Free modules over principal rings

Let A be a principal entire ring. Recall the notion of rank.

If F is a free module over A, then there a set I and a basis $x_i \in F$ for each $i \in I$, such that one has a direct sum decomposition

$$\bigoplus_{i \in I} Ax_i \simeq F.$$

The cardinality of I is called the rank of F, or the dimension of F.

Theorem

Suppose that F is a free module of finite rank over A, and $M \subset F$ is a submodule. Then, M is also free and its rank is at most the rank of F.

strategy of the proof

Let n be the rank of F, and take a basis $x_1, \cdots, x_n$ of F. Define, for each $r = 1, \cdots, n$,

$$M_r = M \cap \bigoplus_{i=1}^{r} Ax_i.$$

We adopt the convention that $M_0 = 0$. Note that $M_n = M$. We will inductively show that each M_r is free.

induction step

Suppose that M_r is free. We want to show that M_{r+1} is free. By definition, an element $m \in M_{r+1}$ can be uniquely written in the form

$$m = b_1x_1 + \cdots + b_rx_r + a_m x_{r+1}$$

with $b_1, \cdots, b_r, a_m \in A$. Then, $\{a_m \colon m \in M_{r+1}\}$ is an ideal, say $\mathfrak{a}$. By assumption, $\mathfrak{a} = (a)$ for some $a \in A$. Choose $w \in M_{r+1}$ with $a_w = a$. Then, we have

$$M_{r+1} = M_r + Aw.$$

Indeed, for any $m \in M_{r+1}$, $a_m = b_{r+1}a$ for some $b_{r+1} \in A$, and $m - b_{r+1}w \in M_r$.

On the other hand, $M_r \cap Aw = 0$. So $M_{r+1} \simeq M_r \oplus Aw$. Since Aw is either zero or free of rank one, we are done.

initiation

For $r = 0$, $M_0 = 0$ if free.

Question

For principal rings which are not entire, find a counter-example to the theorem. Such rings include $\mathbb{Z}/p^r\mathbb{Z}$ with a prime p and an integer $r \geq 2$, and a ring of the form $k[t]/(t^n)$ with $n \geq 2$.