Sign

The sign map

Theorem

For every $n \geq 1$, there is a homomorphism

$$\epsilon\colon S_n \to \{\pm 1\}$$

which maps every transposition to -1.

To prove the theorem, we introduce some notation. For $\sigma \in S_n$, define $\theta(\sigma)$ to be the cardinality of the set

$$\{1 \leq a \leq n\colon \sigma(a) > a\}.$$

Define $\epsilon(\sigma) := (-1)^{\theta(\sigma)}$. We will show that ϵ is the desired homomorphism.

Proof

Recall that any given permutation σ is a product of transpositions. If

$$\sigma = \tau_1 \cdots \tau_r = \psi_1 \cdots \psi_s$$

where τ's and ψ's are permutations, then one can show $r - s$ is even by showing $\epsilon(\sigma) = (-1)^r = (-1)^s$. This shows that $\epsilon(\sigma) = -1$ when σ is a transposition. It also shows that σ is a homomorphism.

Definition

A permutation σ is called even if $\epsilon(\sigma) = 1$. Otherwise, it is called odd. The alternating group on n-letters, denoted by A_n, is defined to be the collection of all even permutations in S_n. It is a normal subgroup of index two.

Question

Can you read the sign of a permutation from its cycle decomposition?