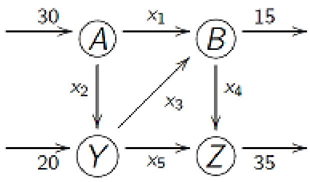
SPRING 2024
DEPARTMENT OF COMPUTER ENGINEERING
MAT222
LINEAR ALGEBRA
MIDTERM SOLUTIONS

April 24, 2024

1) The flow of traffic through a network of telephone towers is shown in the following figure.



Find x_1, x_2, x_3, x_4, x_5 using Gauss-Jordan elimination. **(20 p.)**

Solution: We have

$$\begin{array}{lcl} \text{node } A & : & x_1 + x_2 = 30 \\ \text{node } B & : & x_1 + x_3 = 15 + x_4 \\ \text{node } C & : & x_2 + 20 = x_3 + x_5 \\ \text{node } D & : & x_4 + x_5 = 35 \end{array}$$

These give the system

$$\begin{array}{ccccccc} & & x_1 & + & x_2 & = & 30 \\ x_1 & + & x_3 & - & x_4 & = & 15 \\ -x_2 & + & x_3 & + & x_5 & = & 20 \\ & & x_4 & + & x_5 & = & 35 \end{array}$$

whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 0 & 1 & -1 & 0 & 15 \\ 0 & -1 & 1 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 0 & 1 & -1 & 0 & 15 \\ 0 & -1 & 1 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & -1 & 1 & -1 & 0 & -15 \\ 0 & -1 & 1 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & -1 & 1 & 0 & 15 \\ 0 & -1 & 1 & 0 & 1 & 20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 15 \\ 0 & 1 & -1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 15 \\ 0 & 1 & -1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 50 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{array}{ccccccc} x_1 & + & x_3 & + & x_5 & = & 50 \\ x_2 & - & x_3 & - & x_5 & = & -20 \\ & & x_4 & + & x_5 & = & 35 \end{array}$$

The solution set of this system is

$$\{(50 - t - s, -20 + t + s, t, 35 - s, s) : t, s \in \mathbb{R}\}.$$

As a particular solution we choose $t = s = 10$ and obtain that $x_1 = 30$, $x_2 = 0$, $x_3 = 10$, $x_4 = 25$, and $x_5 = 10$.■

(2) Is a skew-symmetric matrix with an odd integer size invertible? Explain your answer. **(20 p.)**

Solution: Let A be a skew-symmetric matrix. Then $A^T = -A$. Since n is odd, we have

$$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^n \det(A) = -\det(A),$$

which implies that $\det(A) = 0$. Thus, A is not invertible.■

(3) Find a basis of the subspace of $\mathbb{R}^4$ consists of all vectors orthogonal to both $(1, 0, -1, 1)$ and $(0, 1, 2, 3)$. **(20 p.)**

Solution: W need to find all vectors $\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ such that

$$(u_1, u_2, u_3, u_4) \cdot (1, 0, -1, 1) = 0 \quad \text{and} \quad (u_1, u_2, u_3, u_4) \cdot (0, 1, 2, 3) = 0.$$

These equations give the system

$$\begin{array}{ccccccc} u_1 & - & u_3 & + & u_4 & = & 0 \\ u_2 & + & 2u_3 & + & 3u_4 & = & 0 \end{array}$$

whose solution set is

$$\{(t - s, -2t - 3s, t, s) : t, s \in \mathbb{R}\}.$$

Since any solution can be written as

$$t(1, -2, 1, 0) + s(-1, -3, 0, 1),$$

the desired basis is $B = \{(1, -2, 1, 0), (-1, -3, 0, 1)\}$.■

(4) Are null spaces of the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

same? Explain your answer. **(20 p.)**

Solution: The null space $N(A)$ of A consists of all the solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Since the solution set of this system is

$$\{(-2t - 4s, -3t - 5s, t, -6s, s, 0) : t, s \in \mathbb{R}\},$$

an element of the null space $N(A)$ is (for $t = s = -1$)

$$\mathbf{u} = \begin{bmatrix} 6 \\ 8 \\ -1 \\ 6 \\ -1 \\ 0 \end{bmatrix}.$$

But

$$B\mathbf{u} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ -1 \\ 6 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ -1 \end{bmatrix} \neq \mathbf{0},$$

so $\mathbf{u} \notin N(B)$. Thus, null spaces of A and B are not same.■

(5) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be in $\mathbb{R}^3$. Show that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{u}) = \mathbf{v} \times \mathbf{u}$ is a matrix transformation and find its standard matrix. **(20 p.)**

Solution: The cross-product $\mathbf{v} \times \mathbf{u}$ is defined by

$$\mathbf{v} \times \mathbf{u} = \begin{bmatrix} v_2u_3 - v_3u_2 \\ v_3u_1 - v_1u_3 \\ v_1u_2 - v_2u_1 \end{bmatrix}.$$

Since

$$T(\mathbf{u}) = \mathbf{v} \times \mathbf{u} = \begin{bmatrix} v_2u_3 - v_3u_2 \\ v_3u_1 - v_1u_3 \\ v_1u_2 - v_2u_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A\mathbf{u},$$

the transformation T is a matrix transformation with the standard matrix

$$A = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. ■$$