direct sum

The category of (left) modules over a ring A has coproduct. The coproduct is also known as direct sum.

coproduct

The coproduct, or the direct sum, is denoted by

$$\bigoplus_{i \in I} M_i$$

and its typical element is a finite formal sum

$$\sum_{k=1}^{n} m_{i_k}$$

of elements from M_{i_k}'s. In other words, it is the subset of

$$\prod_{i \in I} M_i$$

generated by the images of

$$M_j \to \prod_{i \in I} M_i$$

whose component is zero except at j, where it is the identify.

The universal property of the product is with respect to the natural injections

$$\eta_j \colon M_j \to \bigoplus_{i \in I} M_i$$

$$m_j \mapsto m_j$$

for $j \in I$. If M' is another module and has maps $f_j \colon M_j \to M'$, then it uniquely factors through

$$f \colon \bigoplus_{i \in I} M_i \to M'$$

with $f \circ \eta_j = f_j$.

Question

If I is finite, then the direct sum also satisfies the universal property of the product. What fails if I is infinite?