# internal direct sum

Direct sums can be characterized in terms of submodules. It is called the internal sum.

# definition

Let $M$ be a module over a ring $A$. Let $I$ be a set. Suppose that there is a family of submodules

$$M_j \subset M$$

parametrized by $j \in I$ such that

$$M_j \cap \sum_{i \in I - \{j\}} M_i = 0$$

for every $j \in I$. Moreover, assume

$$\sum_{j \in I} M_j = M.$$

In this case, we say $M$ is the internal direct sum of $M_j$'s.

If $M$ is the internal direct sum of $M_j$'s, then one has a unique morphism

$$\bigoplus M_j \to M$$

by the universal property. We claim that this is an isomorphism. It suffices to show the surjectivity and the injectivity of the map.

The surjectivity follows from $\sum M_j = M$.

The injectivity follows from

$$M_j \cap \sum_{i \in I - \{j\}} M_i = 0$$

for every $j \in I$.

Since direct sum

$$\bigoplus_{j \in I} M_j$$

is defined up to a unique isomorphism, one can say: if $M$ is the internal direct sum of $M_j$'s, then $M$ is the (categorical) direct sum of $M_j$'s.

# Question

Let $p, q$ be two distinct primes. Describe $\mathbb{Z}/pq\mathbb{Z}$ as an internal direct sum of two non-zero proper submodules. What is the correspondonding description as a categorical direct sum?