Kernel

Definition

Let G, G' be group and $f\colon G \to G'$ be a homomorphism. The kernel of f is by definition

$$\ker(f) = \{x \in G\colon f(x) = e'\}.$$

That is to say, $\ker(f) = f^{-1}(e')$.

Let G, G' be group and $f\colon G \to G'$ be a homomorphism.

Proposition

$\ker(f)$ *is a subgroup.*

Proof.

We check the definition.

1. $f(e) = e'$, so $e \in H$.
2. If $h_1, h_2 \in H$, then $f(h_1 h_2) = f(h_1)f(h_2) = e'e' = e'$.
3. If $h \in H$, then $f(h^{-1}) = f(h)^{-1} = (e')^{-1} = e'$.

$\square$

Let G, G' be group and $f\colon G \to G'$ be a homomorphism.

Proposition

f is injective if and only if $\ker(f) = \{e\}$*.*

Proof.

$\Rightarrow$ is trivial. For the other direction, first observe that

$$\begin{aligned} & f(h_1) = f(h_2) \\ \Rightarrow\ & f(h_1 h_2^{-1}) = e' \\ \Rightarrow\ & h_1 h_2^{-1} = e \\ \Rightarrow\ & h_1 = h_2 \end{aligned}$$

$\square$