

distinguished class

Let  $\mathcal{C}$  be a class of field extensions. We say  $\mathcal{C}$  is distinguished if the following conditions are satisfied.

- ▶ Let  $k \subset F \subset E$  be a tower of fields. The extension  $E/k$  is in  $\mathcal{C}$  if and only if both  $F/k$  and  $E/F$  is in  $\mathcal{C}$ .
- ▶ If  $E/k$  is in  $\mathcal{C}$ , and  $F/k$  is any extension of  $k$ , then  $EF/F$  is in  $\mathcal{C}$ .
- ▶ If  $F/k$  and  $E/k$  are in  $\mathcal{C}$  and  $F, E$  are subfields of a common field, then  $EF/k$  is in  $\mathcal{C}$ .

## Proposition

*The class of finite extensions is distinguished.*

## Proof

We check the three conditions in order. First, consider the tower  $k \subset F \subset E$  of fields. Since

$$[E : k] = [E : F][F : k],$$

$E/k$  is finite if and only if both  $E/F$  and  $F/k$  are finite.

Second, assume that  $E/k$  is finite and  $F/k$  is an arbitrary extension such that  $E$  and  $F$  are both contained in a same field. Then, we need to show that  $EF/F$  is finite. If  $E/k$  is generated by  $a_1, \dots, a_n$ , then they generated  $EF$  over  $F$ . Thus  $EF/F$  is finitely generated by algebraic elements, which implies that  $EF/F$  is finite.

Lastly, assume that  $E/k$  and  $F/k$  are extensions contained in some common field. If  $E = k(\alpha_1, \dots, \alpha_n)$  and  $F = k(\beta_1, \dots, \beta_m)$ , then  $EF = k(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ . In particular,  $EF$  is finitely generated by algebraic elements. We conclude that  $EF/k$  is finite.

This completes the proof.

## Question

Show that the class of algebraic extensions is distinguished.