Number of p-Sylow subgroups

Recall:-

Theorem

Let G be a finite group and p be a prime divisor of $|G|$. Let P be a fixed p-Sylow subgroup. Any p-subgroup H is contained in a conjugate of P.

and

Theorem

Two p-Sylow subgroups of G are conjugate to each other.

Lemma

Let G be a finite group and P be a p-Sylow subgroup. If P is normal in G, then G has a unique p-Sylow subgroup

Proof.

Direct from the previous two theorems. $\square$

Theorem

The number of p-Sylow subgroups of G is congruent to 1 *modulo p.*

Proof.

Let S be the set of conjugates of P, on which P acts by conjugation. Then, P is a fixed point. We want to show that this is the only fixed point. If $Q \in S$ is another fixed point with normailizer M, then $P \subset M$. However, M has unique p-Sylow subgroup by the previous lemma. Therefore $P = Q$.

If P has unique fixed point in S, then the assertion of the theorem follows from applying the orbit decomposition formula to the P-set S. □