Hilbert's theorem

Recall that a commutative ring is Noetherian if any ascending chain of ideals stabilize. Equivalently, a commutative ring is Noetherian if any ideal is finitely generated.

Theorem

Let A be a commutative Noethering ring. Then, $A[X]$ is Noetherian.

To prove the theorem, it suffices to show that any ideal $\mathfrak{A} \subset A[X]$ is finitely generated.

Lemma

In the proof, we will use the following fact. Let M be any finitely generated A-module.

Any A-submodule of M is finitely generated.

An auxiliary module

Define $\mathfrak{a} \subset A$ by the condition that $a \in \mathfrak{a}$ if and only if a is the leading coefficient of some $f \in \mathfrak{A}$. Then, $\mathfrak{a}$ is an ideal. By assumption $\mathfrak{a}$ is finitely generated. Let $a_1, a_2, \cdots, a_n \in \mathfrak{a}$ be a generating set. Then, there exists $f_i \in \mathfrak{A}$ whose leading coefficient is a_i. Choose some large N so that $\deg f_i < N$ for all i. Consider the finitely generated A-module

$$A[X]_{<N} := A + AX + AX^2 + \cdots + AX^{N-1}.$$

The intersection

$$\mathfrak{A}_{<N} := A[X]_{<N} \cap \mathfrak{A}$$

is an A-submodule of $A[X]_{<N}$. Then, by the lemma, $\mathfrak{A}_{<N}$ is finitely generated as an A-module. Pick a generating set $g_1, \cdots, g_m \in \mathfrak{A}_{<N}$. We claim that the ideal I generated by $f_1, \cdots, f_n, g_1, \cdots, g_m$ is equal to $\mathfrak{A}$.

Proof - outline

To show $I = \mathfrak{A}$, we will show two claims.

1. If $h \in \mathfrak{A}$ and $\deg h < N$, then $h \in I$.
2. If $v \in \mathfrak{A}$ and $\deg v \geq N$, then there exists $v' \in \mathfrak{A}$ such that $v' - v \in I$.

Once we show these two, the desired equality $\mathfrak{A} = I$ follows by induction on the degree.

Proof - the first claim

Suppose that $h \in \mathfrak{A}$ and $\deg h < N$. By our choice of g_i's we have

$$h = \sum_{i=1}^{m} c_i g_i$$

with $c_i \in A$. This means that $h \in I$.

Proof - the second claim

Let $v \in \mathfrak{A}$ and $\deg v \geq N$. Let b be the leading coefficient of v. By our choice of a_i's, we have

$$b = \sum d_i a_i$$

for some $d_i \in A$. Then,

$$v' = v - \sum d_i \cdot f_i \cdot X^{\deg v - \deg f_i}$$

satisfies $\deg v' < \deg v$ and $v' - v \in I$. The second claim is proved.