Size of a conjugacy class in S_n

In A_n or S_n, a non-trivial conjugacy class has to be large. In a quantitative form, we aim to sketch the proof of:-

Proposition

Let $n \geq 6$. A non-trivial conjugacy class in A_n is of size at least n.

Proposition

Let $n \geq 6$. A non-trivial conjugacy class in S_n is of size at least $2n$.

Note that the former follows from the latter because A_n has index two in S_n.

The case of S_n

Let $\sigma \in S_n$ be a non-trivial permutation, generating the conjugacy class C. Suppose that σ is a k-cycle, $k \geq 2$. Then, it follows from counting the number of k-cycles in S_n that $|C| \geq 2n$ whenever $n \geq 5$.

Now suppose that the cycle decomposition of σ contains a k-cycle with $\frac{n}{2} < k$. Then, the number of cycles of length k is one and the number of conjugacy classes is at least $(k-1)!$. Note that

$$([n/2] + 1)! \geq 2n$$

if $n \geq 6$.

Let σ be a permutation which is not a k-cycle for any $k \geq 2$. Also assume that the largest cycle in σ has length at most $n/2$. We would like to show that $|C| \geq 2n$, by induction on n.

If $n > 6$, one can decompose

$$\{1, 2, \cdots, n\} = \{1, \cdots, n_1\} \cup \{n_1 + 1, \cdots, n_1 + n_2 = n\}$$

and choose $\sigma' \in C$ so that σ' preserves the decomposition and $n_1, n_2 \geq 3$. Using the decomposition we get nontrivial conjugacy classes

$$C \cap S_{n_i} \subset S_{n_i}$$

for $i = 1, 2$.

Using lower bounds of $|C|$ obtained for small values of n by direct computation, one can complete the induction argument.

Question

To complete the argument, establish the lower bounds for the smallest non-trivial conjugacy class in S_n for $n \leq 8$. Can you give a general lower bound better than $2n$?