Examples of Groups

Groups arise from various contexts.

Example 1

Let G be a group and S a non-empty set. The set

$$M(S, G)$$

of all functions from S to G is also a group under point-wise multiplication. Namely, for $f, g \in M(S, G)$, define

$$(fg)(x) = f(x)g(x).$$

This group is commutative if and only if G is commutative.

Example 2

Let S be any set. The set

$$\mathrm{Perm}(S)$$

of all bijections from S into itself is a group under composition.

Example 3

Let V be a vector space over a field. The group

$$\mathrm{GL}(V)$$

of all linear isomorphisms from V into itself is a group under composition. If V has dimension n over a field k, then $\mathrm{GL}(V)$ is often written as $\mathrm{GL}(n, k)$. This group is non-commutative when $n \geq 2$.

Example 4

More generally, if A is an object in a category, then

$$\mathrm{Aut}(A)$$

of isomorphisms from A into itself is a group under composition.

Example 5

- The set of rational numbers with addition.
- The set of non-zero rational numbers with multiplication

Example 6

Let n be a positive integer. The set

$$\{z \in \mathbb{C} : z^n = 1\}$$

is a group with n elements. Every element is a power of

$$e^{\frac{2\pi i}{n}}$$

so it 'generates' the group. Any generator of this group is called a primitive n-th root of unity.

Example 7

If G_1 and G_2 are groups, then the cartesian product

$$G_1 \times G_2$$

is again a group with component-wise multiplication

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

More generally, if there is a family $(G_i)_{i \in I}$ of groups indexed by a set I, the direct product

$$\prod_{i \in I} G_i$$

is a group.