Eisenstein's Criterion

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Let A be a factorial ring and $p \in A$ a prime. A polynomial in $A[X]$

$$f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0$$

is irreducible if

$$\begin{aligned} p &\nmid a_n \\ p &\mid a_{n-1} \\ p &\mid a_{n-2} \\ &\vdots \\ p &\mid a_1 \\ p^2 &\nmid a_{n-1} \end{aligned}$$

are satisfied.

Proof

Let $f = gh$ be a non-trivial factorization of f in $K[X]$. Then, we have $\deg(g), \deg(h) < n$. On the other hand, by the classification of irreducible elements in $A[X]$, we may rewrite it as

$$f = ag_1 h_1$$

with $a = \mathrm{cont}(gh)$, $g_1 = g/\mathrm{cont}(g)$, $h_1 = h/\mathrm{cont}(h)$. Since $p \nmid \mathrm{cont}(f)$, $p \nmid a$. Reducing modulo p, we get

$$\bar{f} = \bar{a}\bar{g}_1 \bar{h}_1$$

and $\bar{a} \neq 0$. Note that $\deg g, \deg h < n$ implies that

$$\deg \bar{g}_1, \deg \bar{g}_2 < n.$$

By assumption, $\bar{f} = \bar{a}_n X^n$ and $\bar{a}_n \neq 0$. It follows that the constant terms of g_1 and h_1 are both divisible by p. This contradicts $p^2 \nmid a_0$.

Question

Let p be a prime number. Show that

$$\frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Q}[X]$ using Eisenstein's criterion. You may want to consider the change of variable $x \mapsto x + 1$.