Applying Hom-functors

Let A be a ring and $F\colon \boldsymbol{Mod}_A \to \mathcal{C}$ be a functor to a category $\mathcal{C}$. Given an exact sequence

$$M' \to M \to M'',$$

one obtains a diagram

$$F(M') \to F(M) \to F(M'')$$

in $\mathcal{C}$.

Let us take
$$F(-) = \operatorname{Hom}_A(N, -)$$
for some A-module N. Here $\operatorname{Hom}_A(N, -)$ denotes the set of A-homomorphisms, so
$$\operatorname{Hom}_A(N, -) = \operatorname{Mor}_{\mathbf{Mod}_A}(N, -)$$
by definition.

For any N, F is left-exact in the following sense.

Proposition

If

$$0 \to M' \to M \to M''$$

is exact, then

$$0 \to Hom_A(N, M') \to Hom_A(N, M) \to Hom_A(N, M'')$$

is exact.

reformulation

Note that the exactness of

$$0 \to \mathrm{Hom}_A(N, M') \xrightarrow{\kappa} \mathrm{Hom}_A(N, M) \xrightarrow{\phi} \mathrm{Hom}_A(N, M'')$$

is equivalent to the assertion that

$$\kappa \colon \mathrm{Hom}_A(N, M') \to \mathrm{Hom}_A(N, M)$$

is the kernel of the map

$$\phi \colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M'')$$

proof

We will show that

$$\kappa \colon \mathrm{Hom}_A(N, M') \to \mathrm{Hom}_A(N, M)$$

is the kernel of the map

$$\phi \colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M'').$$

To show this, it suffices to check the following universal property; if a morphism

$$f \colon X \to \mathrm{Hom}_A(N, M)$$

satisfies $\phi \circ f = 0$, then it factors uniquely to a morphism

$$\tilde{f} \colon X \to \mathrm{Hom}_A(N, M')$$

with $\kappa \circ \tilde{f} = f$. Indeed, for each $x \in X$, $u = f(x) \colon N \to M$ lifts uniquely to some $\tilde{u} \colon N \to M'$, since $M' \to M$ is the kernel of $M \to M''$. Defining $\tilde{f}(x) = \tilde{u}$ we get the desired $\tilde{f}$.

Question

The proof for the left-exactness of $F = \mathrm{Hom}_A(N, -)$ is formal. Turn it into a proof of the left-exactness of $\mathrm{Hom}_A(-, N)$: If

$$M' \to M \to M'' \to 0$$

is exact, then

$$0 \to \mathrm{Hom}_A(M'', N) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N)$$

is exact for any A-module N.