simplicity of A_n

Definition

A group G is simple if a normal subgroup of G is either trivial or G itself.

We aim to sketch the proof of:-

Proposition

For $n \geq 5$, A_n is simple.

Using lower bounds

Recall that for $n \geq 6$, any non-trivial conjugacy class in A_n has at least n elements. Using this, we will show that a non-trivial subgroup $N \subset A_n$ must be equal to A_n, by induction on n.

From now on, let N be a non-trivial proper normal subgroup of A_n.

For each $i = 1, 2, \cdots, n$, define $H_i \subset S_n$ to be the isotropy subgroup of i. Of course, $H_i \xrightarrow{\sim} S_{n-1}$. Define

$$N_i = N \cap H_i$$

which is a normal subgroup of H_i. Suppose that N_i is trivial for all i. This means that N acts on $\{1, 2, \cdots, n\}$ freely; for any integer $k = 1, 2, \cdots, n$, the isotropy group is trivial. In this case,

$$\begin{aligned} N &\to \{1, 2, \cdots, n\} \\ \sigma &\mapsto \sigma(1) \end{aligned}$$

is an embedding and $|N| \leq n$. Since N, as a non-trivial normal subgroup of A_n, contains at least two conjugacy classes of A_n, we obtain a contradiction. Thus, N_i cannot be trivial for all i. Use this to complete the induction argument.

Question

To complete the argument, show that A_n is simple for $n = 5, 6$.