primary parts

Let A be a principal entire ring and $p \in A$ be a prime. Suppose that M is a finitely generated module over A such that $M = M(p)$. We want to show:

Theorem

There exist positive integers $s_1, \cdots, s_n$ such that $M = \bigoplus_{i=1}^{n} A/(p^{s_i})$.

In other words, M is a direct sum of cyclic A-modules; a module is called cyclic if it can be generated by one element.

strategy of proof

Let $s \geq 0$ be the least integer such that $p^s M = 0$. Then, there exists some $x \in M$ such that $p^{s-1}x \neq 0$ but $p^s x = 0$. Consider the short exact sequence

$$0 \to Ax \to M \to M/Ax \to 0$$

Let $\pi \colon M \to M/Ax$ be the projection map. We claim that any element $y \in M/Ax$ can be lifted to $\tilde{y} \in M$, namely $\pi(\tilde{y}) = y$, so that π induces an isomorphism $A\tilde{y} \simeq Ay$.

Once this is shown, one proceeds by induction. For modules generated by one element, there is nothing to show. For general M, pick a direct sum decomposition

$$M/Ax \simeq \bigoplus Ay_i$$

and deduce

$$M \simeq \left(\bigoplus A\tilde{y}_i\right) \oplus Ax.$$

proof of claim

$$0 \to Ax \to M \to M/Ax \to 0$$

Let $y' \in M$ be an arbitrary lift of $y \in M/Ax$. If $r \geq 0$ is the least integer with $p^r y = 0$, then $p^r y' = p^t cx$ for some $p \nmid c$. On the other hand, $s - t \leq s - r$ by our choice of x and s. In other words, $r \leq t$.

There are two cases; $t \geq s$ and $t < s$. If $t \geq s$, then $p^t cx = 0$ and y' satisfies $A\tilde{y} \simeq Ay$. If $t < s$, we try to find another lift

$$\tilde{y} = y' + wx$$

with $w \in A$ so that $p^r \tilde{y} = 0$. To solve for w, consider

$$p^r \tilde{y} = p^r y' + p^r wx = p^t cx + p^r wx = (p^t c + p^r w)x.$$

By choosing $w = -p^{t-r} c$, we obtain $p^r \tilde{y} = 0$ as desired.

uniqueness

Without loss of generality, one can write the decomposition uniquely as

$$M \simeq \bigoplus_{i=1}^{n} A/(p^{s_i})$$

with $s_1 \leq s_2 \leq \cdots \leq s_n$.

Question

Decompose $\mathbb{Z}/(120) \oplus \mathbb{Z}/(45)$ as

$$\bigoplus_{i=1}^{n} \mathbb{Z}/(N_i)$$

with $N_1 \mid N_2 \mid \cdots$.