Discrete Mathematics

Lecture 5: Relations and Functions

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Table of contents

Relations and Functions: Motivation

Relations and Functions

- Functions from a set theoretic approach
- Finite functions
- New counting ideas
- Idea of function complexity

Cartesian Products and Relations

Definition (Cartesian Product)

For sets $A, B \subseteq \mathcal{U}$, the *Cartesian product*, or *cross product*, of A and B is denoted by $A \times B$ and equals $\{(a, b) | a \in A, b \in B\}$.

Definition (Relation)

For sets $A, B \subseteq \mathcal{U}$, any subset of $A \times B$ is called a (binary) *relation from A to B*. Any subset of $A \times A$ is called a (binary) *relation on A*.

Example Relations

Example Relations

(a) Simply $\mathcal{R} = \{(2,4),(5,2),(3,4),(3,5),(4,4)\}$

(b) On set $\mathbb{Z}^+$, define relation $\mathcal{R} = \{(x,y) | x \leq y\}$

(c) From $\mathbb{N}$ to $\mathbb{N}$, define relation $\mathcal{R} = \{(m,n) | n = 7m\}$

(d) Recursively define the same relation $\mathcal{R}$ on $\mathbb{N}$:

1. $(0,0) \in \mathcal{R}$; and
2. If $(s,t) \in \mathcal{R}$, then $(s+1, t+7) \in \mathcal{R}$.

Number of Relations

Question

For finite sets A and B where $|A| = m$ and $|B| = n$, how many relations are there from A to B?

Number of Relations

Number of Relations

For finite sets A and B where $|A| = m$ and $|B| = n$, there are

$$2^{mn}$$

relations from A to B.

Properties of Cartesian Products

Theorem (5.1)

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Function

Definition (Function)

For nonempty sets A, B, a function, or mapping, f from A to B, denoted $f : A \to B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

We often write $f(a) = b$ when (a, b) is an ordered pair in the function f. For $(a, b) \in f$, b is called the image of a under f, whereas a is called the preimage of b under f.

Domain and Range

Definition (Domain and Range)

For the function $f : A \to B$,

- A is the domain of f
- B is the codomain of f
- The subset of B that contain the images of elements of A is called the range, and denoted as $f(A)$.

Properties of Images

Theorem (Properties of Images)

Let $f : A \to B$ with $A_1, A_2 \subseteq A$. Then,

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$
3. $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ *when f is one-to-one (injective).*

Why do we need injectiveness for equality in the intersection case?

Restriction and Extension

Definition (Restriction)

If $f : A \to B$ and $A_1 \subseteq A$, then $f|_{A_1} : A_1 \to B$ is called the restriction of f to A_1 if $f|_{A_1}(a) = f(a)$ for all $a \in A_1$

Definition (Extension)

If $f : A_1 \to B$ and $A_1 \subseteq A$, then a function $g : A \to B$ where $g(a) = f(a)$ for all $a \in A_1$ is called the extension of f to A

Number of Functions

Question

For finite sets A and B where $|A| = m$ and $|B| = n$, how many functions are there from A to B?

Number of Functions

Number of Functions

For finite sets A and B where $|A| = m$ and $|B| = n$, there are

$$|B|^{|A|} = n^m$$

functions from A to B.

Question

Inverting a Function

If we have a function from A to B, can we obtain a function if we interchange the components in the ordered pairs?

One-to-one Function (Injection)

Definition (One-to-one function or injection)

A function $f : A \to B$ is called one-to-one or injective, if each element of B appears at most once as the image of an element in A.

If elements of B has no more than one preimage, the function is an injection.

Number of One-to-One Functions

Number of One-to-One Functions

For finite sets A and B where $|A| = m$ and $|B| = n$, there are

$$n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!} = P(n, m)$$

one-to-one functions from A to B.

Proof.

There are n choices for the first element in B, $(n-1)$ choices for the second, ..., $(n-m+1)$ for the last element in B. $\square$

Surjection

Definition (Surjection)

A function $f : A \to B$ is called onto, or surjective, if $f(A) = B$.

If all elements in B has at least one preimage, the function is a surjection (onto function).

Number of Surjections

Number of Surjections

For sets A and B where $|A| = m$ and $|B| = n$, there are

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

onto functions from A to B.

Number of Surjections

Example

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{w, x, y, z\}$. Applying the formula we get:

$$\binom{4}{4}4^7 - \binom{4}{3}3^7 + \binom{4}{2}2^7 - \binom{4}{1}1^7$$

$$= \sum_{k=0}^{4} (-1)^k \binom{4}{4-k} (4-k)^7 = 8400$$

onto functions from A to B.

Bijection

Definition (Bijection)

A function $f : A \to B$ is said to be bijective or to be a one-to-one correspondence, if f is both one-to-one and onto.

If there is a one-to-one matching between A and B, that leaves no elements unmatched on either side, we call it a bijection.

Identity Function

Definition (Identity Function)

The function $1_A : A \to A$ defined by $1_A(a) = a$ for all $a \in A$ is called the identity function for A.

Definition (Equal Functions)

If $f, g : A \to B$ are two functions where $f(a) = g(a)$ for all $a \in A$ we say that f and g are equal and write $f = g$.

Caveat

Let $f : \mathbb{Z} \to \mathbb{Z}$, and $g : \mathbb{Z} \to \mathbb{Q}$ where $f(x) = x = g(x)$ for all $x \in \mathbb{Z}$. Although they act the same on every integer, $f \neq g$ because their codomains are different.

Composite Function

Definition (Composite Functions)

If $f : A \to B$ and $g : B \to C$ we define the composite function, denoted $g \circ f : A \to C$, by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

Caveat

According to the definition of composite functions, codomain of f must be equal to the domain of g.

Properties of Composite Functions

Theorem

Let $f : A \to B$ and $g : B \to C$.

1. *If f and g are one-to-one, then so is $g \circ f$*
2. *If f and g are onto, then so is $g \circ f$*

Corollary

Let $f : A \to B$ and $g : B \to C$.

1. *If f and g are bijective, then so is $g \circ f$*

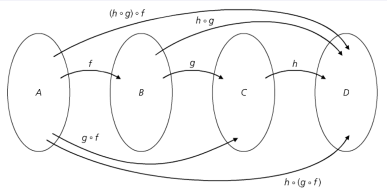
Function Composition is Associative

Theorem

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then,
$(h \circ g) \circ f = h \circ (g \circ f)$

Proof.

Since the two functions have the same domain, A, and codomain D, the result will follow by showing that for every $x \in A$, $((h \circ g) \circ f)(x) = (h \circ (g \circ f)(x))$



Recursive Composition of Functions

Definition (Recursive compositions)

If $f : A \to A$ we define $f^1 = f$, and for $n \in \mathbb{Z}^+$, $f^{n+1} = f \circ (f^n)$.

Example

- With $A = \{1, 2, 3, 4\}$ and $f : A \to A$ defined by $f = \{(1, 2), (2, 2), (3, 1), (4, 3)\}$, we have $f^2 = f \circ f = \{(1, 2), (2, 2), (3, 2), (4, 1)\}$ and $f^3 = f \circ f^2 = f \circ f \circ f = \{(1, 2), (2, 2), (3, 2), (4, 2)\}$.
- What are f^4 and f^5?

Invertible Functions

Definition (Invertible Functions)

If $f : A \to B$ then f is said to be invertible if there is a function $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Theorem (Invertible $\Leftrightarrow$ Bijective)

A function $f : A \to B$ is invertible if and only if it is both one-to-one and onto (or in other words bijective).

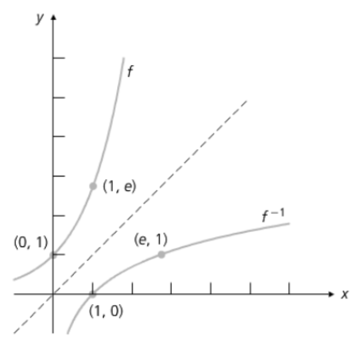
Theorem (Invertible Composition)

If $f : A \to B$, $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Invertible Functions

Example

Let $f : \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = e^x$ where $e = 2.718\ldots$ is Euler's constant. From the graph, we can see that f is one-to-one and onto. So $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$ exists and actually, it is



$$f^{-1} = \{(x, y) | y = e^x\}^c = \{(x, y) | x = e^y\} = \{(x, y) | y = \ln x\}$$

so, $f^{-1}(x) = \ln x$

Special Functions

Definition (Closedness)

For any non-empty sets, A and B, any function $f : A \times A \to B$ is called a binary operation on A. If $B \subseteq A$, then the binary operation is said to be closed (on A). When $B \subseteq A$ we may also say that A is closed under f.

Definition (Unary operation)

A function $g : A \to A$ is called a unary, or monary operation on A.

Closedness

Example

Addition of two integers is closed on integers. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ define $f(m, n) = m + n$.

Example

Is division of rational numbers closed on rational numbers? $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ define $g(m, n) = m/n$.

Commutative and Associative

Definition (Commutative and Associative)

- Let $f : A \times A \to B$ that is f is a binary operation on A. f is said to be commutative if $f(a, b) = f(b, a)$ for all $(a, b) \in A \times A$.
- When $B \subseteq A$ (that is f is closed), f is said to be associative if $f(f(a, b), c) = f(a, f(b, c))$ for all $(a, b, c) \in A \times A \times A$.

Special Relations

Definition (Reflexive relation)

A relation $\mathcal{R}$ on set A is called reflexive if for all $x \in A$, $(x, x) \in \mathcal{R}$.

Example

For instance "equals" ($\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x = y)\}$) is a reflexive relation on real numbers. Because $x = x$ for all $x \in \mathbb{R}$.

Special Relations

Definition (Symmetric relation)

A relation $\mathcal{R}$ on set A is called symmetric if $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$, for all $x, y \in A$.

Example

For instance "equals" ($\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x = y)\}$) is a symmetric relation on real numbers. Because $x = y$ implies $y = x$.

Special Relations

Definition (Transitive relation)

A relation $\mathcal{R}$ on set A is called transitive if for all $x, y, z \in A$, $(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$.

Example

For instance "divides" ($\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x|y)\}$) is a transitive relation on integers. Because $x|y$ and $y|z$ implies $x|z$.

Special Relations

Definition (Antisymmetric relation)

A relation $\mathcal{R}$ on set A is called antisymmetric if $(x, y), (y, x) \in \mathcal{R} \Rightarrow x = y$, for all $x, y \in A$.

Example

For instance "greater than or equal to" ($\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x \geq y)\}$) is an antisymmetric relation. Because $x \geq y$ and $y \geq x$ implies $x = y$.

Special Relations

Definition (Partial order)

A relation $\mathcal{R}$ on set A is called a partial order if $\mathcal{R}$ is **reflexive**, **antisymmetric** and **transitive**.

Example

For instance "greater than or equal to"
($\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x \geq y)\}$) is a partial order.

Special Relations

Definition (Equivalence relation)

A relation $\mathcal{R}$ on set A is called an equivalence relation if $\mathcal{R}$ is **reflexive**, **symmetric** and **transitive**.

Example

For instance "equal to" ($\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x = y)\}$) is an equivalence relation.

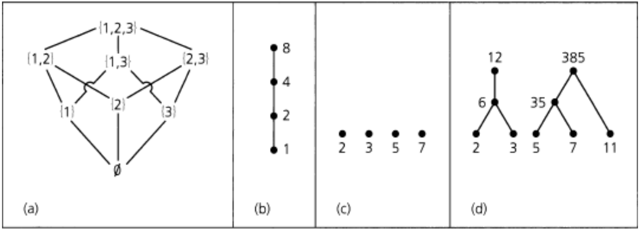
Hasse Diagrams to Represent Partial Orders

Definition (Partial order)

Hasse Diagram is a diagram to represent a partial order $\mathcal{R}$ where we draw lines from x up to y if $(x, y) \in \mathcal{R}$

Example

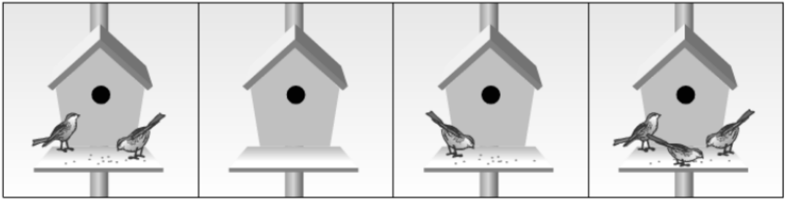
For instance "subset of" and "divides" are both partial orders and we can represent them with the Hasse diagrams below.



The Pigeonhole Principle

Theorem (The Pigeonhole Principle)

If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons in it.



This is also commonly called **Dirichlet's box principle** or **Dirichlet's drawer principle** after Peter Gustav Lejeune Dirichlet, who is believed to have formalized this result first.

The Pigeonhole Principle: Generalization

Theorem (The Pigeonhole Principle – General form)

If m pigeons occupy n pigeonholes, then at least one pigeonhole has $\lfloor (m-1)/n \rfloor + 1$ or more pigeons in it, where $\lfloor * \rfloor$ is the floor function.

The Pigeonhole Principle

Example

There are 81 cities in Turkey. Prove that in a group of 82 Turkish people, there are at least two people who are from the same city.

Hint: Here, we can see the cities as pigeonholes, and people as pigeons.

The Pigeonhole Principle

Example

In a soccer game, how many yellow cards can a soccer referee show without showing any red cards? Note that if a player gets two yellow cards, the player automatically gets a red card and gets dismissed. Also note that in a game, there are 22 players and both teams are allowed to make at most 3 substitutions.

What are our pigeonholes and pigeons now?
How many yellow cards guarantees that there will be a red card?

The Pigeonhole Principle

Example

Consider an apartment of 24 floors. In an apartment meeting, there are 13 people. Show that at there is at least one couple residing in consecutive floors.

What are our pigeonholes and pigeons now?

The Pigeonhole Principle

Example

In an auditorium of capacity 800, how many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first and last initials? (e.g. John Doe has initials J.D.)

The Pigeonhole Principle

Example

Let $S = \{3, 7, 11, 15, \ldots, 95, 99, 103\}$. How many elements must we select from S to ensure that there will be at least two whose sum is 110?

What are our pigeonholes and pigeons?

The Pigeonhole Principle

Example

Any subsets of size 6 from the set $S = \{1, 2, 3, \ldots, 9\}$ must contain two elements whose sum is 10.

What are our pigeonholes and pigeons?

The Pigeonhole Principle

Example

Let triangle ABC be equilateral, having $|AB| = 1$. Show that if we select 10 points in the interior of this triangle, there must be at least two, whose distance apart is less than $1/3$.

The Pigeonhole Principle

Example

If 11 integers are selected from $1, 2, \ldots, 100$, prove that there are at least two, say x and y, such that $0 < \sqrt{x} - \sqrt{y} < 1$.

The Pigeonhole Principle

Example

If 11 integers are selected from $1, 2, \ldots, 100$, prove that there are at least two, say x and y, such that $0 < \sqrt{x} - \sqrt{y} < 1$.

Proof.

Let's list all square roots: $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{99}, \sqrt{100}$. (Approximately, $1, 1.4, 1.7, 2, \ldots, 10$) Selecting 11 elements from $\{1, 2, 3, \ldots, 100\}$ there must be two, say x and y, where $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{y} \rfloor$, then $|\sqrt{x} - \sqrt{y}| < 1$. $\square$

The Pigeonhole Principle

Example

Let $m \in \mathbb{Z}^+$ with m odd. Prove that there exists a positive integer n such that m divides $2^n - 1$

The Pigeonhole Principle

Example

Let $m \in \mathbb{Z}^+$ with m odd. Prove that there exists a positive integer n such that m divides $2^n - 1$

Proof.

Consider numbers from $2^1 - 1$ to $2^{m+1} - 1$

... $\square$