Building a Constitutive Relation for Fiber Bundle Model with Plasticity (unbroken fibers)

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1 Introduction

Suppose after yielding point, the fiber starts yielding but never be broken. In yielding region, $E' = \alpha \cdot E$ where E is elastic modulus and α is in [0, 1] Then we obtain the stress (σ) at a particular strain (ε) that is imposed on our bundle:

$$\sigma = E \cdot \varepsilon \; if \; \varepsilon < \varepsilon_1 \; (for \; intact \; fibers) \tag{1}$$

$$\sigma = E \cdot \varepsilon_1 + \alpha \cdot E \cdot (\varepsilon - \varepsilon_1) \; if \; \varepsilon >= \varepsilon_1 \; (for \; yielding \; fibers) \tag{2}$$

Introduce $f(\varepsilon_1)$ as the probability density function of ε_1 then the probability of fibers whose ε_1 falls in $[\varepsilon_1, \varepsilon_1 + d\varepsilon_1]$ is:

$$\int_0^{\varepsilon} f(\varepsilon_1) d\varepsilon_1 \tag{3}$$

The cummulative distribution function:

$$F(\varepsilon_1) = \int_0^{\varepsilon_1} f(\varepsilon_1') d\varepsilon_1' \tag{4}$$

At an instant imposed strain ε on the bundle, with N is the total number of fibers, the number of yielding fibers is:

$$N \cdot F(\varepsilon) = N \cdot \int_0^{\varepsilon} f(\varepsilon_1) d\varepsilon_1 \tag{5}$$

since $F(\varepsilon) = P(\varepsilon_1 < \varepsilon)$. So the number of intact fibers is:

$$N \cdot [1 - F(\varepsilon)] \tag{6}$$

The force exerted by N fibers at an instant ε is composed of forces by intact fibers and forces by yielding fibers, which is:

$$F = N \cdot [1 - F(\varepsilon)] \cdot (E \cdot \varepsilon) + \int_0^{\varepsilon} N \cdot f(\varepsilon_1) d\varepsilon_1 \cdot [E \cdot \varepsilon_1 + \alpha \cdot E \cdot (\varepsilon - \varepsilon_1)] \quad (7)$$

Divide by N, we obtain the stress-strain relation:

$$\sigma = [1 - F(\varepsilon)] \cdot E \cdot \varepsilon + \int_0^{\varepsilon} f(\varepsilon_1) d\varepsilon_1 \cdot [E \cdot \varepsilon_1 + \alpha \cdot E \cdot (\varepsilon - \varepsilon_1)] \quad (8)$$

2 Mathematical Formulation

Suppose this is Weibull distribution, i.e. its CDF is:

$$F(x) = 1 - e^{-(\frac{x}{\lambda})^m} \qquad x \geq 0 \quad (9)$$

and PDF is:

$$f(x) = \frac{m}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{m-1} \cdot e^{-(\frac{x}{\lambda})^m} \qquad x \geq 0 \quad (10)$$

$$\Rightarrow f(x) = \frac{1}{\lambda} \cdot e^{-\frac{x}{\lambda}} \qquad x \geq 0 \;\&\; m = 1 \quad (11)$$

Then we obtain the stress-strain relation where $E = 1$ and $m = 1$:

$$\sigma = [1 - (1 - e^{-\frac{\varepsilon}{\lambda}})] \cdot \varepsilon + \int_0^{\varepsilon} \frac{1}{\lambda} \cdot e^{-\frac{\varepsilon_1}{\lambda}} d\varepsilon_1 \cdot [\varepsilon_1 + \alpha \cdot (\varepsilon - \varepsilon_1)] \quad (12)$$

$$\Leftrightarrow \sigma = e^{-\frac{\varepsilon}{\lambda}} \cdot \varepsilon + \Big[[\varepsilon_1 + \alpha \cdot (\varepsilon - \varepsilon_1)] \cdot (-e^{-\frac{\varepsilon_1}{\lambda}})\Big]_0^{\varepsilon} - \int_0^{\varepsilon} -e^{-\frac{\varepsilon_1}{\lambda}} \cdot (1 - \alpha) d\varepsilon_1 \quad (13)$$

$$\Leftrightarrow \sigma = e^{-\frac{\varepsilon}{\lambda}} \cdot \varepsilon + \Big[[\varepsilon_1 + \alpha \cdot (\varepsilon - \varepsilon_1)] \cdot (-e^{-\frac{\varepsilon_1}{\lambda}})\Big]_0^{\varepsilon} - \Big[\lambda \cdot (1 - \alpha) \cdot e^{-\frac{\varepsilon_1}{\lambda}}\Big]_0^{\varepsilon} \quad (14)$$

$$\Leftrightarrow \sigma = e^{-\frac{\varepsilon}{\lambda}} \cdot \varepsilon + (-\varepsilon \cdot e^{-\frac{\varepsilon}{\lambda}} + \alpha \cdot \varepsilon) - \lambda \cdot (1 - \alpha) \cdot (e^{-\frac{\varepsilon}{\lambda}} - 1) \quad (15)$$

$$\Leftrightarrow \sigma = \alpha \cdot \varepsilon - \lambda \cdot (1 - \alpha) \cdot (1 - e^{-\frac{\varepsilon}{\lambda}}) \quad (16)$$

3 Results

Using GLE, we can obtain the constitutive relation for fiber bundle model with plasticity (unbroken fibers): constit3.gle