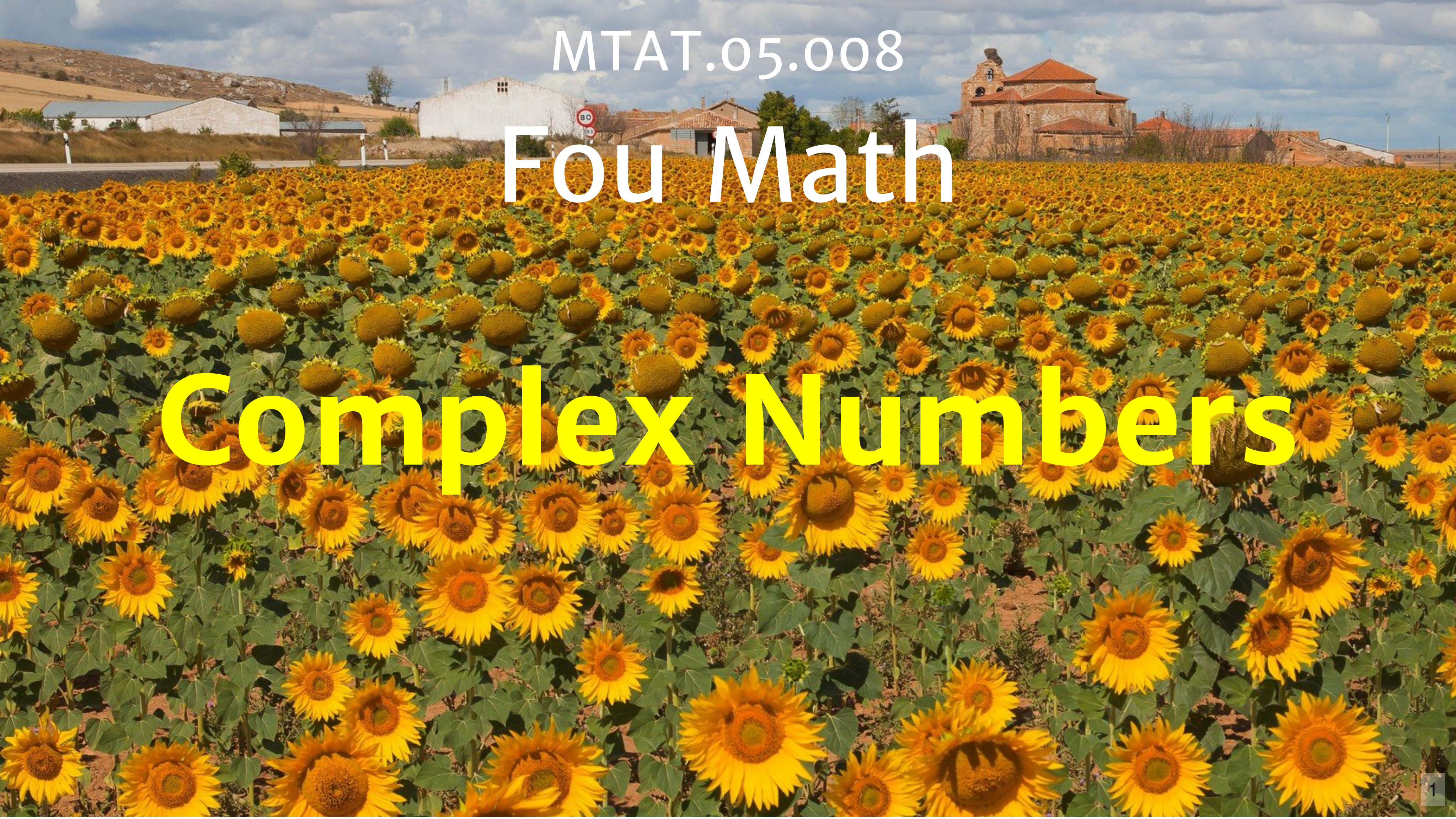

MTAT.05.008

Fou Math

Complex Numbers

What can you do with *real* numbers $\mathbb{R}$

Arithmetic:

"+" operator	addition	$x + y$
unary "-" operator	negation	$-x = 0 - x$
binary "-" operator	subtraction	$x - y = x + (-y)$
0	zero (add. neutral)	$x + 0 = x$
"*" operator	multiplication	$x*y = x \cdot y$
"/" operator	division	$x/y = x \cdot \mathrm{inv}(y)$
	multipl. inverse	$\mathrm{inv}(x) = 1/x$
1	one (mult. neutral)	$x \cdot 1 = x$

Geometry:

abs()	length, abs. value	$\|x\| \geq 0$

What can you do with $\mathbb{R}$?

Rules:

- From previous slide:
  - 0 "works": $\forall x$: $x + 0 = x$
  - 1 "works": $\forall x$: $x \cdot 1 = x$
- *Commutativity* of $+$ and $\cdot$ operators
  $\forall x, y$: $\quad x + y = y + x$ and $x \cdot y = y \cdot x$
- *Associativity* of $+$ and $\cdot$ operators:
  $\forall x, y, z$: $\; x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- *Distributivity* of $\cdot$ over $+$:
  $\forall x, y, z$: $\; x \cdot (y + z) = x \cdot y + x \cdot z$
- *Triangle inequality*: $\forall x, y$: $|x + y| \leq |x| + |y|$
- *C*-property*: $\forall x$: $|x \cdot x| \; = |x|^2$
  follows from multiplicativity of abs: $\forall x, y$: $|x \cdot y| = |x| \cdot |y|$

Why isn't $\mathbb{R}$ enough?

Q: Why do we need complex numbers? Why aren't real numbers enough?

- Math answer:
  Some things don't work with real numbers, because they are not algebraically closed meaning there are polynomials which nave no zero. $X^2 + 1$ is an example: There is no real x such that when you plug it in for X the result is 0. That doesn't seem to be a big deal, but we'll see some bad (for algorithms) consequences later in the course.

- Physics answer 2:
  Because the universe has complex numbers as amplitudes of wave functions in quantum physics.

Definition of $\mathbb{C}$, the complex numbers:

1. $\mathbb{R} \subset \mathbb{C}$ — every real number is (or: can be considered as) a complex number

2. There's a special complex number $\mathbf{i} \in \mathbb{C}$ with the property $\mathbf{i}^2 + 1 = 0$.

   There are 2 with that property: You have to decide which one of the two gets to be Mr. $\mathbf{i}$.)

3. The arithmetic operations and `abs()` are availabe for $\mathbb{C}$.

   They follow the same rules — *with 1 exception!* (and some new rules: next slide)

4. For every $z \in \mathbb{C}$ there is 1 unique pair (x, y) of real numbers such that

$$z = x + \mathbf{i}y$$

x is called the *real part* of z,

y is called the *imaginary part* of z.

Differences in the rules:

- From previous slide:
  - 0 "works": $\forall x$: $x + 0 = x$
  - 1 "works": $\forall x$: $x \cdot 1 = x$
- *Commutativity* of $+$ and $\cdot$ operators
  $\forall x, y$: $\quad x + y = y + x$ and $x \cdot y = y \cdot x$
- *Associativity* of $+$ and $\cdot$ operators:
  $\forall x, y, z$: $\; x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- *Distributivity* of $\cdot$ over $+$:
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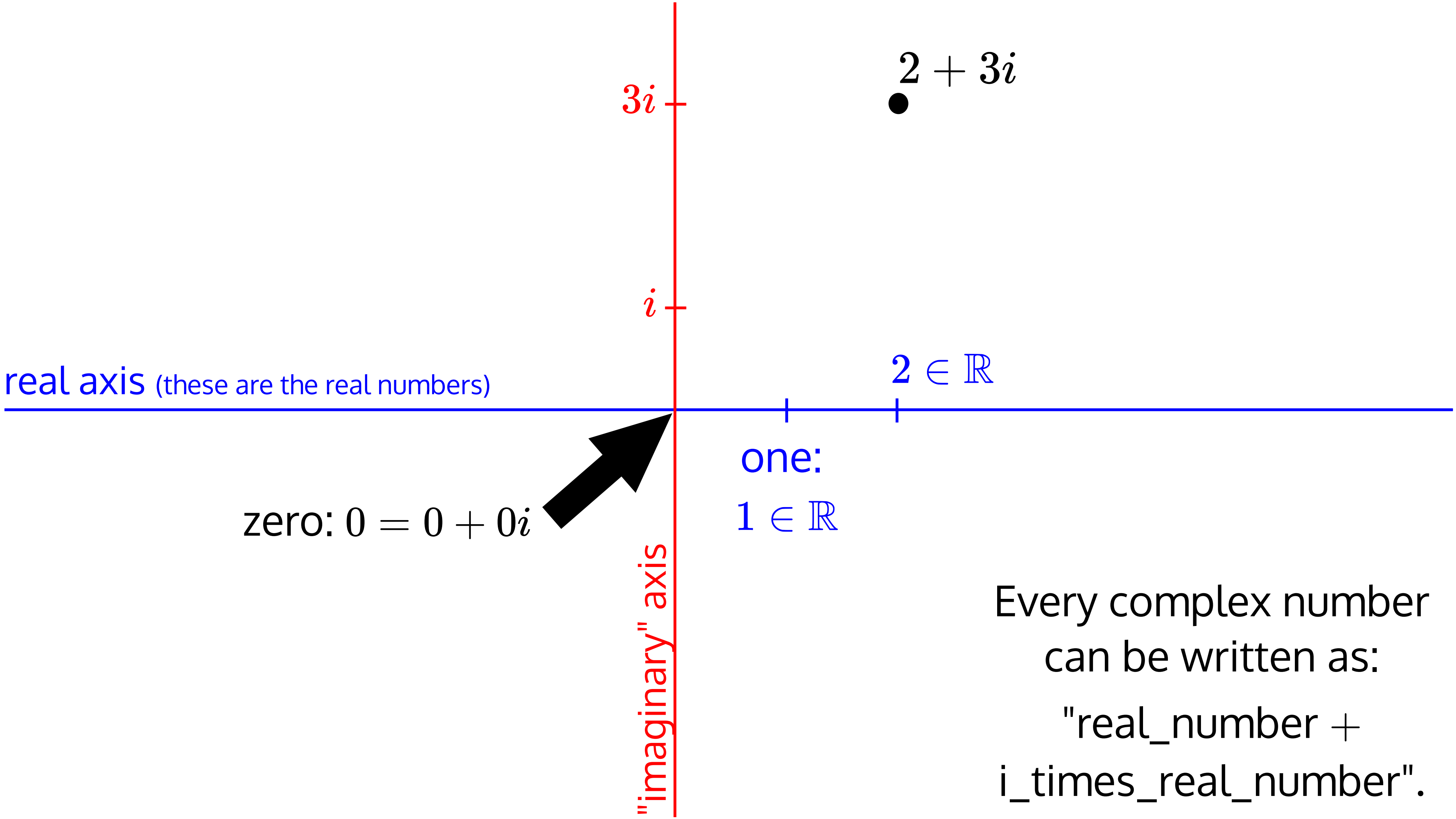
- *Triangle inequality*: $\forall x, y$: $|x + y| \leq |x| + |y|$
- *C*-property*: $\forall x$: $|x^* \cdot x| = |x|^2$
  follows from multiplicativity of abs: $\forall x, y$: $|x \cdot y| = |x| \cdot |y|$

new operator "$*$" *complex conjugate*

The rules are important — but let's not get lost in them.

Let's draw a picture! 🕺



Every complex number can be written as: "real_number $+$ i_times_real_number".

Arithmetic in $\mathbb{C}$

Visualize:	Points in the 2d plane: For complex number $z = x + iy$, draw (x, y)
Zero, one	Origin ($x = 0$, $y = 0$) is "zero": $0 + i \cdot 0 = 0$ $x = 1, y = 0$ is "one": $1 + i \cdot 0 = 1$
Addition	Just follow the rules! (Blackboard)
Subtraction	Just follow the rules! (Blackboard)
Multiplication	Let's see: $\begin{aligned}(x + i \cdot y) \cdot (u + i \cdot v) &= x \cdot u + x \cdot i \cdot v + i \cdot y \cdot u + i \cdot y \cdot i \cdot v \\ &= x \cdot u + i \cdot x \cdot v + i \cdot y \cdot u + i^2 \cdot y \cdot v \\ &= x \cdot u + i \cdot x \cdot v + i \cdot y \cdot u + (-1) \cdot y \cdot v \\ &= x \cdot u - y \cdot v + i \cdot (x \cdot v + y \cdot u)\end{aligned}$

Real and imaginary part functions

We said above: For $z \in \mathbb{C}$, it's real part and imaginary part are unique. In other words:

$$x_1 + iy_1 = z = x_2 + iy_2 \;\; \text{implies} \;\; x_1 = x_2 \;\&\; y_1 = y_2$$

So we can define functions (in the mathematical sense: unique assignment):

$$\Re \colon \mathbb{C} \to \mathbb{R} \colon \Re(x + iy) = x \qquad \text{real part}$$

$$\Im \colon \mathbb{C} \to \mathbb{R} \colon \Im(x + iy) = y \qquad \text{imaginary part}$$

Complex conjugate with it's rules

Definition of conj:

$$(x+iy)^{*} := x - iy$$

In math, usually:

$$\overline{z} = z^{*}$$

New rules:

1. "$\mathbb{R}$-linear": $\forall \alpha, \beta \in \mathbb{R}, \forall z, w \in \mathbb{C}$:
   $$(\alpha z + \beta w)^{*} = \alpha z^{*} + \beta w^{*}$$
2. "Multiplicative": $\forall z, w \in \mathbb{C}$
   $$(w \cdot z)^{*} = z^{*} \cdot w^{*}.$$
3. "Involution": $(z^{*})^{*} = z$

Absolute value (modulus)

Definition of abs:

$$|z| := \sqrt{(\Re z)^2 + (\Im z)^2} = \sqrt{z^* z}$$

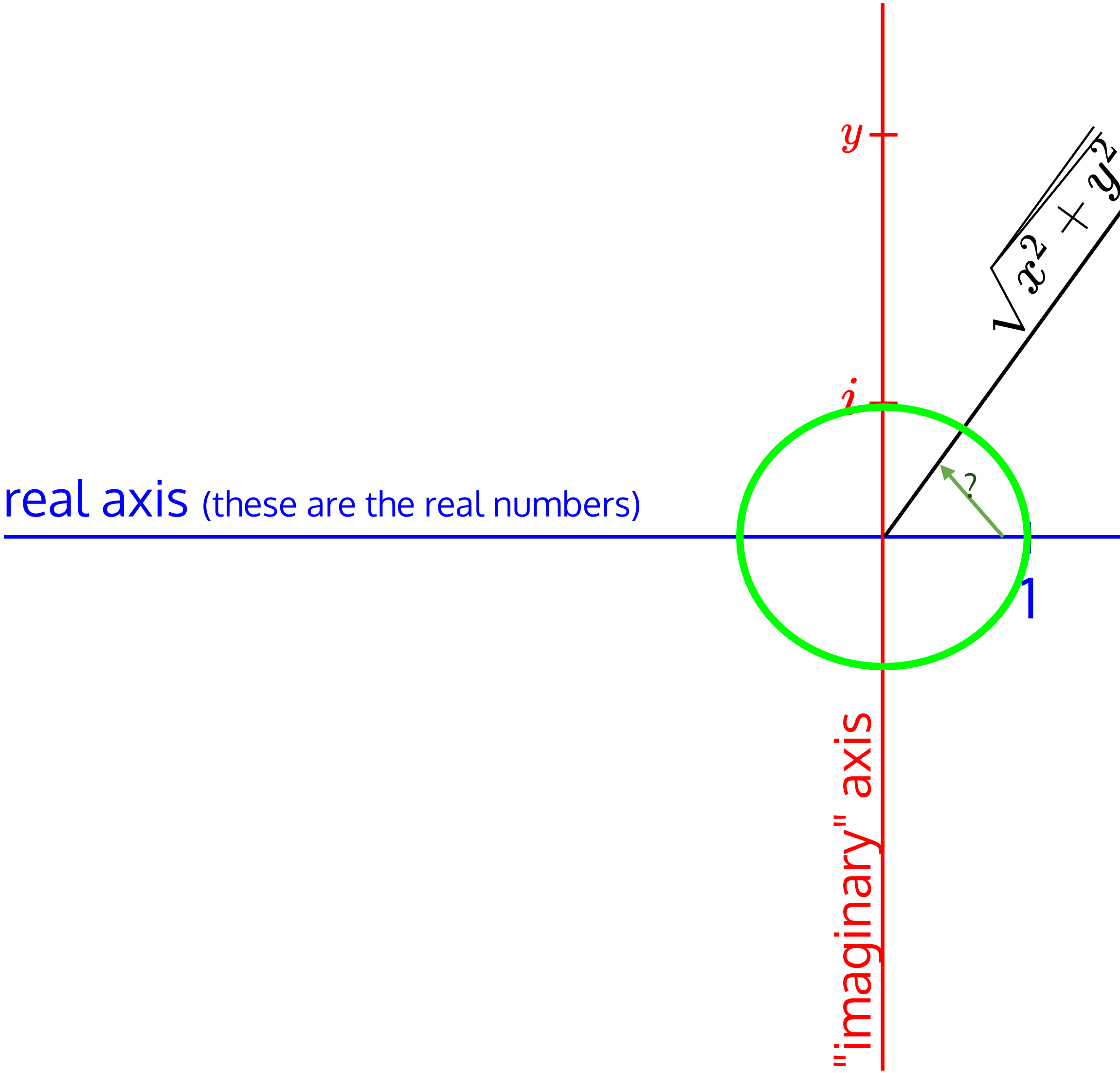
$$\begin{aligned}
&(x+iy)^* \cdot (x+iy) \\
&= (x-iy)(x+iy) \\
&= x^2 - (iy)^2 \\
&= x^2 - i^2 y^2 \\
&= x^2 + y^2
\end{aligned}$$

Verify multiplicativity of abs:

- $|wz| = \sqrt{z^* w^* w z} = \sqrt{w^* w\, z^* z} = \sqrt{w^* w}\sqrt{z^* z} = |w| \cdot |z|$

Verifying the triangle inequality is more involved. We don't do it here.


Too many
letters!!!



Points in the plane can be identified by:

- "argument"

  (=counter-clockwise angle from real axis)
- "modulus" ($=$ **`abs()`**)

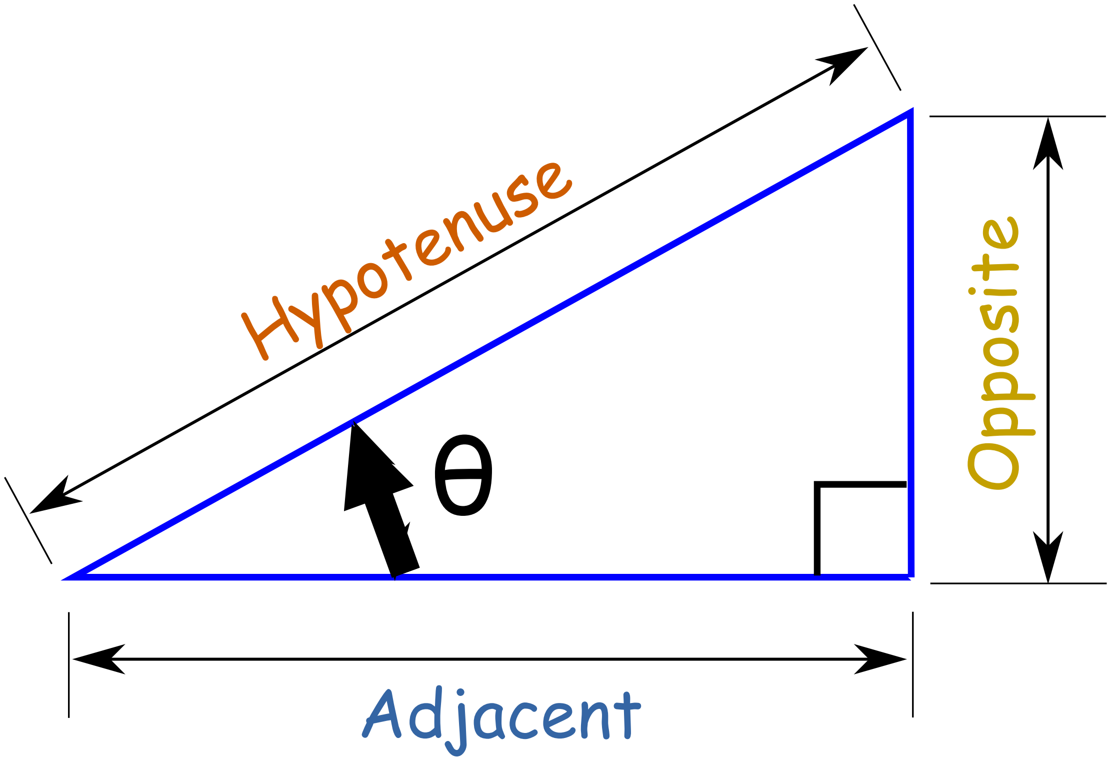
  (=length of vector)

Remember Highschool?

$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

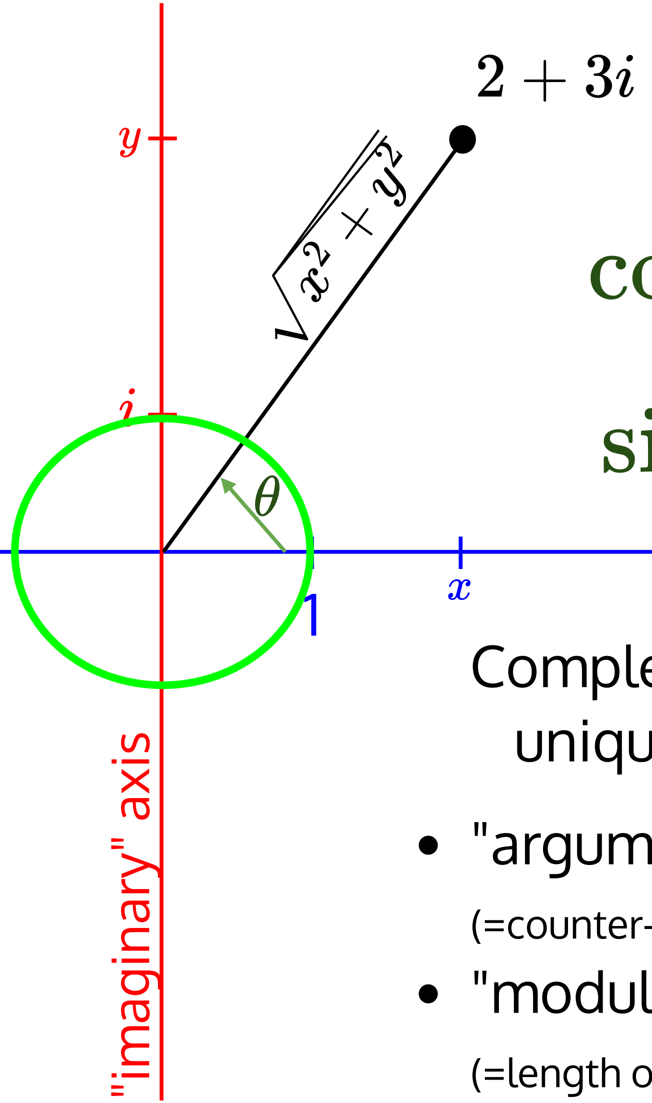


$|z| :=$ length of z

$= \sqrt{x^2 + y^2}$

$= \sqrt{z^* \cdot z}$

$z = x + iy$



$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Complex numbers can be uniquely identified by:

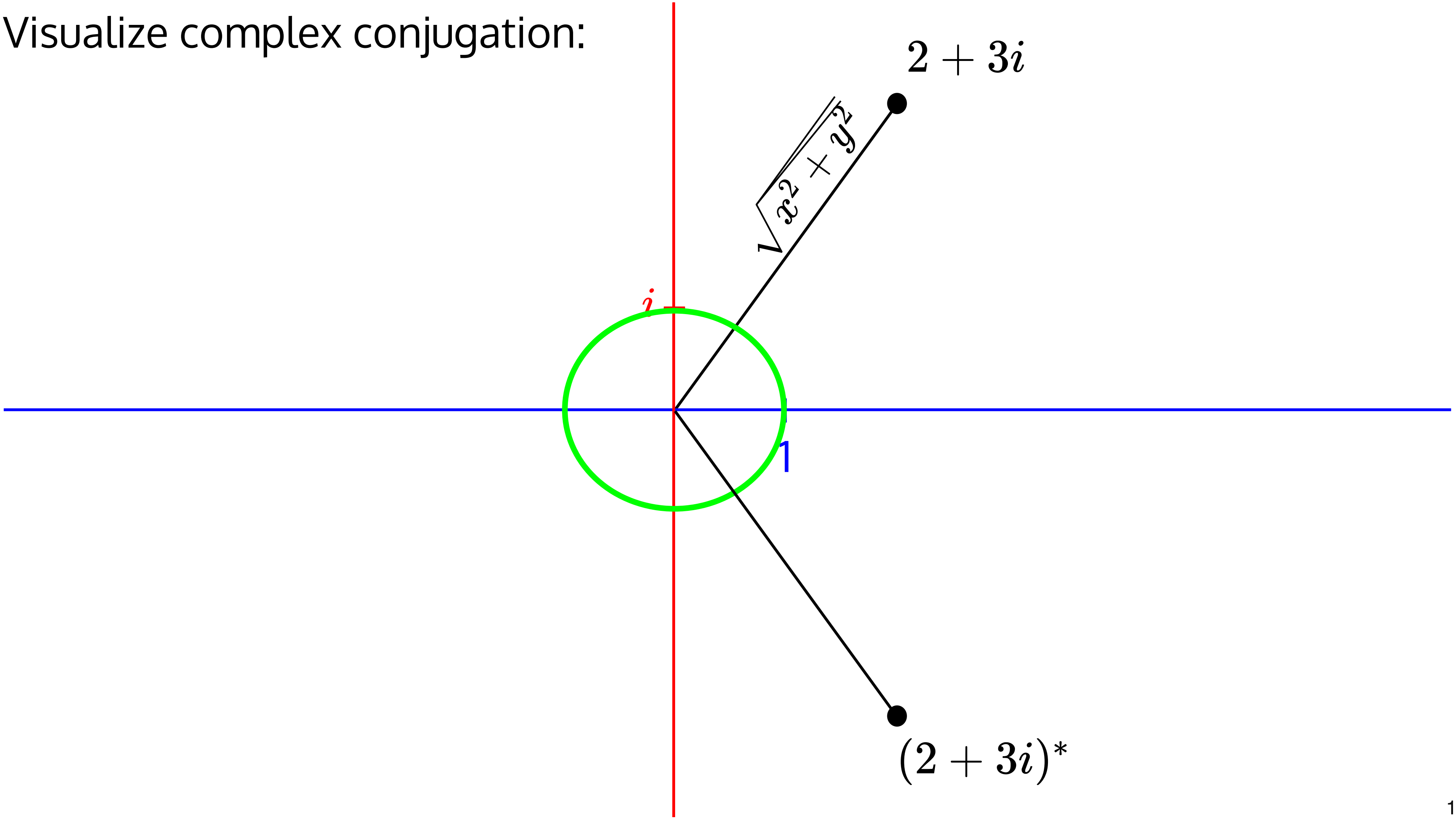
- "argument"

  (=counter-clockwise angle from real axis)

- "modulus"

  (=length of vector)

Visualize complex conjugation:



We still don't know how to divide...

...or do we?

Let's work out how to divide by a real number!

$z = x + iy$, divide by $u \in \mathbb{R} \setminus \{0\}$:

$$z/u = \quad z \cdot \frac{1}{u} = (x + iy) \cdot \frac{1}{u} = x \cdot \frac{1}{u} + iy \cdot \frac{1}{u} = x/u + iy/u$$

in $\mathbb{R}$, we know how to do $\texttt{inv}(u) = 1/u$

The Inverse

$z = x + iy$

Informal mnemonic:

$\frac{1}{z} = \frac{z^*}{z^*} \cdot \frac{1}{z} = \frac{z^*}{z^* z}$

$$(x+iy)^* := x - iy$$

$$z^* z = |z|^2 \qquad = x^2 + y^2$$

Why? Because:

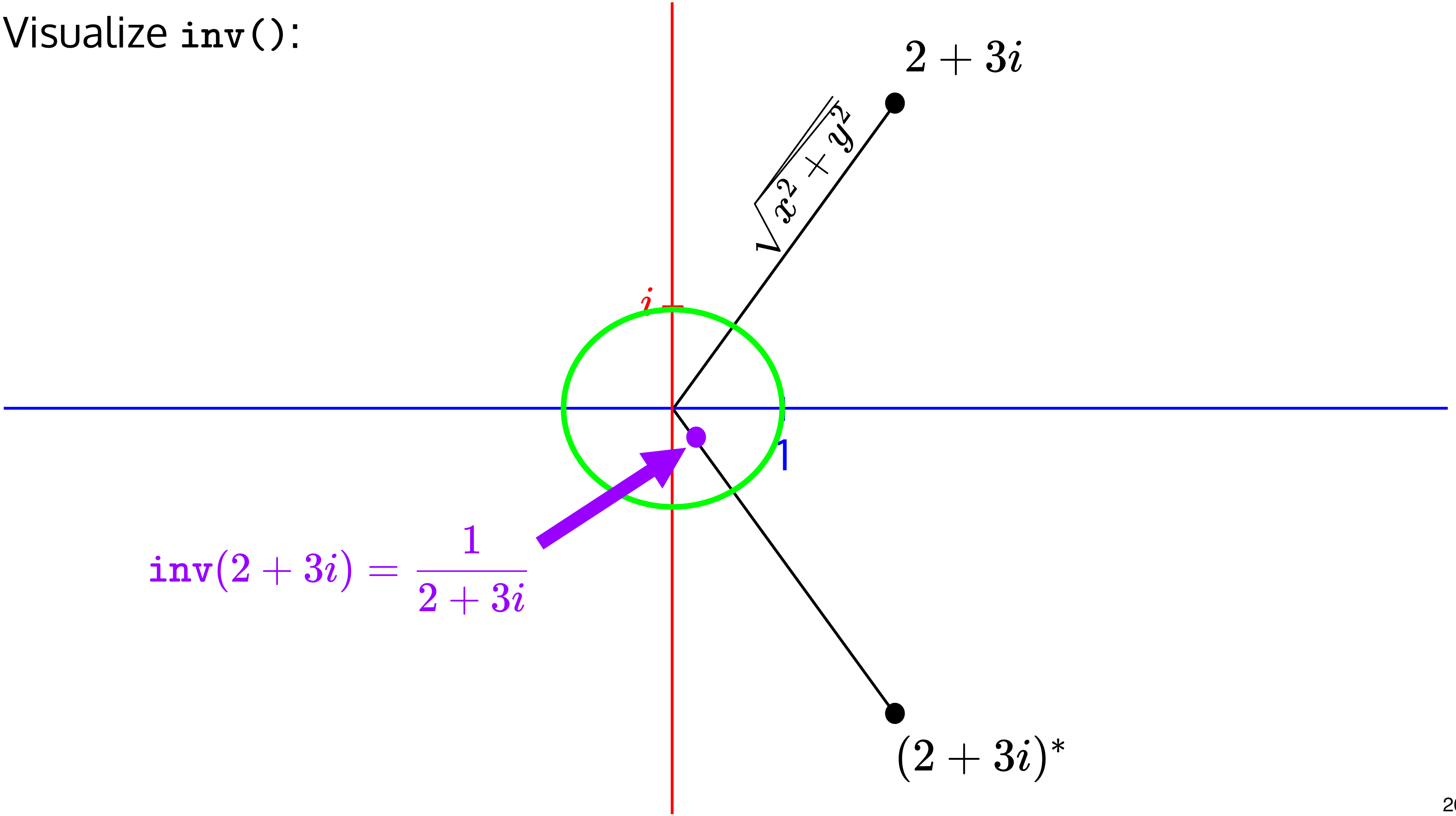
$$\mathrm{inv}(z) \cdot z = \frac{1}{z^* z} z^* \cdot z = \frac{z^* z}{z^* z} = 1$$

$$\boxed{\mathrm{inv}(z) := \frac{1}{|z|^2} \cdot z^*}$$

$$= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

inv. of real number: ✔

Visualize `inv()`:



Next:

- Exponential function
- Visualization of multiplication

And remember: Why do we learn it?

Because we need it for the exciting (quantum) stuff!