splitting

splitting a surjection

Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a short exact sequence. We say the sequence splits if there exists

$$\theta \colon M'' \to M$$

such that $g \circ \theta = 1_{M''}$.

In that case, one has

$$M \simeq \mathrm{im}(f) \oplus \mathrm{im}(\theta).$$

Equivalently, one has

$$M \simeq \ker(g) \oplus \mathrm{im}(\theta).$$

splitting an injection

Given the same exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

assume that there exists

$$\tau \colon M \to M'$$

such that $\tau \circ f = 1_{M'}$.

In that case, one has

$$M \simeq \ker(\tau) \oplus \operatorname{im}(f)$$

or, equivalently,

$$M \simeq \ker(\tau) \oplus \ker(g).$$

relations between splittings

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

Given some splitting θ with $g \circ \theta = 1_{M''}$, one obtains an isomorphism

$$M \simeq \mathrm{im}(f) \oplus \mathrm{im}(\theta).$$

From this, one obtains τ as the projection

$$M \simeq \mathrm{im}(f) \oplus \mathrm{im}(\theta) \to \mathrm{im}(f) \simeq M'.$$

Conversely, one can obtain a splitting θ from some τ.

interpretation of split exact sequences

One can talk about maps between short exact sequences. If a sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is split, then there is a diagram

$$\begin{array}{ccccccccc}
0 & \to & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M' & \xrightarrow{(1,0)} & M' \oplus M'' & \xrightarrow{\pi_2} & M'' & \to & 0
\end{array}$$

where vertical maps are isomorphisms and all squares are commutative.

Question

Is the sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

split?