# Principal, Noetherian, and factorial rings

## Proposition

*Let $A$ be a principal ring. Then, any ascending chain*

$$(a_1) \subset (a_2) \subset \cdots$$

*in $A$ stabilizes.*

## Proof.

The union $\bigcup_{k\geq 1}(a_k)$ is principal, so it of the form $(c)$. Then, $c \in (a_N)$ for some large $N$. Then, $(a_n) = (c)$ for all $n \geq N$. $\square$

## Definition

If any ascending chain of ideals stabilize in a commutative ring $A$, then we say $A$ is Noetherian.

The previous proposition says a principal ring is Noetherian.

## Proposition

*Let $A$ be a Noetherian ring. Then, any non-zero element can be factored into irreducibles.*

## Proof.

Suppose that a non-zero $a \in A$ is not factorizable. Then, $a = a_1 b_1$ with non-units $a_1, b_1$. At least one of $a_1$ or $b_1$ is not factorizable, otherwise $a$ is factorizable. We may assume that $a_1$ is not factorizable. Repeat the process to get $a_n = a_{n+1} b_{n+1}$ with non-units $a_{n+1}$ and $b_{n+1}$ for all $n \geq 1$. The chain

$$(a_1) \subset (a_2) \subset \cdots$$

does not stabilize, a contradiction. $\square$

## Proposition

*Let $A$ be a principal ring. Factorization of a non-zero element is unique.*

## Proof.

It suffices to show that if $p$ is an irreducible element, then $p \mid ab$ implies $p \mid a$ or $p \mid b$. Indeed, if there are two factorizations $a = \prod p_i = \prod q_j$ of a non-zero element $a$, then $p_1$ must divide some $q_j$. Then, one can proceed by induction.

In remains to show the claim: $p \mid ab \Rightarrow p \mid a$ or $p \mid b$. Suppose $p \nmid a$. Under the assumption that $A$ is principal, irreducibility of $p$ implies the maximaility of $(p)$. This yields $(p) + (a) = A$. This means

$$ax + py = 1$$

has a solution with $x, y \in A$. Then, multiplying $b$, we get $px + pby = b$, showing that $p \mid b$.

□

# Question

Find a factorization of 3 in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Is it unique?