Compositum and embeddings

Let E_1, E_2 be fields contained in some large field E. Let σ be an embedding of E into another field L.

Lemma

We have $\sigma(E_1E_2) = \sigma(E_1)\sigma(E_2)$.

To prove the lemma we introduce an explicit generating set for the compositum of two fields F and E contained in some large field. The field generated by fractions of the form

$$\frac{a_1 b_1 + \cdots + a_m b_m}{a'_1 b'_1 + \cdots + a'_n b'_n}$$

with $a_i, a'_j \in E$, $b_i, b'_j \in F$ is EF.

Proof

We show that $\sigma(E_1)\sigma(E_2)$ and $\sigma(E_1E_2)$ are generated by the same set. $\sigma(E_1)\sigma(E_2)$ is generated by elements of the form

$$\frac{a_1^\sigma b_1^\sigma + \cdots + a_m^\sigma b_m^\sigma}{a_1'^\sigma b_1'^\sigma + \cdots + a_n'^\sigma b_n'^\sigma}$$

with $a_i, a_j' \in E$, $b_i, b_j' \in F$. This generator is equal to

$$\sigma\left(\frac{a_1 b_1 + \cdots + a_m b_m}{a_1' b_1' + \cdots + a_n' b_n'}\right)$$

so it is in the generating set of $\sigma(E_1E_2)$.

Question

Let p be a prime and r, s be positive integers. Let F, E be finite fields with p^r and p^s elements, respectively. Find some large field containing both F and E and the degree of their compositum.