Building a Constitutive Relation for Fiber Bundle Model with Plasticity (breakable fibers)

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September 14, 2024

I. Introduction

Consider a fiber bundle model with breakable fibers subjected to an external strain ε. The average yielding threshold of all fibers is $\langle\varepsilon_y\rangle$, and the average breaking threshold of all fibers is $\langle\varepsilon_b\rangle$.

The probability density function of ε_y is $p_y(\varepsilon_y)$ for $\varepsilon_y \in [\langle\varepsilon_y\rangle - \Delta\varepsilon_y, \langle\varepsilon_y\rangle + \Delta\varepsilon_y]$, and the probability density function of ε_b is $p_b(\varepsilon_b)$ for $\varepsilon_b \in [\langle\varepsilon_b\rangle - \Delta\varepsilon_b, \langle\varepsilon_b\rangle + \Delta\varepsilon_b]$. Assume both are uniformly distributed:

$$p_y(\varepsilon_y) = \begin{cases} \frac{1}{2\Delta\varepsilon_y} & \text{if } \langle\varepsilon_y\rangle - \Delta\varepsilon_y \leq \varepsilon_y \leq \langle\varepsilon_y\rangle + \Delta\varepsilon_y \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$p_b(\varepsilon_b) = \begin{cases} \frac{1}{2\Delta\varepsilon_b} & \text{if } \langle\varepsilon_b\rangle - \Delta\varepsilon_b \leq \varepsilon_b \leq \langle\varepsilon_b\rangle + \Delta\varepsilon_b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Their cumulative distribution functions are:

$$P_y(\varepsilon_y) = \begin{cases} 0 & \text{if } \varepsilon_y < \langle\varepsilon_y\rangle - \Delta\varepsilon_y \\ \frac{\varepsilon_y - (\langle\varepsilon_y\rangle - \Delta\varepsilon_y)}{2\Delta\varepsilon_y} & \text{if } \langle\varepsilon_y\rangle - \Delta\varepsilon_y \leq \varepsilon_y \leq \langle\varepsilon_y\rangle + \Delta\varepsilon_y \\ 1 & \text{if } \varepsilon_y > \langle\varepsilon_y\rangle + \Delta\varepsilon_y \end{cases} \tag{3}$$

$$P_b(\varepsilon_b) = \begin{cases} 0 & \text{if } \varepsilon_b < \langle\varepsilon_b\rangle - \Delta\varepsilon_b \\ \frac{\varepsilon_b - (\langle\varepsilon_b\rangle - \Delta\varepsilon_b)}{2\Delta\varepsilon_b} & \text{if } \langle\varepsilon_b\rangle - \Delta\varepsilon_b \leq \varepsilon_b \leq \langle\varepsilon_b\rangle + \Delta\varepsilon_b \\ 1 & \text{if } \varepsilon_b > \langle\varepsilon_b\rangle + \Delta\varepsilon_b \end{cases} \tag{4}$$

There are 3 types of fibers: intact, yielding, and broken. The stress-strain function is:

$$\sigma(\varepsilon) = E\varepsilon[1 - P_y(\varepsilon)] + \left(\int_0^{\varepsilon} [\alpha E(\varepsilon - \varepsilon_y) + E\varepsilon_y] \cdot p_y(\varepsilon_y)\, d\varepsilon_y \right) \cdot [1 - P_b(\varepsilon)] \tag{5}$$

$$= E\varepsilon[1 - P_y(\varepsilon)] + \left[\int_0^{\varepsilon} [(1-\alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot p_y(\varepsilon_y)\, d\varepsilon_y \right] \cdot [1 - P_b(\varepsilon)] \tag{6}$$

where α is the ratio of the elastic modulus of yielding fibers to that of intact fibers, and E is the elastic modulus of intact fibers.

II. Mathematical Formulation

Since the behavior of the uniform distribution varies across different intervals, we need to break the process down and consider each interval separately. For the sake of simplicity[1], consider the two intervals $[\langle\varepsilon_y\rangle - \Delta\varepsilon_y, \langle\varepsilon_y\rangle + \Delta\varepsilon_y]$ and $[\langle\varepsilon_b\rangle - \Delta\varepsilon_b, \langle\varepsilon_b\rangle + \Delta\varepsilon_b]$ do not overlap, i.e., $\langle\varepsilon_y\rangle + \Delta\varepsilon_y < \langle\varepsilon_b\rangle - \Delta\varepsilon_b$.

1. For $0 \leq \varepsilon < \langle\varepsilon_y\rangle - \Delta\varepsilon_y$

In this interval, all fibers are intact, i.e., no fiber has yielded yet. The stress-strain function is:

$$\sigma(\varepsilon) = E\varepsilon \tag{7}$$

2. For $\langle\varepsilon_y\rangle - \Delta\varepsilon_y \leq \varepsilon \leq \langle\varepsilon_y\rangle + \Delta\varepsilon_y$

In this interval, some fibers are yielding, some still are intact. Substitute (3), (1), and (4) in (6). The stress-strain function is:

$$\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon\left[1 - \frac{\varepsilon - (\langle\varepsilon_y\rangle - \Delta\varepsilon_y)}{2\Delta\varepsilon_y}\right] + \left[\int\limits_0^{\langle\varepsilon_y\rangle - \Delta\varepsilon_y} 0 + \int\limits_{\langle\varepsilon_y\rangle - \Delta\varepsilon_y}^{\varepsilon} [(1-\alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot \frac{1}{2\Delta\varepsilon_y}\, d\varepsilon_y\right] \cdot [1 - 0] \\
&= E\varepsilon\left[1 - \frac{\varepsilon - \langle\varepsilon_y\rangle}{2\Delta\varepsilon_y} - \frac{\Delta\varepsilon_y}{2\Delta\varepsilon_y}\right] + \frac{E}{2\Delta\varepsilon_y}\int\limits_{\langle\varepsilon_y\rangle - \Delta\varepsilon_y}^{\varepsilon} [(1-\alpha)\varepsilon_y + \alpha\varepsilon]\, d\varepsilon_y \\
&= E\varepsilon\left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_y\rangle}{2\Delta\varepsilon_y}\right) + \frac{E}{2\Delta\varepsilon_y}\left[\frac{1-\alpha}{2}{\varepsilon_y}^2 + \alpha\varepsilon\varepsilon_y\right]_{\langle\varepsilon_y\rangle - \Delta\varepsilon_y}^{\varepsilon} \\
&= E\varepsilon\left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_y\rangle}{2\Delta\varepsilon_y}\right) + \frac{E}{2\Delta\varepsilon_y}\left[\frac{1-\alpha}{2}\varepsilon^2 + \alpha\varepsilon^2 - \frac{1-\alpha}{2}\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)^2 - \alpha\varepsilon\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)\right] \\
\sigma(\varepsilon) &= E\varepsilon\left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_y\rangle}{2\Delta\varepsilon_y}\right) + \frac{E}{2\Delta\varepsilon_y}\left[\frac{1+\alpha}{2}\varepsilon^2 - \alpha\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)\varepsilon - \frac{1-\alpha}{2}\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)^2\right]
\end{aligned} \tag{8}$$

3. For $\langle\varepsilon_y\rangle + \Delta\varepsilon_y < \varepsilon < \langle\varepsilon_b\rangle - \Delta\varepsilon_b$

In this interval, all fibers are yielding, i.e., no fiber is intact anymore, and no fiber is broken yet. Substitute (3), (1), and (4) into (6). The stress-strain function is:

[1] Is it just for the sake of simplicity or is it actually cannot overlap?

$$
\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon(1-1) + \left[\int\limits_{0}^{\langle\varepsilon_y\rangle-\Delta\varepsilon_y} 0 + \int\limits_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \big[[(1-\alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot \frac{1}{2\Delta\varepsilon_y}\ d\varepsilon_y\big] + \int\limits_{\langle\varepsilon_y\rangle+\Delta\varepsilon_y}^{0} 0 \right] \cdot (1-0) \\
&= \frac{E}{2\Delta\varepsilon_y} \int\limits_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \big[(1-\alpha)E\varepsilon_y + \alpha E\varepsilon\big]\ d\varepsilon_y \\
&= \frac{E}{2\Delta\varepsilon_y} \left[\frac{1-\alpha}{2}{\varepsilon_y}^2 + \alpha\varepsilon\varepsilon_y \right]_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \\
&= \frac{E}{2\Delta\varepsilon_y} \left[\frac{1-\alpha}{2} \Big[\big(\langle\varepsilon_y\rangle + \Delta\varepsilon_y\big)^2 - \big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)^2 \Big] + \alpha\varepsilon 2\Delta\varepsilon_y \right] \\
&= \frac{E}{2\Delta\varepsilon_y} \left(\frac{1-\alpha}{2} \cdot 4\langle\varepsilon_y\rangle\Delta\varepsilon_y + \alpha\varepsilon 2\Delta\varepsilon_y \right) \\
\sigma(\varepsilon) &= E\big[(1-\alpha)\langle\varepsilon_y\rangle + \alpha\varepsilon\big] \qquad (9)
\end{aligned}
$$

4. For $\langle\varepsilon_b\rangle - \Delta\varepsilon_b \le \varepsilon \le \langle\varepsilon_b\rangle + \Delta\varepsilon_b$

In this interval, some fibers are broken, some still are yielding but no fiber is intact. Substitute (3), (1), and (4) into (6). The stress-strain function is:

$$
\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon(1-1) + \left[\int\limits_{0}^{\langle\varepsilon_y\rangle-\Delta\varepsilon_y} 0 + \int\limits_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \big[(1-\alpha)E\varepsilon_y + \alpha E\varepsilon\big] \cdot \frac{1}{2\Delta\varepsilon_y}\ d\varepsilon_y + \int\limits_{\langle\varepsilon_y\rangle+\Delta\varepsilon_y}^{0} 0 \right] \cdot \Big[1 - \frac{\varepsilon - \big(\langle\varepsilon_b\rangle - \Delta\varepsilon_b\big)}{2\Delta\varepsilon_b} \\
&= \left[\frac{E}{2\Delta\varepsilon_y} \int\limits_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \big[(1-\alpha)\varepsilon_y + \alpha\varepsilon\big]\ d\varepsilon_y \right] \cdot \left[1 - \frac{\varepsilon - \big(\langle\varepsilon_b\rangle - \Delta\varepsilon_b\big)}{2\Delta\varepsilon_b} \right] \\
&= \frac{E}{2\Delta\varepsilon_y} \left[\frac{1-\alpha}{2}{\varepsilon_y}^2 + \alpha\varepsilon\varepsilon_y \right]_{\langle\varepsilon_y\rangle-\Delta\varepsilon_y}^{\langle\varepsilon_y\rangle+\Delta\varepsilon_y} \cdot \left(1 - \frac{\varepsilon - \langle\varepsilon_b\rangle}{2\Delta\varepsilon_b} - \frac{1}{2} \right) \\
&= \frac{E}{2\Delta\varepsilon_y} \left[\frac{1-\alpha}{2} \Big(\big(\langle\varepsilon_y\rangle + \Delta\varepsilon_y\big)^2 - \big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)^2 \Big) + \alpha\varepsilon\big(\langle\varepsilon_y\rangle + \Delta\varepsilon_y\big) - \big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big) \right] \cdot \left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_b\rangle}{2\Delta\varepsilon_b} \right) \\
&= \frac{E}{2\Delta\varepsilon_y} \left(\frac{1-\alpha}{2} \cdot 4\langle\varepsilon_y\rangle\Delta\varepsilon_y + \alpha\varepsilon 2\Delta\varepsilon_y \right) \cdot \left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_b\rangle}{2\Delta\varepsilon_b} \right) \\
\sigma(\varepsilon) &= \frac{E}{2}\Big[(1-\alpha)\langle\varepsilon_y\rangle + \alpha\varepsilon\Big] \cdot \left(1 - \frac{\varepsilon - \langle\varepsilon_b\rangle}{\Delta\varepsilon_b} \right) \qquad (10)
\end{aligned}
$$

5. For $\langle\varepsilon_b\rangle + \Delta\varepsilon_b < \varepsilon$

In this interval, all fibers are broken. The stress-strain function is:

$$\sigma(\varepsilon) = 0 \tag{11}$$

III. Results

Now, we have the stress-strain function for each interval. We can combine them to get the final stress-strain function for the entire process:

$$\sigma(\varepsilon) = \begin{cases} E\varepsilon & \text{if } 0 \le \varepsilon < \langle\varepsilon_y\rangle - \Delta\varepsilon_y \\ E\varepsilon\left(\frac{1}{2} - \frac{\varepsilon - \langle\varepsilon_y\rangle}{2\Delta\varepsilon_y}\right) + \frac{E}{2\Delta\varepsilon_y}\left[\frac{1+\alpha}{2}\varepsilon^2 - \alpha\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)\varepsilon - \frac{1-\alpha}{2}\big(\langle\varepsilon_y\rangle - \Delta\varepsilon_y\big)^2\right] & \text{if } \langle\varepsilon_y\rangle - \Delta\varepsilon_y \le \varepsilon \le \langle\varepsilon_y\rangle + \Delta\varepsilon_y \\ E\big[(1-\alpha)\langle\varepsilon_y\rangle + \alpha\varepsilon\big] & \text{if } \langle\varepsilon_y\rangle + \Delta\varepsilon_y < \varepsilon < \langle\varepsilon_b\rangle - \Delta\varepsilon_b \\ \frac{E}{2}\Big[(1-\alpha)\langle\varepsilon_y\rangle + \alpha\varepsilon\Big] \cdot \left(1 - \frac{\varepsilon - \langle\varepsilon_b\rangle}{\Delta\varepsilon_b}\right) & \text{if } \langle\varepsilon_b\rangle - \Delta\varepsilon_b \le \varepsilon \le \langle\varepsilon_b\rangle + \Delta\varepsilon_b \\ 0 & \text{if } \langle\varepsilon_b\rangle + \Delta\varepsilon_b < \varepsilon \end{cases}$$

Using GLE, we can obtain the constitutive relation for fiber bundle model with plasticity (breakable fibers): constit4.gle