Algebraic closure

Definition

A field K is algebraically closed if every polynomial in $K[X]$ of degree at least one has a root in K.

Theorem

Let k be a field. Then, there is an algebraically closed field containing k as a subfield.

To prove the theorem, we will use the following lemma.

Lemma

There is a field E/k such that every polynomial $f \in k[X]$ of degree at least one has a root in E.

proof of the theorem from the lemma

Applying the lemma iteratively, get a sequence of extensions

$$k = E_0 \subset E_1 \subset E_2 \subset \cdots$$

such that every polynomial in $E_i[X]$ of degree at least one has a root in E_{i+1}. Let

$$E = \bigcup_{i \geq 0} E_i$$

and $f \in E[X]$. Then, $f \in E_N[X]$ for sufficiently large N. By construction, f has a root in E_{N+1}. In particular, it has a root in E.

proof of the lemma

Let S be the set of all non-constant polynomials in $k[X]$. To each $f \in S$, introduce a new indeterminate X_f. Consider the ring

$$A = k[X_f \colon f \in S]$$

and its ideal I generated by $f(X_f)$ for all $f \in S$. We claim that I is not the unit ideal. Indeed, if so, we have

$$\sum_{i=1}^{n} g_i f_i(X_{f_i}) = 1$$

for some $g_i \in A$. Consider an extension E/k where each f_i has a root, $i = 1, 2, \cdots, n$. Let $\alpha_1, \cdots, \alpha_n \in E$ be roots of $f_1, \cdots, f_n$, respectively. Then, substituting $X_{f_i} = \alpha_i$, we get $0 = 1$; a contradiction.

Since I is not a unit ideal, there is a maximal ideal, say $\mathfrak{m}$, containing I. Obtain a field homomorphism $\sigma \colon k \to A/\mathfrak{m}$. Identifying k with $\sigma(k)$, we obtain the desired field extension.

Question

Use the theorem to show that there is an algebraic extension k^{a}/k such that k^{a} is algebraically closed. Such a field k^{a} is called an algebraic closure of k.