Friday 29/11/2019 Midterm 2 Duration: 90 minutes

Name:

Student No:

Rules: 1. Write your solutions clearly. 2. All solutions must be written in the given spaces. You can use an extra sheet for drafting but then write your final clear solutions back to the question sheet. Extra papers will be collected but not graded.

P1 [30 points] Pigeonhole Principle (In this question, define your pigeons and pigeonholes clearly.)
(a) Prove that if we select 1001 integers from the set $S = \{1, 2, 3, \ldots, 2000\}$, there exist m, n in the selection where $gcd(m, n) = 1$.

Pigeonholes:

Pigeons:

(b) In a 6/49 lottery ticket, you guess 6 numbers out of 1,2,3,...,49 and you win top prize with 6 correct guesses. If you guess 5 correct numbers, you get some secondary prize. In order to guarantee this secondary prize by guessing at least 5 correct numbers, at least how many tickets should one play?

Pigeonholes:

Pigeons:

(c) Let S be a set of five positive integers the maximum of which is at most 9. (For example S can be $\{1, 3, 4, 8, 9\}$ or $\{3, 4, 5, 7, 8\}$) Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct. (For example if $S = \{1, 3, 4, 8, 9\}$ subsets {1,3} and {4} has the same sum or if $S = \{3, 4, 5, 7, 8\}$ subsets {3,8} and {4,7} has the same sum, etc.)

Pigeonholes:

Pigeons:

P2 [40 points] Inclusion-Exclusion Principle

(a) In a software company, there are 50 tasks and 6 employees to do these tasks. If employees can be assigned at most 15 tasks, how many possible assignments are there?

Define the conditions:

Write the inclusion-exclusion formula and find the solution.

(b) In a group of 8 people, we know that everyone has at least one friend from this group. How many friendship networks are there among this group?

Define the conditions:

Write the inclusion-exclusion formula and find the solution:

(c) In the garage of a building there are 20 parking spaces for 20 cars. On Monday night, all cars are parked to this garage in some random order. On Tuesday night, one notices that they are again parked and no car is parked at the same place as Monday. Then, on Wednesday, once again, all the cars are parked, and no car is parked at the same place as Tuesday. In how many ways can this happen? (Note that a car can park to the same place on Monday and Wednesday though. Otherwise the question would be difficult.) Just write the approximate answer into the box below and get 10 points.

P3 [30 points] Generating Functions Solve each of the questions below using generating functions. For each question, write down the polynomial that represents the problem [5 points], find the corresponding generating function [5 points], and then find the coefficient [5 points].

(a) How many integer solutions are there to the equation $x_1+x_2+x_3+x_4+x_5 = 24$ with the restriction that all of $x_i > 1$ and two of them are odd, the remaining three are even integers?

Polynomial P(X) = Coefficient we need:

Generating function: Coefficient we need:

Calculation:

(b) After a football team wins the champions league, the president of the team remembers his promise about distributing a prize of \$100M (Here, M means millions) to the players according to these rules:

- Everyone gets a multiple of \$1M.
- Goalkeeper and all 4 defenders get at most \$40M each,
- All 4 midfielders get a multiple of \$5M each,
- and both 2 forwards get at least \$20M but at most \$55M each.

Polynomial P(X) = Coefficient we need:

Generating function: Coefficient we need:

Calculation:

P4 [Bonus 20 points] Mixed topic - To solve this question, you should make use of the topics above together. Mike has a lot of pennies (1 cent coins), nickels (5 cents coins) and dimes (10 cents coins). He wants to give 30 cents to each member of a group of people. But he wants to give a different combination of coins to people. For example he will give 3 dimes to one, 2 dimes, 2 nickels to another, or 2 nickels and 20 pennies to another. Then he realizes that he cannot do this. i.e. He has to pay at least two people with the same combination of coins. At least how many people are there in the group?

Table 1: Some generating functions that can be useful. For all $m, n \in \mathbb{Z}^+$, $a \in \mathbb{R}$

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$

3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n$

5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$

6) $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$

7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$

8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots = 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$