primary decomposition

Let A be a principal entire ring, and M be a torsion module over A. For each prime ideal $\mathfrak{p} \subset A$, choose a generator p.

$$M(p) = \{m \in M \colon p^n m = 0 \text{ for some } n \geq 0\}$$

We want to show:

Theorem
There is a decomposition

$$M \simeq \bigoplus_p M(p).$$

strategy of the proof

We will use the Chinese remainder theorem. Any ideal $I \subset A$ is uniquely factored as

$$I = (p_1)^{s_1} \cdot \cdots \cdot (p_n)^{s_n}$$

for distinct primes $p_1, \cdots p_n$ and positive integers $s_1, \cdots, s_n$. Furthremore, one has

$$A/I \simeq \bigoplus_{i=1}^{n} A/p_i^{s_i}.$$

Allowing zero as exponents, one has

$$A/I = \bigoplus_{p} A/(p)^{s_p(I)}$$

with $s_p(I) \geq 0$.

proof - rewritting M

For any A-module M, we have

$$M = \operatorname{Hom}_A(A, M).$$

Since M is torsion, one has

$$\operatorname{Hom}_A(A, M) = \varinjlim_I \operatorname{Hom}_A(A/I, M)$$

where the direct limit is taken over all non-zero ideals of I.

One can naturally identify

$$\operatorname{Hom}_A(A/I, M) \simeq \{m \in M \colon am = 0 \text{ for some } a \in I\}$$

by looking at the image of $1 \in A/I$.

proof - using Chinese remainder theorem

Once can show

$$\varinjlim_{I} \operatorname{Hom}_A(A/I, M) \simeq \varinjlim_{I} \operatorname{Hom}_A(\bigoplus_{p} A/p^{s_p(I)}, M)$$

$$\simeq \varinjlim_{I} \bigoplus_{p} \operatorname{Hom}_A(A/p^{s_p(I)}, M)$$

$$\simeq \bigoplus_{p} \lim_{n \to \infty} \operatorname{Hom}_A(A/p^n, M).$$

Here, one uses the Chienese remainder theorem and change the order of $\varinjlim$ and $\bigoplus$. Since $\lim_{n \to \infty} \operatorname{Hom}_A(A/p^n, M) \simeq M(p)$, we get the desired decomposition.

Show that one can change the order of $\varinjlim$ and $\bigoplus$ by comparing their universal properties.