Hilbert's theorem

Recall that a commutative ring is Noetherian if any ascending chain of ideals stabilize. Equivalently, a commutative ring is Noetherian if any ideal is finitely generated.

Theorem

Let A be a commutative Noethering ring. Then, $A[X]$ is Noetherian.

To prove the theorem, it suffices to show that any ideal $\mathfrak{A} \subset A[X]$ is finitely generated.

Lemma

In the proof, we will use the following fact. Let M be any finitely generated A-module.

Any A-submodule of M is finitely generated.

An auxiliary module

Define $\mathfrak{a} \subset A$ by the condition that $a \in \mathfrak{a}$ if and only if a is the leading coefficient of some $f \in \mathfrak{A}$. Then, $\mathfrak{a}$ is an ideal. By assumption $\mathfrak{a}$ is finitely generated. Let $a_1, a_2, \cdots, a_n \in \mathfrak{a}$ be a generating set. Then, there exists $f_i \in \mathfrak{A}$ whose leading coefficient is a_i. Choose some large N so that $\deg f_i < N$ for all i. Consider the finitely generated A-module

$$A[X]/(X^N)$$

and the image

$$\mathfrak{A} \twoheadrightarrow \mathfrak{A}_N \subset A[X]/(X^N)$$

is an A-submodule. Then, by the lemma, $\mathfrak{A}_N$ is finitely generated as an $\mathfrak{A}$-module. Pick a generating set $\bar{g}_1, \cdots, \bar{g}_m \in \mathfrak{A}_N$ as an A-module, where $g_1, \cdots, g_m \in \mathfrak{A}$.
We claim that the ideal I generated by $f_1, \cdots, f_n, g_1, \cdots, g_m$ is equal to $\mathfrak{A}$.

Proof - step I

Let $h \in \mathfrak{A}$. Since $\bar{g}_1, \cdots, \bar{g}_m$ generate $\mathfrak{A}_N$ as an A-module, we can express

$$h = \sum_{i=1}^{m} c_i g_i + u(X)X^N$$

for some $c_i \in A$ and $u(X) \in A[X]$. In order to show $h \in I$, it suffices to show that

$$u(X)X^N \in I.$$

If $u(X)X^N = 0$, then we are done. If not, $u(X)X^N \in \mathfrak{A}$ and its degree is at least N. So it suffices to show that

$$v \in \mathfrak{A}, \deg(v) \geq N \Rightarrow v \in I.$$

Proof - step II

We would like to show

$$v \in \mathfrak{A}, \deg(v) \geq N \Rightarrow v \in I$$

by contradiction. Assume that there exists $v \in \mathfrak{A} - I$ with $\deg(v) \geq N$. Choose such a v with minimal degree. Let b be the leading coefficient of v. By our choice of a_i's, we have

$$b = \sum d_i a_i$$

for some $d_i \in A$. Then,

$$v' = v - \sum d_i \cdot f_i \cdot X^{\deg v - \deg f_i} \in \mathfrak{A}$$

On the other hand, $v' \notin I$ because in each step we have substracted an element from I. Thus we have obtained an element $v' \in \mathfrak{A} - I$ with $\deg v' < \deg v$, a contradiction.

Question

Instead of $\mathfrak{A}_N$, one could take

$$\mathfrak{A}'_N := \{g \in \mathfrak{A} \colon \deg g \leq N\}.$$

Reproduce the proof with a generating set of $\mathfrak{A}'_N$ as an A-module.