Free modules

definition

Let A be a ring. An A-module M is called a free module if, there exists a set I and a family of elements

$$m_i \in M$$

parametrized by $i \in I$ such that

$$M \simeq \bigoplus_{i \in I} A \cdot m_i$$

and that

$$A \to A \cdot m_i$$

is an isomorphism.
The cardinality of I is called the rank.

Alternatively, one can define a free module over a given set I as a direct sum

$$\bigoplus_I A.$$

Then, a module M is free if and only if it is isomorphic to $\bigoplus_I A$ for some I.

If A is commutative, one can show that the cardinality of I is well-defined. Recall that A has a maximal ideal $\mathfrak{m} \subset A$ and $A/\mathfrak{m}$ is a field. Then,

$$M/\mathfrak{m}M$$

is a module over $A/\mathfrak{m}$. Then, the cardinality of I is characterized as the dimension of $M/\mathfrak{m}M$ as a vector space over $A/\mathfrak{m}$.

Question

Show that the free module over a given set I has the universal property in the following sense; if N is an A-module, then any map of sets $I \to N$ extends uniquely to an A-module homomorphism

$$\bigoplus_I A \to N.$$