FACULTY OF ENGINEERING
DEPARTMENT OF COMPUTER ENGINEERING
MAT 222 LINEAR ALGEBRA AND NUMERICAL METHODS
FINAL EXAM SOLUTIONS

1. Consider the set $\{(c, j+1, k+1), (1, 0, j+1), (k+2, 0, 1)\}$, which is a basis for $\mathbb{R}^3$. Construct an orthogonal basis of $\mathbb{R}^3$ whose first element is $(c, j+1, k+1)$. Then, find the coordinates of the vector (c, j, k) with respect to the new basis.

Solution: Suppose $c = 1, j = 4, k = 6$. Then the given set is $\{(1, 5, 7), (1, 0, 5), (8, 0, 1)\}$, which is a basis for $\mathbb{R}^3$ but not orthogonal. Let us apply Gram-Schmidt process to this basis. Labeling the vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, respectively, we have

$$\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 5) - \left(\frac{12}{25}, \frac{12}{5}, \frac{84}{25}\right) = \left(\frac{13}{25}, -\frac{12}{5}, \frac{41}{25}\right)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \operatorname{proj}_{\mathbf{v}_1}\mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_2}\mathbf{v}_3 = (8, 0, 1) - \left(\frac{1}{5}, 1, \frac{7}{5}\right) - \left(\frac{377}{1090}, -\frac{174}{109}, \frac{1189}{1090}\right) = \left(\frac{1625}{218}, \frac{65}{109}, -\frac{325}{218}\right).$$

Then $\{\mathbf{v}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 5, 7), \left(\frac{13}{25}, -\frac{12}{5}, \frac{41}{25}\right), \left(\frac{1625}{218}, \frac{65}{109}, -\frac{325}{218}\right)\}$ is an orthogonal basis for $\mathbb{R}^3$. The given vector is $\mathbf{v} = (c, j, k) = (1, 4, 6)$. Let us write it as $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Then c_1, c_2, c_3 can be calculated as follows:

$$c_1 = \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{63}{75}, \quad c_2 = \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{19}{218}, \quad c_3 = \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{195}{12675}.$$

2. Consider the data below.

x_i	1	1.5	2	3
y_i	c	a	$j+1$	$k+2$

Find the least squares regression line for this data **using the normal equations or the mean deviation form.** (Hint: First construct the design matrix.)

Solution: Suppose $a = 2, c = 1, j = 4, k = 6$. The design matrix is $X = \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and the observation vector is $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 7 \end{bmatrix}$. The normal equations are $X^T X\beta = X^T\mathbf{y}$. The coefficient matrix of this system and its inverse are

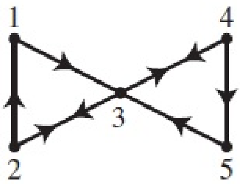
$$X^T X = \begin{bmatrix} 4 & 15/2 \\ 15/2 & 65/4 \end{bmatrix}, \quad (X^T X)^{-1} = \begin{bmatrix} 13/7 & -6/7 \\ -6/7 & 16/35 \end{bmatrix}.$$

As a result, the parameter vector is obtained by

$$\beta = (X^T X)^{-1} X^T\mathbf{y} = \begin{bmatrix} -15/7 \\ 22/7 \end{bmatrix}.$$

This means that the least squares regression line for the given data is $y = \frac{-15}{7} + \frac{22}{7}x$.

3. Consider a miniature version of the Internet that consists of five web pages whose directed graph model is given below.



In the figure, arrows indicate hyperlinks. For example, the arrow pointing from web page 2 toward web page 1 means that there is a hyperlink in web page 2 to web page 1. Now, consider a random surfer who at each time step moves randomly to and from these web pages according to this model.

(a) Let $s = 1 + (k + j \bmod 5)$. If the random surfer is at web page s at time $t = 0$, what is the probability that the surfer will be at web page 3 at time $t = 4$? What is the probability that the surfer will again be at web page s at time $t = 4$? (Hint: Construct the stochastic (transition) matrix first.)

(b) Suppose that the surfer has clicked the mouse 1 million times, thus moving to a new web page 1 million times. In how many of these 1 million times do you expect this surfer to be at web page s? Please find an approximate result.

Solution: (a) The stochastic matrix is as follows:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

Suppose $j = 4, k = 6$. Then $s = 1$. The surfer being at web page 1 is represented by the state vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The probabilites of the state vector at time 4 is computed by

$$\mathbf{x}_4 = P^4 \mathbf{x}_0.$$

Let us compute P^4. We have

$$P^2 = \begin{bmatrix} 0 & 0 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 1/4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P^4 = (P^2)^2 = \begin{bmatrix} 0 & 1/8 & 1/8 & 1/8 & 0 \\ 1/4 & 1/8 & 1/4 & 1/8 & 1/4 \\ 1/2 & 1/2 & 1/4 & 1/2 & 1/2 \\ 1/4 & 1/8 & 1/4 & 1/8 & 1/4 \\ 0 & 1/8 & 1/8 & 1/8 & 0 \end{bmatrix}.$$

So, the probabilistic state vector at time 4 is

$$\mathbf{x}_4 = P^4 \mathbf{x}_0 = \begin{bmatrix} 0 & 1/8 & 1/8 & 1/8 & 0 \\ 1/4 & 1/8 & 1/4 & 1/8 & 1/4 \\ 1/2 & 1/2 & 1/4 & 1/2 & 1/2 \\ 1/4 & 1/8 & 1/4 & 1/8 & 1/4 \\ 0 & 1/8 & 1/8 & 1/8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ 1/2 \\ 1/4 \\ 0 \end{bmatrix}. \tag{1}$$

This means that the probability that the surfer will be at web page 1 is 0, while the probability that the surfer will be at web page 3 is 1/2.

(b) The long term proabilites are determined by the eigenvector corresponding to the eigenvalue 1. For $\lambda = 1$ we have

$$P - \lambda \mathbf{I} = \begin{bmatrix} 0-\lambda & 1/2 & 0 & 0 & 0 \\ 0 & 0-\lambda & 1/2 & 0 & 0 \\ 1 & 1/2 & 0-\lambda & 1/2 & 1 \\ 0 & 0 & 1/2 & 0-\lambda & 0 \\ 0 & 0 & 0 & 1/2 & 0-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & 1/2 & 0 & 0 \\ 1 & 1/2 & -1 & 1/2 & 1 \\ 0 & 0 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & -1 \end{bmatrix}.$$

The nullspace of this matrix can be found to be the set of all multiples of the vector $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$. We must scale it to be a probability vector, i.e. a vector whose entries sum up to 1. Then we have $\mathbf{v} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \\ 0.2 \\ 0.1 \end{bmatrix}$. This shows that in the long run 10 percent of the time the surfer will be at web page 1. Thus, we can comment that if the surfer moves to a new web page 1 million times, approximately 100000 times he/she will be at web page 1.

4. Let $\mathbf{v}_1 = \begin{bmatrix} 1+j \\ k \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} k+c \\ -(j+1) \\ -c(j+1) \end{bmatrix}$. Let A be the 3×2 matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Consider the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find a vector $\mathbf{x} \in \mathbb{R}^2$ such that the product $A\mathbf{x}$ is as closest as possible to $\mathbf{b}$.

(b) Let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Find the component of $\mathbf{b}$ which is orthogonal to V.

Solution: (a,b) Suppose $c = 1, j = 4, k = 6$. Then $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -5 \\ -5 \end{bmatrix}$. Then $A = \begin{bmatrix} 5 & 7 \\ 6 & -5 \\ 1 & -5 \end{bmatrix}$. Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. One method is to find the projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto the column space of A. Then we can answer both parts. Since the columns of A are orthogonal, $\hat{\mathbf{b}}$ is just the sum of the projections of $\mathbf{b}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$. These two projections can be computed as follows:

$$\text{proj}_{\mathbf{v}_1} \mathbf{b} = \left(\frac{15}{31}, \frac{18}{31}, \frac{3}{31} \right), \quad \text{proj}_{\mathbf{v}_2} \mathbf{b} = \left(\frac{14}{99}, -\frac{10}{99}, -\frac{10}{99} \right)$$

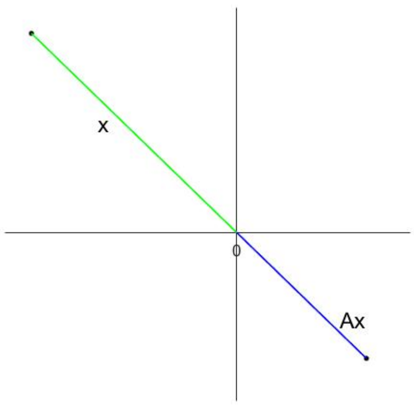
Thus, the projection is

$$\hat{\mathbf{b}} = \text{proj}_{\mathbf{v}_1} \mathbf{b} + \text{proj}_{\mathbf{v}_2} \mathbf{b} = \left(\frac{15}{31}, \frac{18}{31}, \frac{3}{31} \right) + \left(\frac{14}{99}, -\frac{10}{99}, -\frac{10}{99} \right) = \left(\frac{1919}{3069}, \frac{1472}{3069}, -\frac{13}{3069} \right).$$

The solution of the least squares problem in part (a) is just the solution of the system $A\mathbf{x} = \hat{\mathbf{b}}$, which is $\mathbf{x} = \begin{bmatrix} 3/31 \\ 2/99 \end{bmatrix}$. For part (b), the requested vector is just $\mathbf{b} - \hat{\mathbf{b}}$, which is

$$\mathbf{b} - \hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1919}{3069} \\ \frac{1472}{3069} \\ -\frac{13}{3069} \end{bmatrix} = \begin{bmatrix} \frac{1150}{3069} \\ -\frac{1472}{3069} \\ \frac{3082}{3069} \end{bmatrix}.$$

5. Consider the matrix $A = \begin{bmatrix} j+1 & * \\ 1 & k+1 \end{bmatrix}$. Suppose that a particle moves freely on the unit circle in $\mathbb{R}^2$. Let us denote the varying position vector of this particle by $\mathbf{x}$. Suppose also that a second particle is tracing the curve indicated by $A\mathbf{x}$. Suppose further that there is a time instant such that these two particles are collinear with the origin but pointing towards opposite directions. In addition, the second particle is closer to the origin compared to the first particle. See the figure below. Find a value for the missing entry of A such that this scenario is possible. (There are infinitely many answers.) **(20 points)**



Solution: Suppose $j = 4, k = 6$. Denoting the missing entry by x, it is understood that the matrix $A = \begin{bmatrix} 5 & x \\ 1 & 7 \end{bmatrix}$ has an eigenvalue λ between -1 and 0. Let us take a number from this interval, for instance -0.5. Then one of the roots of the characteristic polynomial of A is -0.5. The characteristic polynomial of A is $p(\lambda) = \begin{vmatrix} 5-\lambda & x \\ 1 & 7-\lambda \end{vmatrix} = (5-\lambda)(7-\lambda) - x$. Since -0.5 is a root, we must have $p(-0.5) = (5-0.5)(7-0.5) - x = 0$. This shows that $x = (4.5)(6.5) = 29.25$. Note that this is just one of the possible values of x. A different choice for the eigenvalue λ will produce a different result.

6. Consider the linear transformation T in $\mathbb{R}^2$ that does the following:

(i) First, it rotates any vector by 90 degrees in counter-clockwise direction.

(ii) Then, it takes the resulting vector from step (i) and projects it onto the y-axis.

(a) Find a matrix A such that $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$.

(b) Compute the image of the vector $\begin{bmatrix} j \\ k \end{bmatrix}$ under T.

Solution (a) T can be thought of as the composition of the linear transformations T_1 and T_2, where T_1 rotates any vector by 90 degrees in counter-clockwise direction and T_2 projects any vector onto the y-axis. The matrix for T_1 is $A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the matrix for T_2 is $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then the matrix for T is

$$A = A_2 A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(b) Suppose $j = 4, k = 6$. Then the image of the given vector under T is

$$A \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$