Algebraic extensions

Definition

Let E/F be a field extension. For an element $\alpha \in E$, we say α is algebraic (over F) if

$$1, \alpha, \alpha^2, \cdots$$

are linearly dependent over F. Otherwise, we say α is transcendental.

We say E/F is algebraic if every element of E is algebraic over F.

Let E/F be a field extension, and $\alpha \in E$ be any element. Then, by the universal property of polynomial rings, we have a ring homomorphism

$$\begin{aligned} f \colon F[X] &\to E \\ X &\mapsto \alpha. \end{aligned}$$

α is algebraic over F if and only if $\ker(f)$ is non-zero. In that case, it is generated by a monic polynomial, say $P(X)$. Such $P(X)$ is unique, and called the minimal polynomial of α over F.

Since $F[X]/(P(X))$ is isomorphic to a subring of E, it must be entire. It follows that $P(X)$ is irreducible.

Conversely, if $P(X)$ is an irreducible polynomial, then

$$E := F[X]/P(X)$$

is a field extension. It is necessarily finite and algebraic.

Question

Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.