Defn. Harmonic numbers $H_1 = 1$ $\quad H_2 = 1 + \frac{1}{2}$ $\quad H_3 = 1 + \frac{1}{2} + \frac{1}{3} \ldots$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Thm. $\forall n \in \mathbb{Z}^+$, $\displaystyle\sum_{j=1}^{n} j H_j = \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4}$

Proof. We'll make induction on n.

Idea: Show that thm holds for the smallest element of the universal set that the thm is about.

- **Base step:** For $n=1$ we have $\displaystyle\sum_{j=1}^{1} 1 \cdot H_1 \underset{(\text{Defn})}{=} 1 = \underset{(\text{Thm})}{\frac{1(2)}{2} H_2 - \frac{1 \cdot 2}{4}} = \frac{2}{2} \cdot \frac{3}{2} - \frac{1}{2} = \frac{1}{1}$ ✓

- **Ind. hypothesis:** For $n=k$ we assume $\displaystyle\sum_{j=1}^{k} j H_j = \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4}$

- **Ind. step:** For $n=k+1$ we'll show that $\displaystyle\sum_{j=1}^{k+1} j H_j \overset{?}{=} \frac{(k+1)(k+2)}{2} H_{k+2} - \frac{(k+1)(k+2)}{4}$

(Defn) (Thm)

$H_{k+2} = H_{k+1} + \frac{1}{k+2}$ by defn

$$= \boxed{\sum_{j=1}^{k} j H_j} + (k+1) H_{k+1}$$

$$= \frac{k(k+1)}{2} H_{k+1} - \frac{k(k+1)}{4} + (k+1) H_{k+1} \overset{?}{=} \frac{(k+1)(k+2)}{2} \cdot \left(H_{k+1} + \frac{1}{k+2}\right) - \frac{(k+1)(k+2)}{4}$$

$$= H_{k+1}\left(\frac{k(k+1)}{2} + \frac{2(k+1)}{2}\right) - \frac{k(k+1)}{4} = H_{k+1}\left(\frac{(k+1)(k+2)}{2}\right) + \frac{2(k+1)}{2 \cdot 2} - \frac{(k+1)(k+2)}{4}$$

$$= H_{k+1}\left(\frac{k^2 + k + 2k + 2}{2}\right) - \frac{k(k+1)}{4} = H_{k+1}\left(\frac{k^2+3k+2}{2}\right) + \frac{2k+2 - (k+1)(k+2)}{4}$$

$\frac{k^2+3k+2}{2}$ ✓ $\quad$ ✓ $\quad$ ✓ $\quad \frac{\cancel{2k} + \cancel{2} - k^2 - \cancel{3k} - \cancel{2}}{4} \left(\frac{-k^2 - k}{4}\right)$ ✓

So we are done, our thm is correct. QED

for $n=4$ 2 5 6 7 11 13 14 27 33 46 51 53 54 62 65 72

5

Does 51 exist in this array
⇓
# of comparisons we have to make is at most $n+1$ $= 5$

Proof of binary search thm.

- Base step: For $n=0$ $|A_n| = 2^n = 2^0 = 1$ # of comp we need is obv. 1 $= 0+1 = 1$

$|A_k| = 2^k$

- Ind. hyp.: Assume that for $n=k$ # of comp. we need (at most) is $k+1$
- Ind. step. for $n=k+1$, the size of the array $|A_n| = 2^{k+1} = 2 \cdot 2^k$

Searching x 2^k a_m 2^k 2^{k+1}

If $x = a_m$ # of com 1

$x < a_m$ $k+1 + 1 = k+2$

$x > a_m$ $k+1 + 1 = k+2$

So, the number of comp. needed to find x in an array of size 2^{k+1} is $k+2$ Our thm also tells us it must be $k+1+1 = k+2$ com.

6 · 5

12 · 5

8 · 4

4 ·

2 ·

1 ·

Defn. Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $\forall n \geq 2$

0	1	2	3	4	5	6				
0	1	1	2	3	5	8	13	21	34	55 ...
0	1	1	4	9	25	64	169	441		

Thm. $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$ $\forall n \geq 0$

$n=0$	0
$n=1$	1
$n=2$	2
$n=3$	6
$n=4$	15
$n=5$	40
$n=6$	104
$\vdots$	?

Proof. Base step: For $n=0$

$$\sum_{i=0}^{0} F_i^2 = 0^2 = F_0 F_1 = 0 \cdot 1 = 0 \checkmark$$

Ind. hyp. Assume that for $n = k$,

$$\boxed{\sum_{i=0}^{k} F_i^2 = F_k F_{k+1}}$$

Ind. step. We'll show that for $n = k+1$,

$$\sum_{i=0}^{k+1} F_i^2 = \boxed{\sum_{i=0}^{k} F_i^2} + F_{k+1}^2 \stackrel{?}{=} F_{k+1} F_{k+2}$$

$$= F_k F_{k+1} + F_{k+1}^2 \stackrel{?}{=} F_{k+1} F_{k+2}$$

$$= F_{k+1}(F_k + F_{k+1})$$

$$F_{k+1} \cdot F_{k+2} = \checkmark$$

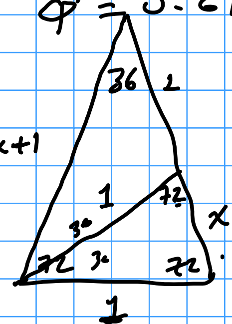
We're done QED.

$\infty \quad \frac{1}{0} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{8}{5} \quad \frac{13}{8} \quad \frac{21}{13} \ldots$

Hypoth $\infty \quad 1 \quad 2 \quad 1.5 \ldots$

$\varphi = 1.618033\ldots = \frac{\sqrt{5}+1}{2}$

$\varphi' = 0.618033 = \frac{\sqrt{5}-1}{2}$



$\frac{1}{x} = \frac{x+1}{1}$

$x(x+1) = 1$

$x^2 + x - 1 = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ $\quad \frac{\sqrt{5}-1}{2}$ $\quad \sqrt{5}$

Defn. Lucas numbers: $L_0 = 2 \quad L_1 = 1 \quad L_n = L_{n-1} + L_{n-2}$

0	1	2	3	4	5	6	7	8	
2	1	3	4	7	11	18	29	47	...

Thm. $\forall n \in \mathbb{Z}^+, \quad L_n = F_{n-1} + F_{n+1}$

ind.	0	1	2	3	4	5	6	7
F	0	1	1	2	3	5	8	13
L	2	1	3	4	7	11	18	29

illustr.

Proof. We make induction on n:

- Base step: For $n=1$ $\quad L_1 = F_0 + F_2$

  $1 = 0 + 1$ ✓

- Ind hyp. Assume that for $m \le k$ for some $k \le n$

  $\boxed{L_m = F_{m-1} + F_{m+1}}$

- Ind. step. for $n = k+1$, we need to show

$$\to L_{k+1} = F_k + F_{k+2}$$

$$L_{k-1} + L_k \stackrel{?}{=} F_{k-1} + F_{k-2} + F_k + F_{k+1}$$

$m=k$

$m=k-1$

So we are done QED.

$a \lfloor b$
$\underline{-\ }$ n
0

$x = y + z \quad \mod b$
$\downarrow \quad \downarrow \quad \downarrow$
$b \quad b \quad b$
$0 \quad 0 \quad 0$

$\gcd(24, 32)$
$= 8$

$3 \mid 9 \qquad 3 \mid 6 \qquad 9x + 6y \mod 3 =$

$25 = 8 \cdot 4 - 7$

$25 = 8 \cdot 3 + 1$
a b quotient remainder

$25 \lfloor 8$
$-24 \quad \cancel{2}\ 3$
1

$25 = 8 \cdot 2 + 9$

$r_0 \qquad r_1 \qquad r_2$

$960 = 1 \cdot 500 + 460$

$500 = 1 \cdot 460 + 40$

$460 = 11 \cdot 40 + 20$

$40 = 2 \cdot \underline{\underline{20}} = \gcd(960, 500)$

$960 = 1 \cdot 600 + 360$

$600 = 1 \cdot 360 + 240$

$360 = 1 \cdot 240 + 120$

$240 = 2 \cdot 120$ //

$\underbrace{16 \cdot 6 \cdot 10}$, $6 \cdot 10 \cdot 10$

$\to 2^6 \cdot 3 \cdot 5 \quad \to 2^3 \cdot 3 \cdot 5^2$

$2^3 \cdot 3 \cdot 5 = 120$

56 — - - - - -

46

230