Orbit decomposition formula

Let G be a group and S be a G-set. Decompose

$$S = \coprod_{i \in I} Gs_i$$

into disjoint G-orbits.

Theorem
We have

$$|S| = \sum_{i \in I} (G : G_{s_i}).$$

Proof.
Decompose S into disjoint G-orbits, each of which is of the form G/G_{s_i}. $\qquad\square$

Let G act on G by conjugation. Let C be the set of conjugacy classes of G.

Corollary

We have

$$(G:1) = \sum_{x \in C} (G:G_x).$$

Proof.

Apply the orbit decomposition formula. □