Time evolution of a membrane

Project for Computational Physics II, WS19/20 — *March 2, 2020*

1 Grading scheme

Weight	Assessed competence
20%	Physical Transcription + Planning
35%	Implementation
15%	Testing
30%	Running + Numerical Analysis + Visualization + Physical Analysis

Deadline: 30.03.2020

2 Physical system

We consider an elastic square membrane in tension: think for instance of the membrane of a drum. At rest, the membrane lies on a plane and each of its points is identified by a pair of coordinates $\mathbf{x} = (x_1, x_2)$ with $0 \leq x_{1,2} \leq 1$ (we assume some arbitrary physical units).

When the membrane vibrates, it is displaced from the rest plane. At time t, let $h_t(\mathbf{x})$ be the the displacement of the point $\mathbf{x}$ in the orthogonal direction to the rest plane. We assume that the membrane is held fixed at the boundary, i.e. we assume the boundary conditions $h_t(\mathbf{x}) = 0$ if $\mathbf{x}$ belongs to the boundary of the membrane.

The Hamiltonian of the membrane is given by

$$H(\pi, h) = \int_0^1 dx_1 dx_2 \left\{ \frac{1}{2}\pi^2(\mathbf{x}) + \frac{1}{2}\sum_{\mu=1}^{2}(\partial_\mu h)^2(\mathbf{x}) \right\} = \int_0^1 dx_1 dx_2 \left\{ \frac{1}{2}\pi^2(\mathbf{x}) - \frac{1}{2} h \nabla^2 h(\mathbf{x}) \right\} . \tag{1}$$

We will also consider the possibility to immers the membrane in a translational-invariant external potential $V(h)$ which yields the following Hamiltonian

$$H(\pi, h) = \int_0^1 dx_1 dx_2 \left\{ \frac{1}{2}\pi^2(\mathbf{x}) - \frac{1}{2} h \nabla^2 h(\mathbf{x}) + V(h(\mathbf{x})) \right\} . \tag{2}$$

The mathematics of this problem is the same as the mathematics of a classical real scalar field in 1+2 dimensions (the generalization to more dimensions is trivial).

The system is discretized by introducing a lattice spacing $a = 1/N$ and by restricting the coordinates to a lattice

$$n_\mu = \frac{x_\mu}{a} \in \{0, 1, 2, \ldots, N\} , \quad \text{for } \mu = 1, 2 . \tag{3}$$

Notice that N is the number of intervals and $N + 1$ is the number of points (this is different from the periodic case).

The discretized Hamiltonian can be defined as

$$\hat{H}(\pi, h) = \sum_{n_1, n_2 = 0}^{N} a^2 \left\{ \frac{1}{2}\pi^2(\mathbf{n}) - \frac{1}{2} h \hat{\nabla}^2 h(\mathbf{n}) + V(h(\mathbf{n})) \right\} . \tag{4}$$

We will take for $\hat{\nabla}^2$ the usual discretization of the Laplacian with the correct boundary conditions.

The equation of motions can be written in the following form

$$\dot{h}_t(\mathbf{n}) = \frac{1}{a^2} \frac{\partial H}{\partial \pi(\mathbf{n})}(\pi_t, h_t) = \pi_t(\mathbf{n}) , \tag{5}$$

$$\dot{\pi}_t(\mathbf{x}) = -\frac{1}{a^2} \frac{\partial H}{\partial h(\mathbf{n})}(\pi_t, h_t) = \hat{\nabla}^2 h(\mathbf{n}) - V'(h(\mathbf{n})) . \tag{6}$$

These equations of motion can be solved iteratively with approximated integrators. We will use a particular class of integrators which are suitable for equations of motion of the Hamiltonian form: the *symplectic integrators*.

We will consider a potential of the type (this kind of potential is found e.g. in the Hamiltonian of the Higgs particle)

$$V(h) = \frac{\lambda}{4!} h^4 . \tag{7}$$

3 Project goals

- Write a code that solves the equations of motion for a given initial condition up to some time T.
- Develop and write tests for the various parts of the code.
- Two symplectic integrators should be used: the 2nd order leapfrog integrator and the 4th order Yoshida integrator. Their description can be found on `https://en.wikipedia.org/wiki/Leapfrog_integration`. More details can be found e.g. in `https://escholarship.org/content/qt35h9v2k9/qt35h9v2k9.pdf`.
- The code should give the possibility to save (π_t, h_t) on disk at regular time intervals (which scan be larger than the integration step). This should be used to produce an animation of the time evolution.
- The code should calculate some observables on the fly: the energy, the value of (π_t, h_t) at the center of the membrane. This should be used to build plots of these observables as a function of time. What other observables may be interesting to calculate?
- In the following, we fix the initial condition to an approximation of the pyramid with vertex in $\mathbf{x} = (1/4, 1/4)$ and $h(1/4, 1/4) = 1$.
- If τ is the integration step, the energy is conserved in the $\tau \to 0$ limit. However, at non-zero values of τ, energy conservation is violated. Define the violation of energy conservation as

  $$\Delta H_t = H(\pi_t, h_t) - H(\pi_0, h_0) . \tag{8}$$

  Fix $N = 20$, $\lambda = 0.1$, $T = 12$, and calculate (and plot) the violation of energy conservation as a function of t for several values of τ, and for both integrators. Investigate how energy conservation is recovered in the $\tau \to 0$ limit. (This is also a test of the algorithms).
- Fix $N = 20$, $\lambda = 0.1$, $T = 12$, and calculate (and plot) the displacement of the membrane in the center, i.e. $h_t(1/2, 1/2)$, and its momentum , i.e. $\pi_t(1/2, 1/2)$ as a function of t for several values of τ, and for both integrators. Investigate how the two integrators give the same result in the $\tau \to 0$ limit.
- The continuous physical system is obtained in the $N \to \infty$ and $\tau \to 0$ limits. Using only the 4th order integrator, investigate quantitatively how large N needs to be and how small τ needs to be to get a good approximation of the continuous system and of the continuous evolution up to a maximum time $T = 12$ (for $\lambda = 0.1$). For the chosen values of N and τ, plot all observables, and produce the animations of the membrane vibration.