Discrete Mathematics

Lecture 9: Generating Functions

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January 3, 2024

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Motivation

Question

In how many ways can a mother distribute 12 oranges to her children Ayse, Burak and Can such that Ayse gets at least 4, Burak and Can gets at least 2, and Can gets at most 4?

Solution ideas

- Combinations with repetition + Inclusion-Exclusion

$$x_1 + x_2 + x_3 = 12, x_1 \geq 4, x_2 \geq 2, 2 \leq x_3 \leq 4$$

A Simpler, Familiar Question

Question

How many solutions are there for the equation below?

$$x_1 + x_2 + x_3 = 12, \forall i : 0 \leq x_i \leq 6$$

Solution ideas

- Normally we solve it using Inc-Exc principle.
- But we can also attack this question algebraically.

A Simpler, Familiar Question

Question

How many solutions are there for the equation below?
$x_1 + x_2 + x_3 = 12, \forall i : 0 \leq x_i \leq 6$

Solution

$$(x^0+x^1+x^2+x^3+\ldots+x^6)(x^0+x^1+\ldots+x^6)(x^0+x^1+\ldots+x^6)$$

- What is the coefficient of x^{12} in the expansion of this multiplication?
- Note that this is the same question because there is a 1-to-1 relation between
  - Solutions to the equation system above
  - and number of ways getting x^{12} in the multiplication
  - e.g.: $(x_1 = 2, x_2 = 6, x_3 = 4) \iff (x^2 \cdot x^6 \cdot x^4 = x^{12})$

Back to our question

Question

In how many ways can a mother distribute 12 oranges to her children Ayse, Burak and Can such that Ayse gets at least 4, Burak and Can gets at least 2, and Can gets at most 4?

Solution ideas

- Combinations with repetition + Inclusion-Exclusion

$$x_1 + x_2 + x_3 = 12, x_1 \geq 4, x_2 \geq 2, 2 \leq x_3 \leq 4$$

- Generating Functions

$$(x^4+x^5+x^6+x^7+\dots)(x^2+x^3+x^4+x^5+\dots)(x^2+x^3+x^4)$$

Coefficient of x^{12} =? We will solve this later...

Using Generating Functions

Using Generating Functions to solve problems

Here are the steps to solve a question with GFs

- Understand the question and see that it is suitable for generating functions (there is no better way of solving it) (Easy)
- Find out the polynomial for the question (Easy)
- Turn the polynomial into generating functions (Easy)
- Make any possible algebraic simplifications (Easy)
- Use binomial theorem to find the result (Easy)

Polynomial Extraction Example 1

Question

In how many ways can we select 20 balls among red, blue, black balls where we have:

- Even red balls
- At least 14 blue balls
- Less then 5 black balls?

Solution

Our question can be translated into this problem: What is $[x^{20}]$ (the coefficient of x^{20}) in $(1+x^2+x^4+\ldots+x^{20})(x^{14}+x^{15}+\ldots+x^{20})(1+x+\ldots+x^4)$?

Polynomial Extraction Example 2

Question

In how many ways can we get n cents using an indefinite number of pennies, nickels and dimes:

- Penny: 1 cent coin
- Nickel: 5 cents coin
- Dime: 10 cents coin

Solution

Our question can be translated into this problem: What is $[x^n]$ (the coefficient of x^n) in $(1 + x + x^2 + x^3 + \dots)(1 + x^5 + x^{10} + \dots)(1 + x^{10} + \dots)$?

Generating Functions Definition

Definition

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Then, the function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

Example Generating Functions

$$
\begin{array}{rcr}
\frac{1}{1-x} & 1 + x + x^2 + x^3 + \dots & (1, 1, 1, \dots) \\
\frac{1-x^{n+1}}{1-x} & 1 + x + x^2 + x^3 + \dots + x^n & (1, 1, \dots, 1) \\
\frac{1}{1+x} & 1 - x + x^2 - x^3 + \dots & (1, -1, 1, -1, \dots) \\
\frac{1}{1-ax} & 1 + ax + a^2x^2 + a^3x^3 + \dots & (1, a, a^2, \dots) \\
\frac{1}{(1-x)^2} & \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + \dots & (1, 2, 3, \dots)
\end{array}
$$

Multiplication

Multiplication

Multiplication means right shifting for generating functions. For example

$$(1, 1, 1, 1, \dots) \leftrightarrow 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$(0, 1, 1, 1, \dots) \leftrightarrow x + x^2 + x^3 + \dots = \frac{x}{1-x}$$

$$(0, 0, 1, 1, \dots) \leftrightarrow x^2 + x^3 + \dots = \frac{x^2}{1-x}$$

Substitution

Substitution

When we have arithmetic progressions with gaps, we can apply substitution idea. For example:

$$(1, 0, 1, 0, \dots) \leftrightarrow 1 + x^2 + x^4 + \dots = ?$$

$$y = x^2$$

$$(1, 0, 1, 0, \dots) \leftrightarrow 1 + y + y^2 + y^3 + \dots = \frac{1}{1 - y}$$

$$(1, 0, 1, 0, \dots) \leftrightarrow 1 + x^2 + x^4 + \dots = \frac{1}{1 - x^2}$$

Finding Coefficients

Question

In how many ways can a mother distribute 12 oranges to her children Ayse, Burak and Can such that Ayse gets at least 4, Burak and Can gets at least 2, and Can gets at most 4?

Solution ideas

- Combinations with repetition + Inclusion-Exclusion

$$x_1 + x_2 + x_3 = 12, x_1 \geq 4, x_2 \geq 2, 2 \leq x_3 \leq 4$$

- Generating Functions

$$(x^4+x^5+x^6+x^7+\ldots)(x^2+x^3+x^4+x^5+\ldots)(x^2+x^3+x^4)$$

Coefficient of $x^{12} =$? We will solve this **now**

Extended Binomial Theorem

Interestingly, we can extend the Binomial Theorem for 1) negative 2) not necessarily integer numbers.

Extended Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Finding Coefficients

Dice

If we roll a die 15 times, in how many ways can we end up with a total of 40?

- Polynomial:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^{15}$$

Coefficient of $x^{40} = ?$

Using GF to solve recurrence relations

Question

Solve $a_k = 3a_{k-1}$ for $k = 1, 2, \ldots$ with initial cond. $a_0 = 2$

Solution

- Let the GF for $\{a_k\}$ be $G(x) = \sum_{k=0}^{\infty} a_k x^k$.
- Then, $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$.
- Using the recurrence relation,

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2 \end{aligned}$$

Using GF to solve recurrence relations

Solution

- Then, $G(x) - 3xG(x) = (1-3x)G(x) = 2$.
- Solving for $G(x) = 2/(1-3x)$, using the generating function $1/(1-ax)$, we have:

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

- $\{a_k\} = (2, 6, 18, 54, 162, \ldots)$
- But note that you can find the 100th term directly as $2 \cdot 3^{100} x^k$ now.

Using GF to solve recurrence relations

Question

Solve $a_n = -a_{n-1} + 6a_{n-2}$ for $n \geq 2$ with $a_0 = -1, a_1 = 8$.

Solution

- Let the GF for $\{a_n\}$ be $G(x) = \sum_{k=0}^{\infty} a_k x^k$.
- Then,

$$\begin{aligned} G(x) &= a_0 + a_1 x + a_2 x^2 + \dots \\ xG(x) &= \phantom{a_0 +{}} a_0 x + a_1 x^2 + \dots \\ x^2 G(x) &= \phantom{a_0 + a_1 x +{}} a_0 x^2 + \dots \end{aligned}$$

Using GF to solve recurrence relations

Solution

- Then, we multiply last one with -6:

$$\begin{aligned} G(x) &= a_0 + a_1x + a_2x^2 + \dots \\ xG(x) &= \quad a_0x + a_1x^2 + \dots \\ -6x^2G(x) &= \quad\quad -6a_0x^2 + \dots \end{aligned}$$

- and all terms with $x^2, x^3, \dots$ are cancelled according to the recurrence relation. (Since $a_2 + a_1 - 6a_0 = 0$ and so on...)
- So we get:

$$G(x) + xG(x) - 6x^2G(x) = a_0 + a_1x + a_0x$$

Using GF to solve recurrence relations

Solution

- From $G(x) + xG(x) - 6x^2G(x) = a_0 + a_1x + a_0x$,

$$G(x) = (7x - 1)/(1 + x - 6x^2)$$

- So,

$$G(X) = \frac{-1 + 7x}{(1 - 2x)(1 + 3x)} = \frac{A}{1 - 2x} + \frac{B}{1 + 3x}$$

- With a little computation, we find $A = -2, B = 1$
- Then, $G(x) = \frac{-2}{1+3x} + \frac{1}{1-2x}$.

Using GF to solve recurrence relations

Solution

- Then from $G(x) = \frac{-2}{1+3x} + \frac{1}{1-2x}$, $a_n = (-2)(-3)^n + 2^n$.
- Check:

$$
\begin{aligned}
a_0 &= -2+1 &&= -1 \\
a_1 &= 6+2 &&= 8 \\
a_2 &= -18+4 &&= -14 = -8-6 \\
a_3 &= 54+8 &&= 62 \quad = 14+48 \\
&\ldots
\end{aligned}
$$