Maximal ideals

Definition

Let A be a ring. A two-sided ideal $\mathfrak{m} \subset A$ is maximal if there is no ideal of A which properly contains $\mathfrak{m}$.

Proposition

In a commutative ring, a maximal ideal is prime.

Proof.

Let $\mathfrak{m} \subset A$ be a maximal ideal. By properness, $\mathfrak{m} \neq A$, so $1 \in A - \mathfrak{m}$. For $a, b \in A - \mathfrak{m}$, we need to show that $ab \notin \mathfrak{m}$. If $ab \in \mathfrak{m}$, then, $I = aA/\mathfrak{m} \subset A/\mathfrak{m}$ is a non-zero proper ideal. The inverse image of I along the natural map

$$A \to A/\mathfrak{m}$$

say J, is another proper ideal that properly contains $\mathfrak{m}$. This contradicts the maximality of $\mathfrak{m}$. $\square$

Question

Let k be a field, $n \geq 2$, and $A = M_n(k)$ be the matrix algebra. Is zero ideal maximal as a two-sided one? Using this, show that

$$pM_n(\mathbb{Z}) \subset M_n(\mathbb{Z})$$

is maximal two-sided ideal which is not prime.