Local rings and localization

Let A be a commutative ring. If $\mathfrak{p} \subset A$ is a prime ideal, then, $S = A - \mathfrak{p}$ is a multiplicatively closed subset.

We denote the corresponding localized ring by

$$A_{\mathfrak{p}} := S^{-1}A.$$

Using the natural map

$$i\colon A \to A_{\mathfrak{p}}$$

define

$$\mathfrak{m} := i(\mathfrak{p})A_{\mathfrak{p}}.$$

It is an ideal of $A_{\mathfrak{p}}$ generated by $i(\mathfrak{p})$.

Proposition

$(A_{\mathfrak{p}}, \mathfrak{m})$ *is a local ring.*

Proof.

Note that $\mathfrak{m} = S^{-1}\mathfrak{p}$ consists of $\frac{a}{s}$ with $a \in \mathfrak{p}$. If $\frac{a}{s} \in A_{\mathfrak{p}} - \mathfrak{m}$, then $a \notin \mathfrak{p}$ so $s \in S$ and $\frac{a}{s} \in A_{\mathfrak{p}}^{\times}$. This means we have

$$A_{\mathfrak{p}} = \mathfrak{m} \dot{\cup} A_{\mathfrak{p}}^{\times}.$$

It follows that $\mathfrak{m}$ is maximal ideal. Indeed, an ideal which properly contains $\mathfrak{m}$ contains a unit, so it must be equal to $A_{\mathfrak{p}}$. Also, any proper ideal must be disjoint from $A_{\mathfrak{p}}^{\times}$, so it must be contained in $\mathfrak{m}$. $\square$

Question

Let p be a prime. What is the maximal ideal of $\mathbb{Z}_{(p)}$?