lifting normal extensions

Theorem

A normal extension remains normal under lifting, compositum, and intersection.

lifting

Consider the diagram of fields:

$$\begin{tikzcd}[arrows=-] & KF & \\ K \arrow[ur] & & F \arrow[ul] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

If K/k is normal, it is the splitting field of a family of the polynomials, say $(f_i)_{i \in I}$. Then, KF/F is the splitting field of the same family $(f_i)_{i \in I}$ of polynomials.

compositum

Consider the diagram of fields:

$$
\begin{tikzcd}[column sep=small]
 & KF \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & F \arrow[dl, dash] \\
 & k &
\end{tikzcd}
$$

If K/k and F/k are normal, then they are the splitting field of the families of the polynomials, say $(f_i)_{i \in I}$ and $(g_j)_{j \in J}$, respectively. Then, KF/F is the splitting field of the composite family $(f_i)_{i \in I} \cup (g_j)_{j \in J}$ of polynomials.

intersection

Consider the diagram of fields:

$$\begin{tikzcd}[arrows=dash]
 & k^{\mathrm{a}} \arrow[dl] \arrow[dr] & \\
K \arrow[dr] & & F \arrow[dl] \\
 & K \cap F \arrow[d] & \\
 & k &
\end{tikzcd}$$

Any embedding $K \to k^{\mathrm{a}}$ has image in K. Similarly, any embedding $F \to k^{\mathrm{a}}$ has image in F. Since an embedding $K \cap F \to k^{\mathrm{a}}$ can be extended to an embeeding of either F or K, its image is contained in both F and K.

Question

In the diagram

$$\begin{array}{c} E \\ | \\ F \\ | \\ k \end{array}$$

with E/k normal, show that E/F is normal.