10/01/2024 Final Exam Duration: 90 minutes

Name: Student No:

P1 [15pts] How many positive integers n less than 600 satisfy $\gcd(n, 600) = 1$? Show your calculation.

since $\gcd(600,600)>1$ note that 600 will be gone anyway

Let conditions c_i be
c_1: div by 2
c_2: div by 3
c_3: div by 5

then, the answer is $\bar{N} = S_0 - S_1 + S_2 - S_3$ where $S_0 = 600$ (all num 1 to 600)

$S_1 = \frac{600}{2} + \frac{600}{3} + \frac{600}{5} = 300+200+120 = 620$ $S_3 = \frac{600}{2\cdot3\cdot5} = 20$

$S_2 = \frac{600}{2\cdot3} + \frac{600}{2\cdot5} + \frac{600}{3\cdot5} = 100+60+40 = 200$

Then, $\bar{N} = 600 - 620 + 200 - 20 = 160$ //

P2 [15pts] In how many ways can Alice select 9 balls from a bag that contains 3 red, 3 blue, 3 green and 3 white balls. Show your calculation and clearly write your final formula.

$x_1+x_2+x_3+x_4 = 9$, $\forall i, 0 \le x_i \le 3$. Let $c_i: x_i \geq 4$. Then, the answer is $\bar{N}$ where

$\bar{N} = S_0 - S_1 + S_2 - \underbrace{S_3}_{=0} + \underbrace{S_4}_{=0} = \binom{12}{3} - \binom{4}{1}\binom{8}{3} + \binom{4}{2}\binom{4}{3}$ //

P3 [15pts] Alice, Bob, and Charlie each create a random permutation of the numbers from 1 to 20. Then, they examine these permutations and find that there are no indices containing the same number in any pair of permutations. (For example, there is no match like the number three in the second index in [15,3,11,...] and [17,3,5,...]).

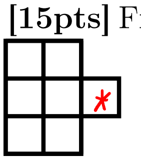
(a)Using the derangement formula, find a very simple upper bound for the probability of this occurring. Explain your calculation and reasoning.

According to derangement formula, P(A has no coincidence with B) = $1/e$. So are P(A-C) & P(B-C).
So, an obvious upper bound is $(1/e)^3$ because we want no coincidences which means we want all three events happening at the same time.

(b) Why is this not an exact solution but just an upper bound?

Because when there are no coincidences between A-B and B-C, the prob. of no coincidence A-C is actually smaller than $1/e$.

P4 [15pts] Find the rook polynomial for the board below. Clearly write it in the format $1+ax+bx^2+\cdots$.

$R($  $, x) = xR(\boxplus, x) + R(\quad, x)$

$\boxplus$ is easy: $1+4x+2x^2$

$\to$ $\boxminus$ is easy: $1+2x$

$= x(1+4x+2x^2) + xR(\boxminus, x) + R(\quad, x)$

$= x+4x^2+2x^3 + x+2x^2 + xR(\boxminus, x) + R(\boxplus, x)$

$= 2x+6x^2+2x^3 + \underbrace{x(1+2x)}_{x+2x^2} + 1+4x+2x^2 = 1+7x+10x^2+2x^3$ //

P5 [20 points] A florist needs to prepare a bouquet that contains 20 flowers with

- at least 4 roses
- positive even number of tulips
- odd number of daisies
- zero or one orchid
- zero or one lily
- a positive multiple of five of jasmines

The answer is the coefficient of x^{20} in the poly. mult:

$$\underbrace{(x^4+x^5+\cdots)}_{\text{(roses)}}\underbrace{(x^2+x^4+\cdots)}_{\text{(tulips)}}\underbrace{(x+x^3+x^5+\cdots)}_{\text{(daisies)}}\underbrace{(1+x)}_{\text{(orch)}}\underbrace{(1+x)}_{\text{(lily)}}\underbrace{(x^5+x^{10}+\cdots)}_{\text{(jasm)}}$$

$$= x^4 \cdot x^2 \cdot x \cdot x^5 \,(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)^2(1+x)^2(1+x^5+x^{10}+\cdots)$$

$$= x^{12}\cdot\frac{1}{1-x}\cdot\left(\frac{1}{1-x^2}\right)^2(1+x)^2\cdot(1+x^5+x^{10}+\cdots) = x^{12}\cdot(1-x)^{-3}(1+x^5+\cdots)$$

So, we need coeff of x^8 in $(1+x^5+\cdots)(1-x)^{-3}$

$1 \quad \binom{-3}{8}(-x)^8$

$x^5 \quad \binom{-3}{3}(-x)^3$

$\binom{10}{8}+\binom{5}{3}(-1)^3(-1)\; x^8$

$\Rightarrow \binom{10}{8}+\binom{5}{3}.$

In how many ways can the florist do this? Make your calculations in the space above ↑↑, write your final answer clearly.

P6 [20 points] Let the series $\{a_n\}$ be defined with the recursive definition: $a_0 = 0, a_1 = 1$ and $\forall n \geq 2 : a_n = 5a_{n-1} - 6a_{n-2}$. By using generating functions, find a closed formula for a_n (A formula that depends only on n, so that if we need a_{10000} we can just substitute n with 10000 and calculate the result.) Make your calculations below and clearly write the final result.

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then,

$-5x G(x) = \quad -5a_0 x - 5a_1 x^2 - \cdots$

$+6x^2 G(x) = \quad 6a_0 x^2 + \cdots$

$= 0$ (due to the definition)

$+$

$G(x)(1-5x+6x^2) = 0 + x - 0\cdot x = x.$

$$G(x) = \frac{x}{(1-2x)(1-3x)}$$

Let $G(x) = \dfrac{A}{1-2x} + \dfrac{B}{1-3x}$. Then $A(1-3x)+B(1-2x) = x$

So, $A+B=0$, $-3A-2B=1$

$A=-1$, $B=1$

So, $G(x) = \dfrac{1}{1-3x} - \dfrac{1}{1-2x}$

Generates 3^n — Generates 2^n

Thus, the term $a_n = 3^n - 2^n$

Check:

Defn	Formula
$a_2 = 5\cdot1 - 6\cdot0 = 5$	$a_2 = 3^2 - 2^2 = 5$
$a_3 = 5\cdot5 - 6\cdot1 = 19$	$a_3 = 3^3 - 2^3 = 19.$
$a_4 = 5\cdot19 - 6\cdot5 = 65\ldots$	$a_4 = 3^4 - 2^4 = 65$
	$\vdots$

Table 1: Some generating functions that can be useful. For all $m, n \in \mathbb{Z}^+$, $a \in \mathbb{R}$

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$

3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n$

5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$

6) $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$