1 Inductive Proofs

Prove each of the following claims by induction

Claim 1. *The sum of the first n even numbers is* n^2+n. *That is,* $\sum_{i=1}^{n}(2i) = n^2+n$

Base Case:
$\sum_{i=1}^{1}(2i) = 2$

$P(n) = n^2 + n$

$P(1) = (1)^2 + 1$

$P(1) = 2$

The sum meets the base case.

Inductive Case:

Assume $\sum_{i=1}^{n}(2i) = n^2 + n$
$\sum_{i=1}^{n+1}(2i) = \sum_{i=1}^{n}(2i) + 2(n+1)$
$= n^2 + n + 2n + 2$

$=n^2 + 3n + 2$

$P(n) = n^2 + n$
$P(n+1) = (n+1)^2 + (n+1)$

$= n^2 + 3n + 2$

P(n+1) is true, so by PMI P(n) is true for all $n \geq 1$.

Claim 2. $\sum_{i=1}^{n}\frac{2}{3^i} = 1 - \frac{1}{3^n}$

Base Case:

$\sum_{i=1}^{1} \frac{2}{3^1} = \frac{2}{3}$

$P(n) = 1 - \frac{1}{3^n}$

$P(1) = 1 - \frac{1}{3^1}$

$P(1) = \frac{2}{3}$

The sum meets the base case.

Inductive Case:

Assume $\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}$

$\sum_{i=1}^{n+1} \frac{2}{3^i} = \sum_{i=1}^{m} \frac{2}{3^i} + \frac{2}{3^{n+1}}$

$= (1 - \frac{1}{3^n}) + \frac{2}{3^{n+1}}$

$= 1 - \frac{1}{3^{n+1}}$

$P(n) = 1 - \frac{1}{3^n}$

$P(n+1) = 1 - \frac{1}{3^{n+1}}$

P(n+1) is true, so by PMI P(n) is true for all $n \geq 1$.

Claim 3. *For every $n \geq 1$, $5^n - 1$ is divisible by 4. In other words, for every $n \geq 1$, there exists some integer constant z_n such that $5^n - 1 = 4z_n$. Note that each power of 5 has a different z.*

For n = 1:

$5^1 - 1 = 4z_1$
$4z_1 = 4$
$z_1 = 1$

For n = 2:
$5^2 - 1 = 4z_2$
$4z_2 = 24$
$z_2 = 6$

For n = 3:
$5^3 - 1 = 4z_3$
$4z_3 = 124$
$z_3 = 31$

From solving the first three z_n values given that $n \geq 1$, shows us that the sum of the z_n values is $\sum_{i=1}^{n} \frac{5^i-1}{4} = \frac{5^{n+1}-4n-5}{16}$

Base Case:

$\sum_{i=1}^{1} \frac{5^1-1}{4} = 1$

$P(n) = \frac{5^{n+1}-4n-5}{16}$

$P(1) = \frac{5^2-4(1)-5}{16}$

$P(1) = \frac{25-4-5}{16}$

$P(1) = 1$

The sum of the z_n values meets the base case

Inductive Case:

Assume $\sum_{i=1}^{n} \frac{5^n-1}{4} = \frac{5^{n+1}-4n-5}{16}$

$\sum_{i=1}^{n+1} \frac{5^n-1}{4} = \sum_{i=1}^{n} \frac{5^n-1}{4} + \frac{5^{n+1}-1}{4}$

$\sum_{i=1}^{n+1} \frac{5^n-1}{4} = \frac{5^{n+1}-4n-5}{16} + \frac{5^{n+1}-1}{4}$

$= \frac{5^{n+1}-4n-5}{16} + \frac{4(5^{n+1}-1)}{16}$

$= \frac{5^{n+2}-4n-4-5}{16}$

$= \frac{5^{n+2}-4n-9}{16}$

$P(n) = \frac{5^{n+1}-4n-5}{16}$

$P(n+1) = \frac{5^{n+2}-4(n+1)-5}{16}$

$= \frac{5^{n+2}-4n-9}{16}$

P(n+1) is true, so by PMI P(n) is true for all $n \geq 1$.

2 Recursive Invariants

The function `minPos`, given below in pseudocode, takes as input an array A of size n numbers and returns the smallest positive number in the array. If no positive numbers appear in the array, it returns zero. (Note that zero is neither positive nor negative.) Using induction, prove that the `minPos` function works correctly. Clearly state your recursive invariant at the beginning of your proof.

```
Function minPos(A,n)
  If n = 0 Then
    Return infinity (or largest possible value)
  Else
    best <- minPos(A,n-1)
    If A[n-1] < best And A[n-1] > 0 Then
      best <- A[n-1]
    EndIf
    Return best
  EndIf
EndFunction
```

Recursive Invariant:
$P(n)$ = minPos(A,n) returns infinity or the smallest possible positive number, given that $n \geq 0$.

Base Case:
$n = 0$
minPos(A,0) $= \infty$
Therefore $P(0)$ is true.

Inductive Case:
Suppose that $P(n)$ is true, minPos(A,n) returns infinity or the smallest possible positive number.

Case 1: minPos(A,n) $= \infty$
There are no positive values in A

- Subcase 1a: A[n] ≤ 0
Because n > 0, by assumption, best = minPos(A,n) $= \infty$
Since A[n] ≤ 0, the if function was skipped
minPos(A,n+1) $= \infty$ and the invariant holds true.

- Subcase 1b: A[n] > 0
By assumption, best = minPos(A,n) $= \infty$
Because A[n] $<$ best and A[n] > 0, best = A[n].
So, minPos(A,n+1) = A[n].
Because A[n] is the smallest possible positive value in the array, the invariant

holds true.

Case 2: minPos(A,n) $< \infty$ (or the smallest possible positive number) (SP denotes the smallest positive value)

- Subcase 2a: A[n] ≤ 0
best = SP and because A[n] ≤ 0, the function returns SP. minPos(A,n+1) = SP, so the invariant holds true.

- Subcase 2b: A[n] > 0
If A[n] $\geq$ SP:
best = SP. Because A[n] $\geq$ SP, the if condition was not met, and the function should return SP
Since SP is the smallest positive value in the array, the invariant holds true.

If A[n] $<$ SP:
best = SP,
Since A[n] $<$ best and A[n] > 0, best = A[n]
minPos(A,n+1) = A[n]
Since A[n] is the smallest positive value in the array, the invariant holds true.
Therefore, in all cases, P(n+1) is true. By PMI, P(n) is true for all $n \geq 0$.