torsion-free modules over principal rings

For an entire ring A and an A-module M, let

$$M_{\mathrm{tor}} = \{m \in M \colon am = 0 \text{ for some } a \in A - \{0\}\}$$

be the torsion submodule of M. We say M is torsion-free if $M_{\mathrm{tor}} = 0$.

Theorem

Suppose that A is a principal entire ring, and M be a finitely generated module over A. If M is torsion-free, then M is free.

Note that any free module over A is torsion-free.

proof

Choose a finite set of generators $y_1, \cdots, y_n$ for M as an A-module. Let $v_1, \cdots, v_m$ be a maximal linearly independent subset of $\{y_1, \cdots, y_n\}$. Then, by maximality, for each y_i one has

$$a_i y_i + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m = 0$$

for some $a_i \neq 0$. This means $a_i y_i \in \sum_{j=1}^m Av_j$. Taking $a = a_1 \cdots a_n$, one obtains $ay_i \in \sum_{j=1}^m Av_j$ for all i. One gets a map

$$\begin{aligned} M &\to aM \subset \sum_{j=1}^m Av_j \\ m &\mapsto am \end{aligned}$$

which is injective since M is torsion-free. On the other hand, the linear independence of $v_1, \cdots, v_m$ implies that the sum is in fact a direct sum. M is isomorphic to a submodule of a free module of rank m, so it is free as well.

Question

Show that $\mathbb{Q}$ is not free over $\mathbb{Z}$, showing that the finite generation is necessary.