MTAT.05.008 Fou Math

Assoc. Prof. Dirk Oliver Theis

Homework # 01

Handed out:	Thu, Sep 1, 2022
Due:	Thu, Sep 8, 2022 (by 16:15)
As PDF by email to	`dotheis@ut.ee`
subject:	`FOUMATH-HW01-`*lastname*

Rules and Remarks

- Each problem has 25 points.
- If there's an error in the statement of a problem rendering the problem too easy, too difficult, too stupid, or plain wrong, the problem will be turned into a bonus problem. If the problem became too easy through the error, solving the easy problem won't give credits, but "repairing" the problem (i.e., understanding what was "meant") and then solving it will.
- Problems turned into bonus problems will be replaced so that the total number of problems over the course of the semester will stay the same.
- No group work, no copying from others, no copying from the internet. Violations will be reported to the Dean's Office.
- This first set of problems tests your basic math background (real numbers, functions, convergence/series, discrete math).

Problem 1

Let $f\colon\ [0,\infty[\ \to \mathbb{R}$ be a function with the following properties:

$$f(0) \geq 0 \tag{1a}$$

$$\text{for all } t \in [0,1] \text{ and } x, y \in [0,\infty[\text{:} \quad f((1-t)x + ty) \geq (1-t)f(x) + tf(y) \tag{1b}$$

Prove that for all $x, y \geq 0$:

$$f(x+y) \leq f(x) + f(y). \tag{1c}$$

Problem 2

Let n, k be positive integers. Determine, with proof, the number of ways in which the number n can be written as a sum of k nonnegative integers. In other words, what is the number of elements of the set

$$\Big\{x \in \mathbb{Z}_+^k \;\Big|\; \sum_{j=1}^{k} x_j = n\Big\}, \tag{2}$$

(where $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$)?

Problem 3

For $\varepsilon > 0, x > 0$, consider the equation

$$x \cdot \Big(1 + \ln\big(\tfrac{1}{\varepsilon\sqrt{x}}\big)\Big) = 1. \tag{3}$$

Prove that for each fixed $\varepsilon > 0$, if $\varepsilon < 1$, then the equation has two distinct solutions $x_1, x_2 > 0$.

Problem 4

(a) Show that for all positive real numbers x, y the inequality

$$\sqrt{xy} \leq (x + y)/2 \tag{4a}$$

holds.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, such that the series

$$\sum_{n=0}^{\infty} a_n, \tag{4b}$$

converges. Prove that the series

$$\sum_{n=0}^{\infty} \sqrt{a_{n+1}a_n} \tag{4c}$$

converges, too.