Maximality of p-Sylow subgroups

Maximality

Theorem

Let G be a non-trivial finite group and p be a prime divisor of $|G|$. Let P be a fixed p-Sylow subgroup. Any p-subgroup H of G is contained in a conjugate of P.

Proof

Let N be the normalizer of P in G; the largest subgroup which contains P as a normal subgroup. On the other hand, let S be the set of all conjugates of P. By definition $P \in S$ and G acts transitively on S. The stabilizer of P is equal to N. Now, the orbit-stabilizer formula implies that $(G : N) = |S|$.

Now we let H act on S. To show the theorem, it suffices to show that H has a fixed point. Indeed, if Q is a fixed point, then H is contained in the normalizer of Q, say M. The map

$$H \to M/Q$$

is trivial since $|H|$ is relatively prime to $|M/Q|$, whence $H \subset Q$.

It remains to show that H has a fixed point in S. This follows from the orbit decomposition formula, since H is a p-group and $p \nmid |S|$.