Embeddings of fields

Let E/F be an extension of fields. If

$$\sigma\colon F \hookrightarrow L$$

is an embedding, then denote by

$$F^{\sigma}$$

the image of F in L. Sometimes we use σF instead of F^{σ}. Then, an embedding

$$\tau\colon E \to L$$

is over σ if $\tau|_F = \sigma$. We also say that τ extends σ.
If σ is identity on F, then we simply say that τ is an embedding over F.

Lemma

Let E/k be an algebraic extension. If $\sigma\colon E \to E$ is an embedding of E into itself over k. Then, σ is an automorphism.

Proof

Since any homomorphism between fields is injective, it suffices to show that σ is surjective. Let $\alpha \in E$ be an arbitrary element. We will show that α is in the image of σ.

Let $P(X)$ be the minimal polynomial of α over k. Consider $F = k(\alpha\colon \alpha \in E, P(\alpha) = 0)$, the field generated by all roots of $P(X)$ in E. Note that $P(\alpha) = 0$ impies that $P(\sigma(\alpha)) = 0$ because σ is identity on k. This means that $F^\sigma \subset F$. Note that $P(X)$ has a finite number of roots in E, so F/k is finite. Since $F \to F^\sigma$ is an injective k-linear map between finite dimensional vectors spaces of an equal dimension, $F \simeq F^\sigma$ and α is in the image of σ.

Question

Find a counter-example to the lemma when E/k is not algebraic.