

Connected sum of surfaces

Suppose that X, Y are two connected topological surfaces, compact without boundary. Then, the connected sum

$$X \# Y$$

is a new manifold obtained by “gluing” X and Y .

Choose a point $x \in X$ and a small closed disk D_x around it. Remove the interior of D_x from X . Do the same to a chosen point of $y \in Y$. Then, glue the two ‘punctured’ manifolds along the circles ∂D_x and ∂D_y .

The above operation turns the homeomorphism classes of such surfaces into a commutative monoid. Its element can be uniquely written in the form $n\tau + m\pi$, $n \geq 0$, $0 \leq m \leq 3$, where τ is the class of a torus and π is the class of a projective plane. They satisfy $3\pi = \tau + \pi$.