Localization

Let A be a commutative ring. Take a multiplicatively closed subset $S \subset A$. From this, one can form a ring of fractions $S^{-1}A$. Its element is represented by a pair (a, s) with $a \in A$ and $s \in S$, and two pairs (s_1, a_1) and (s_2, a_2) represent the same element if

$$s(s_1 a_2 - s_2 a_1) = 0 \quad \text{for some } s \in S. \tag{1}$$

One can check that (1) is an equivalence relation.

The equivalence class containing (s, a) is often written as a fraction

$$\frac{a}{s} \in S^{-1}A.$$

$S^{-1}A$ is a ring

The addition

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2}$$

and the multiplication

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$$

turns $S^{-1}A$ into a ring.

The ring $S^{-1}A$ recognizes A in terms of the map

$$\begin{aligned} i\colon A &\longrightarrow S^{-1}A \\ a &\longmapsto \frac{a}{1} \end{aligned}$$

which is a ring homomorphism.

Proposition

If S contains no zero-divisors of A, then i is injective.

Universal property

Suppose $f\colon A \to B$ is a ring homomorphism such that $f(S) \subset B^\times$, $B^\times \subset B$ being the group of invertible elements. Then, there exists a unique homomorphism

$$\tilde{f}\colon S^{-1}A \to B$$

such that $f = \tilde{f} \circ i$. Diagrammatically, we have:-

$$\begin{tikzcd}
 & S^{-1}A \arrow[dd, dashed, "\tilde{f}"] \\
A \arrow[ur, "i"] \arrow[dr, "f"'] & \\
 & B
\end{tikzcd}$$

Question

Consider $N = pq$, where p, q are distinct primes. Take two multiplicatively closed subsets $S_1 = \{1, N, N^2, \cdots\}$ and $S_2 = \{p^a q^b : a, b, \in \mathbb{Z}_{\geq 0}\}$ of $\mathbb{Z}$. Find an isomorphism $S_1^{-1}\mathbb{Z} \to S_2^{-1}\mathbb{Z}$.