STA 35C Statistical Data Science III

Midterm exam 1 solution

Instructor: Dogyoon Song

Problem 1: Solution (20 points + 2 bonus)

(a) $\mathbb{E}[X]$ **and** $\mathrm{Var}(X)$ **for** $X \sim \mathrm{Binomial}(2, \frac{1}{3})$**.**

- $\mathbb{E}[X] = n\,p = 2 \times \frac{1}{3} = \frac{2}{3}$.
- $\mathrm{Var}(X) = n\,p\,(1-p) = 2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$.

Method 2: Alternatively, $X = X_1 + X_2$ where $X_1, X_2 \sim \mathrm{Bernoulli}(\frac{1}{3})$ i.i.d. Since $\mathbb{E}[X_1] = \frac{1}{3}$ and $\mathrm{Var}(X_1) = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$, it follows that

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \tfrac{2}{3}, \quad \text{and} \quad \mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = \tfrac{4}{9}.$$

Method 3: Otherwise, we can directly use the PMF to obtain

$$\mathbb{E}[X] = \sum_{x=0}^{2} x\; p_X(x) = 0 \cdot \binom{2}{0} \cdot \left(\frac{2}{3}\right)^2 + 1 \cdot \binom{2}{1} \cdot \frac{1}{3} \cdot \frac{2}{3} + 2 \cdot \binom{2}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{3}$$

Similarly, we can compute $\mathbb{E}[X^2] = \frac{8}{9}$, and thus, $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{4}{9}$.

(b) $\mathbb{E}[W]$ **and** $\mathrm{Var}(W)$ **for** $W = X + 2\,Y + 2$**.**
Given: $\mathbb{E}[X] = \frac{2}{3}, \quad \mathrm{Var}(X) = \frac{4}{9}, \quad \mathbb{E}[Y] = 9, \quad \mathrm{Var}(Y) = 9, \quad \mathrm{corr}(X, Y) = 0.3$.

- $\mathbb{E}[W] = \mathbb{E}[X] + 2\,\mathbb{E}[Y] + 2 = \frac{2}{3} + 2 \times 9 + 2 = \frac{2}{3} + 18 + 2 = \frac{62}{3}$.
- $\mathrm{Cov}(X, Y) = \rho\,\sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(Y)} = 0.3 \times \sqrt{\frac{4}{9}} \times \sqrt{9} = 0.3 \times \frac{2}{3} \times 3 = 0.6$.

$\therefore \mathrm{Var}(W) = \mathrm{Var}(X) + 4\,\mathrm{Var}(Y) + 4\,\mathrm{Cov}(X, Y) = \frac{4}{9} + 4 \times 9 + 4 \times 0.6 = \frac{4}{9} + 36 + 2.4 = 38.4 + 0.444\ldots \approx 38.84$.

(c) Bayesian Update: Factories A vs. B.

(i) **(5 points)** *Probability a randomly chosen box is from A* and *has exactly one defective ($X = 1$):*
Each box is A or B with prob. $\frac{1}{2}$. If a box is from A, $p = \frac{1}{3}$. Then

$$\Pr(X = 1 \mid A) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1 = 2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}.$$

Thus

$$\Pr(\text{A and } X = 1) = \Pr(A) \times \Pr(X = 1 \mid A) = \frac{1}{2} \times \frac{4}{9} = \frac{4}{18} = \frac{2}{9} \approx 0.2222.$$

(ii) (5 points) *Posterior* $\Pr(A \mid X = 1)$ *if* $p(B) = \frac{1}{2}, p = \frac{1}{10}$ *in B:*

$$\Pr(X = 1 \mid B) = \binom{2}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^1 = 2 \times 0.1 \times 0.9 = 0.18.$$

$$\Pr(\text{B and } X = 1) = \tfrac{1}{2} \times 0.18 = 0.09.$$

Therefore,

$$\Pr(A \mid X = 1) = \frac{\Pr(A,\ X = 1)}{\Pr(A,\ X = 1) + \Pr(B,\ X = 1)} = \frac{\frac{2}{9}}{\frac{2}{9} + 0.09} = \frac{0.2222}{0.2222 + 0.09} = \frac{0.2222}{0.3122} \approx 0.712.$$

(iii*) (*2 bonus points) Now with four factories (A,B,C,D) of unknown priors. If $X = 1$,

- Factory C ($p = 1$) always yields $X = 2$. So $\Pr(X = 1 \mid C) = 0$.
- Factory D ($p = 0$) always yields $X = 0$. So $\Pr(X = 1 \mid D) = 0$.

Hence the posterior of D is 0 if $X = 1$, regardless of the prior.

Problem 2: Solution (25 points)

(a) Four Scenarios (12 points).

(i) Nutritionist (3 pts)

- X = (age, weight, exercise), Y = daily protein intake.
- *Regression* problem (continuous Y).
- Primarily *prediction* to forecast intake.

(ii) Market Analyst (3 pts)

- X = browsing habits, Y = phone plan {A,B,C}.
- *Classification* problem (categorical Y).
- Goal is *prediction* for the new user.

(iii) Admissions Officer (3 pts)

- X = homework grades, Y = final exam score (numeric).
- *Regression* problem.
- Focus on *inference*: which assignments matter most.

(iv) Real Estate Agent (3 pts)

- X = (location, bedrooms, area, building age), Y = monthly rent.
- *Regression* problem.
- Goal is *inference*: find the factor(s) significantly affecting rent.

(b) Model Comparison (13 points).

(i) **(5 points)** Evaluate predictive performance via a metric (e.g., MSE).

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}\big(y_i - f(x_i)\big)^2.$$

A lower MSE indicates better prediction. (Ideally we want to check on a test set too, but we cannot.)

(ii) **(4 points)** If Model (2) outperforms Model (1) *only* on training data, it might overfit. We might still prefer Model (1) for interpretability or simpler structure.

(iii) **(4 points)** If Model (2) also excels on test data, it likely generalizes well. Model (1) might be chosen for inference/interpretability or practical concerns (e.g., simpler to explain or cheaper to implement).

Problem 3: Solution (40 points + 2 bonus)

We have Y = puzzle-solving time (minutes), X_1 = indicator for ≥ 2 yrs experience, X_2 = memory score.

(a) Prediction (15 points).

(i) **(5 points)**

$$\hat{Y}_A = 11.2 - 5 \times (1) = 6.2, \qquad \hat{Y}_B = 10 - 0.6 \times (7) = 5.8.$$

(ii) **(5 points)** Model A: $R^2 = 0.50$, Model B: $R^2 = 0.64$. B is better at explaining Y's variance. R^2 is fraction of Y's variation (variance) explained by the model. Equivalently, $R^2 = 1 - \frac{RSS}{TSS}$.

(iii) **(5 points)** A model with both X_1, X_2 yields $R^2 = 0.70$. This *might* be better, capturing both factors, evidenced by the increase in R^2. However, this is *not necessarily* better if overfitting, and adjusted R^2 might not have increased by much, as R^2 might have increased just by chance with additional predictors.

(b) Coefficients & Inference (15 points). We fit (1) $Y \sim X_1$ and (2) $Y \sim X_1 + X_2$ with:

$$\text{(1) Simple: } \hat{\beta}_1 = -5,\ SE = 1.67, \quad \text{(2) Multiple: } \hat{\beta}_1 = -1.6,\ SE = 1.6.$$

(i) **(5 points)**

- **Simple model:** $t = \frac{-5}{1.67} \approx -3.0 \Rightarrow p \approx 0.003 < 0.05$ (significant).
- **Multiple model:** $t = \frac{-1.6}{1.6} = -1.0 \Rightarrow p \approx 0.317 > 0.05$ (not significant).

(ii) **(5 points)**

- **Simple:** $\beta_1 = -5$ means participants with 2+ years' experience solve the puzzle about 5 minutes faster than $X_1 = 0$ on average.
- **Multiple:** $\beta_1 = -1.6$ indicates participants with 2+ years' experience solve the puzzle about only 1.6 minutes faster than $X_1 = 0$ on average, once short-term memory score (X_2) is controlled.

(iii) **(5 points)** If X_1 correlates with X_2, omitting X_2 can inflate X_1's effect by including its indirect influence via X_2. For example, people with higher memory scores may be likelier to enjoy puzzles and thus more experience, or puzzle experience might improve memory. Controlling for X_2 isolates X_1's direct effect, thereby mitigating confounding.

(iv*) **(*2 bonus points)** In this model, $\beta_1 = -7$ is the intercept difference at $X_2 = 0$. Thus, at X_2 fixed at 0, participants with 2+ years' experience solve puzzles 7 minutes faster on average than those without.

(c) Adding More Predictors (10 points).

(i) **(4 points)** Since Y–X_2 relation seems nonlinear, we can consider adding a higher-order term in X_2, e.g., X_2^2.
* Note that X_1 is a dummy variable, and Y is numeric (not categorical); seeing two clusters is normal, and classification methods (e.g., logistic) don't apply.

(ii) **(3 points)** As X_3 seems uncorrelated with Y, it might not help to explain Y. Possibly skip X_3.

(iii) **(3 points)** Although X_4 seems strongly associated with Y, it also strongly correlates with X_2. Including both can cause collinearity, and we should choose only one or carefully interpret; perhaps skip X_4.

Problem 4: Solution (35 points + 4 bonus)

(a) 2D Logistic Regression (20 points).

(i) **(5 points) Compute $\hat{p}(x_{\text{test}})$**
With $\hat{\beta}_0 = -2,\ \hat{\beta}_1 = -1,\ \hat{\beta}_2 = 2$, for $x_{\text{test}} = (1,1)$:

$$\log\left(\tfrac{p}{1-p}\right) = -2 + (-1)\cdot 1 + 2 \cdot 1 = -1.$$

$$p = \frac{1}{1+e^1} \approx 0.269 < 0.5 \implies \hat{y}_{\text{test}} = 0.$$

(ii) **(5 points) Decision Boundary**
The given decision rule predicts $\hat{y} = 1$ if and only if

$$\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \geq \log\left(\tfrac{p^*}{1-p^*}\right) = 0,$$

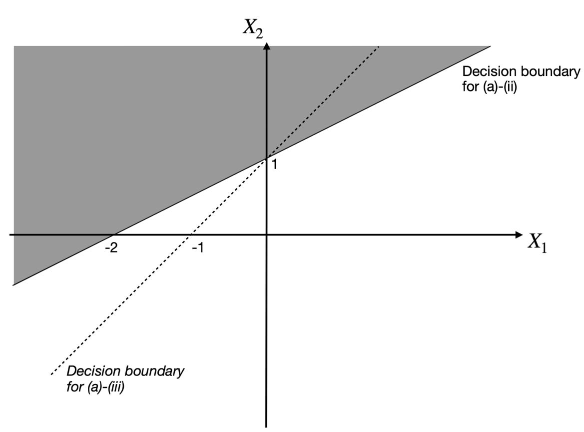
which is equivalent to

$$x_2 \geq -\tfrac{\hat{\beta}_1}{\hat{\beta}_2} x_1 - \tfrac{\hat{\beta}_0}{\hat{\beta}_2} = \frac{1}{2} x_1 + 1.$$

In the last equality, we plugged in $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) = (-2, -1, 2)$.

(iii*) **(*2 bonus)** Changing to $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) = (-1, -1, 1)$, we now predict $\hat{y} = 1$ if and only if

$$x_2 \geq x_1 + 1.$$



(iv) (5 points) Confusion Matrix

	Pred = 1	Pred = 0
$Y = 1$	35	5
$Y = 0$	15	45

TPR $= \frac{35}{35+5} = 0.875$, FPR $= \frac{15}{15+45} = 0.25$.

(v) (5 points) Lowering p^* With $p^* = 0.1$ vs. 0.5, more borderline cases $\to$ "1." False positives $\uparrow$, false negatives $\downarrow$.

(b) 1D LDA (15 points). Each species has a normal distribution with the *same* variance but different means, and we apply LDA to classify by weight.

(i) (4 points) Sample Means & Pooled s

Species A: $\{1.5, 2.5\} \implies \bar{x}_A = 2.0$.

Species B: $\{2.0, 3.0, 4.0\} \implies \bar{x}_B = 3.0$.

Pooled covariance:

$$s^2 = \frac{\left[(1.5-2)^2 + (2.5-2)^2\right] + \left[(2-3)^2 + (3-3)^2 + (4-3)^2\right]}{5-2} = \frac{0.5+2}{3} = \frac{5}{6} \approx 0.8333.$$

(ii) (4 points) Linear Discriminants

$\pi_A = \frac{2}{5}$, $\pi_B = \frac{3}{5}$. Then $\log(\frac{2}{5}) \approx -0.916$, $\log(\frac{3}{5}) \approx -0.511$.

Thus, the linear discriminant functions in this problem reduce to:

$$\delta_A(x) = \frac{x\,\bar{x}_A}{s^2} - \frac{\bar{x}_A^2}{2\,s^2} + \log(\pi_A) = \frac{12}{5}(x-1) + \log\left(\tfrac{2}{5}\right),$$
$$\delta_B(x) = \frac{x\,\bar{x}_B}{s^2} - \frac{\bar{x}_B^2}{2\,s^2} + \log(\pi_B) = \frac{18}{5}\left(x - \tfrac{3}{2}\right) + \log\left(\tfrac{3}{5}\right).$$

(iii) (4 points) Predict at $x_{\text{new}} = 2.5$

From the linear discriminant functions above, we get

$$\delta_A(x) - \delta_B(x) = -\frac{6}{5}x + 3 + \log\left(\frac{2}{3}\right). \tag{1}$$

Inserting $x = 2.5$, we get

$$\delta_A(2.5) - \delta_B(2.5) = -3 + 3 + \log\left(\frac{2}{3}\right) \approx -0.405 < 0.$$

So classify **Species B**.

(iv) (3 points) Adding 4 More A's

With additional data points, A has $2 + 4 = 6$ crabs vs. B has 3, so $\pi_A = \frac{6}{9} = 0.6667$, $\pi_B = 0.3333$. $\log(\frac{0.6667}{0.3333}) = \log(2) \approx 0.693$, a positive shift. Following the same steps as above in (ii)–(iii), we get

$$\delta_A(2.5) - \delta_B(2.5) = -3 + 3 + \log\left(\frac{6}{3}\right) = \log 2 \approx 0.693 > 0.$$

Now we predict **Species A** instead of Species B.

(v*) (*2 bonus points)

Requiring $\Pr(A \mid x) \geq p^* > 0.5$ translates to $\delta_A(x) - \delta_B(x) \geq \log(\frac{p^*}{1-p^*})$. Thus, with $p^* = 0.9$,

$$\text{predict } \hat{Y} = A \quad \text{if and only if} \quad \delta_A(x) - \delta_B(x) > 2\log(3) \approx 2.197.$$

Following (1) in (iii),

$$\delta_A(2) - \delta_B(2) = -\frac{6}{5} \times 2 + 3 + \log\left(\frac{2}{3}\right) \approx 0.195 < 2.197.$$

Thus, we would classify the crab with x'_{new} as **Species B** under $p^* = 0.9$ to avoid missing B crabs; note that it would have been **Species A** with the original threshold $p^* = 0.5$ as $\delta_A(2) - \delta_B(2) \approx 0.195 > 0$.