

distinguished class

Let \mathcal{C} be a class of field extensions. We say \mathcal{C} is distinguished if the following conditions are satisfied.

- ▶ Let $k \subset F \subset E$ be a tower of fields. The extension E/k is in \mathcal{C} if and only if both F/k and E/F is in \mathcal{C} .
- ▶ If E/k is in \mathcal{C} , and F/k is any extension of k , then EF/F is in \mathcal{C} .
- ▶ If F/k and E/k are in \mathcal{C} and F, E are subfields of a common field, then EF/k is in \mathcal{C} .

Proposition

The class of finite extensions is distinguished.

Proof

We check the three conditions in order. First, consider the tower $k \subset F \subset E$ of fields. Since

$$[E : k] = [E : F][F : k],$$

E/k is finite if and only if both E/F and F/k are finite.

Second, assume that E/k is finite and F/k is an arbitrary extension such that E and F are both contained in a same field. Then, we need to show that EF/F is finite. If E/k is generated by a_1, \dots, a_n , then they generated EF over F . Thus EF/F is finitely generated by algebraic elements, which implies that EF/F is finite.

Lastly, assume that E/k and F/k are extensions contained in some common field. If $E = k(\alpha_1, \dots, \alpha_n)$ and $F = k(\beta_1, \dots, \beta_m)$, then $EF = k(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$. In particular, EF is finitely generated by algebraic elements. We conclude that EF/k is finite.

This completes the proof.

Question

Show that the class of algebraic extensions is distinguished.