Discrete Mathematics

Lecture 8: Principle of Inclusion and Exclusion

Murat Ak

Akdeniz University

December 20, 2023

Table of contents

The Principle

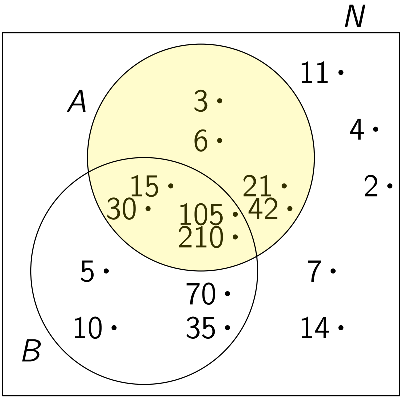
Question

How many numbers between 1 and 100 are not divisible by either 3 or 5?

Solution

- We consider all 100 numbers from 1 to 100. +100
- We eliminate multiples of 3. $\{3, 6, 9, 12, \ldots, 99\}$, which makes 33 numbers in total. -33
- We eliminate multiples of 5. $\{5, 10, 15, 25, \ldots, 100\}$, which makes 20 numbers in total. -20
- But we eliminated multiples of 15 twice! $\{15, 30, \ldots, 90\}$ So we have to take them back again once. +6

The Principle



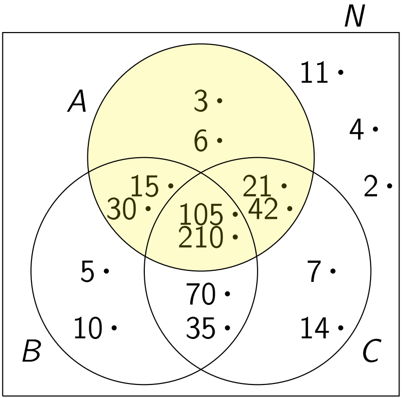
Here is a Venn diagram of the situation and the highlighted set A is the set of numbers divisible by 3.

The Principle

Question

How many numbers between 1 and 1000 are not divisible by either 3,5 or 7?

The Principle



Here is a Venn diagram of the situation and the highlighted set A is the set of numbers divisible by 3.

The Principle

Question

How about the numbers between 1 and 100 are not divisible by either 3,5,7 or 11?

General formula, conditions, etc.

The Principle

Idea

When we want to count the number of situations where none of a set of conditions hold, we can use the Principle of Inclusion and Exclusion.

So, if you encounter a problem which you can boil down to a count of situations where no conditions are satisfied, you can use PIE.

The Principle

Theorem (8.1 The principle of inclusion and exclusion)

Consider a set S, with $|S| = n$, and conditions $c_i, 1 \leq i \leq t$, satisfied by some of the elements of S. The number of elements of S that satisfy none of the conditions $c_i, 1 \leq i \leq t$ is denoted by $\bar{N} = N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \ldots \bar{c}_t)$ where

$$\bar{N} = N - \sum_{1 \leq i \leq t} N(c_i) + \sum_{1 \leq i < j \leq t} N(c_i c_j) - \ldots + (-1)^t N(c_1 c_2 c_3 \ldots c_t)$$

The Principle: None of the conditions

Proof idea of Theorem 8.1.

$|A \cup B| = |A| + |B| - |A \cap B|$.

$|A \cup B \cup C| =$

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Book's proof focuses on how many times an element is counted.

- If an element satisfies no conditions: It is counted once.
- If it satisfies more than one condition: It is counted zero times in total.

$\square$

Important terminology

Remember c_i denotes the i^{th} condition and $N(c_i)$ denotes the number of elements that satisfy c_i. To keep our calculations simpler, we write:

- $S_0 = N$
- $S_1 = \sum_{1 \leq i \leq t} N(c_i)$
- $S_2 = \sum_{1 \leq i < j \leq t} N(c_i c_j)$
- and in general:
- $S_k = \sum N(c_{i_1} c_{i_2} \ldots c_{i_k}), 1 \leq k \leq t$.

The Principle: At least one condition

Corollary (8.1)

The number of elements of S that satisfy at least one of the conditions $c_i, 1 \leq i \leq t$ is denoted by $N(c_1 \vee c_2 \vee c_3 \ldots c_t) = N - \bar{N}$.

Examples

8.5 (Number of solutions to equations with upper bounds on variables)

Let $x_1 + x_2 + x_3 + x_4 = 18$ where $x_i \geq 0$ for all $1 \leq i \leq 4$.
How many solutions are there?
What if we have another condition $x_i \leq 7$ for all $1 \leq i \leq 4$?

- Think about conditions so that
- The number we are looking for is the number of solutions where no conditions are satisfied

Examples

8.9 (An example where calculating $N(c_i)$s is *slightly* harder.)

Six married couples are to be seated around a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?

Examples

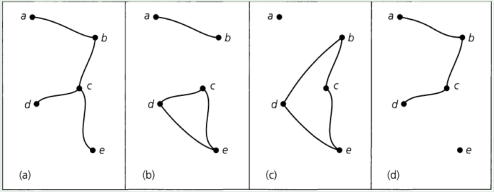
8.8 (An example where conditions might not be perfectly symmetrical)

Euler's phi function. For $n \in \mathbb{Z}^+, n \geq 2$, let $\phi(n)$ be the number of positive integers m, where $1 \leq m \leq n$ and $gcd(m, n) = 1$. In other words $\phi(n) =$ [number of positive integers smaller than n, and also relatively prime to n].
Example: $\phi(9) = 6$ because of $\{1, 2, 4, 5, 7, 8\}$

Examples

Example 8.10 Connecting villages a visual example

In a countryside there are five villages. An engineer wants to devise a system of two-way roads between these villages so that no village remains isolated. (In the figure, a and b are allowed whereas c and d are not.



Exactly m of the conditions

Theorem (8.2 Exactly m of the conditions)

The number of elements of S that satisfy exactly m of the conditions $c_i, 1 \leq i \leq t$ is denoted by
$E_m = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \ldots + (-1)^{t-m} \binom{t}{m} S_t$.

At least m of the conditions

Corollary (8.2 At least m of the conditions)

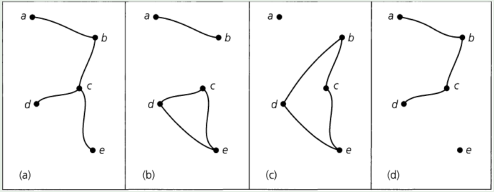
The number of elements of S that satisfy at least m of the conditions $c_i, 1 \leq i \leq t$ is denoted by
$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \ldots + (-1)^{t-m} \binom{t-1}{m-1} S_t.$

Examples

Example 8.10 Connecting villages a visual example

Remember this example.

- How many road system isolate exactly 2 villages?
- How many isolate at least 2?
- How many isolate at most 2?



Derangements: Worst Bet Ever

Question

You guess the final table of Turkish Soccer Super Leage after week 34. If you guess only 1 team's place correctly, you will win. What is your chance? (Assume you know nothing about the teams.)

Question

Suppose I go to theatre together with 3000 people in a city with a population of 3 million. If I know 1000 people in the city, what is the probability of me knowing noone in the theatre

Question

Suppose in a town of 1000 people, only 1 people is not sick and in a particular day, every sick person makes contact with one person and if that is the healthy one, that person also becomes sick. What is the probability of the healthy person to remain healthy after one day?

Derangements: Nothing Is In Its Right Place

Definition (Background reminder : McLaurin Series)

From elementary calculus we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so if we substitute x by -1, we get:

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!},$$

Question: Horse racing

Suppose that while at the racetrack, Ralph bets on each of the ten horses in a race to come in according to how they are favored. In how many ways, can they reach the finish line so that he loses all of his bets?

Question: Party Hats

Suppose that when 10 people come to a party they leave their hats at the entrance. During the party, power outage happens, and when they are leaving in the dark, they just grab a random hat. What is the probability that nobody gets their own hat?

Question: Gift giving 1

Suppose we have 10 students in a class. In the last week of the year, they want to have a gift-giving party where every student will give a gift to another student. To decide who buys a gift for whom, they write their names and numbers on papers, fold them, and place them in a box. Starting with the 1st student, each student draws a paper. If a student draws their own name, they draw another paper and then return their own to the box.

Question 1: Is there a problem with this procedure?

Question: Gift giving 2

What is the probability of the problem actually happening?

Question: Gift giving 3

Propose a simple solution to this problem. (You can compromise a little randomization)