Existence of Free groups

Let S be a set. Let G be a group and

$$f\colon S \to G$$

be a set map such that $f(S)$ generates G. The lemma below is a simple exercise with cardinalities.

Lemma

Let $\kappa = \max\{|S|, |\mathbb{Z}|\}$. *Then,* $|G| \leq \kappa$.

Proof.

Every element in G is a finite product of elements from $f(S)$ and $f(S)^{-1}$. Thus, there is a surjection

$$\coprod_{n \geq 1} S^n \to G.$$

On the other hand, one has $|\coprod_{n\geq 1} S^n| \leq \kappa$. $\square$

a large group

Let S be an arbitary set, and T be a set with cardinality $\kappa = \max\{|S|, |\mathbb{Z}|$
Then, for each non-empty subset $H \subset T$, the collection Γ_H of all group structures on H forms a set. Consider the product

$$F_0 := \prod_{\emptyset \neq H \subset T} \prod_{f\colon S \to H} \prod_{\gamma \in \Gamma_H} H_{\gamma,f}$$

where the component $H_{\gamma,f}$ is a copy of H with group structure given by γ. One has a natural set map

$$\phi\colon S \to F_0$$

which is componentwise given by the map $f\colon S \to H_{\gamma,f}$.

extension property

Let G be an arbitrary group equipped with a set map

$$\psi \colon S \to G$$

The image of ψ generates a subgroup $G_0 \subset G$ whose cardinality is bounded by κ. Thus, we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, equal] & H_{\gamma,f} \arrow[d, "\sim"] \\ S \arrow[r, "\psi"] & G_0 \end{tikzcd}$$

for some γ and f. In particular we get an extension of ψ to F_0;

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, equal] & F_0 \arrow[r] \arrow[d] & H_{\gamma,f} \arrow[dl] \\ S \arrow[r, "\psi"] & G_0 & \end{tikzcd}$$

universal property

Let $F \subset F_0$ be the subgroup generated by $\phi(S)$. The extension property just discussed yields η such that the following diagram is commutative;

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, equal] & F \arrow[d, "\eta"] \\ S \arrow[r, "\psi"] & G \end{tikzcd}$$

Moreover, η is unique because $\phi(S)$ generates F.

Definition

The free group on S denotes the group F equipped with $S \to F$. It is often denoted by $F = F(S)$.

Question

Let $S = S_1 \coprod S_2$. Show that the maps $F(S_i) \to F(S)$ for $i = 1, 2$ turn $F(S)$ into the coproduct of $F(S_1)$ and $F(S_2)$.