Existence of p-Sylow subgroups

Let G be a finite abelian group of order $m > 1$. Let p be a fixed prime divisor of m.

Lemma

G has a subgroup of order p.

Proof.

We use induction on $|G|$. If G has prime order, then we are done. Suppose that $|G|$ is not a prime. In this case, there is a subgroup $H \subset G$ of prime order. If $|H| = p$, then we are done, so assume $|H|$ is relatively prime to p. Otherwise, since any subgroup of G is normal, one can form a smaller subgroup G/H to which the induction hypothesis can be applied. Let gH be an element of order p in G/H. Then, $g^{|H|}$ has order p and we are done. $\square$

Theorem

Let G be a finite group and p a prime factor of $|G|$. There is a p-Sylow subgroup of G

We will prove the theorem by contradiction. So we will assume the contrary, and take a counterexample G with minimal $|G|$. It suffices to show two things.

1. The center Z of G has order divisible by p.
2. If Z has order divisible by p, then we get a contradiction.

The center Z of G

Let us apply the orbit decomposition formula to G acting on itself by conjugation(the class formula). That is to say, choose a set C of representative for the conjugacy classes of G, and get

$$(G : 1) = \sum_{x \in C} (G : G_x).$$

We have $G = G_x$ precisely when x is in the center Z of G, so the above can be rewritten as

$$(G : 1) = (Z : 1) + \sum_{x \in C - Z} (G : G_x).$$

On the other hand, the minimality of G implies that each $(G : G_x)$ is a multiple of p when $x \in C - Z$. Reduing both sides modulo p, we get $p||Z|$.

Getting the contradiction

If Z has order divisible by p, there is a subgroup $H \subset Z$ of order p, by the lemma we proved earlier. Note that Z is normal and abelian in G, so H is also normal in G. The quotient group $K = G/H$ has a p-Sylow subgroup by the minimality of G, say $K_0 \subset K$. Let

$$f\colon G \to K$$

be the projection and let $G_0 := f^{-1}(K_0)$. Then, one can show that G_0 has order $p|G_0|$ and is a p-Sylow subgroup of G.