$A[X]$ is factorial

Theorem

Let A be a factorial ring. Then, $A[X]$ is factorial.

existence

Let $f \in A[X]$ be a non-zero polynomial. Factor

$$f = c \cdot p_1 p_2 \cdots p_r$$

in $K[X]$, where $c \in K^\times$ and p_i is irreducible in $K[X]$. Replacing p_i by $p_i/\mathrm{cont}(p_i)$, we may assume that $c \in A$ and $p_1, \cdots, p_r$ have content one. By factoring c into irreducibles of A, one obtains a factorization of f.

uniqueness

Let

$$\begin{aligned} f &= u \cdot c_1 c_2 \cdots c_\rho \cdot p_1 p_2 \cdots p_r \\ &= v \cdot d_1 d_2 \cdots d_\sigma \cdot q_1 p_2 \cdots q_s \end{aligned}$$

be two factorizations in $A[X]$. Here $u, v \in A^\times$, $c_i, d_i \in A$, $p_i, q_i \in A[X]$. Comparing contents and using the assumption that A is factorial, we may assume that $\rho = \sigma$. Then, using the fact that $K[X]$ is factorial, conclude that $r = s$, and that, by rearranging the factors, $p_i = a_i q_i$ with $a_i \in K^\times$. Again, considering contents, we have $a_i \in A^\times$. This show the uniqueness.

Question

Conclude that $k[X_1, \cdots, X_n]$ is factorial for any field k and any integer $n \geq 1$.