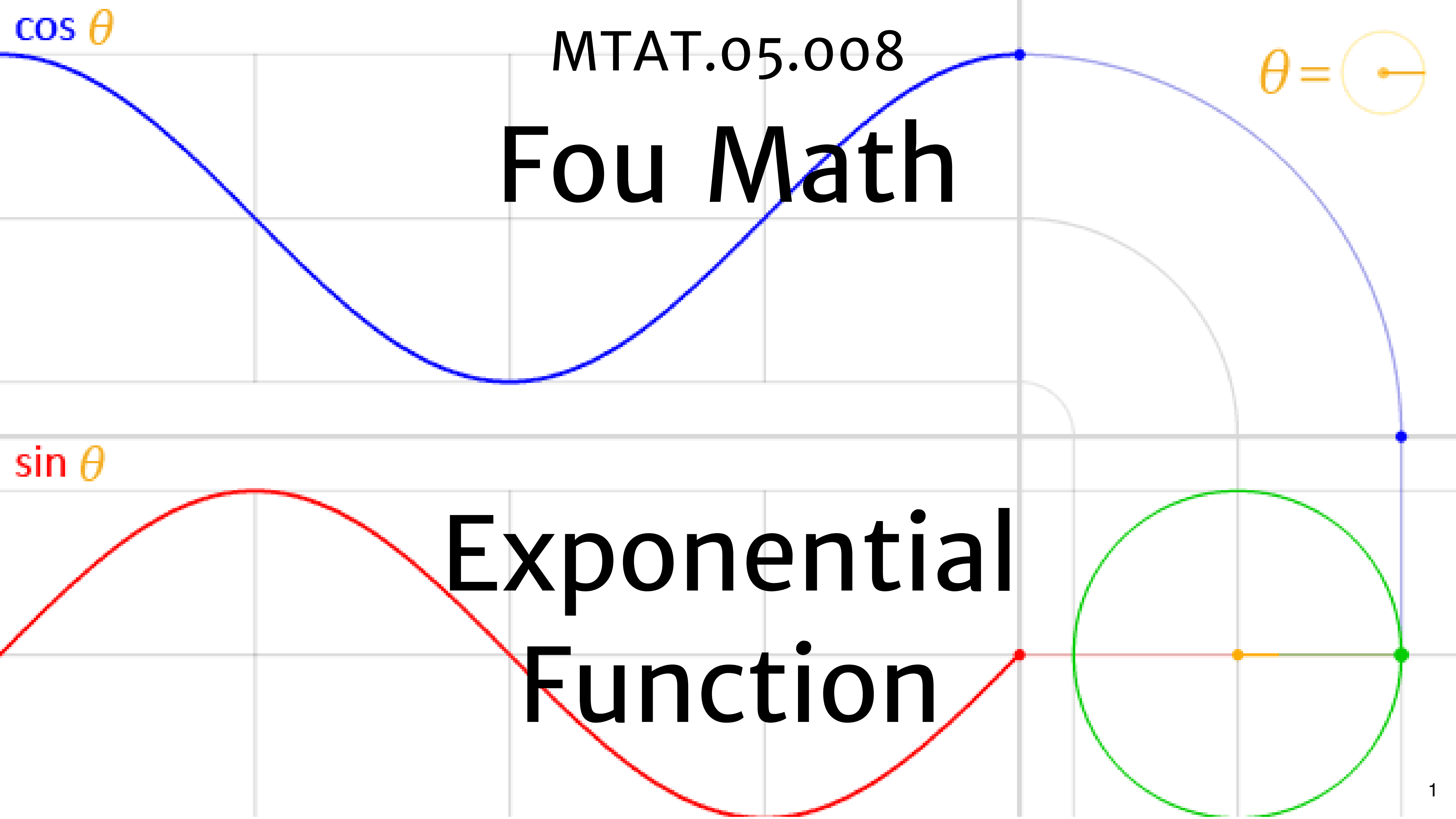
MTAT.05.008

Fou Math

Exponential Function



Exponential function

$X^0 = 1$ ALWAYS!
Even if $X = 0$.

$$\begin{aligned} e^a &:= \exp(a) \\ &:= 1 + a + a^2/2 + a^3/6 + \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} a^k := \lim_{K \to \infty} \sum_{k=0}^{K} \frac{1}{k!} a^k \end{aligned}$$

This *converges* for every a:

- For every a there exists a b (the limit) such that: You can get arbitrarily close to b by summing sufficiently many of the terms. In math:
  - For every $\epsilon > 0$ (the error) there's a $K_{\text{sufficient}}$ such that

$$\forall K \geq K_{\text{sufficient}} : \quad \left| \sum_{k=0}^{K} \frac{a^k}{k!} \quad - b \right| \leq \epsilon$$

- That justifies saying "$b = \sum_{k=0}^{\infty} \frac{a^k}{k!}$"

Does this converge?

1. $\sum_{k=0}^{\infty} (\frac{1}{2})^k$ Yes: Distance from 2 decreases towards zero
2. $\sum_{k=0}^{\infty} (+1)^k$ ***No!*** "Converges" to ∞ which is not allowed since $\infty \notin \mathbb{C}$.
3. $\sum_{k=0}^{\infty} (-1)^k$ ***No!*** Gets arbitrarily close to both 0 and 1, but doesn't "stay" with any of them for all $K \geq K_{\text{sufficient}}$
4. $\sum_{k=0}^{\infty} i^k$ ***No!*** Gets arbitrarily close to both $i, i-1, -1$, but doesn't "stay" with any of them for all $K \geq K_{\text{sufficient}}$
5. $\sum_{k=0}^{\infty} z^k$ ***It depends!*** $\sum_{k=1}^{K} z^k = \frac{1 - z^{K+1}}{1 - z}$ "geometric sum" $\to \frac{1}{1-z}$ if $|z| < 1$; for $z = 1, -1, i$, see above!

Exponential function

$$
\begin{aligned}
e^a &:= \exp(a) \\
&:= 1 + a + a^2/2 + a^3/6 + \ldots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} a^k := \lim_{K\to\infty} \sum_{k=0}^{K} \frac{1}{k!} a^k
\end{aligned}
$$

The "a" can be from any class A of objects ("type") which allow:

1. Multiplication: Operator `*(a::A, b::A)::A = ...`
2. Scalar-multiplication: Operator `*(α::ℝ, b::A)::A = ...`
3. Addition: Operator `+(a::A, b::A)::A = ...`
4. Length / norm / absolute value: `abs(a::A)::ℝ` or `norm(a::A)::ℝ`

$|a|$ $\|a\|$

Today, we'll only do

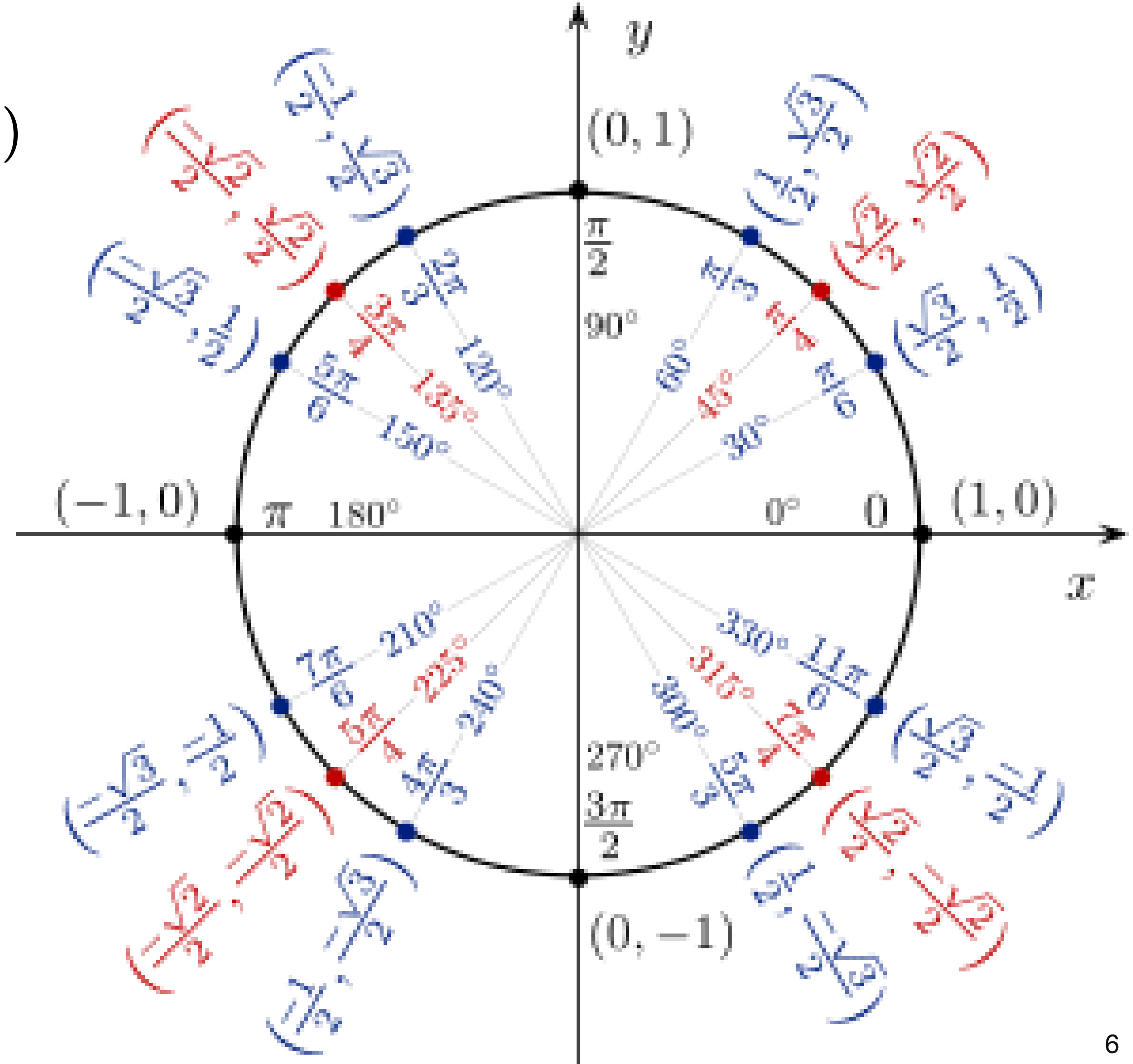
$$A = \mathbb{C}$$

... but I'll point out the points where things will be different once we allow more general "types" A.

Some calculations

- Let's compute $e := e^1 = \exp(1)$
- Let's compute e^i

- $e^{0 \cdot i} = e^0 = 1$
- $e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$
- $e^{i\pi/2} = i$
- $e^{i\pi} = -1$
- $e^{3\pi i/2} - i$
- $e^{2\pi i} = 1$



The two important properties of Exp()

Theorem. For all $w, z \in \mathbb{C}$: $e^{w+z} = e^w \cdot e^z$.

Proposition. For all $z \in \mathbb{C}$: $(e^z)^* = e^{(z^*)}$.

The two important properties of Exp()

Proposition. For all $z \in \mathbb{C}$: $\quad (e^z)^* = e^{(z^*)}$.

The 2nd one is easy:

$$
\begin{aligned}
(e^z)^* &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} z^k\right)^* && \text{plug in def.} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{k!} z^k\right)^* && \texttt{conj}() \text{ is } \mathbb{R}\text{-linear} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(z^k\right)^* && \texttt{conj}() \text{ is } \mathbb{R}\text{-linear} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (z^*)^k && \texttt{conj}() \text{ is multiplicative} \\
&= e^{(z^*)} && \text{plug in def.}
\end{aligned}
$$

The most important property of Exp()

Theorem. For all $w, z \in \mathbb{C}$: $\quad e^{w+z} = e^w \cdot e^z$.

Proof (1)

Lemma (Binomial Theorem). For all $w, z \in \mathbb{C}$:

$$(w+z)^n = \sum_{j=0}^{n} \binom{n}{j} w^j z^{n-j}.$$

Example: $$\begin{aligned}(w+z)^2 &= (w+z)(w+z) \\ &= w^2 + wz + zw + z^2 \\ &= w^2 + wz + wz + z^2 \\ &= w^2 + 2wz + z^2\end{aligned}$$

Proof (2)

$$
\begin{aligned}
e^{w+z} &= \sum_{k=0}^{\infty} \frac{1}{k!}(w+z)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} w^{\ell} z^{k-\ell} && \text{Binomial Theorem} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} w^{\ell} z^{k-\ell} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{\ell!} w^{\ell} \frac{1}{(k-\ell)!} z^{k-\ell} \\
&= \sum_{s=0}^{\infty} \sum_{(a,b)\in\mathbb{N}^2, a+b=s} \frac{1}{a!} w^a \frac{1}{b!} z^b && \text{rename vars} \\
&= \sum_{a=0}^{\infty} \frac{1}{a!} w^a \sum_{b=0}^{\infty} \frac{1}{b!} z^b = e^w \cdot e^z
\end{aligned}
$$

Fun consequences

1. Look at the sum: For all $x \in \mathbb{R}$: $e^x \in [0, \infty) \subset \mathbb{R}$

2. For all $z \in \mathbb{C}$: $|e^z|^2 = \dots$
   - $(e^z)^* \cdot e^z = e^{(z^*)} \cdot e^z = e^{z^*+z} = e^{\Re(z) \cdot 2} = e^{\Re(z)+\Re(z)} = e^{\Re(z)} \cdot e^{\Re(z)} = \left(e^{\Re(z)}\right)^2$
   - Hence: $|e^{x+iy}| = e^x$
   - In particular: $|e^{iy}| = e^0 = 1$

3. For all $z = x + iy \in \mathbb{C}$: $e^{x+iy} = \dots$
   - $e^z = e^x \cdot e^{iy} = |e^z| \cdot e^{iy}$

Same thing, written down differently:

For a given $z \in \mathbb{C}$ let $w := \ln(|z|) + i\arg(z)$

$$e^w = e^{\Re(w) + i\Im(w)} = e^{\Re(w)} \cdot e^{i\Im(w)} = e^{\ln|z|} \cdot e^{i\arg(z)} = |z| \cdot e^{i\arg(z)} = z$$

Why??

Sine and Cosine

$$\begin{aligned}\cos(a) &:= 1 - a^2/2 + a^4/4! - a^6/6! + \ldots \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} a^{2j}\end{aligned}$$

$$\begin{aligned}\sin(a) &:= a^1 - a^3/3! + a^5/5! - a^7/7! + \ldots \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} a^{2j+1}\end{aligned}$$

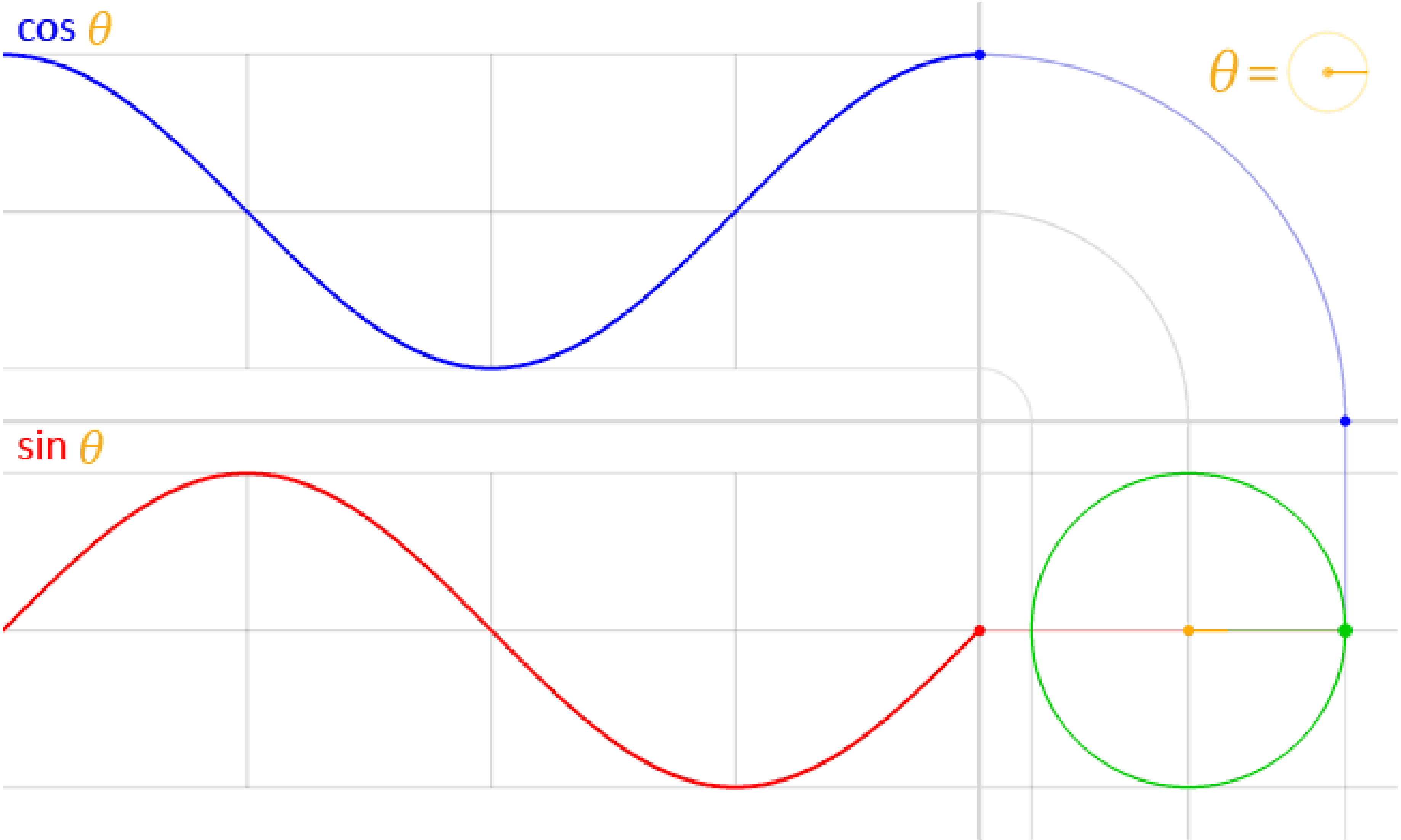
Sine and Cosine, and Exp

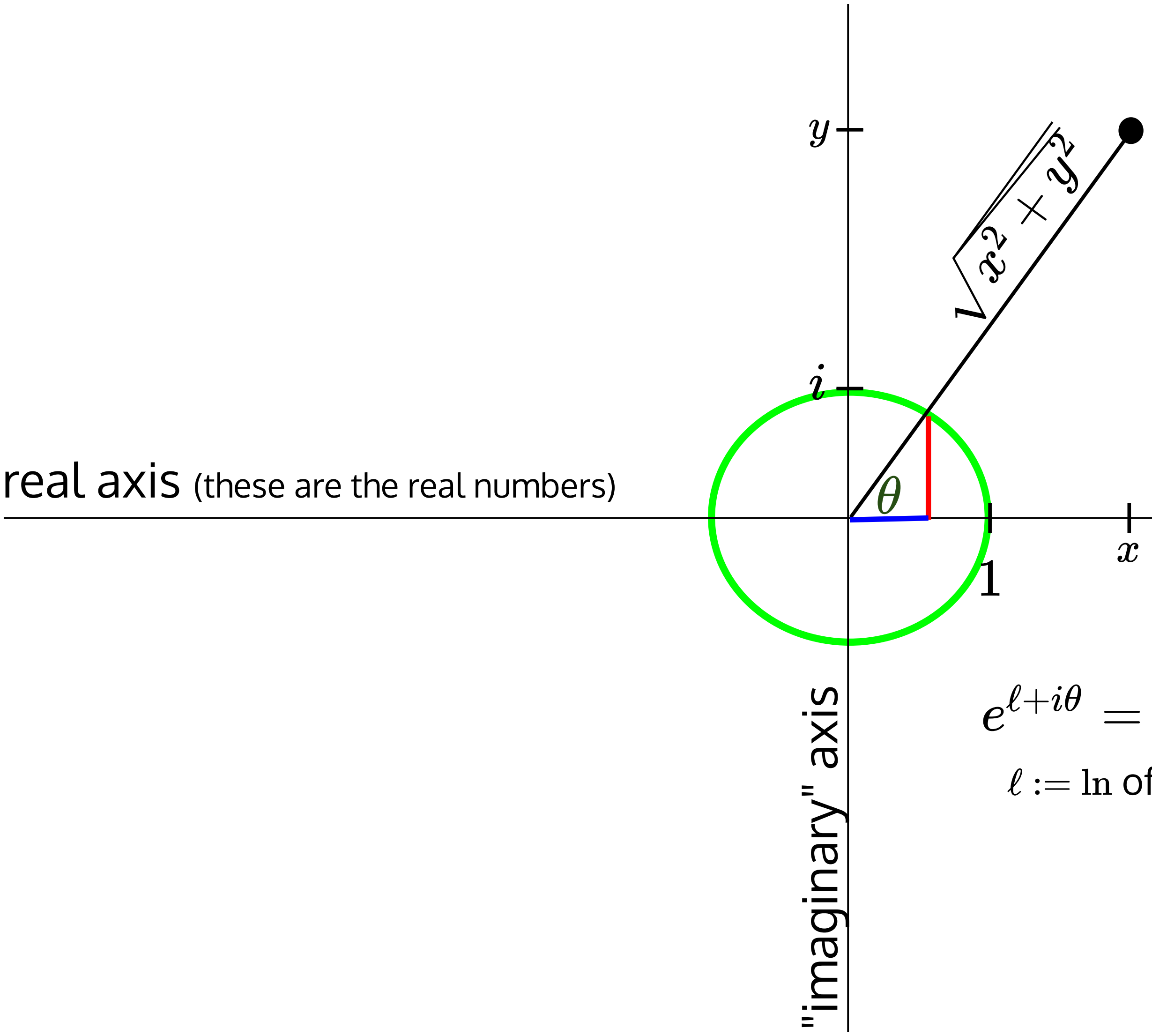
$$
\begin{aligned}
&\cos(a) + i\sin(a) \\
&= 1 - a^2/2 + a^4/4! - a^6/6! + \dots \\
&+ \quad ia^1 - ia^3/3! + ia^5/5! - ia^7/7! + \dots \\
&= 1 + (ia)^1 + (ia)^2/2 + (ia)^3/3! + (ia)^4/4! + (ia)^5/5! + \dots \\
&= \exp(ia)
\end{aligned}
$$

Euler Formula. $e^{ia} = \cos(a) + i\sin(a)$.

$$\boxed{e^{ia} = \cos(a) + i\sin(a)}$$

$$e^{w} = e^{u+iv} = \underbrace{e^{u}}_{\text{modulus}} \cdot e^{i\underbrace{\overset{\in \mathbb{R}}{\downarrow}v}_{\text{argument}}} = e^{u} \cdot (\cos v + i \sin v)$$


cos θ
$\theta =$
sin θ

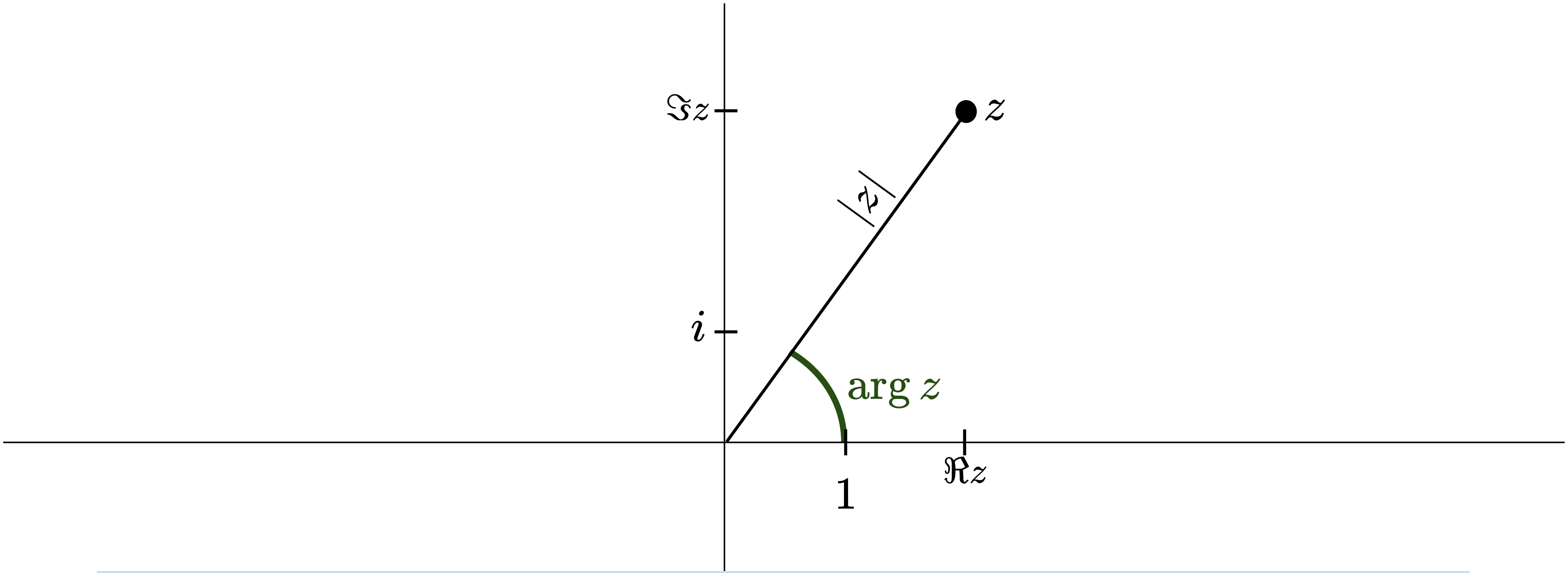


$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$e^{\ell+i\theta} = e^{\ell}\cdot e^{i\theta} = e^{\ell}\cdot(\cos\theta + i\sin\theta)$$

$\ell :=$ ln of modulus



$$z = |z| \cdot e^{i \arg z}$$

$\mathbb{C}$-Multiplication — Geometric view

Let $w, z \in \mathbb{C}$

$r := |w|$, $s := |z|$

$w = re^{i\theta}$, $z = se^{i\varphi}$

$$\boldsymbol{w \cdot z = re^{i\theta} \cdot se^{i\varphi} = rs\, e^{i(\theta+\varphi)}}$$

- Multiply the muduli
- Add the arguments

$$\boldsymbol{w/z = re^{i\theta}/\left(se^{i\varphi}\right) = r/s \cdot e^{i(\theta-\varphi)}}$$

- Divide the muduli
- Subtract the arguments

How does π come in?

Math fact: $e^{i\pi/2} = i$

Proof sketch:

1. Fact: $x \mapsto \cos(x)$ is strictly decreasing on interval $[0, 2]$
2. Fact: $\cos(0) = 1$, $\cos(2) < 0$

Consequence: There exists a *unique* real number x_0 w/

$$\cos(x_0) = 0.$$

Because the person inventing it was hungry, he called x_0 "half a pie".

What about π?

Why is $e^{2\pi i} = 1$?

Because:

$$e^{2\pi i} = e^{\pi i/2+\pi i/2+\pi i/2+\pi i/2} = i^4 = 1$$

Simple facts to remember

For $z \in \mathbb{C}$:

- $\Re z := \dfrac{z + z^*}{2}$. This is the "real part" of z, since $\Re(x + iy) = x$

- $\Im z := \dfrac{z - z^*}{2i}$. This is the "imaginary part" of z, since $\Im(x + iy) = y$

- Note that $\cos\theta = \Re(e^{ix\theta}) = (e^{i\theta} + e^{-i\theta})/2$

  and $\sin\theta = \Im(e^{i\theta}) = (e^{i\theta} - e^{-i\theta})/2i$

What you have learned about $\mathbb{C}$ ...

1. How to identify complex numbers with points on the 2d plane
   - Cartesian coordinates ↔ $x + iy$ expressions
   - Polar coordinates ↔ $r\, e^{i\theta}$ expressions
2. How addition (and subtraction) works:
   - Arithmetically in $x + iy$ expressions
   - Geometrically as 2d vectors
3. How multiplication works:
   - Arithmetically in $x + iy$ expressions
   - Arithmetically in $r\, e^{i\theta}$ expressions
   - Geometrically in 2d via polar coordinates
4. How multiplicative inverse / division works:
   - Arithmetically with complex conjugation
   - Arithmetically in $r\, e^{i\theta}$ expressions
   - Geometrically in 2d with scaling and reflection