splitting fields

Let k be a field and $f \in k[X]$ be a non-constant polynomial.

Definition

A splitting field L of f means a field extension L/k such that f splits into linear factors in $L[X]$, and that L is generated by the roots of f over k.

Theorem

Let L be a splitting field of a non-constant polynomial f. If E is another splitting field of f, then there exists an isomorphism $E \simeq L$. Moreover, if k^{a} is an algebraic closure of k and $k \subset L \subset k^{\mathrm{a}}$, then any embedding

$$E \to k^{\mathrm{a}}$$

over k induces an isomorhpism $E \simeq L$.

proof

Suppose that E/k is another splitting field. Let L^{a} be an algebraic closure of L. Then, it is an algebraic closure of k as well. Then, there exists an embedding

$$\sigma \colon E \to L^{\mathrm{a}}$$

over k. If $\beta_1, \cdots, \beta_r$ are the roots of f in E with multiplicity, then, $\sigma(\beta_1), \cdots, \sigma(\beta_r)$ are the roots of f in L^{a} with multiplicity. It follows that $\sigma(E)$ is generated by the roots of f. Since L is a splitting field, $\sigma(E) = L$ and σ iduces $E \simeq L$.

More generally, a splitting field for a family $(f_i)_{i \in I}$ of polynomials is an extension L/k such that each f_i splits into linear factors in L and that L is generated by all the roots of f_i. Here, the set I is allowed to be infinite.

Question

Show that the theorem holds for splitting fields for a family $(f_i)_{i \in I}$ of polynomials. Namely, if E, K are two splitting fields for the same family, then any embedding $E \to K^{\mathrm{a}}$ over k induces an isomorphism $E \simeq K$.