Extensions with roots of polynomials

Theorem

If k is a field and $f_1, \cdots, f_n \in k[X]$ are polynomials of degree at least one, then there exists an extension E/k in which each f_i has a root, $i = 1, 2, \cdots, n$.

We may assume that each f_i is irreducible. Moreover, by induction, it suffices to show the theorem form $n = 1$.

proof when $n = 1$ and f_1 is irreducible

Assume $n = 1$ and $f = f_1$ is irreducible. Let $F = k[X]/f(X)$. There is a natural injective homomorphism

$$\sigma\colon k \hookrightarrow F$$

between fields.

Strictly speaking, this is not an extension since F does not contain k. To solve this problem, consider $E = k \coprod (F - \sigma(k))$. Then, there is a bijection $\tau\colon E \simeq F$ such that

$$\tau|_k = \sigma$$

$$\tau|_{F-\sigma(k)} = \mathrm{id}_{F-\sigma(k)}.$$

Using τ, turn F into a field that contains k.

Question

Let $F = \mathbb{Q}[T]/(T^3 - 3)$. Find the factorization of $X^3 - 3$ in $F[X]$.