

two-sided ideals and
homomorphisms

Let A, B be rings. A homomorphism from A to B is a map

$$\phi: A \rightarrow B$$

such that they are monoid homomorphisms in two ways; for all $x, y \in A$,

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y), \\ \phi(0) &= 0\end{aligned}$$

and

$$\begin{aligned}\phi(xy) &= \phi(x)\phi(y), \\ \phi(1) &= 1.\end{aligned}$$

Let A be a ring. Given a two-sided ideal $\mathfrak{a} \subset A$, the set

$$A/\mathfrak{a}$$

of additive cosets of \mathfrak{a} is again a ring. It is called a factor ring. It is equipped with a natural map $A \rightarrow A/\mathfrak{a}$.

If B is another ring and $\phi: A \rightarrow B$ is a ring homomorphism, then its kernel

$$\ker(\phi) := \{a \in A: \phi(a) = 0\}$$

is a two-sided ideal. We have

$$\ker(A \rightarrow A/\mathfrak{a}) = \mathfrak{a}.$$

So kernels of ring homomorphisms are precisely the two-sided ideals.

Universal property

If $\phi: A \rightarrow A'$ is a ring homomorphism with $\phi(\mathfrak{a}) = 0$, then there exists a unique $\tilde{\phi}$ making the diagram below commutative:-

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ & \searrow & \nearrow \pi \\ & A/\mathfrak{a} & \end{array}$$

The diagram shows a commutative square. The top horizontal arrow is labeled ϕ . The bottom horizontal arrow is labeled π . The left vertical arrow points from A down to A/\mathfrak{a} . The right vertical dashed arrow is labeled $\tilde{\phi}$.

Can you interpret it as a universal property?

Question

Can you interpret \mathbb{C} as a factor ring of $\mathbb{R}[x]$?