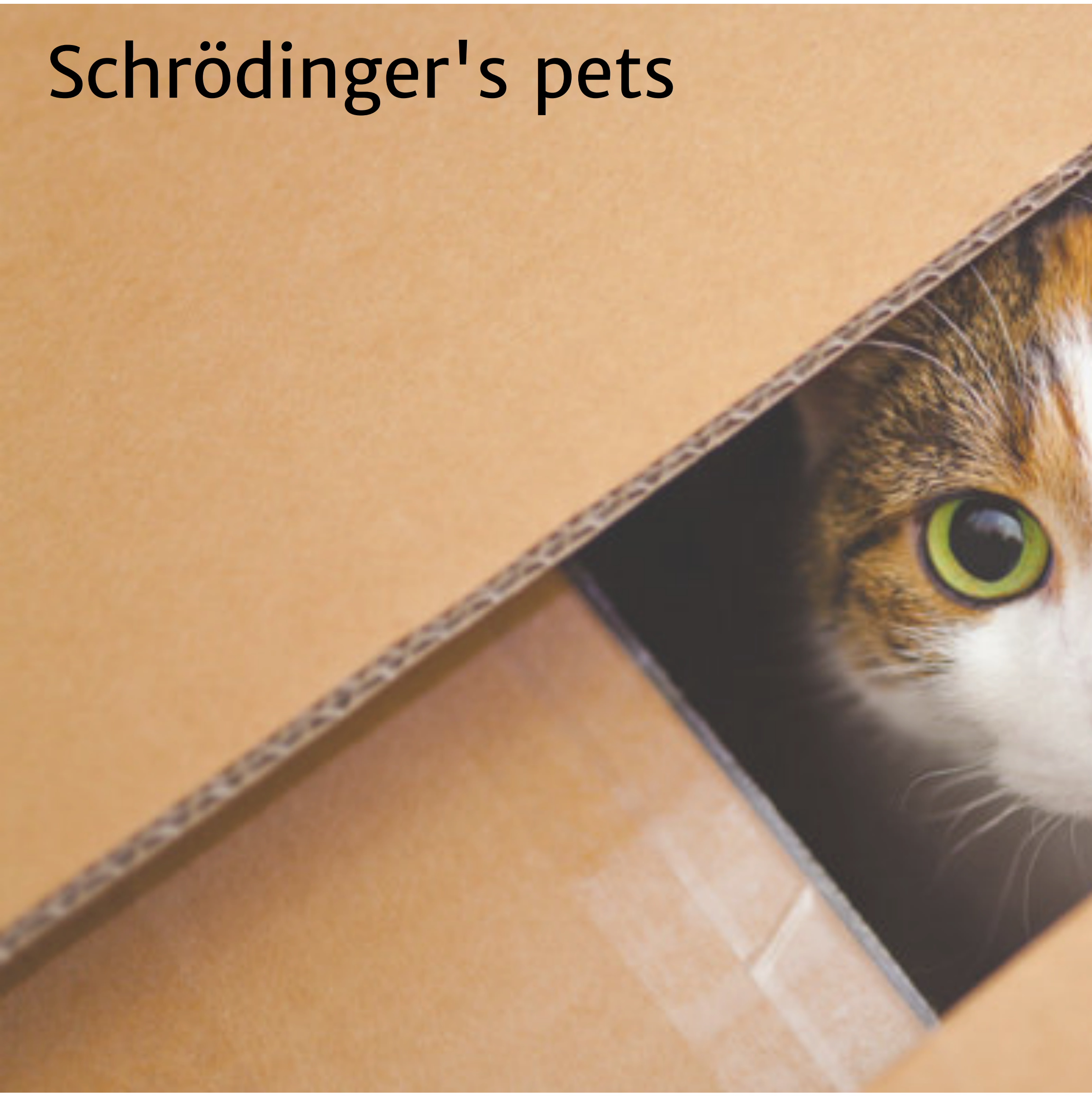
Tensor Proc
Hilbert Sp

One reason for learning Hilbert spac
Postulate of quantum

For every closed quantum system, th
space $\mathcal{H}$ such that the states that the
correspond to norm-1

Schrödinger's pets



Schrödinger's pets

Schrödinger's cat (in a quantum box)
can be in the following states:

1. Dead ψ_{dead}
2. Alive ψ_{alive}

Physicists tell us that, as these two state
measurement, the two vectors must be

3. Any superposition $\alpha\psi_{\text{dead}} + \beta\psi_{\text{alive}}$ (with

➜ 2-dim Hilbert space $\mathcal{H}_{\text{cat}}$

Physics w

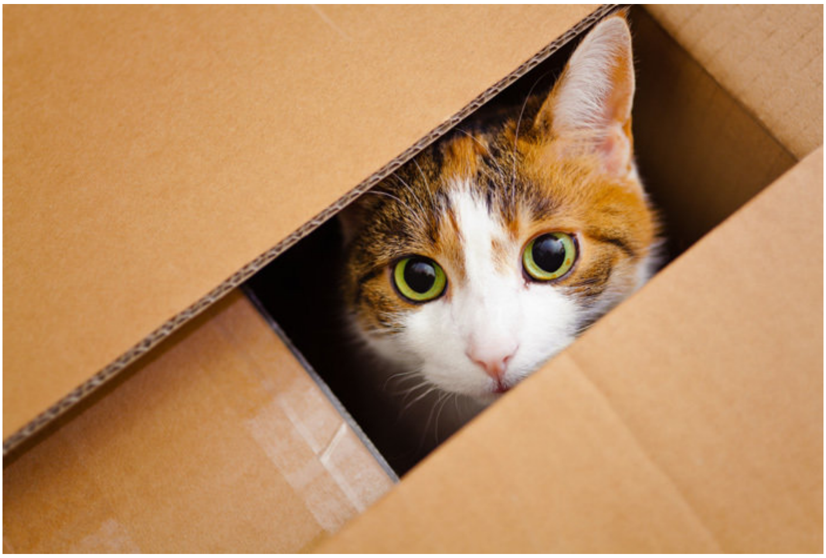
Schrödinger's pets

Schrödinger's dog (in a quantum box) can be in the following states:

1. Drooling ϕ_{drool}
2. Peeing ϕ_{pee}
3. Sleeping ϕ_{sleep}
4. Any superposition $\alpha\phi_{\mathrm{drool}} + \beta\phi_{\mathrm{pee}} + \gamma\phi_{\mathrm{sle}}$ (with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$) of the three.

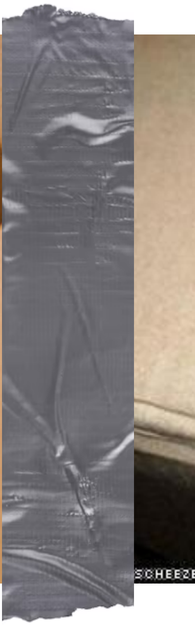
➜ 3-dim Hilbert space $\mathcal{H}_{\mathrm{dog}}$

Schrödinger's pets

The cat and the dog are two separate qua
own Hilbert space. By default, both sys
exchange of information between them. B
somehow, and consider them toget

Schrödinger's pets



The cat and the dog are two separate qua
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What's the Hilbert space of th

Schrödinger's pets

The combined system can be in the states:

1. cat dead, dog drooling	$\psi_{\text{dead}} \otimes$
2. cat dead, dog peeing	$\psi_{\text{dead}} \otimes$
3. cat dead, dog sleeping	$\psi_{\text{dead}} \otimes$
4. cat alive, dog drooling	$\psi_{\text{alive}} \otimes$
5. cat alive, dog peeing	$\psi_{\text{alive}} \otimes$
6. cat alive, dog sleeping	$\psi_{\text{alive}} \otimes$

The "$\otimes$" (tensor product) is a new type o
(multiplication), but unlike multiplication,
leave the symbol there instead of

Schrödinger's pets

Superpositions:

$\alpha \cdot (\psi_{\text{dead}} \otimes \phi_{\text{drool}}) + \beta \cdot (\psi_{\text{dead}} \otimes \phi_{\text{pee}}) + \gamma \cdot (\psi_{\text{dead}} \otimes \phi_{\text{sleep}}) + \delta \cdot (\psi_{\text{a}}$

These (with arbitrary $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in$
Hilbert space that is the "tensor pr

Confusion warning:

- Tensor product of vectors
- Tensor product of spaces

Definition of tensor product of

Definition. Let $\mathcal{H}_1, \mathcal{H}_2$ be $\mathbb{K}$ Hilbert spaces.
a $\mathbb{K}$ Hilbert space

- whose elements are $\sum_{j=1}^{m} \alpha_j \cdot (\psi_j^1 \otimes \psi_j^2)$,

  for arbitrary $m \in \mathbb{Z}_+$, $\psi_1^1, \ldots, \psi_m^1 \in \mathcal{H}_1$, ψ

- satisfying the following laws that we kn

1. $\otimes$ is bi-$\mathbb{K}$-linear:
   - $(\alpha\phi^1 + \beta\psi^1) \otimes \phi^2 = \alpha \cdot (\phi^1 \otimes \phi^2) + \beta$
   - $\phi^1 \otimes (\alpha\phi^2 + \beta\psi^2) = \alpha \cdot (\phi^1 \otimes \phi^2) + \beta$
2. Inner product: $\left(\phi^1 \otimes \phi^2 \mid \psi^1 \otimes \psi^2\right) = (\phi^1|$

For vector arit

the $\otimes$-operator be

like multiplicatio

the left box contains the cat,

the dog — so in $\psi \otimes \phi$, you can

because ψ contains informatio

contains information

Definition of tensor product of

Definition. Let $\mathcal{H}_j$, $j = 1, \ldots, r$ be $\mathbb{K}$ Hilbert sp
a $\mathbb{K}$ Hilbert space

- whose elements are $\sum_{j=1}^{m} \alpha_j \cdot (\psi_j^1 \otimes \cdots \otimes \psi_j^r)$
  for arbitrary $m \in \mathbb{Z}_+$, $\psi_1^1, \ldots, \psi_m^1 \in \mathcal{H}_1$, ..., ψ

- satisfying the following laws that we know

1. $\otimes$ is r-$\mathbb{K}$-linear:
   - $(\alpha\phi^1 + \beta\psi^1) \otimes \phi^2 \otimes \cdots \otimes \phi^r = \alpha \cdot (\phi^1 \otimes \phi^2 \otimes$
   - ...
   - $\phi^1 \otimes \cdots \otimes \phi^{r-1} \otimes (\alpha\phi^r + \beta\psi^r) = \alpha \cdot (\phi^1 \otimes \cdots$
2. Inner product: $(\phi^1 \otimes \cdots \otimes \phi^r \mid \psi^1 \otimes \cdots \otimes \psi$

For vector arit

the $\otimes$-operator be

like multiplicati

in $\psi^1 \otimes \psi^2 \otimes \cdots \otimes \psi^r$, the ψ^j is

$j\mathrm{th}$ quantum system, so can

Back to cats & dogs: ONBs of t

Proposition.
Let $\mathcal{H}_1, \ldots, \mathcal{H}_r$ *be* $\mathbb{K}$*-Hilbert spaces, and for*
ONB of $\mathcal{H}_j$*. Then the following is an ONB o*

$$b^1_{k_1} \otimes \cdots \otimes b^r_{k_r} \text{ for } k \in$$

Notatio

1. $\psi_1 \in \mathcal{H}_1$, ..., $\psi_r \in \mathcal{H}_r$ vectors ➜ $\psi_1 \otimes$
2. $\mathcal{H}_1, \ldots, \mathcal{H}_r$ $\mathbb{K}$-Hilbert spaces ➜ "$\mathcal{H}_1 \otimes$
3. "$\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_r$" $= \mathbf{Span}(\{\psi_1 \otimes \cdots \otimes$

Tensor products of linear opera

We already know what a linear operator (mapping, funct
of how the linear operators are *vectors*, i.e., elem

Let $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $B\colon \mathcal{K}_1 \to \mathcal{K}_2$ b

There's a linear operator

$\mathcal{H}_1 \otimes \mathcal{K}_2 \to \mathcal{H}_2 \otimes$

which does this:

$$(A \otimes B)(\phi \otimes \psi) = (A$$

$\in \mathcal{H}_1$ $\in \mathcal{K}_1$

Excursion into the Realm Of M

Take $\mathbb{K}$-Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_r$, and consider $\mathcal{H}_1 \otimes \cdots \otimes$

There is a simple & convenient way of defin

$$T \colon \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_r \to$$

Universal Property.

Let f be a function $\mathcal{H}_1 \times \cdots \times \mathcal{H}_r \to$

If f is r-multi-$\mathbb{K}$-linear, then there's a

$\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_r \to \mathcal{G}$ satisfying:

$$T_f(\psi_1 \otimes \cdots \otimes \psi_r) :=$$

Excursion into the Realm Of M

Why does f need to be multi-linear?

$$
\begin{aligned}
& f(\psi_1, \ldots, \psi_{j-1}, \alpha\phi_j + \beta\psi_j, \psi_{j+1}, \ldots \\
&= T_f(\psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes (\alpha\phi_j + \beta \\
&= T_f\Big(\alpha \cdot \psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \phi_j \otimes \\
&\qquad + \beta \cdot \psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \psi_j \\
&= \alpha \cdot T_f(\psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \phi_j \otimes \\
&\quad + \beta \cdot T_f(\psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \psi_j \otimes \\
&= \alpha \cdot f(\psi_1, \ldots, \psi_{j-1}, \phi_j, \psi_{j+1}, \ldots \\
&\quad + \beta \cdot f(\psi_1, \ldots, \psi_{j-1}, \psi_j, \psi_{j+1}, \ldots
\end{aligned}
$$

Hilbert spac