**FACULTY OF ENGINEERING**
**DEPARTMENT OF COMPUTER ENGINEERING**
**MAT 222 LINEAR ALGEBRA AND NUMERICAL METHODS**
**MIDTERM EXAM**

4th May 2023

**IMPORTANT:** In problems $1, 3, 5, 6$ consider your student ID as $a$-$b$-$c$-$d$-$e$-$f$-$g$-$h$-$i$-$j$-$k$.

**1.** Consider the following linear system in the unknowns $x, y, z$. Find a relation between $u$ and $v$ such that the system has infinitely many solutions.

$$\begin{aligned} x - 4y - z &= k + 1 \\ -x + 3y + 3z &= u \\ x - 6y + 3z &= v \end{aligned}$$

**Solution:** Suppose that $k = 6$. For the system to have infinitely many solutions, one of the equations must be a linear combination of the other two. It can easily be verified that $x - 6y + 3z = 3(x - 4y - z) + 2(-x + 3y + 3z)$. Therefore, if the same relation is also satisfied by the right-hand sides of the equations, the system will have infinitely many solutions. Thus, the required relation is $3(k + 1) + 2u = v$. Replacing $k$ by 6 we obtain $v - 2u = 7$.

**2.** Muhittin, who is preparing for a sporting competition, is having a breakfast containing four food types every day. The macro contents of these food types are given below.

| | Carbohydrate (g) | Protein (g) | Fat (g) |
|---|---|---|---|
| Mix A | 3 | 1 | 2 |
| Mix B | 1 | 4 | 9 |
| Mix C | 14 | 6 | 1 |
| Mix D | 39 | 2 | 1 |

Muhittin wants to take exactly 50 grams of carbohydrate, 15 grams of protein and 12.5 grams of fat in the breakfast. How many units of each food type must he take to achieve his dietary goal? Your answer should consist of **nonnegative** numbers which may or may not be integers. (There may be infinitely many solutions.)

**Solution:** Suppose Muhittin will take $x_1$ units of Mix A, $x_2$ units of Mix B, $x_3$ units of Mix C and $x_4$ units of Mix D to reach his dietary goal. The resulting equations are as follows:

$$\begin{aligned} 3x_1 + x_2 + 14x_3 + 39x_4 &= 50 \\ x_1 + 4x_2 + 6x_3 + 2x_4 &= 15 \\ 2x_1 + 9x_2 + x_3 + x_4 &= 12.5 \end{aligned}$$

After Gaussian elimination, an echelon form of the above system is $\left[\begin{array}{cccc|c} 1 & 4 & 6 & 2 & 15 \\ 0 & 1 & -11 & -3 & -17.5 \\ 0 & 0 & -125 & 0 & -187.5 \end{array}\right]$, giving rise to the equations

$$\begin{aligned} x_1 + 4x_2 + 6x_3 + 2x_4 &= 15 \\ x_2 - 11x_3 - 3x_4 &= -17.5 \\ -125x_3 &= -187.5 \end{aligned}$$

From these equations it follows that $x_3 = 1.5, x_2 = 3x_4 - 1, x_1 = 10 - 14x_4$ and $x_4$ is free variable. Setting $x_4 = 0.5$, for example, gives a solution with every variable positive. In this case we have $x_1 = 3, x_2 = 0.5, x_3 = 1.5, x_4 = 0.5$. There are other solutions, as well.

**3.** Consider the following linear system. Find the value of $x$ in terms of $u$ and $v$. (Hint: In this problem, Cramer's rule may be easier than Gaussian elimination.) **(20 points)**

$$
\begin{aligned}
&(1+u)x + y + z + t = k + 2 \\
&x + (1-u)y + z + t = k + 2 \\
&x + y + (1+v)z + t = 0 \\
&x + y + z + (1-v)t = 0
\end{aligned}
$$

**Solution:** Let us apply Cramer's Rule. Let $A$ be the coefficient matrix and $A_x$ be the matrix formed after the replacement of the first column of $A$ by the right-hand side of the system. Let us first compute $|A|$.

$$
\begin{bmatrix} 1+u & 1 & 1 & 1 \\ 1 & 1-u & 1 & 1 \\ 1 & 1 & 1+v & 1 \\ 1 & 1 & 1 & 1-v \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & u^2 & -u & -u \\ 1 & 1-u & 1 & 1 \\ 0 & u & v & 0 \\ 0 & u & 0 & -v \end{bmatrix}
$$

Here we used the second row to eliminate the other entries in the first column. The resulting determinant is equal to $|A|$. Using cofactor expansion along the first column, we have

$$
\begin{aligned}
|A| &= 1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} u^2 & -u & -u \\ u & v & 0 \\ u & 0 & -v \end{vmatrix} = -1 \cdot \left( u \cdot (-1)^{3+1} \begin{vmatrix} -u & -u \\ v & 0 \end{vmatrix} - v \cdot (-1)^{3+3} \begin{vmatrix} u^2 & -u \\ u & v \end{vmatrix} \right) \\
&= -(u \cdot uv - v(u^2 v + u^2)) = u^2 v^2
\end{aligned}
$$

By similar simplification one can compute the determinant of $A_x$ by $|A_x| = (k+2)uv^2$. Assuming $k = 6$, $x$ is calculated by $x = \dfrac{|A_x|}{|A|} = \dfrac{8uv^2}{u^2v^2} = \dfrac{8}{u}$.

**4.** Prove the following.

Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}$ be vectors in $V$. If $\mathbf{u}$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, then $\dim(\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}) < \dim(\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}\})$. **(20 points)**

**Solution:** Let $A = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and $B = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}\}$. Since $A$ and $B$ are both vector spaces and $A \subset B$, $A$ is a subspace of $B$. It only remains to show that they are different spaces. But this is obvious since $\mathbf{u} \in B$ and $\mathbf{u} \notin A$. Thus, we have the following:

$$A \text{ is a subspace of } B \text{ and } A \neq B.$$

This shows that $\dim(A) < \dim(B)$.

**5.** Consider the linear equations $ax + cy + fz = 1$ and $(h-3)x + (j+2)y + (k+1)z = 2$. Write down a third equation in the unknowns $x, y, z$ such that the resulting $3 \times 3$ system has a unique solution. Note that your solution should not consist of trial and error. **(20 points)**

**Solution:** Assume $a = 2, c = 1, f = 4, h = 3, j = 4, k = 6$. Let the missing equation be $c_1x + c_2y + c_3z = p$. Then, the coefficient matrix of the resulting system becomes

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 6 & 7 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The system has a unique solution if and only if the determinant of $A$ is zero. This also means that the rows of $A$ are inearly independent, hence the set $\{(2,1,4),(0,6,7),(c_1,c_2,c_3)\}$ is a basis for $\mathbb{R}^3$. Thus, if we extend the set $\{(2,1,4),(0,6,7)\}$ to a basis for $\mathbb{R}^3$, then we can take the third vector as $(c_1,c_2,c_3)$. One method is to consider the nullspace of the matrix

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 6 & 7 \end{bmatrix}.$$

If $(x_1,x_2,x_3)^T$ is in the nullspace, then we have the equations

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 0 \\ 6x_2 + 7x_3 &= 0 \end{aligned},$$

which gives solutions of the form $t \begin{bmatrix} -17 \\ -14 \\ 12 \end{bmatrix}$. So we can take $(c_1,c_2,c_3) = (-17,-14,12)$. The right-hand side $p$ can be anything, let us take $p = 1$. As a result, we can take the third equation to be

$$-17x - 14y + 12z = 1.$$

**6.** Consider the matrix $\mathbf{D} = \begin{bmatrix} 2 & 0 & 2a \\ 1 & -1 & a-k-1 \\ 1 & 3 & a+3k+3 \end{bmatrix}$. Find a basis for the column space of $\mathbf{D}$. Then consider a vector $\mathbf{v}$ of length 3 whose first two components are $2(j+1)$ and $d+1$, i.e. $\mathbf{v}$ is of the form $\begin{bmatrix} 2(j+1) \\ d+1 \\ * \end{bmatrix}$. Find the missing entry of $\mathbf{v}$ such that it becomes an element of the column space of $D$. **(20 points)**.

**Solution:** Assume $a = 2, d = 1, j = 4, k = 6$. Notice that 3rd column is a linear combination of the other two since we have $\begin{bmatrix} 2a \\ a-k-1 \\ a+3k+3 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (k+1) \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$. So a basis for $C(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$.

Then, if the vector $\begin{bmatrix} 2(j+1) \\ d+1 \\ * \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ * \end{bmatrix}$ is in $C(A)$, there are constants $c_1$ and $c_2$ such that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ * \end{bmatrix} \longrightarrow \begin{aligned} 2c_1 &= 10 \\ c_1 - c_2 &= 2 \\ c_1 + 3c_2 &= * \end{aligned}$$

From this we can conclude that $c_1 = 5, c_2 = 3$ and $* = 14$.