Completion

Definition

Let I be a directed system; it is a set with partial order $\leq$ such that for any pair $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. Consider a family G_i of groups indexed by I, together with maps

$$f_{ij} \colon G_j \to G_i$$

for any $i \leq j$. Assume that they are compatible in that

$$f_{ii} = 1$$

and

$$f_{ij} f_{jk} = f_{ik}.$$

Define the inverse limit of the family G_i as

$$\varprojlim G_i = \left\{ (g_i) \in \prod_i G_i \colon f_{ij}(g_j) = g_i \right\}.$$

Completion

Suppose that there is a decreasing family $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ of normal subgroups. This forms a directed system. The inverse limit

$$\widehat{G} = \varprojlim G/G_r$$

is called the completion of G.

One can interpret an element of $\widehat{G}$ as a Cauchy sequence.

Question

Let p be a prime. Let $\mathbb{Z}_p$ be the completion of $\mathbb{Z}$ obtained by the family of subgroups $\mathbb{Z} \supset p\mathbb{Z} \supset p^2\mathbb{Z} \supset \cdots$. What is the torsion subgroup of $\mathbb{Z}_p$?