

Hilbert's theorem

Recall that a commutative ring is Noetherian if any ascending chain of ideals stabilize. Equivalently, a commutative ring is Noetherian if any ideal is finitely generated.

Theorem

Let A be a commutative Noetherian ring. Then, $A[X]$ is Noetherian.

To prove the theorem, it suffices to show that any ideal $\mathfrak{A} \subset A[X]$ is finitely generated.

Lemma

In the proof, we will use the following fact. Let M be any finitely generated A -module.

Any A -submodule of M is finitely generated.

An auxiliary module

Define $\mathfrak{a} \subset A$ by the condition that $a \in \mathfrak{a}$ if and only if a is the leading coefficient of some $f \in \mathfrak{A}$. Then, \mathfrak{a} is an ideal. By assumption \mathfrak{a} is finitely generated. Let $a_1, a_2, \dots, a_n \in \mathfrak{a}$ be a generating set. Then, there exists $f_i \in \mathfrak{A}$ whose leading coefficient is a_i . Choose some large N so that $\deg f_i < N$ for all i . Consider the finitely generated A -module

$$A[X]/X^N$$

and the image

$$\mathfrak{A} \twoheadrightarrow \mathfrak{A}_N \subset A[X]/(X^N)$$

is an A -submodule. Then, by the lemma, \mathfrak{A}_N is finitely generated as an \mathfrak{A} -module. Pick a generating set $\bar{g}_1, \dots, \bar{g}_m \in \mathfrak{A}_N$ as an A -module, where $g_1, \dots, g_m \in \mathfrak{A}$.

We claim that the ideal I generated by $f_1, \dots, f_n, g_1, \dots, g_m$ is equal to \mathfrak{A} .

Proof

Suppose, on the contrary, that $\mathfrak{A} - I$ is non-empty. Pick an element with minimal degree, say h . By our construction,

$$h' = h - \sum c_i g_i \in (X^N)$$

for some $c_i \in A$. The leading coefficient of h' , say b is in \mathfrak{a} , so our construction implies that

$$b = \sum d_i a_i$$

for some $d_i \in A$. Then,

$$h'' = h' - \sum d_i \cdot f_i \cdot X^{N - \deg f_i} \in \mathfrak{A}$$

On the other hand, $h'' \notin I$ because in each step we have subtracted an element from I . Thus we have obtained an element $h'' \in \mathfrak{A} - I$ with $\deg h'' < \deg h$, a contradiction.

Question

Deduce that $k[X_1, \dots, X_n]$ is Noetherian for any field k and any $N \geq 1$.