separable extensions

Definitions

Let k be a field.

An element α algebraic over k is separable if the minimal polynomial f has no multiple roots.

An algebraic extension E/k is separable if every element of E is separable over k.

Characterization of finite separable extensions

If F/k is a finite extension, and k^{a} is a fixed algebraic closure of k, then the number of embeddings

$$F \to k^{\mathrm{a}}$$

is at most $[F : k]$. If we denote this number by $[F : k]_s$, then it amounts to saying that

$$[F : k]_s \leq [F : k].$$

The equality holds true if and only if E/k is separable.

For a tower $E/F/k$ of finite extensions, we have

$$[E : F]_s [F : k]_s = [E : k]_s.$$

A criterion for separable extensions

Suppose that $F = k(\alpha_1, \cdots, \alpha_n)$. If each α_i is separable over k, then F/k is separable.

Theorem

Separable extensions form a distinguished class.

To prove it, we check the following three conditions.

1. Let $E/F/k$ be a tower of fields. E/k is separable if and only if both E/F and F/k are seperable.
2. If E/k is separable and F/k is an extension, then EF/F is separable.
3. If F/k and E/k is separable, then EF/k is separable.

condition 1

Let $E/F/k$ be a tower of fields. E/k is separable if and only if both E/F and F/k are seperable.

Assume that E/k is separable. Then, for any $\alpha \in E$, its minimal polynomial over k, say f_α, has no multiple roots. Its minimal polynomial over F is a factor of f_α, so it has no multiple roots. If $\alpha \in F$, its minimal polynomial over k is f_α, so it has no multiple roots.

Assume that both E/F and F/k are separable. To show E/k is separable, its suffices to show that any $\alpha \in E$ is separable over k. Consider an intermediate finite extension F'/k generated by the coefficients of the minimal polynomial of α over F. Consider the tower $k/F'/F'(\alpha)$ of finite extensions. By

$$\begin{aligned}[F'(\alpha) : k]_s \leq [F'(\alpha) : k] &= [F'(\alpha) : F'][F' : k] \\ &= [F'(\alpha) : F']_s[F' : k]_s = [F'(\alpha) : k]_s,\end{aligned}$$

we conclude that $[F'(\alpha) : k]_s = [F'(\alpha) : k]$.

condition 2

If E/k is separable and F/k is an extension, then EF/F is separable.

Note that EF/F is generated by elements of E over F. Each $\alpha \in E$ is separable over k, so it is separable over F. This shows that EF/F is generated by seperable elements. Conclude that EF/F is separable.

condition 3

If F/k and E/k is separable, then EF/k is separable.

This follows from the first two conditions.

Question

Prove the characterization and the criterion.