

# Tensor Product Hilbert Space

One reason for learning Hilbert space  
Postulate of quantum

*For every closed quantum system, there exists a Hilbert space  $\mathcal{H}$  such that the states that the system can be in correspond to norm-1 vectors in  $\mathcal{H}$ .*

Schrödinger's pets



# Schrödinger's pets

Schrödinger's cat (in a quantum box) can be in the following states:

1. Dead  $\psi_{\text{dead}}$
2. Alive  $\psi_{\text{alive}}$

Physicists tell us that, as these two states are not eigenstates of the measurement, the two vectors must be orthogonal.

3. Any superposition  $\alpha\psi_{\text{dead}} + \beta\psi_{\text{alive}}$  (with  $|\alpha|^2 + |\beta|^2 = 1$ )

→ 2-dim Hilbert space  $\mathcal{H}_{\text{cat}}$

Physics w

# Schrödinger's pets

Schrödinger's dog (in a quantum box) can be in the following states:

1. Drooling  $\phi_{\text{drool}}$
2. Peeing  $\phi_{\text{pee}}$
3. Sleeping  $\phi_{\text{sleep}}$
4. Any superposition  $\alpha\phi_{\text{drool}} + \beta\phi_{\text{pee}} + \gamma\phi_{\text{sleep}}$   
(with  $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ ) of the three.

→ 3-dim Hilbert space  $\mathcal{H}_{\text{dog}}$

# Schrödinger's pets



The cat and the dog are two separate quantum systems with their own Hilbert space. By default, both systems do not exchange of information between them. But if we somehow, and consider them together,

# Schrödinger's pets



The cat and the dog are two separate quantum systems, each with its own Hilbert space. By default, both systems have no exchange of information between them. But somehow, and consider them together.

*What's the Hilbert space of the combined system?*

# Schrödinger's pets

The combined system can be in the states:

- |                            |  |
|----------------------------|--|
| 1. cat dead, dog drooling  | $\psi_{\text{dead}} \otimes \phi_{\text{drooling}}$  |
| 2. cat dead, dog peeing    | $\psi_{\text{dead}} \otimes \phi_{\text{peeing}}$    |
| 3. cat dead, dog sleeping  | $\psi_{\text{dead}} \otimes \phi_{\text{sleeping}}$  |
| 4. cat alive, dog drooling | $\psi_{\text{alive}} \otimes \phi_{\text{drooling}}$ |
| 5. cat alive, dog peeing   | $\psi_{\text{alive}} \otimes \phi_{\text{peeing}}$   |
| 6. cat alive, dog sleeping | $\psi_{\text{alive}} \otimes \phi_{\text{sleeping}}$ |

The " $\otimes$ " (tensor product) is a new type of multiplication, but unlike multiplication, *leave the symbol there instead of*

# Schrödinger's pets

Superpositions:

$$\alpha \cdot (\psi_{\text{dead}} \otimes \phi_{\text{drool}}) + \beta \cdot (\psi_{\text{dead}} \otimes \phi_{\text{pee}}) + \gamma \cdot (\psi_{\text{dead}} \otimes \phi_{\text{sleep}}) + \delta \cdot (\psi_{\text{dead}} \otimes \phi_{\text{wake}})$$

These (with arbitrary  $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{C}$ ) form a finite-dimensional Hilbert space that is the "tensor product" of the two spaces above.

Confusion warning:

- Tensor product of vectors
- Tensor product of spaces



# Definition of tensor product of

**Definition.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be  $\mathbb{K}$  Hilbert spaces.  $\mathcal{H}_1 \otimes \mathcal{H}_2$  is a  $\mathbb{K}$  Hilbert space

- whose elements are  $\sum_{j=1}^m \alpha_j \cdot (\psi_j^1 \otimes \psi_j^2)$ ,  
for arbitrary  $m \in \mathbb{Z}_+, \psi_1^1, \dots, \psi_m^1 \in \mathcal{H}_1, \psi_1^2, \dots, \psi_m^2 \in \mathcal{H}_2$
- satisfying the following laws that we know:
  1.  $\otimes$  is bi- $\mathbb{K}$ -linear:
    - $(\alpha\phi^1 + \beta\psi^1) \otimes \phi^2 = \alpha \cdot (\phi^1 \otimes \phi^2) + \beta \cdot (\psi^1 \otimes \phi^2)$
    - $\phi^1 \otimes (\alpha\phi^2 + \beta\psi^2) = \alpha \cdot (\phi^1 \otimes \phi^2) + \beta \cdot (\phi^1 \otimes \psi^2)$
  2. Inner product:  $(\phi^1 \otimes \phi^2 \mid \psi^1 \otimes \psi^2) = (\phi^1 \mid \psi^1) \cdot (\phi^2 \mid \psi^2)$

For vector arithmetic  
the  $\otimes$ -operator behaves  
like multiplication

the left box contains the cat,  
the dog — so in  $\psi \otimes \phi$ , you can  
because  $\psi$  contains information  
contains information.

# Definition of tensor product of

**Definition.** Let  $\mathcal{H}_j, j = 1, \dots, r$  be  $\mathbb{K}$  Hilbert spaces and  $\mathcal{H}$  a  $\mathbb{K}$  Hilbert space

- whose elements are  $\sum_{j=1}^m \alpha_j \cdot (\psi_j^1 \otimes \dots \otimes \psi_j^r)$  for arbitrary  $m \in \mathbb{Z}_+, \psi_1^1, \dots, \psi_m^1 \in \mathcal{H}_1, \dots, \psi_1^r, \dots, \psi_m^r \in \mathcal{H}_r$

- satisfying the following laws that we know

1.  $\otimes$  is  $r$ - $\mathbb{K}$ -linear:

- $(\alpha\phi^1 + \beta\psi^1) \otimes \phi^2 \otimes \dots \otimes \phi^r = \alpha \cdot (\phi^1 \otimes \phi^2 \otimes \dots \otimes \phi^r) + \beta \cdot (\psi^1 \otimes \phi^2 \otimes \dots \otimes \phi^r)$
- ...
- $\phi^1 \otimes \dots \otimes \phi^{r-1} \otimes (\alpha\phi^r + \beta\psi^r) = \alpha \cdot (\phi^1 \otimes \dots \otimes \phi^r) + \beta \cdot (\phi^1 \otimes \dots \otimes \psi^r)$

2. Inner product:  $(\phi^1 \otimes \dots \otimes \phi^r \mid \psi^1 \otimes \dots \otimes \psi^r) = (\phi^1 \mid \psi^1) \cdot \dots \cdot (\phi^r \mid \psi^r)$

For vector arithmetic,  
the  $\otimes$ -operator behaves  
like multiplication

in  $\psi^1 \otimes \psi^2 \otimes \dots \otimes \psi^r$ , the  $\psi^j$  is  
 $j$ th quantum system, so can

## Back to cats & dogs: ONBs of $\mathcal{H}$

### Proposition.

*Let  $\mathcal{H}_1, \dots, \mathcal{H}_r$  be  $\mathbb{K}$ -Hilbert spaces, and for  $j = 1, \dots, r$  let  $\{b_{k_j}^j\}_{k_j \in \mathbb{N}}$  be an ONB of  $\mathcal{H}_j$ . Then the following is an ONB of  $\mathcal{H}$ :*

$$\{b_{k_1}^1 \otimes \dots \otimes b_{k_r}^r \text{ for } k = (k_1, \dots, k_r) \in \prod_{j=1}^r \mathbb{N}\}$$

# Notation

1.  $\psi_1 \in \mathcal{H}_1, \dots, \psi_r \in \mathcal{H}_r$  vectors  $\rightarrow \psi_1 \otimes \dots \otimes \psi_r$
2.  $\mathcal{H}_1, \dots, \mathcal{H}_r$   $\mathbb{K}$ -Hilbert spaces  $\rightarrow "$  $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r$ "
3.  $"\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r" = \text{Span}(\{\psi_1 \otimes \dots \otimes \psi_r\})$

# Tensor products of linear operators

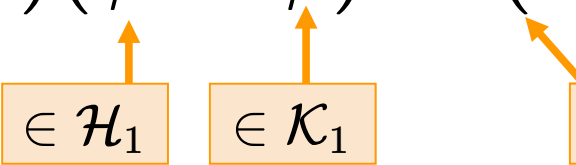
We already know what a linear operator (mapping, function) is, and we know how the linear operators are *vectors*, i.e., elements of a vector space.

Let  $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$  and  $B: \mathcal{K}_1 \rightarrow \mathcal{K}_2$  be linear operators.

There's a linear operator

$$\mathcal{H}_1 \otimes \mathcal{K}_2 \rightarrow \mathcal{H}_2 \otimes \mathcal{K}_2$$

which does this:

$$(A \otimes B)(\phi \otimes \psi) = (A\phi) \otimes (B\psi)$$


The diagram shows three orange boxes at the bottom. The first box contains  $\in \mathcal{H}_1$  and has an orange arrow pointing up to the  $\phi$  in the equation above. The second box contains  $\in \mathcal{K}_1$  and has an orange arrow pointing up to the  $\psi$  in the equation above. A third box is partially visible on the right with an orange arrow pointing up to the  $\otimes$  symbol between  $A$  and  $B$  in the equation above.

# Excursion into the Realm Of M

Take  $\mathbb{K}$ -Hilbert spaces  $\mathcal{H}_1, \dots, \mathcal{H}_r$ , and consider  $\mathcal{H}_1 \otimes \dots \otimes$

There is a simple & convenient way of defin

$$T: \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r \rightarrow$$

## Universal Property.

*Let  $f$  be a function  $\mathcal{H}_1 \times \dots \times \mathcal{H}_r \rightarrow$*

*If  $f$  is  $r$ -multi- $\mathbb{K}$ -linear, then there's a*

$\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r \rightarrow \mathcal{G}$  satisfying:

$$T_f(\psi_1 \otimes \dots \otimes \psi_r) :=$$



# Excursion into the Realm Of M

Why does  $f$  need to be multi-linear?

$$\begin{aligned} & f(\psi_1, \dots, \psi_{j-1}, \alpha\phi_j + \beta\psi_j, \psi_{j+1}, \dots) \\ &= T_f(\psi_1 \otimes \dots \otimes \psi_{j-1} \otimes (\alpha\phi_j + \beta\psi_j) \otimes \psi_{j+1} \otimes \dots) \\ &= T_f\left(\alpha \cdot \psi_1 \otimes \dots \otimes \psi_{j-1} \otimes \phi_j \otimes \psi_{j+1} \otimes \dots \right. \\ &\quad \left. + \beta \cdot \psi_1 \otimes \dots \otimes \psi_{j-1} \otimes \psi_j \otimes \psi_{j+1} \otimes \dots\right) \\ &= \alpha \cdot T_f(\psi_1 \otimes \dots \otimes \psi_{j-1} \otimes \phi_j \otimes \psi_{j+1} \otimes \dots) \\ &\quad + \beta \cdot T_f(\psi_1 \otimes \dots \otimes \psi_{j-1} \otimes \psi_j \otimes \psi_{j+1} \otimes \dots) \\ &= \alpha \cdot f(\psi_1, \dots, \psi_{j-1}, \phi_j, \psi_{j+1}, \dots) \\ &\quad + \beta \cdot f(\psi_1, \dots, \psi_{j-1}, \psi_j, \psi_{j+1}, \dots) \end{aligned}$$

Hilbert space

