & \downarrow \quad \downarrow \\ b & & b \quad b \\ 0 & & 0 \quad 0 \end{array}$$

mod b

$$\gcd(24, 32) = 8$$

$$3 \mid 9$$

$$3 \mid 6$$

$$9x + 6y \equiv 3 \pmod{3}$$

$$25 = 8 \cdot 4 - 7$$

$$\frac{25}{a} = \frac{8}{b} \cdot \frac{3}{\text{quotient}} + \frac{1}{\text{remainder}}$$

$$\begin{array}{r} 25 \overline{) 18} \\ - 24 \\ \hline 3 \end{array}$$

$$25 = 8 \cdot 2 + 9$$

$$r_0 \quad r_1 \quad r_2$$
$$960 = 1 \cdot 500 + 460$$

$$500 = 1.460 + 40$$

$$460 = 11 \cdot 40 + 20$$

$$40 = 2 \cdot \underline{20} = \gcd(960, 500)$$

$$960 = 1.600 + 360$$

$$600 = 1 \cdot 360 + 240$$

$$360 = 1 \cdot 240 + 120$$

$$240 = 2 \cdot 120 //$$

$$\begin{array}{r} 16.6.10 \\ \hline \rightarrow 2^6.3.5 \\ \quad \quad 1 \end{array} \quad , \quad \begin{array}{r} 6.10.10 \\ \hline \rightarrow 2^3.3.5^2 \\ \quad \quad 1 \end{array}$$
$$2^3 \cdot 3 \cdot 5 = 120$$

$$\begin{array}{r} 56 \\ 16 \end{array} \quad \begin{array}{r} - \\ - \\ - \\ - \\ - \\ - \end{array} \quad \begin{array}{r} 230 \end{array}$$