Existence of a maximal ideal

Proposition

Any non-zero ring A has a maximal ideal.

Proof.

Proper ideals are those without 1. Therefore, the union of a totally ordered family of proper ideals is again proper. On the other hand, the zero ideal is proper if the ring is non-zero. From this, one can apply Zorn's lemma. □

Remark

Sometimes, one uses the notion of a ring without unity. It is also known as 'rng'. Such an object may not have a maximal ideal.

Question

Can you find a maximal ideal of $\mathbb{R}[x]$? Can you classify them?