# Localizations and factor rings

Let $A$ be a commutative ring. Let $S \subset A$ be a multiplicatively closed subset, and $I \subset A$ be an ideal. Using the homomorphism

$$\psi \colon A \to S^{-1}A$$

define the ideal

$$J = \psi(I)S^{-1}A.$$

One can show that $J$ consists of the fractions of the form $\frac{a}{s}$ where $a \in I$ and $s \in S$. Thus, we often write $J = S^{-1}I$.

On the other hand, let the image of $S$ under

$$A \to A/I$$

be $S_I$.

We would like to show

$$S_I^{-1}(A/I) \simeq \left(S^{-1}A\right) / \left(S^{-1}I\right)$$

as commutative rings.

Let ***Comm*** be the category of commutative rings. By Yoneda's lemma it suffices to compare the functors

$$\boldsymbol{Comm} \to \boldsymbol{Set}$$

represented by them.

## $S_I^{-1}(A/I)$

For any commutative ring $B$

$$\mathrm{Mor}_{\boldsymbol{Comm}}\left(S_I^{-1}(A/I), B\right) \tag{1}$$

consists of the set of homomorphisms $\phi_I \colon A/I \to B$ such that

$$\text{every element of } \phi_I(S_I) \text{ is invertible.} \tag{2}$$

On the other hand, $\mathrm{Mor}_{\boldsymbol{Comm}}(A/I, B)$ consists of the morphisms $\phi \colon A \to B$ such that

$$\phi(I) = 0. \tag{3}$$

We combine (2) and (3); (1) consists of the morphisms $\phi \colon A \to B$ such that $\phi(I) = 0$ and that $\phi(S)$ is invertible.

# $\left(S^{-1}A\right) / \left(S^{-1}I\right)$

For any commutative ring $B$

$$\mathrm{Mor}_{\boldsymbol{Comm}}\left(\left(S^{-1}A\right) / \left(S^{-1}I\right), B\right) \tag{4}$$

consists of the set of homomorphisms $\phi_S \colon S^{-1}A \to B$ such that

$$\phi_S(I_S) = 0. \tag{5}$$

On the other hand, $\mathrm{Mor}_{\boldsymbol{Comm}}(S^{-1}A, B)$ consists of the morphisms $\phi \colon A \to B$ such that

$$\text{every element of } \phi(S) \text{ is invertible.} \tag{6}$$

We combine (5) and (6); (4) consists of the morphisms $\phi \colon A \to B$ such that $\phi(I) = 0$ and that $\phi(S)$ is invertible.

# Conclusion

We have a natural bijection

$$\mathrm{Mor}_{\boldsymbol{Comm}}\left(S_I^{-1}\left(A/I\right), B\right) \simeq \mathrm{Mor}_{\boldsymbol{Comm}}\left(\left(S^{-1}A\right) / \left(S^{-1}I\right), B\right)$$

so Yoneda's lemma applies. We conclude that

$$S_I^{-1}\left(A/I\right) \simeq \left(S^{-1}A\right) / \left(S^{-1}I\right).$$

# Question

Can you describe the isomorphism provided by Yoneda's lemma?