Irreducibility and primes

Let A be an entire ring.

Proposition

Suppose that (a) is a non-zero prime ideal. Then, a is irreducible.

Proof.

If a is not irreducible, then one has $a = bc$ with non-units $b, c \in A$. Since b is not a unit, $c \notin (a)$. Similarly, $b \notin (a)$. This contradicts the assumption since $bc \in (a)$. $\square$

For a field k, then $k[x^2, x^3]$ is a counterexample to the converse. Here $k[x^2, x^3]$ denotes the subring of $k[x]$ generated by x^2 and x^3. As a vector space over k, it is generated by monomials of degree not equal to one.

Then, x^3 is irreducible, essentially because $x \notin k[x^2, x^3]$.

However, $x^2 \cdot x^2 \in (x^3)$ and $x^2 \notin (x^3)$. This shows that (x^3) is not a prime ideal.

Show that there is an isomorphism

$$k[x^2, x^3] \simeq k[u, v]/(u^2 - v^3).$$