Monoids

Definition

A monoid is a set S equipped with a map

$$S \times S \to S$$
$$(x, y) \mapsto xy$$

satisfying the following conditions.

1. There is an $e \in S$ such that $ex = x$ and $xe = x$ holds for all $x \in S$.
2. For every triple $x, y, z, \in S$, we have $(xy)z = x(yz)$.

The identity element is often written as 1.

Commutative monoids

Let S be a monoid. If

$$xy = yx$$

for all $x, y \in S$, then we say S is commutative, or abelian. In that case, we often adopt the additive notation

$$S \times S \to S$$
$$(x, y) \mapsto x + y$$

instead of $(x, y) \mapsto xy$.

A convention

The product over the empty set is the identity.