

Irreducible elements in $A[X]$

Let A be a factorial ring, and K its quotient field. We want a classification of irreducible elements in $A[X]$ in terms of those in $K[X]$ and A .

irreducible elements in A

Lemma

If $a \in A$ is irreducible, then it remains irreducible in $A[X]$.

Proof.

If $a = fg$ is a factorization in $A[X]$, then we necessarily have $f, g \in A$.
So either f or g is a unit. □

irreducible elements in $K[X]$ with content one

Lemma

If $f \in A[X]$ is irreducible and is of degree at least one, then it is irreducible in $K[X]$ and has content one.

Proof.

Suppose $f = gh$ in $K[X]$. Then, $\text{cont}(f) = \text{cont}(g)\text{cont}(h)$. Rewrite $f = gh$ as

$$\frac{f}{\text{cont}(f)} = \frac{g}{\text{cont}(g)} \frac{h}{\text{cont}(h)}$$

which is a factorization of $\frac{f}{\text{cont}(f)}$ in $A[X]$. By assumption, either $\frac{g}{\text{cont}(g)}$ or $\frac{h}{\text{cont}(h)}$ is a unit in $A[X]$. Since $A[X]^\times = A^\times$, either $g \in K^\times$ or $f \in K^\times$.

On the other hand, $\text{cont}(f) \neq 1$ implies that some prime p divides f which contradicts the assumption. This completes the proof. □

Question

Show that the irreducible elements in $A[X]$ are classified by those in two previous lemmas.