Univariate polynomial ring over a field

Theorem

Let k be a field. $k[X]$ is principal. In particular, it is factorial.

Proof

Let $I \subset k[X]$ be an ideal. The zero ideal is principal so we may assume that $I \neq 0$. Choose

$$f \in I$$

such that $\deg f = \min_{g \in I}\{\deg g > 0\}$. For any other $g \in I$, one has

$$g = fq + r$$

with $\deg r < \deg f$. On the other hand, $r = g - fq$ so $r \in I$. This means $r = 0$. This shows that $I = (f)$.

Generally, a Euclidean function on a domain A is a function

$$\phi\colon A - \{0\} \to \mathbb{Z}_{\geq 0}$$

such that the division algorithm works. For this it suffices to require: for $f, g \in A$, $f \neq 0$ one has

$$g = fq + r$$

for some $q, r \in A$ with $\phi(r) < \phi(f)$. We adopt the convention that $\phi(0) = -\infty$.

Question

Repeating the proof, show that a domain with Euclidean function is principal.