Gauss' Lemma

Let A be a factorial ring, and K its quotient field. For a prime element p of A, define

$$\begin{aligned} \operatorname{ord}_p \colon K^\times &\longrightarrow \mathbb{Z} \\ a &\longmapsto \max\left\{n \colon ap^{-n} \in A\right\} \end{aligned}$$

and call $\operatorname{ord}_p(a)$ the order of a at p.

Content

For a polynomial of degree $n \geq 0$

$$f = a_0 + a_1 X + \cdots + a_n X^n$$

define

$$\mathrm{ord}_p(f) = \min_{0 \leq i \leq n} \mathrm{ord}_p a_i$$

and define the content of f to be

$$\mathrm{cont}(f) = \prod_p p^{\mathrm{ord_p(f)}}$$

where the product is taken over all prime elements up to multiplication by unit.

By convention, $\mathrm{ord}_p(0) = \infty$.

Gauss' Lemma

The following identity is due to Gauss. For non-zero polynomials $f, g \in K[X]$, we have

$$\mathrm{cont}(f)\mathrm{cont}(g) = \mathrm{cont}(fg).$$

To prove it, it suffices to consider the case where

$$\mathrm{cont}(f) = \mathrm{cont}(g) = 1.$$

Moreover, for each prime p, it suffices to show that

$$\mathrm{ord}_p(f) = \mathrm{ord}_p(g) = 0 \quad \Rightarrow \quad \mathrm{ord}_p(fg) = 0.$$

Proof

For any $h \in A[X]$, let $\bar{h} \in A/(p)[X]$ be the natural image of h.
Then, $\bar{h} = 0$ if and only if $\mathrm{ord}_p(h) \geq 1$.
We give a proof of Gauss' lemma. If

$$\mathrm{ord}_p(f) = \mathrm{ord}_p(g) = 0,$$

then

$$\bar{f} \neq 0, \bar{g} \neq 0.$$

This in turn implies that

$$\bar{f} \cdot \bar{g} \neq 0$$

because $A/(p)$ is entire whence $A/(p)[X]$ is entire, too.

Since $\bar{f} \cdot \bar{g} \neq 0$ and $\bar{f} \cdot \bar{g} = \bar{fg}$, we conclude that $\mathrm{ord}_p(fg) = 0$.

Question

Show that $A[X]$ is entire when A is entire.