Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory
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Appendix C

Notation

In this section we discuss the conventions, abbreviations, mathematical symbols, and symbols used in the book.

C.1 Conventions

The following conventions have been used:

1. Boldface roman denotes a vector or matrix.
2. The symbol $|\cdot|$ means the magnitude of the scalar contained within.
3. The symbol $\|\cdot\|$ denotes the Euclidean norm of the vector or matrix contained within.
4. Multiple integrals are frequently written as,

$$\int d\tau f(\tau) \int dt g(t,\tau) \triangleq \int f(\tau) \left\{ \int dt g(t,\tau) \right\} d\tau,$$

   that is, an integral is inside all integrals to its left unless a multiplication is specifically indicated by parentheses.
5. $E[\cdot]$ denotes the statistical expectation of the quantity in the bracket.
6. The probability density of x is denoted by $p_x(\cdot)$ and the probability distribution by $P_x(\cdot)$. The probability of an event, A, is denoted by $Pr[A]$. The probability density of x, given that the random variable a has a value A, is denoted by $p_{x|a}(X|A)$.

7. A vertical line in an expression means "such that" or "given that"; that is, $Pr[A|x \leq X]$ is the probability that event A occurs given that the random variable x is less than or equal to the value of X.

We list acronyms used in the text.

C.2 Acronyms

AAR	adaptive angular response
ACCR	asymptotic conditional CRB
AIC	Akaike Information Criterion
AM	alternating maximization
AMF	adaptive matched filter
AML	asymptotic ML estimator
AP	alternating projection
AR	auto-regressive (model)
ARMA	auto-regressive moving average (model)
ASNR	array signal-to-noise ratio
BW_{NN}	null-null bandwidth
BS	beamspace
CCRB	conditional CRB
CFAR	constant false alarm rate
CM	constant modulus
CML	conditional ML estimator
CMT	covariance matrix taper
CRB	Cramér-Rao bound
DEMT	Detection, Estimation, and Modulation Theory
DFT	discrete Fourier transform
DISPARE	distributed signal parameter estimator
DL	diagonal loading
DMI	direct matrix inversion
DMR	dominant-mode rejection (beamformer)
DOA	direction of arrival
DOF	degrees of freedom
DPSS	discrete prolate spheroidal sequence
DSPE	distributed signal parameter estimation

EV	eigenvalue
EM	expectation maximization
ES	eigenspace
ESPRIT	estimation of signal parameter via rotational invariance technique
FB	forward–backward (averaging)
FBSS	forward-backward spatial smoothing
FCA	filled circular array
FIM	Fisher information matrix
FIR	finite impulse response
FO	forward-only (averaging)
GL	grating lobe
GLRT	generalized LRT
GSC	generalized sidelobe canceller
HPBW	half-power bandwidth
HCRB	hybrid CRB
HSST	Householder subspace transformation
ICASSP	International Conference on Acoustic, Speech, and Signal Processing
IDFT	inverse discrete Fourier transform
IMODE	iterative MODE
INR	interference-to-noise ratio
IQML	iterative quadratic maximum likelihood
IQML-QC	IQML with quadratic constraint
IR	invisible region
JASA	Journal Acoustic Society of America
LCMP	linear constrained minimum power (beamformer)
LCMV	linear constrained minimum variance (beamformer)
LEO	low-earth orbit
LMS	least mean square
LNR	loading-to-noise ratio

LP	linear prediction
LRT	likelihood ratio test
LS	least squares
LS-ESPRIT	least squares ESPRIT
ML	maximum likelihood
MA	moving average (model)
MAP	maximum *a posteriori* probability
MBA	multiple beam antenna
MDL	minimum description length
MMSE	minimum mean-square error
MODE	method of direction estimation
MPDR	minimum power distortionless response
MPQR	minimum power quiescent response
MRA	(i) minimum redundancy array (ii) maximum response axis
MRLA	minimum redundancy linear array
MUSIC	multiple signal characterization
MVDR	minimum variance distortionless response
MVQR	minimum variance quiescent response
PM	Parks-McClellan
QC	quadratic constraint
QP	quiescent pattern
QRD	QR decomposition
RMSE	root mean-square error
RLS	recursive least square
SALP	subaperture linear prediction
SCMV	soft constraint minimum variance
SH	sequential hypothesis
SHA	standard hexagonal array
SINR	signal-to-interference and noise ratio
SLA	standard linear array ($d = \lambda/2$)
SLL	sidelobe level
SMI	sample matrix inversion
SNR	signal-to-noise ratio
SRA	standard rectangular array ($d_x = d_y = \lambda/2$)
STAP	space-time adaptive processing

SS	(i) spatial smoothing (ii) steady state
SVD	singular value decomposition
TAM	Toeplitz approximation method
TDL	tapped delay line
TDRSS	Tracking and Data Relay Satellite System
TLS	total least squares
TLS-ESPRIT	total least squares ESPRIT
TNA	thermal noise algorithm
UCA	uniform circular array
UHA	uniform hexagonal array
ULA	uniform linear array
UML	unconditional ML estimator
URA	uniform rectangular array
VR	visible region
WES	weighted eigenspace
WSF	weighted subspace fitting

C.3 Mathematical Symbols

$\mathbf{a}^H$, $\mathbf{A}^H$	conjugate transpose (A.6)
$\mathbf{a}^T$, $\mathbf{A}^T$	transpose (A.4)
$\mathbf{A}^{-1}$	inverse (A.40)
$\Vert\mathbf{a}\Vert$	Euclidean norm of $\mathbf{a}$ (A.36)
$\mathbf{A}^*$	conjugate of $\mathbf{A}$
$[\mathbf{A}]_{ij}$	ij element of $\mathbf{A}$ (A.1)
$[\mathbf{A}]_k^{(r)}$	kth row of $\mathbf{A}$
$[\mathbf{A}]_k^{(c)}$	kth column of $\mathbf{A}$
$\square$	Khatri-Rao product (A.96)
$\Vert\mathbf{A}\Vert_F$	Frobenius (Euclidean) norm of $\mathbf{A}$ (A.39)
$CN[\mathbf{m}, \mathbf{\Lambda}]$	probability density of complex Gaussian (normal) vector
$\otimes$	Kronecker product (A.80)
$\triangleq$	defined as
$\det \mathbf{A}$, $\vert\mathbf{A}\vert$	determinant of $\mathbf{A}$ (A.20)

dim[$\mathbf{A}$]	number of free parameters in $\mathbf{A}$
$\doteq$	equal to first order
$\odot$	Hadamard product (A.71)
log	logarithm: base 10
ln	logarithm: base e
$(n)!$	n factorial
$\nabla_{\boldsymbol{\theta}}$	partial derivative matrix operator
$\propto$	proportional to
$o(N)$	negligible as $N \to \infty$
$O(N)$	proportional to N as $N \to \infty$
orth[$\mathbf{A}$]	columns of $\mathbf{A}$ are orthonormalized
$Re\{\cdot\}$	real part
$Im\{\cdot\}$	imaginary part
rank ($\mathbf{A}$)	rank of $\mathbf{A}$ (A.34)
tr ($\mathbf{A}$)	trace of $\mathbf{A}$ (A.26)
$vec(\mathbf{A})$	stacked vector (A.112)
$\hat{\mathbf{x}}$	estimate of $\mathbf{x}$
$t \to T^-$	t approaches T from below
$A + B \triangleq A \cup B$	A or B or both
$l.i.m.$	limit in the mean
$\int_{-\infty}^{\infty} d\mathbf{x}$	an integral over the same dimension as the vector
diag$[a_1, a_2, \cdots, a_N]$	diagonal matrix with elements a_1, a_2, $\cdots$, a_N
$\int_{\Omega} d\mathbf{x}$	integral over the set Ω
$\mathbf{0}$	matrix with all zero elements
$\mathbf{1}$	unity vector: $[1\ 1\ \cdots\ 1]^T$
$\begin{pmatrix} N \\ k \end{pmatrix}$	binomial coefficient $\left(= \frac{N!}{k!(N-k)!}\right)$

C.4 Symbols

We list symbols that are used in the text. The equation number indicates where the symbol was defined. If the symbol was defined in the text, a nearby equation number is listed. Symbols that are obvious modifications of other symbols are not included.

$\mathbf{a}$	signal direction vector (2.15)
$\mathbf{A}(K)$	data matrix ($\mathbf{A}(K) = \sqrt{K}\tilde{\mathbf{X}}^T$) (7.271)

A_c	conventional array gain (6.32)
A_{iso}	array gain: isotropic noise (2.144)
A_{mpdr}	array gain: MPDR beamformer (6.73)
A_{mvdr}	array gain: MVDR beamformer (6.28)
$\mathbf{A}_{\mu}(K)$	exponentially weighted data matrix (7.269)
$\mathbf{A}_L$	left circulant matrix (A.150)
A_{lcmp}	array gain: LCMP beamformer (6.390)
A_{lcmv}	array gain: LCMV beamformer (6.389)
A_o	optimum array gain (6.28)
$\mathbf{A}_R$	right circulant matrix (A.148)
A_w	array gain for spatially white noise input (2.185)
$AF_{\mathbf{k}}(\mathbf{k})$	array factor in $\mathbf{k}$-space (2.234)
$AF_{\psi}(\psi)$	array factor in ψ-space (2.234)
$AF_u(u)$	array factor in u-space (2.234)
$AR(p)$	autoregressive process of order p (5.311)
$ARMA(p,q)$	autoregressive moving average process of order (p,q) (5.303)
α	step size in LMS algorithm (7.405)
$\alpha(K)$	step size in LMS algorithm (7.405)
$\mathbf{B}$	(i) blocking matrix (6.361) (ii) matrix in polynomial parameterization (8.490) (iii) beam pattern matrix (2.82) (iv) output of DFT (3.96)
$B_a(\psi)$	asymmetric beam pattern (3.285)
$\mathbf{B}_{bs}$	beamspace matrix (3.327)
$\mathbf{B}_{bsbl}$	beamspace blocking matrix (6.530)
$\mathbf{b}_{bs,m}^H$	weight vector for mth beamspace beam (6.499)
$\mathbf{B}_{bsx}^H$	x-component of beamspace matrix, planar array (9.372)
$\mathbf{B}_{bsy}^H$	y-component of beamspace matrix, planar array (9.374)
$B_{du}(u)$	desired beam pattern in u-space
B_f	ratio of signal bandwidth to center frequency (5.288)
$\mathbf{B}_{no}$	non-orthogonal beamspace matrix (3.329)
$B_{mvdr}(\psi)$	beam pattern of MVDR beamformer (6.93)
$B_R(\psi)$	beam pattern using rectangular window (3.213)
$B_{ue}(u)$	element beam pattern (2.232)
$B(u_x, u_y)$	beam pattern in u_x, u_y-space (4.11)
$B(\theta, \phi)$	beam pattern in (θ, ϕ)-space (2.38)

$B(\omega:\theta,\phi)$	beam pattern in (θ,ϕ)-space (2.38)
$B_v(v)$	beam pattern in v-space (3.170)
$B(\psi_x,\psi_y)$	2-D beam pattern in ψ_x,ψ_y space (4.11)
$B_z(z)$	beam pattern polynomial in z-plane (3.42)
$B_{\mathbf{z}}(z_1,z_2)$	2-D z-transform (4.93)
$B_c(\mathbf{k}:\mathbf{k}_T)$	conventional beam pattern in $\mathbf{k}$-space (2.124)
$B_d(\psi)$	desired beam pattern (3.82)
$B_{eig,i}$	ith eigenbeam (7.126)
$B^{(n)}(\mathbf{k})$	nominal beam pattern (2.196)
$\mathbf{B}_{PM}$	phase mode excitation beamformer
$B_\psi(\psi)$	beam pattern in ψ-space (2.71)
B_s	bandwidth of signal (Hz) (2.40)
$B_T(u)$	beam pattern using Taylor weighting (3.173)
$B_{TAY}(u)$	Taylor beam pattern: circular aperture (4.220)
$B_\theta(\theta)$	beam pattern in θ-space (2.69)
$B_u(u)$	beam pattern in u-space (2.70)
$B_{\psi c}(\psi:\psi_T)$	conventional beam pattern in ψ-space (2.125)
$B_{\theta c}(\theta:\theta_T)$	conventional beam pattern in θ-space (2.131)
$B_{uc}(u:u_T)$	conventional beam pattern in u-space (2.126)
BW_{NN}	bandwidth: null-to-null (2.105)
c	velocity of propagation (2.14)
$\mathbf{C}$	constraint matrix $N\times M_c$ (3.256)
$\mathbf{C}_{ACR}(\boldsymbol{\psi})$	approximate Cramér-Rao bound on $\boldsymbol{\psi}$ (8.110)
$\mathbf{C}_{BCR}(\boldsymbol{\theta})$	Bayesian Cramér-Rao bound $(=\mathbf{J}_B^{-1})$ (8.53)
$\mathbf{C}_0,\mathbf{C}_1,\mathbf{C}_2$	constraint matrices using derivatives (3.251),(3.253),(3.255)
$\mathbf{C}_{CR}(\boldsymbol{\psi})$	Cramér-Rao bound on $\boldsymbol{\psi}$ (8.94)
$\mathbf{C}_{HCR}(\boldsymbol{\phi},\boldsymbol{\rho})$	hybrid Cramér-Rao bound (8.676)
$\mathbf{C}_{CCR}(\boldsymbol{\psi})$	conditional Cramér-Rao bound on $\boldsymbol{\psi}$ (8.235)
$\mathbf{C}_{ACCR}(\boldsymbol{\psi})$	asymptotic conditional Cramér-Rao bound on $\boldsymbol{\psi}$ (8.236)
$CRB_{bs}(\boldsymbol{\psi})$	beamspace Cramér-Rao bound on $\boldsymbol{\psi}$ (8.273)
$C_{CR,bs}(\boldsymbol{\psi})$	beamspace Cramér-Rao bound on $\boldsymbol{\psi}$ (8.616)
$\mathbf{C}_{CR}(\boldsymbol{\theta})$	Cramér-Rao bound matrix (8.25)
$\mathbf{C}_K$	scaled spectral matrix (7.3)
$\mathbf{C}_{\mathbf{x}}$	sample spectral matrix (7.10)
$\mathbf{C}_{\mathbf{x},fb}$	forward-backward averaged sample spectral matrix (7.40)
$\mathbf{C}_{\mathbf{x},Re}$	real sample spectral matrix (9.56)
$\mathbf{C}_{MU}$	error covariance matrix using MUSIC algorithm (9.246)
$\mathbf{C}_{WES}$	error covariance matrix, weighted eigenspace (9.263)

$\mathbf{C}_{BMU}$	error covariance matrix, beamspace MUSIC (9.300)
$\mathbf{C}_0$	constraint matrix, zero-order nulls (3.251)
χ^2	complex chi-squared random variable (7.16)
d	distance between sensors, uniformly spaced array (2.3)
D	(i) directivity of an array or aperture (2.144), (ii) number of source signals (also d) (6.143)
$\mathbf{D}$	(i) diagonal matrix of $e^{j\psi_m}$ terms (6.616) (ii) matrix of derivative vectors (8.97)
$\mathbf{d}_n(u)$	nth derivative of $\mathbf{v}(u)$ with respect to u (3.260)
$D(\theta, \phi)$	difference beam pattern in (θ, ϕ) space (4.236)
d_{cir}	distance along circumference (4.157)
$D_{MU}(z)$	MUSIC polynomial
$D_{\alpha}(\theta, \phi)$	complementary difference beams (4.240)
$D_{\beta}(\theta, \phi)$	complementary difference beams (4.243)
$\text{diag}[a_1, \cdots, a_N]$	$N \times N$ diagonal matrix whose elements are $a_1, \cdots, a_N$ (A.119)
δ_p	filter's upper peak of ripple (3.222)
δ_s	filter's lower peak of ripple (3.222)
Δg_i	amplitude perturbation (2.193)
$\overline{\Delta k^2}$	normalized mean-square width of response (3.3)
$\overline{\Delta l^2}$	normalized mean-square width of weighting (3.3)
$\Delta \mathbf{p}_i$	perturbation of sensor position (2.195)
$\Delta p_{xi}, \Delta p_{yi},$ Δp_{zi}	perturbation of sensor position ($x, y,$ and z components) (2.195)
$\Delta \phi_i$	perturbation of sensor phase (2.194)
$\Delta \phi_n$	phase error, nth sensor (2.194)
$\Delta \psi_i$	error in ψ direction in estimate of ith root (9.199)
Δr_i	radial error in estimate of ith root (9.199)
ΔT	segment of observation interval (5.1)
ΔT_{ij}	travel time between i and j elements (2.43)
ΔT_{max}	maximum travel time across array (2.43)
Δu_1	HPBW (half-power beamwidth) (2.100)
Δu_2	BW_{NN} (null-null beamwidth) (2.105)
Δu_s	sampling interval in u-space (3.74)
Δz_i	error in estimate of ith root (9.199)
DI	directivity index (2.167)

d_s	displacement between subarrays in ESPRIT algorithm (9.101)
$\mathbf{e}_n$	vector with 1 in nth position and zero elsewhere (A.104)
$\mathbf{E}_{ij}$	matrix with 1 in ij element and zero elsewhere (A.107)
$e_{pm}(\psi)$	error function in Parks-McClellan algorithm (3.222)
EFF_{MU}	efficiency of MUSIC algorithm (9.251)
η	noise level across band (8.373)
η_m	noise level at mw_0 (8.373)
$\mathbf{F}$	$N \times N$ matrix in discrete Fourier transform (3.97)
$\mathbf{F}(\omega, \mathbf{p})$	Fourier transform of $\mathbf{f}(t, \mathbf{p})$ (2.12)
$F(\boldsymbol{\psi})$	cost function in $\boldsymbol{\psi}$-space (8.389)
$\mathbf{F}(k)$	kth frequency-domain snapshot of complex source-signal vector (8.200)
$F_{map}(\boldsymbol{\theta}, \boldsymbol{\rho})$	MAP cost function (8.690)
$\mathbf{f}(t, \mathbf{p}_n)$	signal input to array at time t (2.6)
$f_{c_n}(t)$	real part of $f(t)$
$F_x(\tau : \mathbf{k})$	temporal correlation-spatial wavenumber spectrum (5.92)
F_{aml}	AML cost function (8.390)
F_{cml}	CML cost function (8.391)
F_{WSF}	WSF cost function (8.392)
F_{MODE}	MODE cost function (8.392)
$\mathbf{F}(w, \mathbf{p})$	Fourier transform of signal at $\mathbf{p}$ (2.12)
$\tilde{f}(t)$	complex envelope (5.71)
$\mathbf{g}$	value of constraints (6.279)
$\mathbf{g}(K)$	gain vector (7.157)
g_i	amplitude of complex weight w_i (2.190)
$\Gamma_N(k)$	generalization of Gamma function (7.13)
$\mathbf{H}$	matrix in CRB, $\mathbf{D}^H \mathbf{P}_{\mathbf{V}}^{\perp} \mathbf{D}$ (8.101)
H_0, H_1	hypotheses 0 and 1 (10.1)
$\mathbf{h}(\tau)$	impulse response of vector filter (2.9)
$\mathbf{H}(\mathbf{y})$	transformation in EM algorithms (8.454)
$\mathbf{H}_o(w)$	MMSE weight vector (6.42)
$\mathbf{H}_w$	whitening matrix (6.517)
$\mathbf{H}_G$	Gaussian approximation to Hessian (8.426)
$\mathbf{H}_k$	Hessian matrix at kth iteration (8.401)

$I_0(x)$	modified Bessel function of zero order (3.36)
$\mathbf{J}$	(i) exchange matrix (A.125)
	(ii) Fisher information matrix (8.26)
$\mathbf{J}_B$	Bayesian version of Fisher information matrix (8.49)
$\mathbf{J}_D$	data information matrix (8.50)
$\mathbf{J}_P$	prior information matrix (8.51)
$\mathbf{J}_{\psi\psi}$	sub-matrix of Fisher information matrix
	for ψ parameter (8.89)
$\mathbf{J}_s$	selection matrix (9.103)
$\tilde{\mathbf{J}}_s$	identity matrix component of $\mathbf{J}_s$ (9.103)
$\tilde{\mathbf{J}}_s^{(m_s)}$	weighted diagonal matrix in ESPRIT
	selection matrix (9.132)
$J(\mathbf{b})$	cost function for IQML (8.496) and MODE (8.532)
$\mathbf{k}$	wavenumber (2.24)
K	number of snapshots (5.1)
$\mathbf{k}_T$	steering direction in k-space (2.119)
k_0	magnitude of wavenumber (2.57)
k_r	radial component of wavenumber (5.131)
$K_x(t_1,t_2:\mathbf{p}_1,\mathbf{p}_2)$	space-time covariance function (5.85)
$K_x(\tau:\mathbf{p}_1,\mathbf{p}_2)$	space-time covariance function;
	wide-sense stationary process (5.86)
$K_x(t_1,t_2:\mathbf{p}_\Delta)$	space-time covariance function ;
	homogeneous process (5.88)
$K_x(\tau:\mathbf{p}_\Delta)$	space-time covariance function; homogeneous,
	wide-sense stationary process (5.90)
$\mathbf{K}_\mathbf{x}(\boldsymbol{\theta})$	covariance matrix of $\mathbf{x}$ (as a function of $\boldsymbol{\theta}$) (8.8)
k_z	z-component of wavenumber $\mathbf{k}$ (2.56)
L	(i) length of linear aperture (2.5)
	(ii) number of subarrays in linear
¿	spatial smoothing (6.614)
$\mathbf{L}$	lower triangular matrix (A.151)
$L(\boldsymbol{\theta})$	likelihood function (8.10)
$L_d(d)$	likelihood function (as a function of d,
	the number of signals) (7.502)
$L(\psi,\mathbf{F})$	likelihood function (conditional model) (8.336)

$L(\boldsymbol{\psi}, \mathbf{S_f})$	likelihood function (stochastic model) (8.277)
$L_{bs}(\boldsymbol{\psi}, \mathbf{F})$	likelihood function in beamspace (conditional model) (8.627)
λ	(i) wavelength (2.24), (ii) Lagrange multiplier (2.163)
$\boldsymbol{\lambda}$	Lagrange multiplier vector (3.262)
$\lambda(\omega)$	Lagrange multiplier as a function of frequency
λ_i	ith eigenvalue of matrix (A.196), (5.208)
λ_l	wavelength of lowest frequency (3.335)
λ_u	wavelength of highest frequency (3.335)
λ_{max}	maximum eigenvalue (5.213)
λ_{min}	maximum eigenvalue (5.213)
$\hat{\lambda}_i$	estimate of ith eigenvalue (7.21)
$\hat{\lambda}_{fb}$	estimated eigenvalue with FB averaged sample spectral matrix (9.60)
$\hat{\lambda}_{Re}$	estimated eigenvalue with real sample spectral matrix (9.63)
$\mathbf{\Lambda}_S$	diagonal matrix of signal subspace eigenvalues (5.238)
$\mathbf{\Lambda}_N$	diagonal matrix of noise eigenvalues (5.239)
$\hat{\mathbf{\Lambda}}_S$	diagonal matrix of estimated signal subspace eigenvalues (8.361)
$\hat{\mathbf{\Lambda}}_N$	diagonal matrix of estimated noise eigenvalues (8.361)
$\hat{\mathbf{\Lambda}}_{S,Re}$	real estimated signal subspace eigenvalue matrix (9.64)
$\hat{\mathbf{\Lambda}}_{N,Re}$	real estimated noise subspace eigenvalue matrix (9.64)
$\mathbf{M}$	matrix representing data (8.362)
M	(i) number of sensors in linear subarray (spatial smoothing) (6.614) (ii) number of sensors in y-direction: rectangular planar array (4.11) (iii) number of frequency bins (5.7)
$\mathcal{M}$	misadjustment in LMS beamformer (7.452)
m_s	parameter in weighted ESPRIT (9.132)
$\mathbf{m_x}(\boldsymbol{\theta})$	mean of $\mathbf{x}$ (as a function of $\boldsymbol{\theta}$) (8.8)
μ	exponential weight (7.138)
$\boldsymbol{\mu}(K)$	diagonal exponential weighting matrix (7.270)
N	number of elements in the array (2.5)
n	index of elements (2.3)

$\tilde{n}$	symmetric index (3.11)
N_H	(i) number of holes (3.299)
	(ii) number of elements in standard hexagonal array (4.256)
N_R	number of redundancies (3.299)
ω_Δ	frequency separation (5.6)
ω	radian frequency (2.10)
ω_c	center frequency (rad/sec) (2.39)
Ω_A	beam solid angle (4.31)
$p(d)$	penalty function (7.503)
p_{z_n}	position of elements on z-axis (2.3)
$\mathbf{P}(K)$	inverse of $\mathbf{\Phi}^{-1}(K)$ (7.154)
$P(\omega,\theta,\phi)$	beam power pattern in frequency-angle space (6.306)
$P(\theta,\phi)$	beam power pattern with ω suppressed (6.306)
$P_{xx}(z)$	output spectrum in z-domain, ARMA models (5.302)
P_D	probability of detection (7.511)
$\mathbf{p}_\Delta$	position difference (2.192)
P_{do}	desired output power (6.593)
P_F	probability of false alarm (7.511)
P_{Io}	interference output power (6.597)
p_n	position of nth sensor (1-D) (2.53)
$\mathbf{p}_n$	position of nth element (3-D) (2.6);
	in rectangular coordinates, $\mathbf{p}_n = [p_{xn}, p_{yn}, p_{zn}]^T$
P_{no}	noise output power (6.598)
$\mathbf{P_V}$	projection matrix with respect to $\mathbf{V}$ (A.177)
$\mathbf{P_V^\perp}$	orthogonal projection matrix (A.178)
$P_x(\omega : \mathbf{k})$	frequency-wavenumber spectrum (5.93)
$\hat{P}_B(\psi)$	beamscan spectrum (9.2)
$\hat{P}_{BW}(\psi)$	weighted beamscan spectrum (9.5)
$\hat{P}_{B,fb}(\psi)$	beamscan spectrum with FB averaging (9.9)
$\hat{P}_{mvdr}(\psi)$	MVDR spectrum (9.13)
$Pr(\epsilon)$	probability of error
ϕ	angle measured counterclockwise from positive x-axis (2.17)
$\mathbf{\Phi}$	diagonal matrix of phase terms in ESPRIT (9.107)
ϕ_i	phase of complex weight w_i (2.190)
$\boldsymbol{\phi}_i$	ith eigenvector of matrix (A.192), (A.196), (5.205)
$\mathbf{\Phi}(K)$	exponentially weighted sample spectral matrix (7.138)
$\mathbf{\Phi_z}(K)$	exponentially weighted sample spectral matrix in

	generalized sidelobe canceller (7.196)
$\hat{\boldsymbol{\phi}}_i$	estimate of ith eigenvector (7.21)
$\hat{\boldsymbol{\phi}}_{Re}$	estimate of real eigenvector (9.59)
$\hat{\boldsymbol{\phi}}_{fb}$	estimate of real eigenvector using FB averaging (9.60)
$\tilde{\mathbf{\Phi}}(K)$	exponentially weighted sample spectral matrix using FB averaging (7.246)
$\overline{\mathbf{\Phi}}(K)$	real exponentially weighted sample spectral matrix (7.256)
Ψ_H	half-power point beamwidth (4.29)
$\hat{\boldsymbol{\psi}}_{aml}$	asymptotic ML estimate of $\boldsymbol{\psi}$ (8.310), (8.315)
$\hat{\boldsymbol{\psi}}_{cml}$	conditional ML estimate (8.346)
$\hat{\boldsymbol{\psi}}_{cml,bs}$	CML estimate in beamspace (8.630)
ψ_p	design parameter in Parks-McClellan algorithm (3.223)
ψ_s	design parameter in Parks-McClellan algorithm (3.223)
ψ_x	x-component of $\boldsymbol{\psi}$ (4.2)
ψ_y	y-component of $\boldsymbol{\psi}$ (4.3)
$\hat{\psi}_{MODE}$	MODE estimate of $\boldsymbol{\psi}$ (8.367)–(8.370)
$\widehat{\mathbf{\Psi}}_{bs,ULS}$	matrix in beamspace unitary LS-ESPRIT (9.324)
$\widehat{\mathbf{\Psi}}_{bs,UTLS}$	matrix in beamspace unitary TLS-ESPRIT (9.325)
$\widehat{\mathbf{\Psi}}_{LS}$	matrix in LS-ESPRIT (9.120)
$\widehat{\mathbf{\Psi}}_{TLS}$	matrix in TLS-ESPRIT (9.122)
$\widehat{\mathbf{\Psi}}_{ULS}$	matrix in unitary LS-ESPRIT (9.163)
$\widehat{\mathbf{\Psi}}_{UTLS}$	matrix in unitary TLS-ESPRIT (9.165)
$\mathbf{Q}$	(i) specific unitary matrix (A.161), (A.162), (7.58), (7.59), (7.248), (7.249) (ii) matrix in quadratic form (6.337)
$\widehat{Q}_{MU}(\psi)$	MUSIC null spectrum (9.43)
$\widehat{Q}_{mvdr}(\psi)$	MVDR null spectrum (9.14)
$\widehat{Q}_{mvdr,fb}$	MVDR null spectrum with FB averaging
$\mathbf{Q}_N$	unitary matrix with dimension $N \times N$ (9.137)
$\widetilde{\mathbf{Q}}_{\mathbf{x}}$	matrix in IQML algorithm (8.508)
$\widetilde{\mathbf{Q}}_D$	matrix in IMODE algorithm (8.536)
$\widehat{\mathbf{Q}}_z(z)$	null polynomial (9.20)
$\widehat{\mathbf{Q}}_{MU,z}(z)$	root MUSIC polynomial (9.48)
$\widehat{\mathbf{Q}}_{MU,fb,z}(z)$	root MUSIC polynomial with FB averaging (9.52)
$\widehat{\mathbf{Q}}_{MU,U,z}(z)$	unitary root MUSIC polynomial (9.65)
$\mathbf{Q}_{MN}(\psi)$	Min-Norm null spectrum (9.95)
$\widehat{\mathbf{Q}}_{WES}(\psi)$	weighted eigenspace null spectrum (9.258)

$\widehat{\mathbf{Q}}_{bs,MU}(\psi)$	beamspace MUSIC null spectra (9.295)
R	ratio of main-lobe height to sidelobe height (3.143)
$R[m]$	discrete rectangular window (3.210)
$\mathbf{R}_e(K)$	weight error correlation matrix (7.438)
$\tilde{\mathbf{R}}(K)$	upper triangular matrix in QRD (7.282)
$\mathbf{R}_{\boldsymbol{\epsilon}}$	correlation matrix of errors (8.52)
$\mathbf{R}_{\mathbf{f}}$	source-signal correlation matrix (8.68)
$\mathbf{R}_{\mathbf{x}}$	input correlation matrix (8.68)
$\mathbf{R}_{\mathbf{n}}$	noise correlation matrix (8.68)
$R_W(\omega : \|m\|)$	co-array of linear array (5.187)
$R_W(\omega : \mathbf{p})$	aperture autocorrelation function (5.184)
$\tilde{\mathbf{r}}_{\mathbf{y}}$	correlation vector for AR process (5.318)
$\tilde{\mathbf{R}}_{\mathbf{y}}$	correlation matrix for AR process (5.316)
$\tilde{\mathbf{R}}_{\mathbf{y}}^A$	augmented correlation matrix for AR process (5.320)
$\boldsymbol{\rho}$	perturbation vector (6.240)
$\boldsymbol{\rho}(\omega)$	normalized spectral matrix (6.27)
ρ_{s1}	spatial correlation coefficient (6.80)
$S_o(w : \theta, \phi)$	spectral density on sphere (5.127)
$\mathbf{S}_{d\mathbf{x}}^H$	cross-spectral matrix between D and $\mathbf{X}$ (6.41)
$\mathbf{S}_{\mathbf{f}}$	source-signal spectral matrix (5.67), (5.222)
$S_f(\omega)$	input signal frequency spectrum at each sensor (2.170)
$\mathbf{S}_{\mathbf{x},fb}$	forward–backward spatial spectral matrix (7.45)
$\mathbf{S}_{\mathbf{n}}$	noise spectral matrix (6.37)
$S_n(\omega)$	input noise frequency spectrum at each sensor (2.170)
$\mathbf{S}_{\mathbf{x}}(\omega)$	spectral matrix of vector process (5.174)
$\mathbf{S}_{y_s}(\omega)$	array output signal spectrum (2.180)
$S_x(\omega : \mathbf{p}_\Delta)$	temporal frequency-spatial correlation function (5.91)
$\mathbf{S}_{\mathbf{x}}(\omega_m)$	spatial spectral matrix at frequency ω_m (5.175)
$\tilde{\mathbf{S}}_{x,L}$	spectral matrix augmented with diagonal loading (7.115)
$\hat{\mathbf{S}}_{\mathbf{x},fb}$	estimated spatial spectral matrix using forward–backward averaging (7.45)
$\hat{\mathbf{S}}_{\mathbf{x}}$	estimated spatial spectral matrix(7.9)
$\mathbf{S}_{\mathbf{x},bs}$	beamspace spatial spectral matrix(6.509)
$\mathbf{S}_M^{(i)}$	forward spatial spectral matrix of ith subarray (6.617)
$\mathbf{S}_{MB}^{(i)}$	backward spatial spectral matrix of ith subarray (6.618)
$\mathbf{S}_{SSFB}$	forward–backward spatially smoothed spectral matrix (6.622)

$\mathbf{S}_{SS}$	spatially smoothed spectral matrix (6.628)
$\hat{\mathbf{S}}_{\mathbf{x},ml}$	AML estimate of $\mathbf{x}$ (8.299)
$\hat{\mathbf{S}}_{\mathbf{f},ml}$	AML estimate of $\mathbf{f}$ (8.296)
$\hat{\mathbf{S}}_{SS}$	sample spectral matrix using spatial smoothing (9.288)
$\hat{\mathbf{S}}_{FBSS}$	sample spectral matrix using forward–backward spatial smoothing (9.289)
$\mathbf{\Sigma}$	normalized signal spectral matrix (8.103)
σ_w^2	spectral height; white noise (5.191)
$\mathbf{T}$	transformation matrix (5.232)
t_{AMF}	test statistic: adaptive matched filter (10.21)
t_e	test statistic (10.5)
T_o	quadratic constraint (2.211)
τ_n	delay at nth sensor relative to a sensor at the origin (2.14)
θ	angle measured from positive z-axis (2.2)
$\bar{\theta}$	complement of θ, $\bar{\theta} = \frac{\pi}{2} - \theta$ (2.2)
θ_H	half-power beamwidth in θ-space, 1-D (2.136)
Θ_H	half-power point beamwidth, 2-D (4.27)
$\hat{\boldsymbol{\theta}}_{map}$	MAP estimate of $\boldsymbol{\theta}$ (8.22)
$\hat{\boldsymbol{\theta}}_{ml}$	ML estimate of $\boldsymbol{\theta}$ (8.15)
$\boldsymbol{\theta}_w$	wanted parameter (8.39)
$\boldsymbol{\theta}_u$	unwanted parameter (8.39)
$\boldsymbol{\theta}_1$	nonrandom parameter (8.58)
$\boldsymbol{\theta}_2$	random parameter (8.58)
θ_R, θ_L	right and left half-power point in θ-space (2.134) (2.135)
$T_n(x)$	nth degree Chebychev polynomial (3.133)
T_{se}	sensitivity function (2.206)
$\mathbf{u}$	direction cosine vector, $\mathbf{u} = [u_x, u_y, u_z]^T$ (2.20)
$\mathbf{U}$	(i) upper triangular matrix (A.153); (ii) left singular vector (A.261); (iii) $N \times N$ matrix of eigenvectors (6.439)
$\mathbf{U}_N$	matrix of noise subspace eigenvectors (5.234)
$\widehat{\mathbf{U}}_N$	estimate of $\mathbf{U}_N$ (8.361)
u_R, u_L	right and left half-power points in u-space (2.132) (2.133)
$\mathbf{U}_{RS}$	real signal subspace matrix (9.158)
$\mathbf{U}_S, \mathbf{U}_s$	matrix of signal subspace eigenvectors (5.231)
$\widehat{\mathbf{U}}_S$	estimate of $\mathbf{U}_S$ (8.361)
u_x, u_y, u_z	direction cosines with respect to x, y,

	and z axes (2.17)(2.18)(2.19)
$\mathbf{U}_Q$	$N \times N$ matrix of eigenvectors (6.339)
$\mathbf{U}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(n)})$	function in EM algorithm (8.456)
$\mathbf{u}_r$	radial directional cosine (4.6)
$\mathbf{U}_{dm}$	dominant mode subspace (6.487)
$\hat{\mathbf{U}}_{bs,S}$	estimated beamspace signal subspace (9.295)
$\hat{\mathbf{U}}_{bs,N}$	estimated beamspace noise subspace (9.295)
$\hat{\mathbf{U}}_{FBSS,S}$	estimated signal subspace using FB spatial smoothing (9.291)
$\hat{\mathbf{U}}_{FBSS,N}$	estimated noise subspace using FB spatial smoothing (9.291)
$\mathbf{U}_P$	weight matrix in MUSIC analysis (9.216)
$\hat{\mathbf{U}}_{S,Re}$	real estimated signal subspace (9.64)
$\hat{\mathbf{U}}_{S,Re}$	real estimated noise subspace (9.64)
$\mathbf{U}_{s1}$	first subarray signal subspace (9.109)
$\hat{\mathbf{U}}_{s1}$	estimated first subarray signal subspace (9.117)
$\mathbf{U}_{s2}$	second subarray signal subspace (9.110)
$\hat{\mathbf{U}}_{s2}$	estimated second subarray signal subspace (9.118)
$\Upsilon(\omega, \mathbf{k})$	frequency-wavenumber response function (2.37)
$\Upsilon_z(z)$	frequency-wavenumber response function in z-plane (2.63)
$\Upsilon_\psi(\psi)$	frequency-wavenumber response function in ψ-space (2.61)
$\Upsilon_e(\omega, \mathbf{k})$	frequency-wavenumber response function of sensor element (2.234)
$\Upsilon_u(u)$	frequency-wavenumber response function in u-space (3.72)
$\mathbf{V}$	right singular vector (A.261)
$\mathbf{v}_{bs}(\psi)$	array manifold vector in beamspace (6.502)
$vec_H(u_x, u_y)$	array manifold vector for standard hexagonal array (4.257)
$\mathbf{v}_{\mathbf{k}}(\mathbf{k})$	array manifold vector: $\mathbf{k}$-space (2.28)
$\mathbf{v}^{(k)}(\psi)$	kth derivative of array manifold vector (7.514)
$\mathbf{V}(\boldsymbol{\theta}, \boldsymbol{\rho})$	perturbed array manifold vector (8.73)
$\mathbf{V}^\dagger$	Moore-Penrose pseudo-inverse of $\mathbf{V}$ (A.185)–(A.187), (8.326)
$\mathbf{v}_\psi(\psi)$	array manifold vector: ψ-space (1-D) (2.72)
$\mathbf{V}_{\boldsymbol{\psi}}(\boldsymbol{\psi})$	array manifold matrix (planar array) (4.52)
$\mathbf{V}_R(\boldsymbol{\psi})$	real array manifold matrix (9.152), (9.340)

$\mathbf{v}_R(\psi)$	real array manifold vector (9.140)
$\mathbf{v}_u(u)$	array manifold vector: u-space (1-D) (2.68)
$\mathbf{v}_\theta(\theta)$	array manifold vector: θ-space (1-D) (2.67)
$\mathbf{v}_S(\psi)$	array manifold vector for reference subarray (9.290)
$\mathbf{V}_M$	array manifold matrix; M-element subarray
$\mathbf{V}_{\boldsymbol{\psi}}(\boldsymbol{\psi})$	array manifold matrix, linear array (8.67)
$\mathbf{V}$	array manifold matrix, linear array ($\boldsymbol{\psi}$ suppressed) (5.221)
$\mathbf{V}_{bs}$	beamspace array manifold matrix, linear array ($\boldsymbol{\psi}$ suppressed) (6.515)
$\mathbf{V}_I$	beamspace array manifold matrix of interference, linear array ($\boldsymbol{\psi}$ suppressed) (6.104)
$\mathbf{W}$	weight matrix, planar array (4.56)
$\mathbf{w}$	complex weight vector (2.49)
$\mathcal{W}_N(K, \mathbf{S_x})$	complex Wishart density (7.11)
$\mathbf{W}(\omega)$	(i) weight vector in frequency domain (6.6) (ii) $D \times N$ matrix processor (6.144)
$\mathbf{W}_o(\omega)$	optimum vector in frequency domain (6.12)
$w_a^*(z)$	aperture weighting function (2.214)
$\mathbf{w}_a$	adaptive weight vector in sidelobe canceller (6.373)
$\mathbf{W}_{ao}$	asymptotically optimal weighting (8.368)
$\mathbf{w}_d$	desired weight vector (3.262)
$\mathbf{w}_{dq}$	desired quiescent weight vector (6.409)
w_i^n	nominal weight (2.190)
w_n^*	complex weighting of sensor output (2.49)
$\mathbf{w}^n$	nominal weight vector (2.190)
$\mathbf{W}^n(\omega)$.	nominal matrix filter
$\mathbf{w}_o$	optimum weight vector (3.264)
$\mathbf{w}_e$	error weight vector (3.271)
$\mathbf{w}_{bs}$	weight vector in beamspace (3.334)
$\bar{\mathbf{w}}_{dq}$	normalized desired quiescent weight vector (6.409)
$\mathbf{w}_{lcmv}^H$	LCMV weight vector (6.357)
$\mathbf{w}_{lcmp}^H$	LCMP weight vector (6.358)
$\mathbf{w}_{dm}^H$	weight vector, dominant-mode beamformer (7.522)
$\mathbf{W}_{mvdr}^H$	minimum variance distortionless response weight vector (6.23)
$\mathbf{w}_{mvdr}^H$	minimum variance distortionless response weight vector (6.74)

$\mathbf{w}_{mpdr}^H$	minimum power distortionless response weight vector (6.71)
$\mathbf{w}_{mpdr,dl}^H$	minimum power distortionless response weight vector with diagonal loading (6.270)
$\mathbf{w}_{mpdr,es}^H$	MPDR weight vector in eigenspace (6.449)
$\mathbf{w}_{lcmp,es}^H$	LCMP weight vector in eigenspace (6.464)
$W_{pm}(\psi)$	weighting function in Parks-McClellan–Rabiner algorithm (3.224)
$\mathbf{w}_q$	quiescent weight vector (6.360)
$w_R(r)$	radial weighting function: circular aperture (4.192)
$\mathbf{w}_{bs}$	beamspace weight vector
$\hat{\mathbf{w}}_{mvdr,smi}^H$	weight vector for SMI implementation of MVDR beamformer (7.83)
$\hat{\mathbf{w}}_{mpdr,smi}^H$	weight vector for SMI implementation of MPDR beamformer (7.85)
$\hat{\mathbf{w}}_{lse}(K)$	least squares error weight vector (7.149)
$\hat{\mathbf{w}}_{rls}(K)$	weight vector in RLS algorithm (7.431)
$\hat{\mathbf{w}}_{lms}(K)$	weight vector in LMS algorithm (7.432)
$\mathbf{X}(k)$(also $\mathbf{X}_k$)	frequency-domain snapshot vector (7.2)
$\tilde{\mathbf{X}}$	$N \times K$ data matrix (7.4)
$\tilde{\mathbf{X}}_{fb}$	$N \times 2K$ forward–backward data matrix (7.48)
$\mathbf{X}_{\Delta T}(\omega_m)$	finite interval Fourier transform (5.4)
$\mathbf{X}(\omega_m, k)$	kth frequency-domain snapshot at frequency ω_m
$\xi_{lms}(K)$	mean-square error: LMS beamformer at Kth iteration (7.448)
$\xi_\mu(K)$	weighted summation of squared errors (7.148)
$\xi_{ex}(\infty)$	steady state excess mean-square error (LMS) (7.451)
$\xi_{ex}(K)$	excess mean-square error (LMS) (7.449)
$\xi_N(K)$	weighted summation of squared errors (7.132)
$\xi_Y(K)$	weighted summation of squared outputs (7.133)
ξ_o	MMSE (7.324)
ξ	MSE (6.39)
$\xi_o(\omega)$	MMSE as a function of ω (6.48)
$\xi_{sd}(K)$	transient MSE using steepest descent (7.368)
$y(t)$	array output (2.7)
$Y(\omega)$	Fourier transform of array output (2.10)

z_k	kth zero (3.58), (3.59)
$\mathbf{z}_n$	coordinate of nth element on z-axis (2.4)
z_λ	normalized distance on z-axis: $= z/\lambda$ (3.72)