Cosets

Let G be a group equipped with a subgroup H.

Definition

A subset $A \subset G$ is called a left coset (of H) if

$$A = aH := \{ah\colon h \in H\}$$

for some $a \in G$.

If A is a left coset, then such a is called a coset representative.

If a is a representative, then $a \in A$ because $e \in H$.

G is partitioned into left cosets.

Right cosets are defined in a similar way, giving rise to another partition of G.

Let G be a group equipped with a subgroup H.

Proposition

The set of all left cosets of H has the same cardinality as the set of all right cosets of H.

Proof.

Turn a left coset xH into Hx^{-1}, a right coset. $\square$

Definition

The index of H in G, denoted by $(G : H)$, is defined to be the cardinality of the (left) cosets of H.

Let G be a group equipped with a subgroup H.

Proposition

Two elements $x, y \in G$ belong to the same left coset if and only if $x^{-1}y \in H$.

In other words, the relation

$$x \sim y \text{ if } x^{-1}y \in H$$

gives rise to the partition of G into left cosets.