Localization and primes

For a homomorphism

$$\psi\colon A \to B$$

between commutative rings, one has the associated map between sets of prime ideals. If $\mathfrak{q} \subset B$ is a prime ideal, then $\mathfrak{p} = \psi^{-1}(\mathfrak{q})$ is also a prime ideal.

Let $S \subset A$ be a multiplicative subset and B be of the form $S^{-1}A$.

Proposition

A prime ideal $\mathfrak{p} \subset A$ is of the form $\psi^{-1}(\mathfrak{q})$ if and only if $\mathfrak{p} \cap S = \emptyset$. Moreover, if $\mathfrak{p} \cap S = \emptyset$, one can choose $\mathfrak{q}$ to be $\psi(\mathfrak{p})S^{-1}A$.

Let $\mathfrak{p} \cap S = \emptyset$ and take $\mathfrak{q} = \psi(\mathfrak{p})S^{-1}A$. It suffices to show that there is an injective homomorphism

$$A/\mathfrak{p} \hookrightarrow S^{-1}A/\mathfrak{q}.$$

The ring $S^{-1}A/\mathfrak{q}$ is isomorphic to

$$S_{\mathfrak{p}}^{-1}(A/\mathfrak{p})$$

where $S_{\mathfrak{p}}$ is the image of S under $A \to A/\mathfrak{p}$. Since $\mathfrak{p} \cap S = \emptyset$, and $A/\mathfrak{p}$ is an integral domain, the map

$$A/\mathfrak{p} \to S_{\mathfrak{p}}^{-1}(A/\mathfrak{p})$$

is injective.

Question

Let $N \geq 2$ be in integer and $S = \{1, N, N^2, \cdots\}$. Describe the set of prime ideals of $S^{-1}\mathbb{Z}$ as a subset of the set of prime ideals of $\mathbb{Z}$.