Chapter 12

Target Detection in the Presence of Reverberation and Noise

When a sonar system goes active, some of the transmitted (radiated) acoustic power may be scattered by an object of interest. Scatter from an object of interest measured at a receiver is referred to as a *target return*. However, some of the transmitted acoustic power may also be scattered by unintentional objects (i.e., objects of no interest) such as a school of fish, marine mammals, air bubbles, or other particulate matter (examples of volume reverberation), the ocean surface (surface reverberation), and the ocean bottom (bottom reverberation). Scatter from objects of no interest measured at a receiver is referred to as a *reverberation return*. Since the transmitted electrical signals used in active sonar systems are amplitude-and-angle-modulated carriers (i.e., real *bandpass* signals), it will be more convenient to do the analysis in this chapter using complex envelopes (see Chapter 11).

12.1 A Binary Hypothesis-Testing Problem

In this section we shall consider the following *binary hypothesis-testing problem* of trying to detect a target return in the presence of *interference* (reverberation plus noise) for a monostatic (backscatter) geometry:

$$\mathrm{H}_0: \quad \tilde{r}(t) = \tilde{z}(t) \tag{12.1-1}$$

$$\mathrm{H}_1: \quad \tilde{r}(t) = \tilde{y}_{Trgt}(t) + \tilde{z}(t) \tag{12.1-2}$$

where

$$\tilde{z}(t) = \tilde{y}_{Rev}(t) + \tilde{y}_{n_a}(t) + \tilde{n}_r(t) \,. \tag{12.1-3}$$

Hypothesis H_0, the null hypothesis, states that the complex envelope of the received signal, $\tilde{r}(t)$, is equal to the sum of the complex envelopes of the *reverberation return*, $\tilde{y}_{Rev}(t)$, ambient noise, $\tilde{y}_{n_a}(t)$, and receiver noise, $\tilde{n}_r(t)$. Hypothesis H_1 states that the complex envelope of the received signal is equal to the sum of the complex envelopes of the *target return*, $\tilde{y}_{Trgt}(t)$, reverberation return, ambient noise, and receiver noise. The functions $\tilde{y}_{Trgt}(t)$, $\tilde{y}_{Rev}(t)$, $\tilde{y}_{n_a}(t)$, and $\tilde{n}_r(t)$ are all random processes. Therefore, $\tilde{r}(t)$ is also a random process. The complex envelope of the received signal is due to processing the output electrical signals (in volts) from all the elements in a linear array. The linear array is composed of an odd number N of identical, equally-spaced, complex-weighted, omnidirectional point-elements lying along the X axis. The problem is to decide which hypothesis is correct.

The mathematical model that we shall use for the complex envelope of the target return is given by

$$\tilde{y}_{Trgt}(t) = \sum_{i=-N'}^{N'} \tilde{y}_{Trgt}(t, x_i), \tag{12.1-4}$$

where (see Appendix 12A)

$$\tilde{y}_{Trgt}(t, x_i) = a_i w_{Trgt} \tilde{x}(t - \tau_{Trgt}) \exp\left[+j2\pi\eta_{D,Trgt}(t - \tau_{Trgt}) \right] \exp(-j2\pi f_c \tau_{Trgt}) \tag{12.1-5}$$

is the complex envelope of the output electrical signal from element i in the linear array due to the target after complex weighting for $t \geq \tau_{Trgt}$, a_i is the real, dimensionless, amplitude weight applied to element i,

$$w_{Trgt} = \frac{1}{(4\pi r_{Trgt})^2} \mathcal{S}_T(f_c) S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0}) \exp[-2\alpha(f_c) r_{Trgt}] \mathcal{S}_R(f_c) \tag{12.1-6}$$

is a dimensionless, complex, *nonzero-mean*, Gaussian, random variable (RV); $(r_{Trgt}, \theta_{Trgt}, \psi_{Trgt})$ are the spherical coordinates of the center of mass of the target, measured from the center of the array (see Fig. 12.1-1); $S_{Trgt}(\bullet)$ is the *target scattering function* in meters, and is treated as a complex, *nonzero-mean*, Gaussian RV (a statistical model of a scattering function is discussed in Section 10.6); $\theta_{Trgt\,0} = 180° - \theta_{Trgt}$, $\psi_{Trgt\,0} = 180° + \psi_{Trgt}$, $\mathcal{S}_T(f_c)$ and $\mathcal{S}_R(f_c)$ are the transmitter and receiver sensitivity functions of the identical, omnidirectional point-elements with units of $\left(\text{m}^3/\text{sec}\right)/\text{V}$ and $\text{V}/\left(\text{m}^2/\text{sec}\right)$, respectively; $\alpha(f_c)$ is the attenuation coefficient of seawater in nepers per meter, f_c is the carrier frequency in hertz, $\tilde{x}(t)$ is the complex envelope of the transmitted electrical signal in volts, τ_{Trgt} and $\eta_{D,Trgt}$ are the round-trip time delay in seconds and Doppler shift in hertz of the target measured from the center of the array, and

$$N' = (N-1)/2\,. \tag{12.1-7}$$

The parameters τ_{Trgt} and $\eta_{D,Trgt}$ shall be treated as *unknown*, *nonrandom constants* to be *estimated*. Equation (12.1-5) indicates that the complex envelopes of the output electrical signals from all the elements in the array due to the target

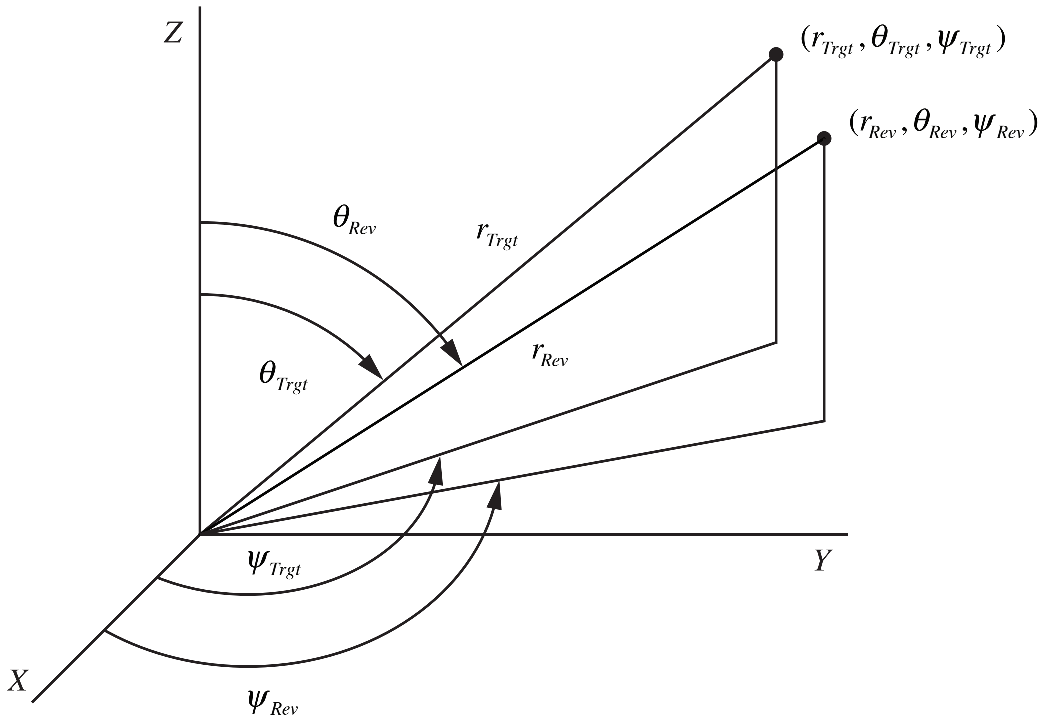

Figure 12.1-1 Spherical coordinates $(r_{Trgt}, \theta_{Trgt}, \psi_{Trgt})$ and $(r_{Rev}, \theta_{Rev}, \psi_{Rev})$ of the center of mass of the target and object responsible for the reverberation, respectively.

after complex weighting are *in-phase*. The target return $\tilde{y}_{Trgt}(t)$ is a complex, *nonzero-mean*, Gaussian, random process.

The mathematical model that we shall use for the complex envelope of the reverberation return is given by

$$\tilde{y}_{Rev}(t) = \sum_{i=-N'}^{N'} \tilde{y}_{Rev}(t, x_i), \tag{12.1-8}$$

where (see Appendix 12A)

$$\tilde{y}_{Rev}(t, x_i) = a_i w_{Rev} \tilde{x}\left(t - [\tau_{i,Rev} + \tau'_{i,Trgt}]\right) \exp\left\{+j2\pi\eta_{D,Rev}\left[t - (\tau_{i,Rev} + \tau'_{i,Trgt})\right]\right\} \times \exp\left[-j2\pi f_c(\tau_{i,Rev} + \tau'_{i,Trgt})\right] \tag{12.1-9}$$

is the complex envelope of the output electrical signal from element i in the linear array due to reverberation after complex weighting for $t \geq \tau_{i,Rev} + \tau'_{i,Trgt}$,

$$w_{Rev} = \frac{1}{(4\pi r_{Rev})^2} \mathcal{S}_T(f_c) S_{Rev}(f_c, \theta_{Rev}, \psi_{Rev}, \theta_{Rev\,0}, \psi_{Rev\,0}) \exp[-2\alpha(f_c) r_{Rev}] \mathcal{S}_R(f_c) \tag{12.1-10}$$

is a dimensionless, complex, *nonzero-mean*, Gaussian RV; $(r_{Rev}, \theta_{Rev}, \psi_{Rev})$ are the spherical coordinates of the center of mass of the object responsible for the reverberation, measured from the center of the array (see Fig. 12.1-1); $S_{Rev}(\bullet)$ is the *reverberation scattering function* in meters, and is treated as a complex, *nonzero-mean*, Gaussian RV (as mentioned earlier, a statistical model of a scattering function is discussed in Section 10.6); $\theta_{Rev\,0} = 180° - \theta_{Rev}$, $\psi_{Rev\,0} = 180° + \psi_{Rev}$, $\tau_{i,Rev}$ is the round-trip time delay in seconds from element i in the linear array to the object responsible for the reverberation, $\tau'_{i,Trgt}$ is the time-delay applied to element i via phase weighting in order to co-phase the output electrical signals from all the elements in the array due to the target (*not* the reverberation), and $\eta_{D,Rev}$ is the Doppler shift in hertz of the object responsible for the reverberation measured from the center of the array. Since $\tau_{i,Rev}$ and $\tau'_{i,Trgt}$ depend on index i, (12.1-9) indicates that the complex envelopes of the output electrical signals from all the elements in the array due to reverberation are *out-of-phase*. The reverberation return $\tilde{y}_{Rev}(t)$ is a complex, *nonzero-mean*, Gaussian, random process.

The mathematical model that we shall use for the complex envelope of ambient noise is given by

$$\tilde{y}_{n_a}(t) = \sum_{i=-N'}^{N'} \tilde{y}_{n_a}(t, x_i), \tag{12.1-11}$$

where (see Section 7.4)

$$\tilde{y}_{n_a}(t, x_i) = a_i \tilde{n}_a(t - \tau'_{i,Trgt}, x_i) \tag{12.1-12}$$

is the complex envelope of the output electrical signal from element i in the linear array due to ambient noise after complex weighting. Since $\tau'_{i,Trgt}$ depends on index i, (12.1-12) indicates that the complex envelopes of the output electrical signals from all the elements in the array due to ambient noise are *out-of-phase*. The complex envelope $\tilde{y}_{n_a}(t)$ is a complex, zero-mean, Gaussian, random process.

And finally, the mathematical model that we shall use for the complex envelope of receiver noise is given by

$$\tilde{n}_r(t) = \sum_{i=-N'}^{N'} \tilde{n}_r(t, x_i), \tag{12.1-13}$$

where (see Section 7.4)

$$\tilde{n}_r(t, x_i) = a_i \tilde{n}(t - \tau'_{i,Trgt}, x_i) \tag{12.1-14}$$

is the complex envelope of receiver noise at the output of element i in the linear array after complex weighting. Since $\tau'_{i,Trgt}$ depends on index i, (12.1-14) indicates that the complex envelopes of the output electrical signals from all the elements in the array due to receiver noise are *out-of-phase*. The complex envelope $\tilde{n}_r(t)$ is a complex, zero-mean, Gaussian, random process.

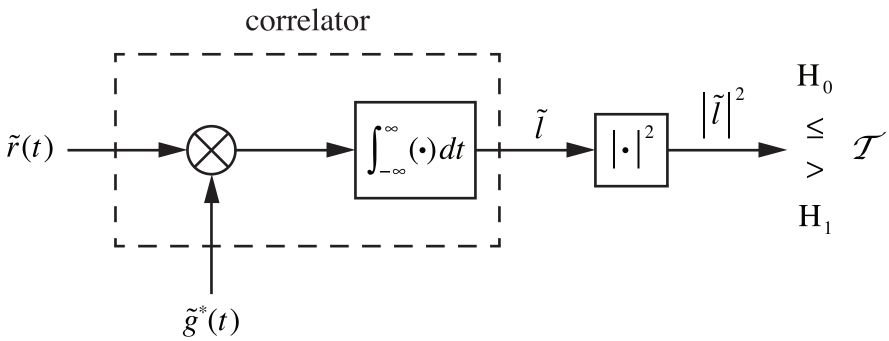

Figure 12.1-2 Correlator receiver followed by a magnitude-squared operation.

In order to decide whether or not a target return is present, we shall process $\tilde{r}(t)$ with the receiver shown in Fig. 12.1-2. The deterministic function $\tilde{g}(t)$ is referred to as the *processing waveform*, as of yet unspecified. The receiver performs the following test: choose hypothesis H_0 if

$$\left|\tilde{l}\right|^2 \leq \mathcal{T}, \tag{12.1-15}$$

and choose hypothesis H_1 if

$$\left|\tilde{l}\right|^2 > \mathcal{T}, \tag{12.1-16}$$

where

$$\tilde{l} = \langle \tilde{r}(t), \tilde{g}(t) \rangle \triangleq \int_{-\infty}^{\infty} \tilde{r}(t)\tilde{g}^*(t)\,dt \tag{12.1-17}$$

is the output from the correlator and is, in fact, the *inner product* of $\tilde{r}(t)$ and $\tilde{g}(t)$. It is also a complex, Gaussian RV. The inner product of two functions of time, denoted by $\langle \bullet, \bullet \rangle$, is analogous to the inner product of two vectors. The inner product is a linear operator, that is, it satisfies the principle of superposition. The real RV, $\left|\tilde{l}\right|^2$, is *non-Gaussian* and is referred to as the *sufficient statistic* or *test* *statistic* because it contains all the information necessary to make a decision. The *decision threshold* $\mathcal{T}$ is commonly chosen to satisfy the *Neyman-Pearson criterion*, which is to *maximize the probability of detection for a given probability of false alarm*. The Neyman-Pearson criterion is sometimes referred to as the *constant false-alarm-rate* (CFAR) criterion. The decision threshold $\mathcal{T}$ required to satisfy the Neyman-Pearson criterion will be determined in Section 12.3.

12.2 The Signal-to-Interference Ratio

In this section we shall derive the output *signal-to-interference ratio* (SIR) of the correlator shown in Fig. 12.1-2, which is defined as follows:

$$\text{SIR} \triangleq \frac{E\left\{\left|\tilde{l}_{Trgt}\right|^2\right\}}{E\left\{\left|\tilde{l}_Z\right|^2\right\}}, \tag{12.2-1}$$

where

$$\tilde{l}_{Trgt} = \left\langle \tilde{y}_{Trgt}(t), \tilde{g}(t) \right\rangle = \int_{-\infty}^{\infty} \tilde{y}_{Trgt}(t)\,\tilde{g}^*(t)\,dt \tag{12.2-2}$$

is the output from the correlator due to processing the complex envelope of the target return, and

$$\tilde{l}_Z = \left\langle \tilde{z}(t), \tilde{g}(t) \right\rangle = \int_{-\infty}^{\infty} \tilde{z}(t)\,\tilde{g}^*(t)\,dt \tag{12.2-3}$$

is the output from the correlator due to processing the sum of the complex envelopes of the reverberation return, ambient noise, and receiver noise [see (12.1-3)]. Both $\tilde{l}_{Trgt}$ and $\tilde{l}_Z$ are complex, *nonzero-mean*, Gaussian RVs. When modeling reverberation and noise, the term SIR is used rather than signal-to-noise ratio (SNR). Substituting (12.1-3) into (12.2-3) yields

$$\tilde{l}_Z = \tilde{l}_{Rev} + \tilde{l}_{n_a} + \tilde{l}_{n_r}, \tag{12.2-4}$$

where

$$\tilde{l}_{Rev} = \left\langle \tilde{y}_{Rev}(t), \tilde{g}(t) \right\rangle = \int_{-\infty}^{\infty} \tilde{y}_{Rev}(t)\,\tilde{g}^*(t)\,dt \tag{12.2-5}$$

is the output from the correlator due to processing the complex envelope of the reverberation return,

$$\tilde{l}_{n_a} = \left\langle \tilde{y}_{n_a}(t), \tilde{g}(t) \right\rangle = \int_{-\infty}^{\infty} \tilde{y}_{n_a}(t)\,\tilde{g}^*(t)\,dt \tag{12.2-6}$$

is the output from the correlator due to processing the complex envelope of the ambient noise, and

$$\tilde{l}_{n_r} = \langle \tilde{n}_r(t), \tilde{g}(t) \rangle = \int_{-\infty}^{\infty} \tilde{n}_r(t)\,\tilde{g}^*(t)\,dt \tag{12.2-7}$$

is the output from the correlator due to processing the complex envelope of the receiver noise, where $\tilde{l}_{Rev}$ is a complex, *nonzero-mean*, Gaussian RV, and both $\tilde{l}_{n_a}$ and $\tilde{l}_{n_r}$ are complex, zero-mean, Gaussian RVs.

In order to derive the SIR, we also need to specify the processing waveform $\tilde{g}(t)$. Substituting (12.1-5) into (12.1-4) yields

$$\tilde{y}_{Trgt}(t) = A_1 w_{Trgt}\,\tilde{x}(t - \tau_{Trgt}) \exp(+j2\pi \eta_{D,Trgt}\, t) \exp\left[-j2\pi(f_c + \eta_{D,Trgt})\tau_{Trgt}\right], \tag{12.2-8}$$

where

$$A_1 = \sum_{i=-N'}^{N'} a_i \,. \tag{12.2-9}$$

Therefore, let

$$\tilde{g}(t) = \tilde{x}(t - \hat{\tau}_{Trgt}) \exp(+j2\pi \hat{\eta}_{D,Trgt}\, t)\,, \tag{12.2-10}$$

which is similar in form to the target return $\tilde{y}_{Trgt}(t)$ given by (12.2-8), where $\hat{\tau}_{Trgt}$ and $\hat{\eta}_{D,Trgt}$ are *estimates* of the round-trip time delay τ_{Trgt} and Doppler shift $\eta_{D,Trgt}$ of the target, respectively.

Now that we have an equation for the processing waveform $\tilde{g}(t)$, let us compute $\tilde{l}_{Trgt}$. Substituting (12.2-8) and (12.2-10) into (12.2-2) yields

$$\begin{aligned}\tilde{l}_{Trgt} = A_1 w_{Trgt} \int_{-\infty}^{\infty} \tilde{x}(t - \tau_{Trgt})\,\tilde{x}^*(t - \hat{\tau}_{Trgt}) \exp\left[-j2\pi(\hat{\eta}_{D,Trgt} - \eta_{D,Trgt})t\right] dt \times \\ \exp\left[-j2\pi(f_c + \eta_{D,Trgt})\tau_{Trgt}\right].\end{aligned} \tag{12.2-11}$$

If we let $t' = t - \tau_{Trgt}$ in (12.2-11), then

$$\tilde{l}_{Trgt} = A_1 w_{Trgt}\, \mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) \exp\left[-j2\pi(f_c + \hat{\eta}_{D,Trgt})\tau_{Trgt}\right], \tag{12.2-12}$$

where

$$\boxed{\begin{aligned} \mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) &= \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t - e_{\tau,Trgt})\exp(-j2\pi e_{D,Trgt}t)\,dt \\ &= F\left\{\tilde{x}(t)\tilde{x}^*(t - e_{\tau,Trgt})\right\} \\ &= \left\langle \tilde{x}(t), \tilde{x}(t - e_{\tau,Trgt})\exp(+j2\pi e_{D,Trgt}t)\right\rangle \end{aligned}} \tag{12.2-13}$$

is the *unnormalized*, *auto-ambiguity function* of the transmitted electrical signal $x(t)$, with complex envelope $\tilde{x}(t)$;

$$\boxed{e_{\tau,Trgt} = \hat{\tau}_{Trgt} - \tau_{Trgt}} \tag{12.2-14}$$

is the *round-trip time-delay estimation error of the target* in seconds, and

$$\boxed{e_{D,Trgt} = \hat{\eta}_{D,Trgt} - \eta_{D,Trgt}} \tag{12.2-15}$$

is the *Doppler-shift estimation error of the target* in hertz. The auto-ambiguity function is complex in general, and is a function of two estimation errors.

The auto-ambiguity function given by (12.2-13) has its *maximum* value at $e_{\tau,Trgt} = 0$ and $e_{D,Trgt} = 0$ where

$$\begin{aligned} \mathcal{X}(0,0) &= \int_{-\infty}^{\infty} \left|\tilde{x}(t)\right|^2 dt \\ &= \left\langle \tilde{x}(t), \tilde{x}(t)\right\rangle \\ &= E_{\tilde{x}} \end{aligned} \tag{12.2-16}$$

is the energy of $\tilde{x}(t)$ in joules-ohms because $\tilde{x}(t)$ is a voltage signal. With the use of (12.2-16), the *normalized*, auto-ambiguity function is defined as follows:

$$\boxed{\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \triangleq \mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt})/E_{\tilde{x}}} \tag{12.2-17}$$

so that

$$\mathcal{X}_N(0,0) = 1\,. \tag{12.2-18}$$

Substituting (12.2-17) into (12.2-12) and computing the second moment (mean-squared value) of $\tilde{l}_{Trgt}$ yields

$$\boxed{E\left\{\left|\tilde{l}_{Trgt}\right|^2\right\} = A_1^2 E_{\tilde{x}}^2 E\left\{\left|w_{Trgt}\right|^2\right\}\left|\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2} \tag{12.2-19}$$

where A_1 is given by (12.2-9), and from (12.1-6),

$$E\left\{\left|w_{Trgt}\right|^2\right\} = \frac{1}{(4\pi r_{Trgt})^4}\left|\mathcal{S}_T(f_c)\right|^2 E\left\{\left|S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0})\right|^2\right\} \times$$
$$\exp[-4\alpha(f_c) r_{Trgt}]\left|\mathcal{S}_R(f_c)\right|^2,$$

(12.2-20)

where $\theta_{Trgt\,0} = 180° - \theta_{Trgt}$ and $\psi_{Trgt\,0} = 180° + \psi_{Trgt}$. As can be seen from (12.2-19), the mean-squared value of $\tilde{l}_{Trgt}$, which is the numerator of the SIR, is directly proportional to the magnitude-squared of the normalized, auto-ambiguity function. Therefore, if the estimation errors $e_{\tau,Trgt} = 0$ and $e_{D,Trgt} = 0$, then $\left|\mathcal{X}_N(0,0)\right|^2 = 1$ [see (12.2-18)] and the numerator of the SIR is maximized. Conversely, as the magnitudes of the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ increase, $\left|\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2 \rightarrow 0$ and, as a result, the $\text{SIR} \rightarrow 0$, which means that the target won't be detected. Another way to explain this is as follows. Under hypothesis H_1, the output of the correlator $\tilde{l} = \tilde{l}_{Trgt} + \tilde{l}_Z$. However, as can be seen from (12.2-12), as $\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) \rightarrow 0$, $\tilde{l}_{Trgt} \rightarrow 0$. Therefore, $\tilde{l} \rightarrow \tilde{l}_Z$. As a result, the sufficient statistic $\left|\tilde{l}\right|^2$ won't exceed the decision threshold $\mathcal{T}$ and hypothesis H_0 will be chosen. Therefore, the target won't be detected.

The second moment (mean-squared value) of $\tilde{l}_Z$, which is the denominator of the SIR, is given by (see Appendix 12B)

$$\boxed{\begin{aligned} E\left\{\left|\tilde{l}_Z\right|^2\right\} = E_{\tilde{x}}^2 E\left\{\left|w_{Rev}\right|^2\right\}\left|\sum_{i=-N'}^{N'} a_i \mathcal{X}_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2 + \\ A_2 E_{\tilde{x}}\left[\tilde{N}_{a0} + \tilde{N}_{r0}\right] \end{aligned}}$$

(12.2-21)

where [see (12.1-10)]

$$E\left\{\left|w_{Rev}\right|^2\right\} = \frac{1}{(4\pi r_{Rev})^4}\left|\mathcal{S}_T(f_c)\right|^2 E\left\{\left|S_{Rev}(f_c, \theta_{Rev}, \psi_{Rev}, \theta_{Rev\,0}, \psi_{Rev\,0})\right|^2\right\} \times$$
$$\exp[-4\alpha(f_c) r_{Rev}]\left|\mathcal{S}_R(f_c)\right|^2,$$

(12.2-22)

$\theta_{Rev\,0} = 180° - \theta_{Rev}$, $\psi_{Rev\,0} = 180° + \psi_{Rev}$,

$$e_{\tau_i} = e_{\tau,Trgt} + (\tau_{i,Trgt} - \tau_{i,Rev}) \tag{12.2-23}$$

is a round-trip time-delay estimation error in seconds, where $e_{\tau,Trgt}$ is the round-trip time-delay estimation error of the target given by (12.2-14), and $\tau_{i,Trgt}$ and $\tau_{i,Rev}$ are the round-trip time delays in seconds from element i in the linear array to the target and the object responsible for the reverberation, respectively;

$$e_D = \hat{\eta}_{D,Trgt} - \eta_{D,Rev} \tag{12.2-24}$$

is a Doppler-shift estimation error in hertz,

$$A_2 = \sum_{i=-N'}^{N'} a_i^2 \ , \tag{12.2-25}$$

and $\tilde{N}_{a0}$ and $\tilde{N}_{r0}$ are the constant levels in $(\text{W-}\Omega)/\text{Hz}$ (or $\text{J-}\Omega$) of the lowpass (baseband), white-noise power spectra of $\tilde{n}_a(t, x_i)$ and $\tilde{n}(t, x_i)$ $\forall\ i$ before complex weighting. Equation (12.2-21) is based on the following set of statistical assumptions made in Appendix 12B: The random processes $\tilde{n}_a(t, x_i)$ and $\tilde{n}(t, x_i)$ are *zero-mean* and *wide-sense stationary* (WSS) *in time and space* $\forall\ i$, $\tilde{n}_a(t, x_i)$ and $\tilde{n}_a(t, x_j)$ are *uncorrelated* for $i \neq j$, $\tilde{n}(t, x_i)$ and $\tilde{n}(t, x_j)$ are *uncorrelated* for $i \neq j$, and $\tilde{n}_a(t, x_i)$ and $\tilde{n}(t, x_i)$ are *lowpass (baseband)*, *white-noise* random processes in the time-domain $\forall\ i$. Since $\tilde{n}_a(t, x_i)$ and $\tilde{n}(t, x_i)$ are zero-mean and WSS in time $\forall\ i$, the random processes $\tilde{y}_{n_a}(t)$ and $\tilde{n}_r(t)$ are also *zero-mean* and *WSS*. It is also assumed that the random processes $\tilde{y}_{Rev}(t)$ and $\tilde{y}_{n_a}(t)$, $\tilde{y}_{Rev}(t)$ and $\tilde{n}_r(t)$, and $\tilde{y}_{n_a}(t)$ and $\tilde{n}_r(t)$ are *statistically independent*.

Substituting (12.2-19), (12.2-21), (12.2-9), and (12.2-25) into (12.2-1) yields the following expression for the signal-to-interference ratio (SIR):

$$\boxed{\text{SIR} = \frac{\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Trgt}\right|^2\right\}\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2}{\dfrac{E_{\tilde{x}}}{A_2} E\left\{\left|w_{Rev}\right|^2\right\}\left|\displaystyle\sum_{i=-N'}^{N'} a_i \chi_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2 + \tilde{N}_{a0} + \tilde{N}_{r0}}}$$

(12.2-26)

where

$$\mathrm{AG} = \frac{\left[\sum_{i=-N'}^{N'} a_i \right]^2}{\sum_{i=-N'}^{N'} a_i^2} \tag{12.2-27}$$

is the *dimensionless* version of the *array gain* of the linear array (see Section 7.5). If rectangular amplitude weights are used, that is, if $a_i = 1 \;\; \forall \;\; i$, then $\mathrm{AG} = N$ and $A_2 = N$. As can be seen from (12.2-26), the transmitted signal's energy $E_{\tilde{x}}$ multiplies both the target factor $E\left\{ \left| w_{Trgt} \right|^2 \right\}$ in the numerator and the reverberation factor $E\left\{ \left| w_{Rev} \right|^2 \right\}$ in the denominator. Therefore, in a reverberant environment, in order to increase the SIR without increasing the strength of the reverberation return, the AG must be increased. If the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are zero, that is, if $\hat{\tau}_{Trgt} = \tau_{Trgt}$ and $\hat{\eta}_{D,Trgt} = \eta_{D,Trgt}$, then (12.2-26) reduces to [see (12.2-18)]

$$\mathrm{SIR} = \frac{\mathrm{AG}\, E_{\tilde{x}}\, E\left\{ \left| w_{Trgt} \right|^2 \right\}}{\frac{E_{\tilde{x}}}{A_2} E\left\{ \left| w_{Rev} \right|^2 \right\} \left| \sum_{i=-N'}^{N'} a_i \mathcal{X}_N(e_{\tau_i}, e_D) \exp\left[+j2\pi (f_c + \eta_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev}) \right] \right|^2 + \tilde{N}_{a0} + \tilde{N}_{r0}}, \tag{12.2-28}$$

where

$$e_{\tau_i} = \tau_{i,Trgt} - \tau_{i,Rev} \tag{12.2-29}$$

because $e_{\tau,Trgt} = 0$ [see (12.2-23)], and

$$e_D = \eta_{D,Trgt} - \eta_{D,Rev} \tag{12.2-30}$$

because $\hat{\eta}_{D,Trgt} = \eta_{D,Trgt}$ [see (12.2-24)].

If *no* reverberation return is present at the receiver, then the SIR given by (12.2-26) reduces to the *signal-to-noise ratio* (SNR)

$$\boxed{\mathrm{SNR} = \left| \mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \right|^2 \mathrm{SNR}_{\max}} \tag{12.2-31}$$

where

$$\boxed{\mathrm{SNR}_{\max} = \mathrm{AG} \times \mathrm{SNR}_0} \tag{12.2-32}$$

is the maximum value of the SNR, and

$$\boxed{\mathrm{SNR}_0 = \frac{E_{\tilde{x}} E\left\{ \left| w_{Trgt} \right|^2 \right\}}{\tilde{N}_{a0} + \tilde{N}_{r0}}} \tag{12.2-33}$$

is the SNR due to processing the output electrical signal from a single element – element number 0 at the center of the linear array. Note that if the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are zero, then $\mathcal{X}_N(0,0) = 1$ [see (12.2-18)] and $\mathrm{SNR} = \mathrm{SNR}_{\max}$. Therefore, in a reverberation free environment, the SNR can be increased by increasing the transmitted signal's energy $E_{\tilde{x}}$ and/or the AG.

We shall conclude our discussion in this section by examining the following theoretical limiting case of a target co-located with the object responsible for the reverberation and moving with the same velocity vector. If a target is co-located with the object responsible for the reverberation, then the spherical coordinates $(r_{Trgt}, \theta_{Trgt}, \psi_{Trgt})$ and $(r_{Rev}, \theta_{Rev}, \psi_{Rev})$ are equal, and the round-trip time delays from element i in the linear array to the target and object responsible for the reverberation are equal $\forall\ i$, that is,

$$\tau_{i,Rev} = \tau_{i,Trgt}, \qquad i = -N', \ldots, 0, \ldots, N'. \tag{12.2-34}$$

Substituting (12.2-34) into (12.2-23) yields

$$e_{\tau_i} = e_{\tau,Trgt}, \qquad i = -N', \ldots, 0, \ldots, N', \tag{12.2-35}$$

and by substituting (12.2-34), (12.2-35), and (12.2-25) into (12.2-26), the SIR reduces to

$$\mathrm{SIR} = \frac{\mathrm{AG}\, E_{\tilde{x}} E\left\{ \left| w_{Trgt} \right|^2 \right\} \left| \mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \right|^2}{\mathrm{AG}\, E_{\tilde{x}} E\left\{ \left| w_{Rev} \right|^2 \right\} \left| \mathcal{X}_N(e_{\tau,Trgt}, e_D) \right|^2 + \tilde{N}_{a0} + \tilde{N}_{r0}}. \tag{12.2-36}$$

Note that in this case, the AG multiplies both the target factor $E\left\{ \left| w_{Trgt} \right|^2 \right\}$ in the numerator and the reverberation factor $E\left\{ \left| w_{Rev} \right|^2 \right\}$ in the denominator. Since the target is co-located with the object responsible for the reverberation, the directions of arrival of the acoustic fields incident upon the array due to the target and reverberation are identical. Therefore, the phase weights that will co-phase all the output electrical signals due to the target will also co-phase all the output electrical signals due to reverberation. Furthermore, if the Doppler-shifts of the target and object responsible for the reverberation are equal, that is, if

$$\eta_{D,Rev} = \eta_{D,Trgt} \, , \tag{12.2-37}$$

then substituting (12.2-37) into (12.2-24) yields

$$e_D = e_{D,Trgt} \, , \tag{12.2-38}$$

and by substituting (12.2-38) into (12.2-36),

$$\text{SIR} = \frac{\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Trgt}\right|^2\right\}\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2}{\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Rev}\right|^2\right\}\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2 + \tilde{N}_{a0} + \tilde{N}_{r0}} \, . \tag{12.2-39}$$

If the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are zero, then (12.2-39) reduces to [see (12.2-18)]

$$\text{SIR} = \frac{\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Trgt}\right|^2\right\}}{\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Rev}\right|^2\right\} + \tilde{N}_{a0} + \tilde{N}_{r0}} \, , \tag{12.2-40}$$

and if

$$\text{AG}\, E_{\tilde{x}}\, E\left\{\left|w_{Rev}\right|^2\right\} >> \tilde{N}_{a0} + \tilde{N}_{r0} \, , \tag{12.2-41}$$

then (12.2-40) reduces to

$$\text{SIR} = \frac{E\left\{\left|w_{Trgt}\right|^2\right\}}{E\left\{\left|w_{Rev}\right|^2\right\}} = \frac{E\left\{\left|S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0})\right|^2\right\}}{E\left\{\left|S_{Rev}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0})\right|^2\right\}} \tag{12.2-42}$$

since the spherical coordinates $(r_{Trgt}, \theta_{Trgt}, \psi_{Trgt})$ and $(r_{Rev}, \theta_{Rev}, \psi_{Rev})$ are equal. Equation (12.2-42) is the ratio between the mean-squared values of the target and reverberation scattering functions. Approximate examples of this theoretical limiting case are a motionless target lying on or near the ocean bottom (bottom reverberation), or a very slow moving target near the ocean surface (surface reverberation).

12.3 Probability of False Alarm and Decision Threshold

In this section we shall determine the decision threshold $\mathcal{T}$ shown in Fig. 12.1-2 for a Neyman-Pearson decision criterion which is to maximize the probability of detection for a given probability of false alarm. The decision threshold can be obtained by satisfying a probability of false alarm constraint.

We begin by rewriting the binary hypothesis-testing problem given by (12.1-1) and (12.1-2) in terms of the output of the correlator $\tilde{l}$ as follows:

$$\mathrm{H}_0: \quad \tilde{l} = \tilde{l}_Z \tag{12.3-1}$$

$$\mathrm{H}_1: \quad \tilde{l} = \tilde{l}_{Trgt} + \tilde{l}_Z \tag{12.3-2}$$

where $\tilde{l}_Z$ and $\tilde{l}_{Trgt}$ are given by (12.2-4) and (12.2-12), respectively. For hypothesis H_0, $\tilde{l}$ shall be expressed in rectangular form as follows:

$$\mathrm{H}_0: \quad \tilde{l} = \tilde{l}_Z = X_0 + jY_0 . \tag{12.3-3}$$

Since $\tilde{l}$ is a complex, Gaussian, random variable (RV), X_0 and Y_0 are real, Gaussian RVs. In order to compute the probability of false alarm, we first need to determine some of the statistics of $\tilde{l}$ given by (12.3-3).

The mean value (average value) of $\tilde{l}$ given hypothesis H_0 is true is given by

$$E\left\{ \tilde{l} \,\middle|\, \mathrm{H}_0 \right\} = m_{X_0} + jm_{Y_0} , \tag{12.3-4}$$

where

$$m_{X_0} = E\left\{ X_0 \right\} \tag{12.3-5}$$

and

$$m_{Y_0} = E\left\{ Y_0 \right\} . \tag{12.3-6}$$

However, from (12.3-1) and (12.2-4),

$$E\left\{ \tilde{l} \,\middle|\, \mathrm{H}_0 \right\} = E\left\{ \tilde{l}_Z \right\} = E\left\{ \tilde{l}_{Rev} \right\} \tag{12.3-7}$$

since [see (12.2-6)]

$$E\left\{ \tilde{l}_{n_a} \right\} = \int_{-\infty}^{\infty} E\left\{ \tilde{y}_{n_a}(t) \right\} \tilde{g}^*(t)\, dt = 0 \tag{12.3-8}$$

and [see (12.2-7)]

$$E\left\{ \tilde{l}_{n_r} \right\} = \int_{-\infty}^{\infty} E\left\{ \tilde{n}_r(t) \right\} \tilde{g}^*(t)\, dt = 0 \tag{12.3-9}$$

because $\tilde{y}_{n_a}(t)$ and $\tilde{n}_r(t)$ are zero-mean. Substituting (12B-26) into (12.3-7) yields

$$\begin{aligned} E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\} &= E\left\{\tilde{l}_Z\right\} \\ &= E\left\{\tilde{l}_{Rev}\right\} \\ &= E_{\tilde{x}}\,E\left\{w_{Rev}\right\}\exp\left[-j2\pi(f_c+\hat{\eta}_{D,Trgt})\tau_{Trgt}\right]\times \\ &\quad \sum_{i=-N'}^{N'} a_i\,X_N(e_{\tau_i},e_D)\exp\left[+j2\pi(f_c+\hat{\eta}_{D,Trgt})(\tau_{i,Trgt}-\tau_{i,Rev})\right], \end{aligned} \tag{12.3-10}$$

where [see (12.1-10)]

$$\begin{aligned} E\left\{w_{Rev}\right\} = \frac{1}{(4\pi r_{Rev})^2}\,\mathcal{S}_T(f_c)\,E\left\{S_{Rev}(f_c,\theta_{Rev},\psi_{Rev},\theta_{Rev\,0},\psi_{Rev\,0})\right\}\times \\ \exp[-2\alpha(f_c)r_{Rev}]\,\mathcal{S}_R(f_c). \end{aligned} \tag{12.3-11}$$

Therefore, m_{X_0} and m_{Y_0} are the real and imaginary parts of (12.3-10), respectively [see (12.3-4)].

The variance of $\tilde{l}$ given hypothesis H_0 is true is given by

$$\mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right) = E\left\{\left|\tilde{l}\right|^2\middle|\,\mathrm{H}_0\right\} - \left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\}\right|^2, \tag{12.3-12}$$

where [see (12.3-3)]

$$E\left\{\left|\tilde{l}\right|^2\middle|\,\mathrm{H}_0\right\} = E\left\{X_0^2\right\} + E\left\{Y_0^2\right\} \tag{12.3-13}$$

and [see (12.3-4)]

$$\left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\}\right|^2 = m_{X_0}^2 + m_{Y_0}^2 . \tag{12.3-14}$$

Substituting (12.3-13) and (12.3-14) into (12.3-12) yields

$$\mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right) = \sigma_{X_0}^2 + \sigma_{Y_0}^2 , \tag{12.3-15}$$

where

$$\sigma_{X_0}^2 = E\left\{X_0^2\right\} - m_{X_0}^2 \tag{12.3-16}$$

is the variance of X_0, and

$$\sigma_{Y_0}^2 = E\left\{Y_0^2\right\} - m_{Y_0}^2 \tag{12.3-17}$$

is the variance of Y_0. However, from (12.3-1),

$$\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)=\operatorname{Var}\left(\tilde{l}_Z\right)=E\left\{\left|\tilde{l}_Z\right|^2\right\}-\left|E\left\{\tilde{l}_Z\right\}\right|^2, \tag{12.3-18}$$

and by substituting (12.3-7) into (12.3-18),

$$\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)=\operatorname{Var}\left(\tilde{l}_Z\right)=E\left\{\left|\tilde{l}_Z\right|^2\right\}-\left|E\left\{\tilde{l}_{Rev}\right\}\right|^2. \tag{12.3-19}$$

With the use of (12.3-18), (12.3-19), (12.2-21), and (12.3-10), $\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ can also be expressed as

$$\begin{aligned}
\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)&=\operatorname{Var}\left(\tilde{l}_Z\right)\\
&=E\left\{\left|\tilde{l}_Z\right|^2\right\}-\left|E\left\{\tilde{l}_Z\right\}\right|^2\\
&=E\left\{\left|\tilde{l}_Z\right|^2\right\}-\left|E\left\{\tilde{l}_{Rev}\right\}\right|^2\\
&=E_{\tilde{x}}^2\,\sigma_{w_{Rev}}^2\left|\sum_{i=-N'}^{N'}a_i\,\chi_N(e_{\tau_i},e_D)\exp\left[+j2\pi(f_c+\hat{\eta}_{D,Trgt})(\tau_{i,Trgt}-\tau_{i,Rev})\right]\right|^2+\\
&\quad A_2E_{\tilde{x}}\left[\tilde{N}_{a0}+\tilde{N}_{r0}\right],
\end{aligned} \tag{12.3-20}$$

where

$$\sigma_{w_{Rev}}^2=E\left\{\left|w_{Rev}\right|^2\right\}-\left|E\left\{w_{Rev}\right\}\right|^2 \tag{12.3-21}$$

is the variance of w_{Rev}, $E\left\{\left|w_{Rev}\right|^2\right\}$ is given by (12.2-22), $E\left\{w_{Rev}\right\}$ is given by (12.3-11), and A_2 is given by (12.2-25). Also, since [see (12.2-4) and (12.3-3)]

$$\mathrm{H}_0:\quad \tilde{l}=\tilde{l}_Z=\tilde{l}_{Rev}+\tilde{l}_{n_a}+\tilde{l}_{n_r}=X_0+jY_0, \tag{12.3-22}$$

X_0 and Y_0 are *uncorrelated* and *statistically independent* because $\tilde{l}_{Rev}$ and $\tilde{l}_{n_a}$, $\tilde{l}_{Rev}$ and $\tilde{l}_{n_r}$, and $\tilde{l}_{n_a}$ and $\tilde{l}_{n_r}$ are *uncorrelated* and *statistically independent* [see (12B-9), (12B-15), and (12B-20), respectively].

Our next problem is to determine the probability density function (PDF) of the test statistic $\left|\tilde{l}\right|^2$ given hypothesis H_0 is true. Let us begin with the magnitude of $\tilde{l}$, where from (12.3-3),

$$\mathrm{H}_0: \quad \left|\tilde{l}\right| = \left|\tilde{l}_Z\right| = \sqrt{X_0^2 + Y_0^2} \, . \tag{12.3-23}$$

Since X_0 is $N(m_{X_0}, \sigma_{X_0}^2)$ and Y_0 is $N(m_{Y_0}, \sigma_{Y_0}^2)$, $\left|\tilde{l}\right|$ is a *Beckmann* RV[1]. In our case, X_0 and Y_0 are uncorrelated Gaussian RVs with correlation coefficient $\rho_{X_0Y_0} = 0$. However, even if X_0 and Y_0 are correlated (i.e., $-1 \le \rho_{X_0Y_0} \le 1$), $\left|\tilde{l}\right|$ is still a Beckmann RV. The set of equations for the PDF and cumulative distribution function (CDF) for a Beckmann RV are lengthy and are not closed-form expressions[2].

In order to obtain more tractable results that will give us a good insight into our problem, we shall make the following simplifying assumption: let

$$\boxed{\sigma_{Y_0}^2 = \sigma_{X_0}^2 = \sigma_0^2} \tag{12.3-24}$$

Therefore, substituting (12.3-24) into (12.3-15) yields

$$\mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_0\right) = 2\sigma_0^2 \, , \tag{12.3-25}$$

where $\mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_0\right)$ is given by (12.3-20). Since [see (12.3-23)]

$$\mathrm{H}_0: \quad \left|\tilde{l}\right|^2 = \left|\tilde{l}_Z\right|^2 = X_0^2 + Y_0^2 \, , \tag{12.3-26}$$

where X_0 is $N(m_{X_0}, \sigma_0^2)$, Y_0 is $N(m_{Y_0}, \sigma_0^2)$, X_0 and Y_0 have the *same* variance σ_0^2, and X_0 and Y_0 are *statistically independent* because they are uncorrelated Gaussian RVs, $\left|\tilde{l}\right|^2$ is a *noncentral*, *chi-squared* RV with 2 degrees of freedom and PDF[3]

$$\boxed{p_0(\gamma) = \frac{1}{2\sigma_0^2} \exp\left(-\frac{\gamma + \gamma_0}{2\sigma_0^2}\right) I_0\left(\frac{\sqrt{\gamma_0}}{\sigma_0^2}\sqrt{\gamma}\right), \qquad \gamma \ge 0} \tag{12.3-27}$$

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[1] *Mathematica*, Version 10.4.1, Wolfram Research, 2016.

[2] P. Dharmawansa, N. Rajatheva, and C. Tellambura, "Envelope and Phase Distribution of Two Correlated Gaussian Variables," *IEEE Trans. Commun.*, vol. 57, pp. 915-921, April 2009.

[3] T. A. Schonhoff and A. A. Giordano, *Detection and Estimation Theory*, Pearson Prentice Hall, Upper Saddle River, NJ, 2006, pp. 602-604.

where the RV

$$\gamma = \left|\tilde{l}\right|^2 = X_0^2 + Y_0^2 , \tag{12.3-28}$$

the nonrandom constant [see (12.3-14)]

$$\gamma_0 = \left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\}\right|^2 = m_{X_0}^2 + m_{Y_0}^2 \tag{12.3-29}$$

is the *noncentrality parameter*, where m_{X_0} and m_{Y_0} are the real and imaginary parts of (12.3-10), respectively [see (12.3-4)],

$$2\sigma_0^2 = \mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right), \tag{12.3-30}$$

where $\mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ is given by (12.3-20) [see (12.3-25)],

$$I_0(u) = \sum_{n=0}^{\infty} \frac{u^{2n}}{2^{2n}(n!)^2} \tag{12.3-31}$$

is the zeroth-order modified Bessel function of the first kind, and

$$p_0(\gamma) = p_{\gamma|\mathrm{H}_0}\left(\gamma\,\middle|\,\mathrm{H}_0\right). \tag{12.3-32}$$

See Fig. 10.6-3 for plots of a scaled, noncentral, chi-squared PDF with 2 degrees of freedom.

The probability of false alarm P_{FA} is given by

$$P_{\mathrm{FA}} = P\left(\left|\tilde{l}\right|^2 > \mathcal{T}\,\middle|\,\mathrm{H}_0\right), \tag{12.3-33}$$

or

$$P_{\mathrm{FA}} = P\left(\gamma > \mathcal{T}\,\middle|\,\mathrm{H}_0\right) = \int_{\mathcal{T}}^{\infty} p_0(\gamma)\,d\gamma , \tag{12.3-34}$$

where $\mathcal{T}$ is the decision threshold. Substituting (12.3-27) into (12.3-34) yields

$$P_{\mathrm{FA}} = \frac{1}{2\sigma_0^2}\int_{\mathcal{T}}^{\infty} \exp\left(-\frac{\gamma+\gamma_0}{2\sigma_0^2}\right) I_0\left(\frac{\sqrt{\gamma_0}}{\sigma_0^2}\sqrt{\gamma}\right) d\gamma . \tag{12.3-35}$$

If we let

$$x^2 = \gamma / \sigma_0^2 \tag{12.3-36}$$

and

$$\alpha_0^2 = \gamma_0 / \sigma_0^2 \,, \tag{12.3-37}$$

then (12.3-35) can be rewritten as

$$\boxed{P_{\mathrm{FA}} = Q\left(\alpha_0, \sqrt{\mathcal{T}} / \sigma_0\right) = \int_{\sqrt{\mathcal{T}}/\sigma_0}^{\infty} \exp\left[-(x^2 + \alpha_0^2)/2\right] I_0(\alpha_0 x)\, x\, dx} \tag{12.3-38}$$

where

$$\boxed{Q(a,b) = \int_b^{\infty} \exp\left[-(x^2 + a^2)/2\right] I_0(ax)\, x\, dx} \tag{12.3-39}$$

is the *Marcum Q-function*. See Example 12.3-1 for a solution for the decision threshold $\mathcal{T}$ that satisfies a P_{FA} constraint using (12.3-38).

If the reverberation scattering function is *zero-mean*, that is, if

$$E\left\{S_{Rev}(f_c, \theta_{Rev}, \psi_{Rev}, \theta_{Rev\,0}, \psi_{Rev\,0})\right\} = 0 \,, \tag{12.3-40}$$

then [see (12.3-11)]

$$E\left\{w_{Rev}\right\} = 0 \tag{12.3-41}$$

and [see (12.3-10)]

$$E\left\{\tilde{l} \,\middle|\, \mathrm{H}_0\right\} = E\left\{\tilde{l}_Z\right\} = E\left\{\tilde{l}_{Rev}\right\} = 0 \,. \tag{12.3-42}$$

As a result, $m_{X_0} = 0$, $m_{Y_0} = 0$, and $\gamma_0 = 0$ [see (12.3-4) and (12.3-29)]. Therefore, the statistically independent Gaussian RVs X_0 and Y_0 are now $N(0, \sigma_0^2)$, and $\gamma = \left|\tilde{l}\right|^2$ given by (12.3-28) is an *exponential* RV. If $\gamma_0 = 0$, and since $I_0(0) = 1$, the noncentral, chi-squared PDF given by (12.3-27) reduces to the *exponential* PDF

$$\boxed{p_0(\gamma) = \frac{1}{2\sigma_0^2} \exp\left(-\frac{\gamma}{2\sigma_0^2}\right), \qquad \gamma \geq 0} \tag{12.3-43}$$

where $2\sigma_0^2 = \mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_0\right)$ [see (12.3-30)], but now, because of (12.3-41) and

(12.3-42), $\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ given by (12.3-20) reduces to

$$\begin{aligned}
\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right) &= \operatorname{Var}\left(\tilde{l}_Z\right) \\
&= E\left\{\left|\tilde{l}_Z\right|^2\right\} \\
&= E_{\tilde{x}}^2 E\left\{\left|w_{Rev}\right|^2\right\}\left|\sum_{i=-N'}^{N'} a_i \chi_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2 + \\
&\quad A_2 E_{\tilde{x}}\left[\tilde{N}_{a0} + \tilde{N}_{r0}\right].
\end{aligned} \tag{12.3-44}$$

Therefore, in the limit as $\sqrt{\gamma_0}/\sigma_0 \to 0$, or equivalently, as

$$\frac{\gamma_0}{2\sigma_0^2} = \frac{\left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\}\right|^2}{\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)} \to 0\,, \tag{12.3-45}$$

$\gamma = \left|\tilde{l}\right|^2$ approaches being an exponential RV.

If the exponential PDF given by (12.3-43) is substituted into (12.3-34), then

$$P_{\mathrm{FA}} = \frac{1}{2\sigma_0^2}\int_{\mathcal{T}}^{\infty} \exp\left(-\frac{\gamma}{2\sigma_0^2}\right) d\gamma\,, \tag{12.3-46}$$

or

$$\boxed{P_{\mathrm{FA}} = \exp\left(-\frac{\mathcal{T}}{2\sigma_0^2}\right)} \tag{12.3-47}$$

Solving for the decision threshold $\mathcal{T}$ yields

$$\boxed{\mathcal{T} = 2\sigma_0^2 \ln\left(1/P_{\mathrm{FA}}\right)} \tag{12.3-48}$$

where $2\sigma_0^2 = \operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ and $\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ is given by (12.3-44). From (12.3-48) it can be seen that $\mathcal{T}$ is inversely proportional to P_{FA}. Therefore, for a given value of σ_0^2, as the $P_{\mathrm{FA}} \to 0$, $\mathcal{T} \to \infty$; and as the $P_{\mathrm{FA}} \to 1$, $\mathcal{T} \to 0$.

Example 12.3-1 Nonzero-Mean Reverberation Scattering Function

In this example we shall determine the decision threshold $\mathcal{T}$ that satisfies a P_{FA} constraint using (12.3-38). We first need to generate a table of values for the Marcum Q-function $Q(a,b)$ given by (12.3-39) by deciding on several different values for the parameter "a." Toward this end, if we divide (12.3-37) by a factor of 2, then

$$\frac{\alpha_0^2}{2} = \frac{\gamma_0}{2\sigma_0^2} = \frac{\left|E\left\{\tilde{l}\,\middle|\mathrm{H}_0\right\}\right|^2}{\mathrm{Var}\left(\tilde{l}\,\middle|\mathrm{H}_0\right)}, \tag{12.3-49}$$

and by setting $\alpha_0^2/2$ equal to several different representative values as shown in Table 12.3-1, we can solve for α_0, and hence, several different representative values for parameter "a." Values of $Q(a,b)$ for $a=1/2$, $\sqrt{2}/2$, 1, $\sqrt{2}$, 2, $2\sqrt{2}$, and 4 for $b=0$ to 5.5 are given in Table 12C-1 in Appendix 12C. The case $\alpha_0=0$ ($a=0$) has already been solved since $\alpha_0=0$ when $\gamma_0=0$, and when $\gamma_0=0$, the decision threshold is given by (12.3-48). Equation (12.3-49) can be thought of as a measure of the randomness of the output of the correlator $\tilde{l}$ given hypothesis H_0 is true. For example, as $\alpha_0^2/2 \to 0$, $\tilde{l}$ becomes more random. Conversely, as $\alpha_0^2/2 \to \infty$, $\tilde{l}$ becomes more deterministic.

Table 12.3-1 Values of α_0 Used to Determine Values of Parameter "a" in the Marcum Q-Function $Q(a,b)$

$\alpha_0^2/2$	$10\log_{10}\left(\alpha_0^2/2\right)$	α_0
0.125	-9 dB	$1/2$
0.25	-6 dB	$\sqrt{2}/2$
0.5	-3 dB	1
1	0 dB	$\sqrt{2}$
2	3 dB	2
4	6 dB	$2\sqrt{2}$
8	9 dB	4

Now that we have a table of values for the Marcum Q-function, let us determine the decision threshold $\mathcal{T}$ for a $P_{\text{FA}}=0.001$ when $\alpha_0=\sqrt{2}/2$. Since [see (12.3-38)]

$$P_{\mathrm{FA}} = Q\left(\sqrt{2}/2, \sqrt{\mathcal{T}}/\sigma_0\right) = 0.001, \tag{12.3-50}$$

set $a = \alpha_0 = \sqrt{2}/2$, and find the value of b that yields $Q\left(\sqrt{2}/2, b\right) = 0.001$ in Table 12C-1 in Appendix 12C. From Table 12C-1:

	$a = \sqrt{2}/2$
b	$Q(a,b)$
4	0.0012837
4.1	0.0009038

Next, use linear interpolation to find b.

If we let

$$Q(a,b) = mb + c, \tag{12.3-51}$$

then the slope m is

$$\begin{aligned} m &= \frac{Q(a,b_2) - Q(a,b_1)}{b_2 - b_1} \\ &= \frac{Q\left(\sqrt{2}/2, 4.1\right) - Q\left(\sqrt{2}/2, 4\right)}{4.1 - 4} \\ &= \frac{0.0009038 - 0.0012837}{0.1} \\ &= -0.003799 \end{aligned} \tag{12.3-52}$$

and the y-intercept is

$$\begin{aligned} c &= Q(a,b_1) - mb_1 \\ &= Q\left(\sqrt{2}/2, 4\right) - (-0.003799)4 \\ &= 0.0012837 + 0.015196 \\ &= 0.0164797. \end{aligned} \tag{12.3-53}$$

Solving for b from (12.3-51) yields

$$b = \frac{Q(a,b) - c}{m}, \tag{12.3-54}$$

and by substituting (12.3-52), (12.3-53), and $Q(a,b) = 0.001$ into (12.3-54), we obtain

$$b \approx 4.0747 \ . \tag{12.3-55}$$

By referring to (12.3-38) and (12.3-39),

$$b = \sqrt{\mathcal{T}} \big/ \sigma_0 \ , \tag{12.3-56}$$

or

$$\mathcal{T} = b^2 \sigma_0^2 \ , \tag{12.3-57}$$

where $2\sigma_0^2 = \mathrm{Var}\left(\tilde{l} \middle| \mathrm{H}_0 \right)$ [see (12.3-30)] and $\mathrm{Var}\left(\tilde{l} \middle| \mathrm{H}_0 \right)$ is given by (12.3-20). Substituting (12.3-55) into (12.3-57) yields

$$\mathcal{T} = 16.603 \sigma_0^2 \ . \tag{12.3-58}$$

The numerical factor 16.603 in (12.3-58) is based on the specific values of $\alpha_0 = \sqrt{2} \big/ 2$ and $P_{\mathrm{FA}} = 0.001$. The equation for the decision threshold $\mathcal{T}$ can be generalized for any value of α_0 and P_{FA} as follows. By referring to (12.3-54) and noting that $P_{\mathrm{FA}} = Q(a, b)$, (12.3-54) can be rewritten as

$$\boxed{b_{\mathrm{FA}} = \frac{P_{\mathrm{FA}} - c_{\alpha_0}}{m_{\alpha_0}}} \tag{12.3-59}$$

where [see (12.3-52) and (12.3-53)]

$$m_{\alpha_0} = \frac{Q(\alpha_0, b_2) - Q(\alpha_0, b_1)}{b_2 - b_1} \ , \tag{12.3-60}$$

$$c_{\alpha_0} = Q(\alpha_0, b_1) - m_{\alpha_0} b_1 \ , \tag{12.3-61}$$

and

$$Q(\alpha_0, b_2) < P_{\mathrm{FA}} < Q(\alpha_0, b_1) \ . \tag{12.3-62}$$

Therefore, (12.3-57) can be rewritten as

$$\boxed{\mathcal{T} = b_{\mathrm{FA}}^2 \sigma_0^2} \tag{12.3-63}$$

where $2\sigma_0^2 = \mathrm{Var}\left(\tilde{l} \middle| \mathrm{H}_0 \right)$ [see (12.3-30)] and $\mathrm{Var}\left(\tilde{l} \middle| \mathrm{H}_0 \right)$ is given by (12.3-20). As can be seen from (12.3-59), for a given value of α_0, b_{FA} will change value as the P_{FA} changes value. As the P_{FA} decreases, b_{FA} increases, and for a given value of σ_0^2, $\mathcal{T}$ increases. ■

12.4 Probability of Detection and Receiver Operating Characteristic Curves

In this section we shall derive an equation for the probability of detection P_{D}. We begin by expressing the complex, Gaussian, random variable (RV) $\tilde{l}_{Trgt}$ in rectangular form as follows:

$$\tilde{l}_{Trgt} = X + jY \,, \tag{12.4-1}$$

where X and Y are real, Gaussian RVs. Since [see (12.3-3)]

$$\tilde{l}_Z = X_0 + jY_0 \,, \tag{12.4-2}$$

where X_0 and Y_0 are real, Gaussian RVs, substituting (12.4-1) and (12.4-2) into (12.3-2) yields

$$\mathrm{H}_1\text{:} \quad \tilde{l} = \tilde{l}_{Trgt} + \tilde{l}_Z = X_1 + jY_1 \,, \tag{12.4-3}$$

where

$$X_1 = X + X_0 \tag{12.4-4}$$

and

$$Y_1 = Y + Y_0 \tag{12.4-5}$$

are real, Gaussian RVs. In order to compute the P_{D}, we first need to determine some of the statistics of $\tilde{l}$ given by (12.4-3).

The mean value (average value) of $\tilde{l}_{Trgt}$ is given by

$$E\left\{\tilde{l}_{Trgt}\right\} = m_X + jm_Y \,, \tag{12.4-6}$$

where

$$m_X = E\{X\} \tag{12.4-7}$$

and

$$m_Y = E\{Y\} \,. \tag{12.4-8}$$

The mean value of $\tilde{l}$ given hypothesis H_1 is true is given by

$$E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\} = m_{X_1} + jm_{Y_1} \,, \tag{12.4-9}$$

where

$$m_{X_1} = E\left\{X_1\right\} = m_X + m_{X_0} \tag{12.4-10}$$

and

$$m_{Y_1} = E\left\{Y_1\right\} = m_Y + m_{Y_0} \,. \tag{12.4-11}$$

However, from (12.4-3) and (12.3-7),

$$E\left\{\tilde{l}\,\middle|\mathrm{H}_1\right\} = E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_Z\right\} = E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_{Rev}\right\}. \tag{12.4-12}$$

Computing the expected value of $\tilde{l}_{Trgt}$ given by (12.2-12) yields

$$E\left\{\tilde{l}_{Trgt}\right\} = A_1 E_{\tilde{x}} E\left\{w_{Trgt}\right\} \chi_N(e_{\tau,Trgt}, e_{D,Trgt}) \exp\left[-j2\pi(f_c + \hat{\eta}_{D,Trgt})\tau_{Trgt}\right], \tag{12.4-13}$$

where A_1 is given by (12.2-9) and [see (12.1-6)]

$$\begin{aligned} E\left\{w_{Trgt}\right\} = {} & \frac{1}{(4\pi r_{Trgt})^2} \mathcal{S}_T(f_c) E\left\{S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0})\right\} \times \\ & \exp[-2\alpha(f_c) r_{Trgt}] \mathcal{S}_R(f_c). \end{aligned} \tag{12.4-14}$$

Substituting (12.4-13) and (12.3-10) into (12.4-12) yields

$$\begin{aligned} E\left\{\tilde{l}\,\middle|\mathrm{H}_1\right\} &= E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_Z\right\} \\ &= E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_{Rev}\right\} \\ &= A_1 E_{\tilde{x}} E\left\{w_{Trgt}\right\} \chi_N(e_{\tau,Trgt}, e_{D,Trgt}) \exp\left[-j2\pi(f_c + \hat{\eta}_{D,Trgt})\tau_{Trgt}\right] + \\ &\quad E_{\tilde{x}} E\left\{w_{Rev}\right\} \exp\left[-j2\pi(f_c + \hat{\eta}_{D,Trgt})\tau_{Trgt}\right] \times \\ &\quad \sum_{i=-N'}^{N'} a_i \chi_N(e_{\tau_i}, e_D) \exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]. \end{aligned} \tag{12.4-15}$$

Therefore, m_{X_1} and m_{Y_1} are the real and imaginary parts of (12.4-15), respectively [see (12.4-9)].

The variance of $\tilde{l}$ given hypothesis H_1 is true is given by

$$\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)=E\left\{\left|\tilde{l}\right|^2\middle|\,\mathrm{H}_1\right\}-\left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\}\right|^2, \tag{12.4-16}$$

where [see (12.4-3)]

$$E\left\{\left|\tilde{l}\right|^2\middle|\,\mathrm{H}_1\right\}=E\left\{X_1^2\right\}+E\left\{Y_1^2\right\} \tag{12.4-17}$$

and [see (12.4-9)]

$$\left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\}\right|^2=m_{X_1}^2+m_{Y_1}^2 . \tag{12.4-18}$$

Substituting (12.4-17) and (12.4-18) into (12.4-16) yields

$$\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)=\sigma_{X_1}^2+\sigma_{Y_1}^2 , \tag{12.4-19}$$

where

$$\sigma_{X_1}^2=E\left\{X_1^2\right\}-m_{X_1}^2 \tag{12.4-20}$$

is the variance of X_1, and

$$\sigma_{Y_1}^2=E\left\{Y_1^2\right\}-m_{Y_1}^2 \tag{12.4-21}$$

is the variance of Y_1. However, from (12.4-3),

$$\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)=\operatorname{Var}\left(\tilde{l}_{Trgt}\right)+\operatorname{Var}\left(\tilde{l}_Z\right)+2\operatorname{Cov}\left(\tilde{l}_{Trgt},\tilde{l}_Z\right). \tag{12.4-22}$$

Assuming that $\tilde{l}_{Trgt}$ and $\tilde{l}_Z=\tilde{l}_{Rev}+\tilde{l}_{n_a}+\tilde{l}_{n_r}$ are *uncorrelated*, (12.4-22) reduces to

$$\begin{aligned}\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)&=\operatorname{Var}\left(\tilde{l}_{Trgt}\right)+\operatorname{Var}\left(\tilde{l}_Z\right)\\&=\operatorname{Var}\left(\tilde{l}_{Trgt}\right)+\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right),\end{aligned} \tag{12.4-23}$$

where

$$\operatorname{Var}\left(\tilde{l}_{Trgt}\right)=E\left\{\left|\tilde{l}_{Trgt}\right|^2\right\}-\left|E\left\{\tilde{l}_{Trgt}\right\}\right|^2 \tag{12.4-24}$$

and $\operatorname{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right)$ is given by (12.3-20). Substituting (12.2-19) and (12.4-13) into (12.4-24) yields

$$\mathrm{Var}\left(\tilde{l}_{Trgt}\right) = A_1^2 E_{\tilde{x}}^2 \sigma_{w_{Trgt}}^2 \left| \mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \right|^2, \tag{12.4-25}$$

where A_1 is given by (12.2-9),

$$\sigma_{w_{Trgt}}^2 = E\left\{ \left| w_{Trgt} \right|^2 \right\} - \left| E\left\{ w_{Trgt} \right\} \right|^2 \tag{12.4-26}$$

is the variance of w_{Trgt}, $E\left\{ \left| w_{Trgt} \right|^2 \right\}$ is given by (12.2-20), and $E\left\{ w_{Trgt} \right\}$ is given by (12.4-14). Substituting (12.4-25) and (12.3-20) into (12.4-23) yields

$$\begin{aligned} \mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_1 \right) &= \mathrm{Var}\left(\tilde{l}_{Trgt}\right) + \mathrm{Var}\left(\tilde{l}_Z\right) \\ &= \mathrm{Var}\left(\tilde{l}_{Trgt}\right) + \mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_0 \right) \\ &= A_1^2 E_{\tilde{x}}^2 \sigma_{w_{Trgt}}^2 \left| \mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \right|^2 + \\ &\quad E_{\tilde{x}}^2 \sigma_{w_{Rev}}^2 \left| \sum_{i=-N'}^{N'} a_i \mathcal{X}_N(e_{\tau_i}, e_D) \exp\left[+j2\pi (f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev}) \right] \right|^2 + \\ &\quad A_2 E_{\tilde{x}} \left[\tilde{N}_{a0} + \tilde{N}_{r0} \right]. \end{aligned} \tag{12.4-27}$$

Also, since it was assumed that $\tilde{l}_{Trgt}$ and $\tilde{l}_Z$ are uncorrelated, X_1 and Y_1 are *uncorrelated* and *statistically independent* because they are Gaussian RVs [see (12.4-3)].

Our next problem is to determine the probability density function (PDF) of the test statistic $\left|\tilde{l}\right|^2$ given hypothesis H_1 is true. Using the same argument that was made in Section 12.3, in order to obtain more tractable results that will give us a good insight into our problem, we shall make the following simplifying assumption: let

$$\boxed{\sigma_{Y_1}^2 = \sigma_{X_1}^2 = \sigma_1^2} \tag{12.4-28}$$

Therefore, substituting (12.4-28) into (12.4-19) yields

$$\mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_1 \right) = 2\sigma_1^2, \tag{12.4-29}$$

where $\mathrm{Var}\left(\tilde{l} \,\middle|\, \mathrm{H}_1 \right)$ is given by (12.4-27). Since [see (12.4-3)]

$$\mathrm{H}_1:\quad \left|\tilde{l}\right|^2 = \left|\tilde{l}_{Trgt} + \tilde{l}_Z\right|^2 = X_1^2 + Y_1^2\,, \tag{12.4-30}$$

where X_1 is $N(m_{X_1}, \sigma_1^2)$, Y_1 is $N(m_{Y_1}, \sigma_1^2)$, X_1 and Y_1 have the *same* variance σ_1^2, and X_1 and Y_1 are *statistically independent* because they are uncorrelated Gaussian RVs, $\left|\tilde{l}\right|^2$ is a *noncentral*, *chi-squared* RV with 2 degrees of freedom and PDF[3]

$$\boxed{p_1(\gamma) = \frac{1}{2\sigma_1^2}\exp\left(-\frac{\gamma+\gamma_1}{2\sigma_1^2}\right) I_0\left(\frac{\sqrt{\gamma_1}}{\sigma_1^2}\sqrt{\gamma}\right), \qquad \gamma \geq 0} \tag{12.4-31}$$

where the RV

$$\gamma = \left|\tilde{l}\right|^2 = X_1^2 + Y_1^2\,, \tag{12.4-32}$$

the nonrandom constant [see (12.4-9)]

$$\gamma_1 = \left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\}\right|^2 = m_{X_1}^2 + m_{Y_1}^2 \tag{12.4-33}$$

is the *noncentrality parameter*, where m_{X_1} and m_{Y_1} are the real and imaginary parts of (12.4-15), respectively [see (12.4-9)],

$$2\sigma_1^2 = \mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right), \tag{12.4-34}$$

where $\mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)$ is given by (12.4-27) [see (12.4-29)], $I_0(\bullet)$ is the zeroth-order modified Bessel function of the first kind given by (12.3-31), and

$$p_1(\gamma) = p_{\gamma|\mathrm{H}_1}\left(\gamma\,\middle|\,\mathrm{H}_1\right). \tag{12.4-35}$$

See Fig. 10.6-3 for plots of a scaled, noncentral, chi-squared PDF with 2 degrees of freedom.

The probability of detection P_D is given by

$$P_\mathrm{D} = P\left(\left|\tilde{l}\right|^2 > \mathcal{T}\,\middle|\,\mathrm{H}_1\right), \tag{12.4-36}$$

or

$$P_\mathrm{D} = P\left(\gamma > \mathcal{T}\,\middle|\,\mathrm{H}_1\right) = \int_{\mathcal{T}}^{\infty} p_1(\gamma)\,d\gamma\,, \tag{12.4-37}$$

where $\mathcal{T}$ is the decision threshold. Substituting (12.4-31) into (12.4-37) yields

$$P_{\mathrm{D}} = \frac{1}{2\sigma_1^2}\int_{\mathcal{T}}^{\infty} \exp\left(-\frac{\gamma+\gamma_1}{2\sigma_1^2}\right) I_0\left(\frac{\sqrt{\gamma_1}}{\sigma_1^2}\sqrt{\gamma}\right) d\gamma \,. \tag{12.4-38}$$

If we let

$$x^2 = \gamma/\sigma_1^2 \tag{12.4-39}$$

and

$$\alpha_1^2 = \gamma_1/\sigma_1^2 \,, \tag{12.4-40}$$

then (12.4-38) can be rewritten as

$$\boxed{P_{\mathrm{D}} = Q\left(\alpha_1, \sqrt{\mathcal{T}}/\sigma_1\right) = \int_{\sqrt{\mathcal{T}}/\sigma_1}^{\infty} \exp\left[-(x^2+\alpha_1^2)/2\right] I_0(\alpha_1 x)\, x\, dx} \tag{12.4-41}$$

where $Q(a,b)$ is the Marcum Q-function given by (12.3-39). See Example 12.4-1 for a solution for the P_{D} using (12.4-41) and the decision threshold $\mathcal{T}$ obtained in Example 12.3-1.

If the target scattering function is *zero-mean*, that is, if

$$E\left\{S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0})\right\} = 0 \,, \tag{12.4-42}$$

then [see (12.4-14)]

$$E\left\{w_{Trgt}\right\} = 0 \tag{12.4-43}$$

and [see (12.4-13)]

$$E\left\{\tilde{l}_{Trgt}\right\} = 0 \,. \tag{12.4-44}$$

As a result, $m_X = 0$ and $m_Y = 0$ [see (12.4-6)]. If the reverberation scattering function is also *zero-mean*, then $m_{X_0} = 0$ and $m_{Y_0} = 0$ (see Section 12.3). Therefore, substituting $m_X = 0$ and $m_{X_0} = 0$ into (12.4-10) yields $m_{X_1} = 0$, and substituting $m_Y = 0$ and $m_{Y_0} = 0$ into (12.4-11) yields $m_{Y_1} = 0$. And by substituting $m_{X_1} = 0$ and $m_{Y_1} = 0$ into (12.4-9), we obtain [see (12.4-12)]

$$\begin{aligned} E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\} &= E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_Z\right\} \\ &= E\left\{\tilde{l}_{Trgt}\right\} + E\left\{\tilde{l}_{Rev}\right\} \\ &= 0, \end{aligned} \tag{12.4-45}$$

and by substituting $m_{X_1} = 0$ and $m_{Y_1} = 0$ into (12.4-33), we obtain $\gamma_1 = 0$. Therefore, the statistically independent Gaussian RVs X_1 and Y_1 are now $N(0, \sigma_1^2)$, and $\boldsymbol{\gamma} = \left|\tilde{l}\right|^2$ given by (12.4-32) is an *exponential* RV. If $\gamma_1 = 0$, and since $I_0(0) = 1$, the noncentral, chi-squared PDF given by (12.4-31) reduces to the *exponential* PDF

$$\boxed{p_1(\gamma) = \frac{1}{2\sigma_1^2}\exp\left(-\frac{\gamma}{2\sigma_1^2}\right), \qquad \gamma \geq 0} \tag{12.4-46}$$

where $2\sigma_1^2 = \text{Var}\left(\tilde{l}\,\middle|\,\text{H}_1\right)$ [see (12.4-34)], but now, because of (12.3-41) and (12.4-43), $\text{Var}\left(\tilde{l}\,\middle|\,\text{H}_1\right)$ given by (12.4-27) reduces to

$$\begin{aligned}
\text{Var}\left(\tilde{l}\,\middle|\,\text{H}_1\right) &= \text{Var}\left(\tilde{l}_{Trgt}\right) + \text{Var}\left(\tilde{l}_Z\right) \\
&= E\left\{\left|\tilde{l}_{Trgt}\right|^2\right\} + E\left\{\left|\tilde{l}_Z\right|^2\right\} \\
&= A_1^2 E_{\tilde{x}}^2 E\left\{\left|w_{Trgt}\right|^2\right\}\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|^2 + \\
&\quad E_{\tilde{x}}^2 E\left\{\left|w_{Rev}\right|^2\right\}\left|\sum_{i=-N'}^{N'} a_i \chi_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2 + \\
&\quad A_2 E_{\tilde{x}}\left[\tilde{N}_{a0} + \tilde{N}_{r0}\right].
\end{aligned} \tag{12.4-47}$$

Therefore, in the limit as $\sqrt{\gamma_1}/\sigma_1 \to 0$, or equivalently, as

$$\frac{\gamma_1}{2\sigma_1^2} = \frac{\left|E\left\{\tilde{l}\,\middle|\,\text{H}_1\right\}\right|^2}{\text{Var}\left(\tilde{l}\,\middle|\,\text{H}_1\right)} \to 0\,, \tag{12.4-48}$$

$\boldsymbol{\gamma} = \left|\tilde{l}\right|^2$ approaches being an exponential RV.

If the exponential PDF given by (12.4-46) is substituted into (12.4-37), then

$$P_\text{D} = \frac{1}{2\sigma_1^2}\int_{\mathcal{T}}^{\infty} \exp\left(-\frac{\gamma}{2\sigma_1^2}\right)d\gamma\,, \tag{12.4-49}$$

or

$$P_\text{D} = \exp\left(-\frac{\mathcal{T}}{2\sigma_1^2}\right). \tag{12.4-50}$$

Since [see (12.3-30) and (12.3-44)]

$$2\sigma_0^2 = \mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_0\right) = E\left\{\left|\tilde{l}_Z\right|^2\right\} \tag{12.4-51}$$

and [see (12.4-34) and (12.4-47)]

$$2\sigma_1^2 = \mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right) = E\left\{\left|\tilde{l}_{Trgt}\right|^2\right\} + E\left\{\left|\tilde{l}_Z\right|^2\right\}, \tag{12.4-52}$$

(12.4-52) can be rewritten as

$$2\sigma_1^2 = 2\sigma_0^2\,(\mathrm{SIR}+1)\,, \tag{12.4-53}$$

where the signal-to-interference ratio (SIR) is defined by (12.2-1). Substituting (12.4-53) into (12.4-50) yields

$$P_{\mathrm{D}} = \left[\exp\left(-\frac{\mathcal{T}}{2\sigma_0^2}\right)\right]^{\frac{1}{\mathrm{SIR}+1}}, \tag{12.4-54}$$

and by substituting (12.3-47) into (12.4-54), we obtain

$$\boxed{P_{\mathrm{D}} = P_{\mathrm{FA}}^{1/(\mathrm{SIR}+1)}} \tag{12.4-55}$$

where the SIR is given by (12.2-26). From (12.4-55) it can be seen that as the $\mathrm{SIR} \to \infty$, the $P_{\mathrm{D}} \to P_{\mathrm{FA}}^0 = 1$; and as the $\mathrm{SIR} \to 0$, the $P_{\mathrm{D}} \to P_{\mathrm{FA}}$.

Taking the logarithm (base 10) of (12.4-55) yields

$$\log_{10} P_{\mathrm{D}} = \frac{1}{\mathrm{SIR}+1}\log_{10} P_{\mathrm{FA}}\,, \tag{12.4-56}$$

and by solving for the SIR, we obtain

$$\boxed{\mathrm{SIR} = \frac{\log_{10} P_{\mathrm{FA}}}{\log_{10} P_{\mathrm{D}}} - 1} \tag{12.4-57}$$

Equation (12.4-57) can be used to compute the SIR that is required in order to obtain a desired P_{D} for a given P_{FA}. Note that the SIR computed according to (12.4-57) is *dimensionless*.

Example 12.4-1 Nonzero-Mean Target and Reverberation Scattering Functions

In this example we shall determine the P_{D} using (12.4-41) and the decision threshold $\mathcal{T}$ obtained in Example 12.3-1 for $\alpha_0 = \sqrt{2}/2$ and a $P_{\mathrm{FA}} = 0.001$. If we divide (12.4-40) by a factor of 2, then

$$\frac{\alpha_1^2}{2} = \frac{\gamma_1}{2\sigma_1^2} = \frac{\left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_1\right\}\right|^2}{\mathrm{Var}\left(\tilde{l}\,\middle|\,\mathrm{H}_1\right)} . \tag{12.4-58}$$

Several different representative values for α_1, and hence, parameter "a" in the Marcum Q-function $Q(a,b)$, are given in Table 12.4-1. The case $\alpha_1 = 0$ ($a = 0$) has already been solved since $\alpha_1 = 0$ when $\gamma_1 = 0$, and when $\gamma_1 = 0$, the P_{D} is given by (12.4-55). Equation (12.4-58) can be thought of as a measure of the randomness of the output of the correlator $\tilde{l}$ given hypothesis H_1 is true. For example, as $\alpha_1^2/2 \to 0$, $\tilde{l}$ becomes more random. Conversely, as $\alpha_1^2/2 \to \infty$, $\tilde{l}$ becomes more deterministic.

Table 12.4-1 Values of α_1 Used to Determine Values of Parameter "a" in the Marcum Q-Function $Q(a,b)$

$\alpha_1^2/2$	$10\log_{10}\left(\alpha_1^2/2\right)$	α_1
0.125	-9 dB	$1/2$
0.25	-6 dB	$\sqrt{2}/2$
0.5	-3 dB	1
1	0 dB	$\sqrt{2}$
2	3 dB	2
4	6 dB	$2\sqrt{2}$
8	9 dB	4

By referring to (12.4-41), parameter b in $Q(a,b)$ is given by

$$b = \sqrt{\mathcal{T}}/\sigma_1 . \tag{12.4-59}$$

Substituting (12.3-55) into (12.3-56) yields

$$\sqrt{\mathcal{T}} = 4.0747\,\sigma_0\,, \tag{12.4-60}$$

and by substituting (12.4-60) into (12.4-59), we obtain

$$b = 4.0747\,\frac{\sigma_0}{\sigma_1}\,, \tag{12.4-61}$$

where (see Appendix 12D)

$$\boxed{\frac{\sigma_0}{\sigma_1} = \sqrt{\frac{\gamma_0}{\gamma_1}\,\frac{\alpha_1}{\alpha_0}} < 1} \tag{12.4-62}$$

$$\boxed{\sqrt{\gamma_0/\gamma_1} < \alpha_0/\alpha_1} \tag{12.4-63}$$

and

$$\boxed{\sqrt{\gamma_0/\gamma_1} < 1} \tag{12.4-64}$$

In order to determine the P_{D}, we need to choose values for α_1 and $\sqrt{\gamma_0/\gamma_1}$. Since $\alpha_0 = \sqrt{2}/2$ in Example 12.3-1, let $\alpha_1 = 2\sqrt{2}$ so that $\alpha_1 > \alpha_0$ (see Table 12.4-1). Therefore, since $\alpha_0/\alpha_1 = 0.25$, by referring to (12.4-63) and (12.4-64), let $\sqrt{\gamma_0/\gamma_1} = 0.125$. Substituting $\sqrt{\gamma_0/\gamma_1} = 0.125$ and $\alpha_1/\alpha_0 = 4$ into (12.4-62) yields $\sigma_0/\sigma_1 = 0.5$, and by substituting $\sigma_0/\sigma_1 = 0.5$ into (12.4-61), we obtain $b = 2.03735$. Since $a = \alpha_1 = 2\sqrt{2}$, from Table 12C-1 in Appendix 12C:

	$a = 2\sqrt{2}$
b	$Q(a,b)$
2	0.8519364
2.1	0.8262441

Next, use linear interpolation to find the value of $Q\left(2\sqrt{2}, 2.03735\right)$ and, thus, the P_{D}.

If we let

$$Q(a,b) = mb + c\,, \tag{12.4-65}$$

then the slope m is

$$\begin{aligned} m &= \frac{Q(a,b_2)-Q(a,b_1)}{b_2-b_1} \\ &= \frac{Q\left(2\sqrt{2},2.1\right)-Q\left(2\sqrt{2},2\right)}{2.1-2} \\ &= \frac{0.8262441-0.8519364}{0.1} \\ &= -0.256923 \end{aligned} \tag{12.4-66}$$

and the y-intercept is

$$\begin{aligned} c &= Q(a,b_1)-mb_1 \\ &= Q\left(2\sqrt{2},2\right)-(-0.256923)2 \\ &= 0.8519364+0.513846 \\ &= 1.3657824\,. \end{aligned} \tag{12.4-67}$$

Substituting (12.4-66), (12.4-67), and $a=2\sqrt{2}$ into (12.4-65) yields

$$Q\left(2\sqrt{2},b\right)=-0.256923b+1.3657824\,, \tag{12.4-68}$$

and by evaluating (12.4-68) at $b=2.03735$, we obtain

$$P_{\mathrm{D}}=Q\left(2\sqrt{2},2.03735\right)=0.842\,. \tag{12.4-69}$$

The formula for computing the P_{D} can be generalized as follows. By referring to (12.4-59), let

$$b_{\mathrm{D}}=\sqrt{\mathcal{T}}\big/\sigma_1\,. \tag{12.4-70}$$

Substituting (12.3-63) into (12.4-70) yields

$$\boxed{b_{\mathrm{D}}=b_{\mathrm{FA}}\frac{\sigma_0}{\sigma_1}} \tag{12.4-71}$$

where b_{FA} depends on the values of α_0 and P_{FA} and is given by (12.3-59), and σ_0/σ_1 depends on the values of α_0, α_1, and $\sqrt{\gamma_0/\gamma_1}$, and is given by (12.4-62). By referring to (12.4-65) and noting that $P_{\mathrm{D}}=Q(\alpha_1,b_{\mathrm{D}})$, we can write that

$$\boxed{P_{\mathrm{D}} = m_{\alpha_1} b_{\mathrm{D}} + c_{\alpha_1}} \tag{12.4-72}$$

where

$$m_{\alpha_1} = \frac{Q(\alpha_1, b_2) - Q(\alpha_1, b_1)}{b_2 - b_1}\,, \tag{12.4-73}$$

$$c_{\alpha_1} = Q(\alpha_1, b_1) - m_{\alpha_1} b_1\,, \tag{12.4-74}$$

and

$$b_1 < b_{\mathrm{D}} < b_2\,. \tag{12.4-75}$$

Therefore, the P_{D} can be plotted versus the P_{FA} for a fixed value of σ_0/σ_1 (i.e., fixed values of α_0, α_1, and $\sqrt{\gamma_0/\gamma_1}$) because for a fixed value of α_0, b_{FA} will change value as the P_{FA} changes value [see (12.3-59)], and as b_{FA} changes value, b_{D} changes value [see (12.4-71)] and, thus, the P_{D} changes value [see (12.4-72)]. As a result, *receiver operating characteristic* (ROC) *curves* can be generated by plotting the P_{D} versus the P_{FA} for different values of σ_0/σ_1. ■

Example 12.4-2 Receiver Operating Characteristic Curves – Zero-Mean Target Scattering Function and No Reverberation Return

Recall that if the reverberation scattering function is *zero-mean*, then the decision threshold $\mathcal{T}$ is given by (12.3-48). However, if *no* reverberation return is present at the receiver, then (12.3-48) reduces to

$$\boxed{\mathcal{T} = A_2 E_{\tilde{x}}\left[\tilde{N}_{a0} + \tilde{N}_{r0}\right]\ln\left(1/P_{\mathrm{FA}}\right)} \tag{12.4-76}$$

where A_2 is given by (12.2-25), $E_{\tilde{x}}$ is the energy of the complex envelope of the transmitted electrical signal in J-Ω, and $\tilde{N}_{a0}$ and $\tilde{N}_{r0}$ are the constant levels in $(\text{W-}\Omega)/\text{Hz}$ (or J-Ω) of the lowpass (baseband), white-noise power spectra of the complex envelopes of the ambient noise and receiver noise $\tilde{n}_a(t, x_i)$ and $\tilde{n}(t, x_i)$, respectively, $\forall\ i$, before complex weighting (see Appendix 12B). If rectangular amplitude weights are used, then $A_2 = N$ where N is the total odd number of elements in the linear array.

Also recall that if both the target and reverberation scattering functions are *zero-mean*, then the relationship between the P_{D}, P_{FA}, and SIR is given by (12.4-55), where the SIR is given by (12.2-26). However, if *no* reverberation return is present at the receiver, then (12.4-55) and (12.2-26) reduce to

$$\boxed{P_{\mathrm{D}} = P_{\mathrm{FA}}^{1/(\mathrm{SNR}+1)}} \tag{12.4-77}$$

and

$$\boxed{\mathrm{SNR} = \left| \mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) \right|^2 \mathrm{SNR}_{\max}} \tag{12.4-78}$$

respectively, where $\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt})$ is the normalized, auto-ambiguity function of the transmitted electrical signal [see (12.2-17) and (12.2-13)],

$$\boxed{\mathrm{SNR}_{\max} = \mathrm{AG} \times \mathrm{SNR}_0} \tag{12.4-79}$$

is the maximum value of the SNR, AG is the *dimensionless* version of the array gain of the linear array given by (12.2-27),

$$\boxed{\mathrm{SNR}_0 = \frac{E_{\tilde{x}} E\left\{ \left| w_{Trgt} \right|^2 \right\}}{\tilde{N}_{a0} + \tilde{N}_{r0}}} \tag{12.4-80}$$

is the SNR due to processing the output electrical signal from a single element – element number 0 at the center of the linear array – and $E\left\{ \left| w_{Trgt} \right|^2 \right\}$ is given by (12.2-20). If the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are zero, then $\mathcal{X}_N(0,0) = 1$ [see (12.2-18)] and $\mathrm{SNR} = \mathrm{SNR}_{\max}$. If rectangular amplitude weights are used, then $\mathrm{AG} = N$. Equations (12.4-78) through (12.4-80) are identical to (12.2-31) through (12.2-33), respectively.

Taking the logarithm (base 10) of (12.4-77) and solving for the SNR yields

$$\boxed{\mathrm{SNR} = \frac{\log_{10} P_{\mathrm{FA}}}{\log_{10} P_{\mathrm{D}}} - 1} \tag{12.4-81}$$

Equation (12.4-81) can be used to compute the SNR that is required in order to obtain a desired P_{D} for a given P_{FA}. Note that the SNR computed according to (12.4-81) is *dimensionless*.

As was previously mentioned, if the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are zero, then $\mathrm{SNR} = \mathrm{SNR}_{\max}$ where $\mathrm{SNR}_{\max}$ is given by (12.4-79). Substituting $\mathrm{SNR} = \mathrm{SNR}_{\max}$ into (12.4-77) yields the corresponding maximum value for the probability of detection, which is desirable, since we want the probability of declaring that we detected a target with the *correct* round-trip time delay (range) and Doppler-shift values to be maximized. Conversely, we would like the SNR to *decrease* as the magnitude of the estimation errors *increase*. A small SNR results in a small P_{D} [see (12.4-77)], which is desirable, since whenever we make large estimation errors, we want the probability of declaring that we detected a target with the *wrong* round-trip time delay (range) and Doppler-shift values to be *small*. To ensure this kind of performance and, hence, to avoid any *ambiguity* concerning the estimates of τ_{Trgt} and $\eta_{D,Trgt}$, the normalized, auto-ambiguity function should have as narrow a mainlobe about the origin $(e_{\tau,Trgt}=0, e_{D,Trgt}=0)$ as possible so that as the estimation errors increase, the value of $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ decreases rapidly and, as a result, both the SNR and the P_{D} decrease rapidly. In addition, it is also desirable that $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ have sidelobe levels as low as possible. Auto-ambiguity functions are discussed in detail in Chapter 13.

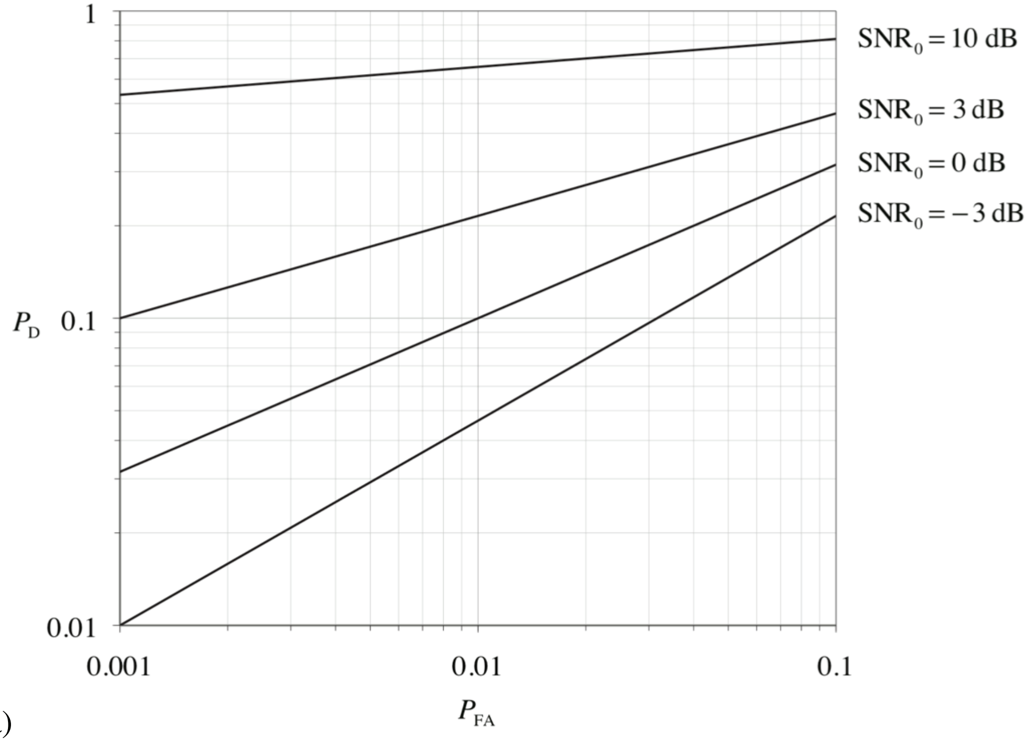
Let us conclude this example by plotting several *receiver operating characteristic* (ROC) curves. For a Neyman-Pearson test, a ROC curve is a plot of the probability of detection P_{D} versus the probability of false alarm P_{FA} for a given value of some parameter. In this example, the parameter of interest is the signal-to-noise ratio (SNR).

In order to simplify the results, assume that the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are *zero* so that $\chi_N(0,0)=1$, and that *rectangular amplitude weights* are used so that $\text{AG}=N$. Therefore, (12.4-78) reduces to

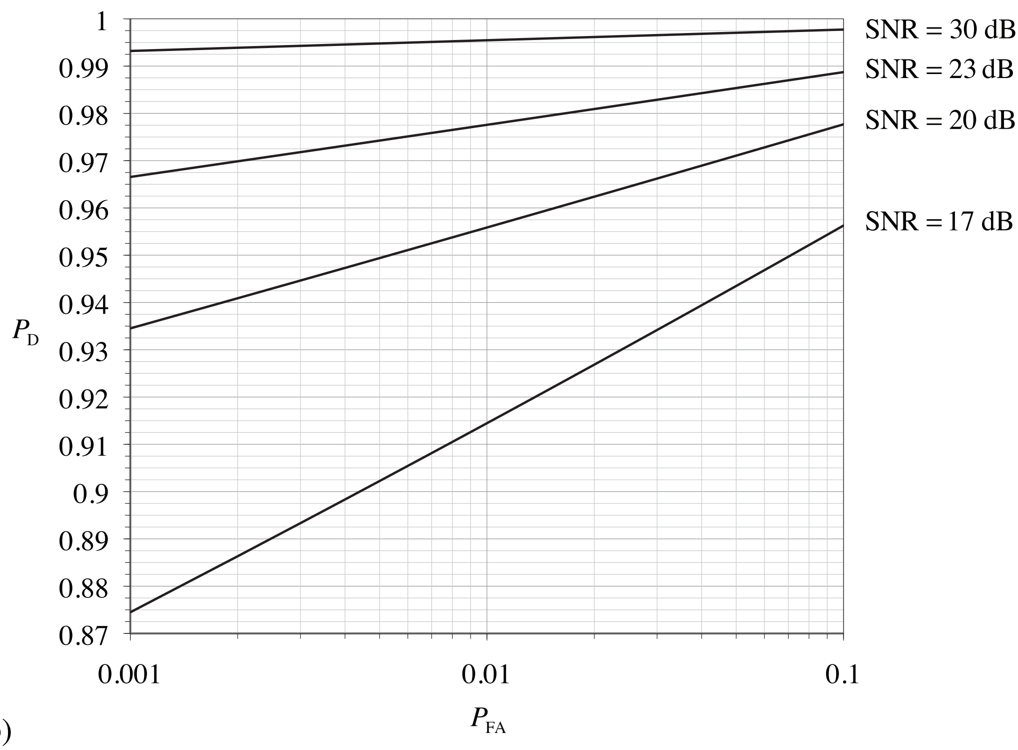
$$\boxed{\text{SNR} = N \times \text{SNR}_0} \tag{12.4-82}$$

First consider the case where $N=1$. Figure 12.4-1 (a) is a plot of (12.4-77) for $\text{SNR}=\text{SNR}_0$ where $\text{SNR}_0=0.5\,(-3.01\text{ dB})$, $\text{SNR}_0=1\,(0\text{ dB})$, $\text{SNR}_0=2\,(3.01\text{ dB})$, and $\text{SNR}_0=10\,(10\text{ dB})$. Now consider the case of a linear array with $N=101$ elements. As a result, the array gain $\text{AG}=101\,(20.04\text{ dB})$. Figure 12.4-1 (b) is a plot of (12.4-77) for $\text{SNR}=101\times\text{SNR}_0$ where SNR_0 is equal to the same values used for Fig. 12.4-1 (a). Therefore, $\text{SNR}=50.5\,(17.03\text{ dB})$, $\text{SNR}=101\,(20.04\text{ dB})$, $\text{SNR}=202\,(23.05\text{ dB})$, and $\text{SNR}=1010\,(30.04\text{ dB})$. By comparing Figs. 12.4-1 (a) and 12.4-1 (b), it can clearly be seen how AG greatly increases the P_{D} for a given P_{FA}. ■

Several additional comments regarding the practical importance of (12.4-82) are in order. Solving for N in (12.4-82) yields



(a)



(b)

Figure 12.4-1 Receiver operating characteristic (ROC) curves for (a) a single element and (b) a linear array of 101 elements.

$$\boxed{N = \text{SNR}/\text{SNR}_0} \tag{12.4-83}$$

which is the total number of elements required in order to obtain a desired P_D for a given P_FA, where SNR is given by (12.4-81) and SNR_0 is given by (12.4-80). Note that the value of N obtained from (12.4-83) may have to be *rounded up* to the nearest *odd* number for our linear array. Once N has been determined, and the interelement spacing d is known (e.g., in order to avoid grating lobes for all possible directions of beam steering, $d < \lambda_{\min}/2$), then the overall length of the linear array L_A that is required in order to satisfy the P_D and P_FA constraints is given by $L_A = (N-1)d$. Once L_A has been determined, the range to the near-field/far-field boundary can be computed for a given operating frequency (wavelength) since $r_\text{NF/FF} = \pi R_A^2/\lambda$, where $R_A = L_A/2$ is the maximum radial extent of a linear array.

Problems

Section 12.2

12-1 Use the Schwarz inequality

$$\left|\int_{-\infty}^{\infty} A(\zeta)B(\zeta)d\zeta\right|^2 \le \int_{-\infty}^{\infty} \left|A(\zeta)\right|^2 d\zeta \int_{-\infty}^{\infty} \left|B(\zeta)\right|^2 d\zeta ,$$

where $A(\zeta)$ and $B(\zeta)$ are finite energy, complex functions in general, to prove that $\left|\chi(e_{\tau,Trgt}, e_{D,Trgt})\right| \le E_{\tilde{x}}$.

Section 12.3

12-2 Show that the variance of w_{Rev} depends on the variance of the reverberation scattering function.

Section 12.4

12-3 Show that the variance of w_{Trgt} depends on the variance of the target scattering function.

12-4 Consider the case where the target scattering function is zero-mean and there is no reverberation. If the $P_\text{FA} = 0.01$, then find the output SNR in *decibels* that is needed for a

(a) $P_D = 0.9$

(b) $P_D = 0.95$

(c) $P_D = 0.99$

(d) Repeat Parts (a) through (c) for a $P_{FA} = 0.001$.

12-5 Consider the case where the target scattering function is zero-mean and there is no reverberation. If the estimation errors $e_{\tau,Trgt}$ and $e_{D,Trgt}$ are *zero* and *rectangular amplitude weights* are used, then

(a) how many array elements (an odd number) are required in order to achieve a $P_D = 0.9$ for a $P_{FA} = 0.01$ if $\text{SNR}_0 = 0.25\,(-6\text{ dB})$?

(b) Find the corresponding value of AG in decibels.

(c) If the frequency of operation is 1 kHz and half-wavelength interelement spacing is used, then what is the overall length of the linear array? Use $c = 1500$ m/sec.

(d) What is the range to the near-field/far-field boundary for this linear array?

Appendix 12A Mathematical Models of the Target Return and Reverberation Return

In order to derive mathematical models of the target return and reverberation return, we first need to derive an equation for the complex envelope, $\tilde{y}_s(t,\mathbf{r}_R)$, of the output electrical signal, $y_s(t,\mathbf{r}_R)$, from an omnidirectional point-element at $\mathbf{r}_R$ for the bistatic scattering problem discussed in Chapter 10 (see Fig. 10.1-1 and Section 10.5). We begin by expressing $y_s(t,\mathbf{r}_R)$ as an inverse Fourier transform [see (10.5-9)]:

$$\begin{aligned} y_s(t,\mathbf{r}_R) &= F_f^{-1}\{Y_s(f,\mathbf{r}_R)\} \\ &= \int_{-\infty}^{\infty} Y_s(f,\mathbf{r}_R)\exp(+j2\pi ft)\,df \\ &= \int_{-\infty}^{0} Y_s(f,\mathbf{r}_R)\exp(+j2\pi ft)\,df + \int_{0}^{\infty} Y_s(f,\mathbf{r}_R)\exp(+j2\pi ft)\,df. \end{aligned} \tag{12A-1}$$

If we let $f' = -f$ in the first integral in (12A-1), then $f = -f'$, $df = -df'$, and when $f = -\infty$, $f' = \infty$; and when $f = 0$, $f' = 0$. Therefore,

$$\begin{aligned}\int_{-\infty}^{0} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df &= -\int_{\infty}^{0} Y_s(-f', \mathbf{r}_R)\exp(-j2\pi f't)\,df' \\ &= \int_{0}^{\infty} Y_s(-f', \mathbf{r}_R)\exp(-j2\pi f't)\,df'.\end{aligned} \tag{12A-2}$$

Replacing f' with f yields

$$\int_{-\infty}^{0} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df = \int_{0}^{\infty} Y_s(-f, \mathbf{r}_R)\exp(-j2\pi ft)\,df\,, \tag{12A-3}$$

and by substituting (12A-3) into (12A-1), we obtain

$$y_s(t, \mathbf{r}_R) = \int_{0}^{\infty} Y_s(-f, \mathbf{r}_R)\exp(-j2\pi ft)\,df + \int_{0}^{\infty} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df\,. \tag{12A-4}$$

Since $y_s(t, \mathbf{r}_R)$ is real, its time-domain Fourier transform satisfies the conjugate symmetry property, that is,

$$Y_s(-f, \mathbf{r}_R) = Y_s^*(f, \mathbf{r}_R)\,. \tag{12A-5}$$

Therefore, substituting (12A-5) into (12A-4) yields

$$y_s(t, \mathbf{r}_R) = \int_{0}^{\infty} \left[Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\right]^* df + \int_{0}^{\infty} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df\,. \tag{12A-6}$$

If we let

$$Z = \int_{0}^{\infty} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df\,, \tag{12A-7}$$

and since

$$2\,\mathrm{Re}\{Z\} = Z + Z^*\,, \tag{12A-8}$$

(12A-6) reduces to

$$y_s(t, \mathbf{r}_R) = 2\,\mathrm{Re}\left\{\int_{0}^{\infty} Y_s(f, \mathbf{r}_R)\exp(+j2\pi ft)\,df\right\}. \tag{12A-9}$$

The next step is to express (12A-9) in terms of the complex frequency spectrum $X(f)$ of the real transmitted electrical signal $x(t)$. Substituting (10.5-7) into (10.5-8) yields

$$Y_s(f, \mathbf{r}_R) = X(f) H\left(f, \mathbf{r}_R \,\middle|\, \mathbf{r}_T\right), \tag{12A-10}$$

where

$$H\left(f, \mathbf{r}_R \,\middle|\, \mathbf{r}_T\right) = \frac{1}{(4\pi)^2 r_{TS} r_{SR}} \mathcal{S}_T(f) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \exp\left[-\alpha(f)(r_{TS} + r_{SR})\right] \times \exp(-j2\pi f \tau) \mathcal{S}_R(f) \tag{12A-11}$$

is the overall complex frequency response for the bistatic scattering problem, and

$$\tau = (r_{TS} + r_{SR})/c \tag{12A-12}$$

is the bistatic time delay. Therefore, substituting (12A-10) into (12A-9) yields

$$y_s(t, \mathbf{r}_R) = 2\,\mathrm{Re}\left\{\int_0^\infty X(f) H\left(f, \mathbf{r}_R \,\middle|\, \mathbf{r}_T\right) \exp(+j2\pi f t)\, df\right\}. \tag{12A-13}$$

We are now in a position to derive the complex envelope of $y_s(t, \mathbf{r}_R)$.

Substituting (11.1-24) into (12A-13) yields

$$y_s(t, \mathbf{r}_R) = \mathrm{Re}\left\{\int_0^\infty \tilde{X}(f - f_c) H\left(f, \mathbf{r}_R \,\middle|\, \mathbf{r}_T\right) \exp(+j2\pi f t)\, df\right\}, \tag{12A-14}$$

where $\tilde{X}(f)$ is the complex frequency spectrum of the lowpass (baseband) complex envelope $\tilde{x}(t)$ of the real bandpass signal $x(t)$. If we let $f' = f - f_c$ in (12A-14), then $f = f' + f_c$, $df = df'$, and when $f = 0$, $f' = -f_c$; and when $f = \infty$, $f' = \infty$. Therefore,

$$y_s(t, \mathbf{r}_R) = \mathrm{Re}\left\{\int_{-f_c}^\infty \tilde{X}(f') H\left(f' + f_c, \mathbf{r}_R \,\middle|\, \mathbf{r}_T\right) \exp(+j2\pi f' t)\, df' \exp(+j2\pi f_c t)\right\}. \tag{12A-15}$$

Since $\tilde{X}(f)$ is lowpass with bandwidth W hertz,

$$\left|\tilde{X}(f)\right| = 0, \qquad |f| \ge W, \tag{12A-16}$$

and since the carrier frequency $f_c > W$ [see (11.2-20)], $-f_c < -W$. Therefore, by replacing f' with f, (12A-15) can be rewritten as

$$y_s(t,\mathbf{r}_R) = \mathrm{Re}\left\{\int_{-W}^{W} \tilde{X}(f)\,H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right)\exp(+j2\pi ft)\,df\,\exp(+j2\pi f_c t)\right\}, \tag{12A-17}$$

or

$$y_s(t,\mathbf{r}_R) = \mathrm{Re}\left\{\int_{-\infty}^{\infty} \tilde{X}(f)\,H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right)\exp(+j2\pi ft)\,df\,\exp(+j2\pi f_c t)\right\}. \tag{12A-18}$$

By comparing (12A-18) with (11.1-6), (12A-18) can be expressed as

$$\boxed{y_s(t,\mathbf{r}_R) = \mathrm{Re}\left\{\tilde{y}_s(t,\mathbf{r}_R)\exp(+j2\pi f_c t)\right\}} \tag{12A-19}$$

where

$$\boxed{\begin{aligned}\tilde{y}_s(t,\mathbf{r}_R) &= F_f^{-1}\left\{\tilde{X}(f)\,H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right)\right\}\\ &= \int_{-\infty}^{\infty} \tilde{X}(f)\,H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right)\exp(+j2\pi ft)\,df\\ &= \int_{-W}^{W} \tilde{X}(f)\,H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right)\exp(+j2\pi ft)\,df\end{aligned}} \tag{12A-20}$$

is the complex envelope of $y_s(t,\mathbf{r}_R)$, and [see (12A-11)]

$$\begin{aligned}H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right) = \frac{1}{(4\pi)^2 r_{TS} r_{SR}} &\mathcal{S}_T(f+f_c)S_S(f+f_c,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\times\\ &\exp\left[-\alpha(f+f_c)(r_{TS}+r_{SR})\right]\exp\left[-j2\pi(f+f_c)\tau\right]\times\\ &\mathcal{S}_R(f+f_c).\end{aligned} \tag{12A-21}$$

In order to obtain a closed-form, analytical solution for the complex envelope $\tilde{y}_s(t,\mathbf{r}_R)$, we shall make the following approximation in the frequency interval $[f_c - W, f_c + W]$:

$$\begin{aligned}H\left(f+f_c,\mathbf{r}_R\,\middle|\,\mathbf{r}_T\right) \approx \frac{1}{(4\pi)^2 r_{TS} r_{SR}} &\mathcal{S}_T(f_c)S_S(f_c,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\times\\ &\exp\left[-\alpha(f_c)(r_{TS}+r_{SR})\right]\exp\left[-j2\pi(f+f_c)\tau\right]\mathcal{S}_R(f_c),\\ &\qquad\qquad f_c - W \le f \le f_c + W.\end{aligned} \tag{12A-22}$$

Therefore, substituting (12A-22) into (12A-20) yields

$$\boxed{\tilde{y}_s(t,\mathbf{r}_R) \approx w_S\, \tilde{x}(t-\tau)\exp(-j2\pi f_c \tau)} \tag{12A-23}$$

for $t \geq \tau$, where

$$\tilde{x}(t-\tau) = F_f^{-1}\left\{\tilde{X}(f)\exp(-j2\pi f\tau)\right\} = \int_{-\infty}^{\infty} \tilde{X}(f)\exp(-j2\pi f\tau)\exp(+j2\pi ft)\,df \tag{12A-24}$$

and

$$\boxed{w_S = \frac{1}{(4\pi)^2 r_{TS} r_{SR}} \mathcal{S}_T(f_c) S_S(f_c,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp\left[-\alpha(f_c)(r_{TS}+r_{SR})\right]\mathcal{S}_R(f_c)} \tag{12A-25}$$

In the case of a monostatic (backscatter) geometry, (12A-23) is still applicable, however (see Section 10.1),

$$r_{SR} = r_{TS}\,. \tag{12A-26}$$

Therefore, w_S given by (12A-25) reduces to

$$\boxed{w_S = \frac{1}{(4\pi r_{TS})^2} \mathcal{S}_T(f_c) S_S(f_c,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp[-2\alpha(f_c) r_{TS}]\mathcal{S}_R(f_c)} \tag{12A-27}$$

where (see Section 10.1 and Figs. 10.1-2 and 10.1-3)

$$\theta_{SR} = 180° - \theta_{TS} \tag{12A-28}$$

and

$$\psi_{SR} = 180° + \psi_{TS}\,, \tag{12A-29}$$

and the bistatic time delay given by (12A-12) reduces to the round-trip time delay

$$\tau = 2r_{TS}/c\,. \tag{12A-30}$$

The complex envelope given by (12A-23) is applicable to both bistatic and monostatic scattering problems when none of the platforms are in motion. However, we need to include the effects of platform motion in our target return and reverberation return models. In other words, we need to include a Doppler shift in the model given by (12A-23). If time-compression/time-expansion effects are ignored, then

$$\boxed{\tilde{y}_s(t,\mathbf{r}_R) \approx w_S\, \tilde{x}(t-\tau)\exp\left[+j2\pi\eta_D(t-\tau)\right]\exp(-j2\pi f_c\tau)} \tag{12A-31}$$

for $t \geq \tau$, where η_D is the *Doppler shift* in hertz. Equation (12A-31) is the simplest mathematical model for the complex envelope of the output electrical signal from an omnidirectional point-element at $\mathbf{r}_R$ for the bistatic scattering problem discussed in Chapter 10 when one, two, or all three of the platforms shown in Fig. 10.1-1 are in motion. Since the complex envelope of the transmitted amplitude-and-angle-modulated carrier is given by [see (11.2-31)]

$$\tilde{x}(t) = a(t)\exp[+j\theta(t)], \tag{12A-32}$$

where $a(t)$ and $\theta(t)$ are real amplitude-modulating and angle-modulating functions, respectively, if we represent w_S in polar form as

$$w_S = \left|w_S\right|\exp\left(+j\angle w_S\right), \tag{12A-33}$$

then substituting (12A-32) and (12A-33) into (12A-31) yields

$$\tilde{y}_s(t,\mathbf{r}_R) \approx \left|w_S\right| a(t-\tau)\exp\left[+j\theta(t-\tau)\right]\exp\left[+j2\pi\eta_D(t-\tau)\right]\exp\left(+j\angle w_S\right)\times \exp(-j2\pi f_c\tau). \tag{12A-34}$$

And by substituting (12A-34) into (12A-19), we obtain the real bandpass signal

$$\boxed{y_s(t,\mathbf{r}_R) \approx \left|w_S\right| a(t-\tau)\cos\left[2\pi(f_c+\eta_D)(t-\tau)+\theta(t-\tau)+\angle w_S\right]} \tag{12A-35}$$

for $t \geq \tau$. We shall now use (12A-31) to model the complex envelopes of the output electrical signals at each element in a linear array for a *backscatter* geometry.

Consider a linear array composed of an odd number N of identical, equally-spaced, complex-weighted, omnidirectional point-elements lying along the X axis. The position vector to element i in the array is given by

$$\mathbf{r}_i = x_i\hat{x} = id\hat{x}, \qquad i = -N',\ldots,0,\ldots,N', \tag{12A-36}$$

where

$$x_i = id \tag{12A-37}$$

is the x coordinate of the center of element i, d is the interelement spacing in meters, and

$$N' = (N-1)/2 . \tag{12A-38}$$

For a *backscatter* geometry,

$$\mathbf{r}_R = \mathbf{r}_T = \mathbf{r}_i , \tag{12A-39}$$

where $\mathbf{r}_i$ is given by (12A-36).

The position vector to a scatterer (target or object) is given by either

$$\mathbf{r}_S = x_S \hat{x} + y_S \hat{y} + z_S \hat{z} \tag{12A-40}$$

or

$$\mathbf{r}_S = r_S \hat{r}_S , \tag{12A-41}$$

where

$$r_S = \left| \mathbf{r}_S \right| = \sqrt{x_S^2 + y_S^2 + z_S^2} \tag{12A-42}$$

is the range to the scatterer in meters measured from the origin of the coordinate system as shown in Fig. 10.1-1 (the center of the linear array in our problem),

$$\hat{r}_S = u_S \hat{x} + v_S \hat{y} + w_S \hat{z} \tag{12A-43}$$

is the unit vector in the direction of $\mathbf{r}_S$, and

$$u_S = \sin\theta_S \cos\psi_S , \tag{12A-44}$$

$$v_S = \sin\theta_S \sin\psi_S , \tag{12A-45}$$

and

$$w_S = \cos\theta_S \tag{12A-46}$$

are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively. The spherical coordinates of the scatterer's location (r_S, θ_S, ψ_S) are *unknown* a priori. Therefore, the range between the transmitter at $\mathbf{r}_T$ and scatterer at $\mathbf{r}_S$ given in Section 10.1 as

$$r_{TS} = \left| \mathbf{r}_S - \mathbf{r}_T \right| = \sqrt{(x_S - x_T)^2 + (y_S - y_T)^2 + (z_S - z_T)^2} \tag{12A-47}$$

can be rewritten for our problem as

$$r_{TS} = r_{iS} , \tag{12A-48}$$

where

$$\begin{aligned} r_{iS} &= \left| \mathbf{r}_S - \mathbf{r}_i \right| \\ &= \sqrt{(x_S - x_i)^2 + (y_S - y_i)^2 + (z_S - z_i)^2} \\ &= \sqrt{(x_S - x_i)^2 + y_S^2 + z_S^2} \end{aligned} \tag{12A-49}$$

since $\mathbf{r}_T = \mathbf{r}_i = (x_i, 0, 0)$ [see (12A-39) and (12A-36)]. By expanding the radicand in (12A-49), and since

$$x_S = r_S u_S \, , \tag{12A-50}$$

(12A-49) can also be expressed as

$$r_{iS} = \sqrt{r_S^2 - 2 r_S u_S x_i + x_i^2} \, , \tag{12A-51}$$

where r_S is given by (12A-42), u_S is given by (12A-44), and x_i is given by (12A-37).

Equation (12A-48) can also be used to rewrite the equations for w_S and τ given by (12A-27) and (12A-30), respectively. Substituting (12A-48) into (12A-27) yields

$$w_S = \frac{1}{(4\pi r_{iS})^2} \mathcal{S}_T(f_c) S_S(f_c, \theta_{iS}, \psi_{iS}, \theta_{Si}, \psi_{Si}) \exp[-2\alpha(f_c) r_{iS}] \mathcal{S}_R(f_c) \, , \tag{12A-52}$$

where [see (12A-28) and (12A-29)]

$$\theta_{Si} = 180° - \theta_{iS} \tag{12A-53}$$

and

$$\psi_{Si} = 180° + \psi_{iS} \, . \tag{12A-54}$$

The factor w_S given by (12A-52) for a backscatter geometry can be approximated by removing its dependency on element number i as follows:

$$\boxed{w_S \approx \frac{1}{(4\pi r_S)^2} \mathcal{S}_T(f_c) S_S(f_c, \theta_S, \psi_S, \theta_{S0}, \psi_{S0}) \exp[-2\alpha(f_c) r_S] \mathcal{S}_R(f_c)} \tag{12A-55}$$

where $r_S = r_{0S}$ is the range to the scatterer measured from the origin of the coordinate system (i.e., from element $i = 0$ at the center of the linear array), $\theta_S = \theta_{0S}$ and $\psi_S = \psi_{0S}$ are the spherical angles measured from the origin of the coordinate system (the center of the linear array) to the scatterer, and $\theta_{S0} = 180° - \theta_S$ and $\psi_{S0} = 180° + \psi_S$ [see (12A-53) and (12A-54), respectively]. And, by substituting (12A-48) into (12A-30), we obtain

$$\boxed{\tau_i = 2r_{iS}/c} \tag{12A-56}$$

which is the round-trip time delay from element i in the array to the scatterer, where the range r_{iS} is given by either (12A-49) or (12A-51). Therefore, the complex envelope of the output electrical signal from element i in the linear array for a backscatter geometry *before* complex weighting is given by [see (12A-31)]

$$\tilde{y}_s(t, x_i) \approx w_S\,\tilde{x}(t - \tau_i)\exp\left[+j2\pi\eta_D(t - \tau_i)\right]\exp(-j2\pi f_c \tau_i)\,, \tag{12A-57}$$

for $t \geq \tau_i$, where w_S is given by (12A-55) and τ_i is given by (12A-56).

In order to obtain a model for the complex envelope after complex weighting, we first have to compute the Fourier transform of (12A-57). Doing so yields

$$\tilde{Y}_s(\eta, x_i) \approx w_S\,\tilde{X}(\eta - \eta_D)\exp\left[-j2\pi(\eta + f_c)\tau_i\right], \tag{12A-58}$$

since

$$F_t\left\{\tilde{x}(t - \tau_i)\exp(+j2\pi\eta_D t)\right\} = \tilde{X}(\eta - \eta_D)\exp(-j2\pi\eta\tau_i)\exp(+j2\pi\eta_D\tau_i)\,, \tag{12A-59}$$

where η represents *output (received)* frequencies in hertz, and because of platform motion, $\eta \neq f$, where f represents *input (transmitted)* frequencies in hertz. If there is no platform motion, then $\eta = f$. Multiplying the right-hand side of (12A-58) by the complex weight

$$c_i(\eta) = a_i \exp[+j\theta_i(\eta)] = a_i \exp\left[-j2\pi(\eta + f_c)\tau_i'\right] \tag{12A-60}$$

yields

$$\tilde{Y}_s(\eta, x_i) \approx a_i w_S\,\tilde{X}(\eta - \eta_D)\exp\left[-j2\pi\eta(\tau_i + \tau_i')\right]\exp\left[-j2\pi f_c(\tau_i + \tau_i')\right], \tag{12A-61}$$

and by inverse Fourier transforming (12A-61), we obtain

$$\boxed{\tilde{y}_s(t, x_i) \approx a_i w_S\,\tilde{x}\left(t - [\tau_i + \tau_i']\right)\exp\left\{+j2\pi\eta_D\left[t - (\tau_i + \tau_i')\right]\right\}\exp\left[-j2\pi f_c(\tau_i + \tau_i')\right]} \tag{12A-62}$$

for $t \geq \tau_i + \tau_i'$. Equation (12A-62) is the complex envelope of the output electrical signal from element i in the linear array for a backscatter geometry *after* complex weighting, where a_i is the amplitude weight applied to element i, w_S is given by (12A-55), τ_i is given by (12A-56), and τ_i' is the time-delay applied to element i via phase weighting in order to co-phase the output electrical signals from all the elements in the array due to the scatterer. Substituting (12A-32) and (12A-33) into (12A-62) yields

$$\tilde{y}_s(t,x_i) \approx a_i |w_S| a(t-[\tau_i+\tau_i']) \exp\left[+j\theta(t-[\tau_i+\tau_i'])\right] \exp\left\{+j2\pi\eta_D\left[t-(\tau_i+\tau_i')\right]\right\} \times \\ \exp(+j\angle w_S)\exp\left[-j2\pi f_c(\tau_i+\tau_i')\right], \tag{12A-63}$$

and by substituting (12A-63) into (12A-19), we obtain the real bandpass signal

$$\boxed{y_s(t,x_i) \approx a_i |w_S| a(t-[\tau_i+\tau_i'])\cos\left[2\pi(f_c+\eta_D)(t-[\tau_i+\tau_i'])+\theta(t-[\tau_i+\tau_i'])+\angle w_S\right]} \tag{12A-64}$$

for $t \geq \tau_i + \tau_i'$.

If the scatterer is in the Fresnel or far-field region of the linear array, and if the far-field beam pattern of the array is focused at the range to the scatterer and steered in the direction of the acoustic field incident upon the array (Fresnel region case), or if the far-field beam pattern is simply steered in the direction of the incident acoustic field (far-field region case), then (see Section 7.3)

$$\tau_i + \tau_i' = \tau_S, \qquad i = -N', \ldots, 0, \ldots, N', \tag{12A-65}$$

where

$$\tau_S = 2r_S/c \tag{12A-66}$$

is the round-trip time delay associated with the path length between the scatterer and the center of the array, and $r_S = r_{0S}$ is the range to the scatterer measured from the center of the array (i.e., from element $i = 0$). Substituting (12A-65) into (12A-62) and (12A-64) yields

$$\boxed{\tilde{y}_s(t,x_i) \approx a_i w_S \tilde{x}(t-\tau_S)\exp\left[+j2\pi\eta_D(t-\tau_S)\right]\exp(-j2\pi f_c\tau_S)} \tag{12A-67}$$

and

$$\boxed{y_s(t,x_i) \approx a_i |w_S| a(t-\tau_S)\cos\left[2\pi(f_c+\eta_D)(t-\tau_S)+\theta(t-\tau_S)+\angle w_S\right]} \tag{12A-68}$$

respectively, for $t \geq \tau_S$, where w_S is given by (12A-55). Therefore, if the scatterer is an object of interest (i.e., a target), and if correct focusing and/or beam steering is done so that [see (12A-65)]

$$\tau_{i,Trgt} + \tau'_{i,Trgt} = \tau_{Trgt}, \qquad i = -N', \ldots, 0, \ldots, N', \tag{12A-69}$$

then

$$\boxed{\tilde{y}_{Trgt}(t, x_i) \approx a_i w_{Trgt} \tilde{x}(t - \tau_{Trgt}) \exp\left[+j2\pi\eta_{D,Trgt}(t - \tau_{Trgt})\right] \exp(-j2\pi f_c \tau_{Trgt})} \tag{12A-70}$$

and

$$\boxed{y_{Trgt}(t, x_i) \approx a_i \left| w_{Trgt} \right| a(t - \tau_{Trgt}) \cos\left[2\pi(f_c + \eta_{D,Trgt})(t - \tau_{Trgt}) + \theta(t - \tau_{Trgt}) + \angle w_{Trgt}\right]} \tag{12A-71}$$

are the lowpass (baseband) complex envelope and real bandpass output electrical signals from element i in the linear array due to a target for $t \geq \tau_{Trgt}$, where τ_{Trgt} and $\eta_{D,Trgt}$ are the round-trip time delay and Doppler shift of the target measured from the center of the array,

$$\boxed{w_{Trgt} \approx \frac{1}{(4\pi r_{Trgt})^2} \mathcal{S}_T(f_c) S_{Trgt}(f_c, \theta_{Trgt}, \psi_{Trgt}, \theta_{Trgt\,0}, \psi_{Trgt\,0}) \exp[-2\alpha(f_c) r_{Trgt}] \mathcal{S}_R(f_c)} \tag{12A-72}$$

is a complex, nonzero-mean, Gaussian, random variable (RV); $(r_{Trgt}, \theta_{Trgt}, \psi_{Trgt})$ are the spherical coordinates of the center of mass of the target, measured from the center of the array; $S_{Trgt}(\bullet)$ is the scattering function of the target, and is treated as a complex, nonzero-mean, Gaussian RV; $\theta_{Trgt\,0} = 180° - \theta_{Trgt}$, and $\psi_{Trgt\,0} = 180° + \psi_{Trgt}$. However, if there is also an object of no interest nearby the target, then

$$\boxed{\begin{aligned}\tilde{y}_{Rev}(t, x_i) \approx a_i w_{Rev} \tilde{x}\left(t - [\tau_{i,Rev} + \tau'_{i,Trgt}]\right) \exp\left\{+j2\pi\eta_{D,Rev}\left[t - (\tau_{i,Rev} + \tau'_{i,Trgt})\right]\right\} \times \\ \exp\left[-j2\pi f_c(\tau_{i,Rev} + \tau'_{i,Trgt})\right]\end{aligned}} \tag{12A-73}$$

and

$$\boxed{\begin{aligned} y_{Rev}(t,x_i) \approx a_i \left| w_{Rev} \right| a\left(t - [\tau_{i,Rev} + \tau'_{i,Trgt}]\right) \times \\ \cos\left[2\pi(f_c + \eta_{D,Rev})\left(t - [\tau_{i,Rev} + \tau'_{i,Trgt}]\right) + \theta\left(t - [\tau_{i,Rev} + \tau'_{i,Trgt}]\right) + \angle w_{Rev}\right] \end{aligned}}$$

(12A-74)

are the lowpass (baseband) complex envelope and real bandpass output electrical signals from element i in the linear array due to reverberation for $t \geq \tau_{i,Rev} + \tau'_{i,Trgt}$, where $\tau_{i,Rev}$ is the round-trip time delay from element i in the linear array to the object responsible for the reverberation [see (12A-56)], $\eta_{D,Rev}$ is the Doppler shift of the object responsible for the reverberation measured from the center of the array,

$$\boxed{w_{Rev} \approx \frac{1}{(4\pi r_{Rev})^2} \mathcal{S}_T(f_c) S_{Rev}(f_c, \theta_{Rev}, \psi_{Rev}, \theta_{Rev\,0}, \psi_{Rev\,0}) \exp[-2\alpha(f_c) r_{Rev}] \mathcal{S}_R(f_c)}$$

(12A-75)

is a complex, nonzero-mean, Gaussian RV, $(r_{Rev}, \theta_{Rev}, \psi_{Rev})$ are the spherical coordinates of the center of mass of the object responsible for the reverberation, measured from the center of the array; $S_{Rev}(\bullet)$ is the scattering function of the object responsible for the reverberation, and is treated as a complex, nonzero-mean, Gaussian RV; $\theta_{Rev\,0} = 180° - \theta_{Rev}$, and $\psi_{Rev\,0} = 180° + \psi_{Rev}$.

Appendix 12B Derivation of the Denominator of the Signal-to-Interference Ratio

In this appendix we shall derive the second moment (mean-squared value) of $\tilde{l}_Z$, which is the denominator of the signal-to-interference ratio (SIR) given by (12.2-1). Computing the magnitude-squared of $\tilde{l}_Z$ given by (12.2-4) yields

$$\left|\tilde{l}_Z\right|^2 = \left|\tilde{l}_{Rev}\right|^2 + \left|\tilde{l}_{n_a}\right|^2 + \left|\tilde{l}_{n_r}\right|^2 + 2\,\text{Re}\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_a}\right\} + 2\,\text{Re}\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_r}\right\} + 2\,\text{Re}\left\{\tilde{l}_{n_a}\tilde{l}^*_{n_r}\right\},$$

(12B-1)

where use was made of the identity

$$Z + Z^* = 2\,\text{Re}\{Z\}. \tag{12B-2}$$

Therefore,

$$E\left\{\left|\tilde{l}_Z\right|^2\right\} = E\left\{\left|\tilde{l}_{Rev}\right|^2\right\} + E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} + E\left\{\left|\tilde{l}_{n_r}\right|^2\right\} + \\ 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_a}\right\}\right\} + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_r}\right\}\right\} + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{n_a}\tilde{l}^*_{n_r}\right\}\right\}. \tag{12B-3}$$

Consider the fourth term in (12B-3). With the use of (12.2-5) and (12.2-6),

$$E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_a}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{g}^*(t)\,R_{\tilde{y}_{Rev}\,\tilde{y}_{n_a}}(t,t')\,\tilde{g}(t')\,dt\,dt' , \tag{12B-4}$$

where

$$R_{\tilde{y}_{Rev}\,\tilde{y}_{n_a}}(t,t') = E\left\{\tilde{y}_{Rev}(t)\,\tilde{y}^*_{n_a}(t')\right\} \tag{12B-5}$$

is the cross-correlation function of $\tilde{y}_{Rev}(t)$ and $\tilde{y}_{n_a}(t)$. If $\tilde{y}_{Rev}(t)$ and $\tilde{y}_{n_a}(t)$ are *statistically independent* random processes and $\tilde{y}_{n_a}(t)$ is *zero-mean*, then

$$R_{\tilde{y}_{Rev}\,\tilde{y}_{n_a}}(t,t') = E\left\{\tilde{y}_{Rev}(t)\right\}E\left\{\tilde{y}^*_{n_a}(t')\right\} = 0 , \tag{12B-6}$$

and, as a result,

$$E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_a}\right\} = 0 . \tag{12B-7}$$

Furthermore, since [see (12.2-6)]

$$E\left\{\tilde{l}_{n_a}\right\} = \int_{-\infty}^{\infty} E\left\{\tilde{y}_{n_a}(t)\right\}\tilde{g}^*(t)\,dt = 0 , \tag{12B-8}$$

$$\mathrm{Cov}\left(\tilde{l}_{Rev}, \tilde{l}_{n_a}\right) = E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_a}\right\} - E\left\{\tilde{l}_{Rev}\right\}\left[E\left\{\tilde{l}_{n_a}\right\}\right]^* = 0 , \tag{12B-9}$$

that is, $\tilde{l}_{Rev}$ and $\tilde{l}_{n_a}$ are *uncorrelated*, and since they are Gaussian RVs, they are also *statistically independent*.

Consider the fifth term in (12B-3). With the use of (12.2-5) and (12.2-7),

$$E\left\{\tilde{l}_{Rev}\,\tilde{l}^*_{n_r}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{g}^*(t)\,R_{\tilde{y}_{Rev}\,\tilde{n}_r}(t,t')\,\tilde{g}(t')\,dt\,dt' , \tag{12B-10}$$

where

$$R_{\tilde{y}_{Rev}\,\tilde{n}_r}(t,t') = E\left\{\tilde{y}_{Rev}(t)\,\tilde{n}^*_r(t')\right\} \tag{12B-11}$$

is the cross-correlation function of $\tilde{y}_{Rev}(t)$ and $\tilde{n}_r(t)$. If $\tilde{y}_{Rev}(t)$ and $\tilde{n}_r(t)$ are *statistically independent* random processes and $\tilde{n}_r(t)$ is *zero-mean*, then

$$R_{\tilde{y}_{Rev}\tilde{n}_r}(t,t') = E\{\tilde{y}_{Rev}(t)\}E\{\tilde{n}_r^*(t')\} = 0\,, \tag{12B-12}$$

and, as a result,

$$E\{\tilde{l}_{Rev}\tilde{l}_{n_r}^*\} = 0\,. \tag{12B-13}$$

Furthermore, since [see (12.2-7)]

$$E\{\tilde{l}_{n_r}\} = \int_{-\infty}^{\infty} E\{\tilde{n}_r(t)\}\tilde{g}^*(t)\,dt = 0\,, \tag{12B-14}$$

$$\text{Cov}\left(\tilde{l}_{Rev},\tilde{l}_{n_r}\right) = E\{\tilde{l}_{Rev}\tilde{l}_{n_r}^*\} - E\{\tilde{l}_{Rev}\}\left[E\{\tilde{l}_{n_r}\}\right]^* = 0\,, \tag{12B-15}$$

that is, $\tilde{l}_{Rev}$ and $\tilde{l}_{n_r}$ are *uncorrelated*, and since they are Gaussian RVs, they are also *statistically independent*.

Consider the sixth term in (12B-3). With the use of (12.2-6) and (12.2-7),

$$E\{\tilde{l}_{n_a}\tilde{l}_{n_r}^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}^*(t)R_{\tilde{y}_{n_a}\tilde{n}_r}(t,t')\tilde{g}(t')\,dt\,dt'\,, \tag{12B-16}$$

where

$$R_{\tilde{y}_{n_a}\tilde{n}_r}(t,t') = E\{\tilde{y}_{n_a}(t)\tilde{n}_r^*(t')\} \tag{12B-17}$$

is the cross-correlation function of $\tilde{y}_{n_a}(t)$ and $\tilde{n}_r(t)$. If $\tilde{y}_{n_a}(t)$ and $\tilde{n}_r(t)$ are *statistically independent* random processes and $\tilde{n}_r(t)$ is *zero-mean*, then

$$R_{\tilde{y}_{n_a}\tilde{n}_r}(t,t') = E\{\tilde{y}_{n_a}(t)\}E\{\tilde{n}_r^*(t')\} = 0\,, \tag{12B-18}$$

and, as a result,

$$E\{\tilde{l}_{n_a}\tilde{l}_{n_r}^*\} = 0\,. \tag{12B-19}$$

Furthermore, since $\tilde{l}_{n_a}$ and $\tilde{l}_{n_r}$ are zero-mean [see (12B-8) and (12B-14)],

$$\text{Cov}\left(\tilde{l}_{n_a},\tilde{l}_{n_r}\right) = E\{\tilde{l}_{n_a}\tilde{l}_{n_r}^*\} - E\{\tilde{l}_{n_a}\}\left[E\{\tilde{l}_{n_r}\}\right]^* = 0\,, \tag{12B-20}$$

that is, $\tilde{l}_{n_a}$ and $\tilde{l}_{n_r}$ are *uncorrelated*, and since they are Gaussian RVs, they are also *statistically independent*. Therefore, substituting (12B-7), (12B-13), and (12B-19) into (12B-3) yields

$$E\left\{\left|\tilde{l}_Z\right|^2\right\} = E\left\{\left|\tilde{l}_{Rev}\right|^2\right\} + E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} + E\left\{\left|\tilde{l}_{n_r}\right|^2\right\}. \tag{12B-21}$$

Consider the first term in (12B-21). Substituting (12.1-9) into (12.1-8) yields

$$\tilde{y}_{Rev}(t) = w_{Rev} \sum_{i=-N'}^{N'} a_i \tilde{x}(t-\Delta_i)\exp(+j2\pi\eta_{D,Rev}t)\exp\left[-j2\pi(f_c+\eta_{D,Rev})\Delta_i\right], \tag{12B-22}$$

where

$$\Delta_i = \tau_{i,Rev} + \tau'_{i,Trgt}\,, \tag{12B-23}$$

and by substituting (12A-69) into (12B-23), we obtain

$$\Delta_i = \tau_{Trgt} + (\tau_{i,Rev} - \tau_{i,Trgt})\,. \tag{12B-24}$$

Substituting (12B-22) and (12.2-10) into (12.2-5) yields

$$\tilde{l}_{Rev} = w_{Rev} \sum_{i=-N'}^{N'} a_i \int_{-\infty}^{\infty} \tilde{x}(t-\Delta_i)\tilde{x}^*(t-\hat{\tau}_{Trgt})\exp\left[-j2\pi(\hat{\eta}_{D,Trgt}-\eta_{D,Rev})t\right]dt \times \exp\left[-j2\pi(f_c+\eta_{D,Rev})\Delta_i\right]. \tag{12B-25}$$

If we let $t' = t - \Delta_i$ in (12B-25), and make use of (12B-24), then

$$\tilde{l}_{Rev} = w_{Rev}\exp\left[-j2\pi(f_c+\hat{\eta}_{D,Trgt})\tau_{Trgt}\right]\times \sum_{i=-N'}^{N'} a_i \mathcal{X}(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c+\hat{\eta}_{D,Trgt})(\tau_{i,Trgt}-\tau_{i,Rev})\right], \tag{12B-26}$$

where

$$\mathcal{X}(e_{\tau_i}, e_D) = \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t-e_{\tau_i})\exp(-j2\pi e_D t)\,dt \tag{12B-27}$$

is the unnormalized, auto-ambiguity function of the transmitted electrical signal $x(t)$, with complex envelope $\tilde{x}(t)$;

$$e_{\tau_i} = e_{\tau,Trgt} + (\tau_{i,Trgt} - \tau_{i,Rev}) \tag{12B-28}$$

is a round-trip time-delay estimation error in seconds, where $e_{\tau,Trgt}$ is the round-trip time-delay estimation error of the target given by (12.2-14), and

$$e_D = \hat{\eta}_{D,Trgt} - \eta_{D,Rev} \tag{12B-29}$$

is a Doppler-shift estimation error in hertz. Therefore,

$$\boxed{E\left\{\left|\tilde{l}_{Rev}\right|^2\right\} = E_{\tilde{x}}^2 E\left\{\left|w_{Rev}\right|^2\right\}\left|\sum_{i=-N'}^{N'} a_i \mathcal{X}_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2} \tag{12B-30}$$

where the normalized, auto-ambiguity function $\mathcal{X}_N(e_{\tau_i}, e_D)$ was obtained by dividing $\mathcal{X}(e_{\tau_i}, e_D)$ by the energy $E_{\tilde{x}}$ of $\tilde{x}(t)$, where $E_{\tilde{x}}$ has units of joules-ohms [see (12.2-17)].

Consider the second term in (12B-21). With the use of (12.2-6),

$$E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}^*(t) R_{\tilde{y}_{n_a}}(t,t')\tilde{g}(t')\,dt\,dt' , \tag{12B-31}$$

where

$$R_{\tilde{y}_{n_a}}(t,t') = E\left\{\tilde{y}_{n_a}(t)\tilde{y}_{n_a}^*(t')\right\} \tag{12B-32}$$

is the autocorrelation function of $\tilde{y}_{n_a}(t)$. Substituting (12.1-12) into (12.1-11) yields

$$\tilde{y}_{n_a}(t) = \sum_{i=-N'}^{N'} a_i \tilde{n}_a(t - \tau'_{i,Trgt}, x_i) , \tag{12B-33}$$

and by substituting (12B-33) into (12B-32), we obtain

$$R_{\tilde{y}_{n_a}}(t,t') = \sum_{i=-N'}^{N'}\sum_{j=-N'}^{N'} a_i a_j R_{\tilde{n}_a}(t - \tau'_{i,Trgt}, t' - \tau'_{j,Trgt}, x_i, x_j) , \tag{12B-34}$$

or

$$\begin{aligned} R_{\tilde{y}_{n_a}}(t,t') = &\sum_{i=-N'}^{N'} a_i^2 R_{\tilde{n}_a}(t - \tau'_{i,Trgt}, t' - \tau'_{i,Trgt}, x_i, x_i) + \\ &\sum_{\substack{i=-N' \\ i\neq j}}^{N'}\sum_{j=-N'}^{N'} a_i a_j R_{\tilde{n}_a}(t - \tau'_{i,Trgt}, t' - \tau'_{j,Trgt}, x_i, x_j), \end{aligned} \tag{12B-35}$$

where

$$R_{\tilde{n}_a}(t - \tau'_{i,Trgt}, t' - \tau'_{j,Trgt}, x_i, x_j) = E\left\{\tilde{n}_a(t - \tau'_{i,Trgt}, x_i)\tilde{n}_a^*(t' - \tau'_{j,Trgt}, x_j)\right\} . \tag{12B-36}$$

If the random process $\tilde{n}_a(t, x_i)$ is *wide-sense stationary* (WSS) *in time and space* $\forall\ i$, then (12B-35) reduces to

$$R_{\tilde{y}_{n_a}}(\Delta t) = R_{\tilde{n}_a}(\Delta t, 0) \sum_{i=-N'}^{N'} a_i^2 + \sum_{\substack{i=-N' \\ i \neq j}}^{N'} \sum_{j=-N'}^{N'} a_i a_j R_{\tilde{n}_a}\left(\Delta t - [\tau'_{i,Trgt} - \tau'_{j,Trgt}], x_i - x_j\right), \tag{12B-37}$$

where $\Delta t = t - t'$. Furthermore, if $\tilde{n}_a(t, x_i)$ is *zero-mean* $\forall\ i$, and $\tilde{n}_a(t, x_i)$ and $\tilde{n}_a(t, x_j)$ are *uncorrelated* for $i \neq j$, then

$$R_{\tilde{n}_a}\left(\Delta t - [\tau'_{i,Trgt} - \tau'_{j,Trgt}], x_i - x_j\right) = 0, \qquad i \neq j, \tag{12B-38}$$

and as a result, (12B-37) reduces to

$$R_{\tilde{y}_{n_a}}(\Delta t) = R_{\tilde{n}_a}(\Delta t, 0) \sum_{i=-N'}^{N'} a_i^2, \tag{12B-39}$$

where

$$R_{\tilde{n}_a}(\Delta t, 0) = E\left\{\tilde{n}_a(t, x_i)\tilde{n}_a^*(t', x_i)\right\} \tag{12B-40}$$

is the autocorrelation function of $\tilde{n}_a(t, x_i)$ at the output of any element i in the linear array. Substituting (12B-39) into (12B-31) yields

$$E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} = \sum_{i=-N'}^{N'} a_i^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}^*(t) R_{\tilde{n}_a}(\Delta t, 0)\tilde{g}(t')\,dt\,dt'. \tag{12B-41}$$

Since $\tilde{n}_a(t, x_i)$ is WSS,

$$R_{\tilde{n}_a}(\Delta t, 0) = F_\eta^{-1}\left\{S_{\tilde{n}_a}(\eta)\right\} = \int_{-\infty}^{\infty} S_{\tilde{n}_a}(\eta)\exp(+j2\pi\eta\Delta t)\,d\eta, \tag{12B-42}$$

where $S_{\tilde{n}_a}(\eta)$ is the power spectrum of $\tilde{n}_a(t, x_i)$ $\forall\ i$. If $\tilde{n}_a(t, x_i)$ is a *lowpass (baseband)*, *white-noise* random process in the time-domain, then

$$S_{\tilde{n}_a}(\eta) = \tilde{N}_{a0}\,\text{rect}\left[\eta/(2B)\right], \tag{12B-43}$$

where the constant $\tilde{N}_{a0}$ has units of $(\text{W-}\Omega)/\text{Hz}$ (or $\text{J-}\Omega$),

$$\text{rect}\left(\frac{\eta}{2B}\right) = \begin{cases} 1, & |\eta| \leq B \\ 0, & |\eta| > B \end{cases} \tag{12B-44}$$

and B is the bandwidth in hertz of the ideal, lowpass filters in a quadrature demodulator used to compute complex envelopes (see Section 11.3). Therefore,

$$R_{\tilde{n}_a}(\Delta t, 0) = 2\tilde{N}_{a0} B \operatorname{sinc}(2B\Delta t), \tag{12B-45}$$

where

$$\operatorname{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}. \tag{12B-46}$$

The constant variance of the zero-mean ambient noise $\tilde{n}_a(t, x_i)$ is given by

$$\sigma^2_{\tilde{n}_a} = R_{\tilde{n}_a}(0,0) = E\left\{\left|\tilde{n}_a(t, x_i)\right|^2\right\} = 2\tilde{N}_{a0} B \tag{12B-47}$$

since $\operatorname{sinc}(0) = 1$. The variance $\sigma^2_{\tilde{n}_a}$ is the average power of $\tilde{n}_a(t, x_i)$ in watts-ohms.

Substituting (12B-45) into (12B-41) and changing the order of integration and rearranging factors yields

$$E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} = 2\tilde{N}_{a0} B \sum_{i=-N'}^{N'} a_i^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{g}(t') \operatorname{sinc}\left[2B(t-t')\right] dt' \, \tilde{g}^*(t)\, dt\,, \tag{12B-48}$$

or

$$E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} = 2\tilde{N}_{a0} B \sum_{i=-N'}^{N'} a_i^2 \int_{-\infty}^{\infty} \tilde{f}(t)\tilde{g}^*(t)\, dt\,, \tag{12B-49}$$

where

$$\tilde{f}(t) = \tilde{g}(t) * \operatorname{sinc}(2Bt) = \int_{-\infty}^{\infty} \tilde{g}(t') \operatorname{sinc}\left[2B(t-t')\right] dt'. \tag{12B-50}$$

By using Parseval's Theorem, the integral on the right-hand side of (12B-49) can be rewritten as

$$\int_{-\infty}^{\infty} \tilde{f}(t)\tilde{g}^*(t)\, dt = \int_{-\infty}^{\infty} \tilde{F}(\eta)\tilde{G}^*(\eta)\, d\eta\,, \tag{12B-51}$$

where, from (12B-50),

$$\tilde{F}(\eta) = \frac{1}{2B}\tilde{G}(\eta)\operatorname{rect}\left(\frac{\eta}{2B}\right). \tag{12B-52}$$

Substituting (12B-52) and (12B-44) into (12B-51) yields

$$\int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{g}^*(t)\,dt = \frac{1}{2B}\int_{-B}^{B} \left|\tilde{G}(\eta)\right|^2 d\eta \,. \tag{12B-53}$$

From (12.2-10),

$$\tilde{G}(\eta) = \tilde{X}(\eta - \hat{\eta}_{D,Trgt})\exp\left[-j2\pi(\eta - \hat{\eta}_{D,Trgt})\hat{\tau}_{Trgt}\right], \tag{12B-54}$$

and by substituting (12B-54) into (12B-53), we obtain

$$\int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{g}^*(t)\,dt = \frac{1}{2B}\int_{-B}^{B} \left|\tilde{X}(\eta - \hat{\eta}_{D,Trgt})\right|^2 d\eta \,. \tag{12B-55}$$

Recall that the frequency spectrum $\tilde{X}(\eta)$ of the complex envelope of the transmitted electrical signal $\tilde{x}(t)$ is lowpass (baseband) with bandwidth W hertz. If the bandwidth B of the ideal, lowpass filters in a quadrature demodulator is computed from the formula

$$\boxed{B = W + \max\left|\hat{\eta}_{D,Trgt}\right|} \tag{12B-56}$$

then (12B-55) can be rewritten as

$$\int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{g}^*(t)\,dt = \frac{1}{2B}\int_{-\infty}^{\infty} \left|\tilde{X}(\eta - \hat{\eta}_{D,Trgt})\right|^2 d\eta = \frac{1}{2B}\int_{-\infty}^{\infty} \left|\tilde{X}(\eta)\right|^2 d\eta \,, \tag{12B-57}$$

and with the use of Parseval's Theorem,

$$\int_{-\infty}^{\infty} \tilde{f}(t)\,\tilde{g}^*(t)\,dt = \frac{1}{2B}\int_{-\infty}^{\infty} \left|\tilde{x}(t)\right|^2 dt = \frac{E_{\tilde{x}}}{2B}\,, \tag{12B-58}$$

where $E_{\tilde{x}}$ is the energy of $\tilde{x}(t)$ in joules-ohms. Therefore, substituting (12B-58) into (12B-49) finally yields

$$\boxed{E\left\{\left|\tilde{l}_{n_a}\right|^2\right\} = E_{\tilde{x}}\tilde{N}_{a0}\sum_{i=-N'}^{N'} a_i^2} \tag{12B-59}$$

Consider the third term in (12B-21). With the use of (12.2-7),

$$E\left\{\left|\tilde{l}_{n_r}\right|^2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}^*(t)\,R_{\tilde{n}_r}(t,t')\,\tilde{g}(t')\,dt\,dt' \,, \tag{12B-60}$$

where

$$R_{\tilde{n}_r}(t,t') = E\left\{\tilde{n}_r(t)\tilde{n}_r^*(t')\right\} \tag{12B-61}$$

is the autocorrelation function of $\tilde{n}_r(t)$. Substituting (12.1-14) into (12.1-13) yields

$$\tilde{n}_r(t) = \sum_{i=-N'}^{N'} a_i \tilde{n}(t-\tau'_{i,Trgt}, x_i), \tag{12B-62}$$

and by substituting (12B-62) into (12B-61), we obtain

$$R_{\tilde{n}_r}(t,t') = \sum_{i=-N'}^{N'} \sum_{j=-N'}^{N'} a_i a_j R_{\tilde{n}}(t-\tau'_{i,Trgt}, t'-\tau'_{j,Trgt}, x_i, x_j), \tag{12B-63}$$

or

$$\begin{aligned} R_{\tilde{n}_r}(t,t') = &\sum_{i=-N'}^{N'} a_i^2 R_{\tilde{n}}(t-\tau'_{i,Trgt}, t'-\tau'_{i,Trgt}, x_i, x_i) + \\ &\sum_{\substack{i=-N' \\ i\neq j}}^{N'} \sum_{j=-N'}^{N'} a_i a_j R_{\tilde{n}}(t-\tau'_{i,Trgt}, t'-\tau'_{j,Trgt}, x_i, x_j), \end{aligned} \tag{12B-64}$$

where

$$R_{\tilde{n}}(t-\tau'_{i,Trgt}, t'-\tau'_{j,Trgt}, x_i, x_j) = E\left\{\tilde{n}(t-\tau'_{i,Trgt}, x_i)\tilde{n}^*(t'-\tau'_{j,Trgt}, x_j)\right\}. \tag{12B-65}$$

If the same assumptions made about the ambient noise are made about the receiver noise, that is, if the random process $\tilde{n}(t,x_i)$ is *zero-mean* and *wide-sense stationary* (WSS) *in time and space* $\forall\ i$, $\tilde{n}(t,x_i)$ and $\tilde{n}(t,x_j)$ are *uncorrelated* for $i \neq j$, and $\tilde{n}(t,x_i)$ is a *lowpass (baseband)*, *white-noise* random process in the time-domain with power spectrum

$$S_{\tilde{n}}(\eta) = \tilde{N}_{r0}\,\text{rect}\left[\eta/(2B)\right] \tag{12B-66}$$

at the output of any element i in the linear array, where the constant $\tilde{N}_{r0}$ has units of $(\text{W-}\Omega)/\text{Hz}$ (or $\text{J-}\Omega$), then by following the same procedure used to derive (12B-37) through (12B-59), the autocorrelation function of the receiver noise $\tilde{n}(t,x_i)$ is given by

$$R_{\tilde{n}}(\Delta t, 0) = E\left\{\tilde{n}(t,x_i)\tilde{n}^*(t',x_i)\right\} = 2\tilde{N}_{r0} B\,\text{sinc}(2B\Delta t) \tag{12B-67}$$

$\forall\ i$, where $\Delta t = t - t'$, the constant variance of the zero-mean receiver noise

$\tilde{n}(t, x_i)$ is given by

$$\sigma_{\tilde{n}}^2 = R_{\tilde{n}}(0,0) = E\left\{\left|\tilde{n}(t, x_i)\right|^2\right\} = 2\tilde{N}_{r0}B \, , \tag{12B-68}$$

where $\sigma_{\tilde{n}}^2$ is the average power of $\tilde{n}(t, x_i)$ in watts-ohms, and

$$\boxed{E\left\{\left|\tilde{l}_{n_r}\right|^2\right\} = E_{\tilde{x}}\tilde{N}_{r0}\sum_{i=-N'}^{N'} a_i^2} \tag{12B-69}$$

where $E_{\tilde{x}}$ is the energy of $\tilde{x}(t)$ in joules-ohms.

Substituting (12B-30), (12B-59), and (12B-69) into (12B-21) yields

$$\boxed{\begin{aligned} E\left\{\left|\tilde{l}_Z\right|^2\right\} = {} & E_{\tilde{x}}^2 E\left\{\left|w_{Rev}\right|^2\right\}\left|\sum_{i=-N'}^{N'} a_i X_N(e_{\tau_i}, e_D)\exp\left[+j2\pi(f_c + \hat{\eta}_{D,Trgt})(\tau_{i,Trgt} - \tau_{i,Rev})\right]\right|^2 + \\ & A_2 E_{\tilde{x}}\left[\tilde{N}_{a0} + \tilde{N}_{r0}\right] \end{aligned}} \tag{12B-70}$$

where

$$A_2 = \sum_{i=-N'}^{N'} a_i^2 \, . \tag{12B-71}$$

Equation (12B-70) is the denominator of the signal-to-interference ratio (SIR).

Appendix 12C Table 12C-1 Marcum Q-Function $Q(a,b)$

Table 12C-1 Marcum Q-Function $Q(a,b)$

	$a=1/2$	$a=\sqrt{2}/2$	$a=1$	$a=\sqrt{2}$	$a=2$	$a=2\sqrt{2}$	$a=4$
b	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.9955972	0.9961133	0.9969711	0.9981606	0.9993216	0.9999077	0.9999983
0.2	0.9825036	0.9845403	0.9879299	0.9926427	0.9972664	0.9996227	0.9999928
0.3	0.9610594	0.9655391	0.9730120	0.9834482	0.9937749	0.9991199	0.9999824
0.4	0.9318151	0.9395302	0.9524414	0.9705849	0.9887516	0.9983574	0.9999652
0.5	0.8955086	0.9070812	0.9265274	0.9540725	0.9820694	0.9972757	0.9999378
0.6	0.8530342	0.8688875	0.8956601	0.9339501	0.9735767	0.9957971	0.9998959
0.7	0.8054067	0.8257489	0.8603040	0.9102852	0.9631057	0.9938250	0.9998328
0.8	0.7537215	0.7785424	0.8209899	0.8831822	0.9504818	0.9912438	0.9997396
0.9	0.6991130	0.7281940	0.7783047	0.8527903	0.9355343	0.9879192	0.9996041
1	0.6427142	0.6756493	0.7328798	0.8193100	0.9181077	0.9836985	0.9994101
1.1	0.5856192	0.6218454	0.6853771	0.7829968	0.8980728	0.9784124	0.9991361
1.2	0.5288489	0.5676836	0.6364742	0.7441625	0.8753379	0.9718777	0.9987544
1.3	0.4733238	0.5140058	0.5868485	0.7031730	0.8498581	0.9639007	0.9982296
1.4	0.4198429	0.4615742	0.5371608	0.6604429	0.8216440	0.9542822	0.9975171
1.5	0.3690690	0.4110555	0.4880400	0.6164276	0.7907678	0.9428235	0.9965616
1.6	0.3215222	0.3630093	0.4400679	0.5716114	0.7573667	0.9293330	0.9952957
1.7	0.2775793	0.3178823	0.3937666	0.5264953	0.7216440	0.9136346	0.9936381
1.8	0.2374792	0.2760064	0.3495881	0.4815821	0.6838668	0.8955758	0.9914930
1.9	0.2013335	0.2376015	0.3079061	0.4373617	0.6443605	0.8750364	0.9887485
2	0.1691406	0.2027822	0.2690121	0.3942969	0.6035010	0.8519364	0.9852765
2.1	0.1408026	0.1715675	0.2331134	0.3528097	0.5617037	0.8262441	0.9809334
2.2	0.1161430	0.1438927	0.2003353	0.3132710	0.5194115	0.7979823	0.9755602
2.3	0.0949259	0.1196235	0.1707252	0.2759910	0.4770802	0.7672322	0.9689856
2.4	0.0768731	0.0985698	0.1442599	0.2412136	0.4351646	0.7341366	0.9610280
2.5	0.0616811	0.0805003	0.1208541	0.2091139	0.3941039	0.6988994	0.9515004
2.6	0.0490350	0.0651560	0.1003710	0.1797978	0.3543082	0.6617828	0.9402153
2.7	0.0386213	0.0522629	0.0826325	0.1533048	0.3161467	0.6231024	0.9269909
2.8	0.0301372	0.0415425	0.0674306	0.1296139	0.2799382	0.5832189	0.9116584
2.9	0.0232984	0.0327216	0.0545376	0.1086497	0.2459437	0.5425284	0.8940693
3	0.0178437	0.0255386	0.0437160	0.0902915	0.2143621	0.5014507	0.8741039
3.1	0.0135384	0.0197498	0.0347266	0.0743825	0.1853284	0.4604161	0.8516786
3.2	0.0101758	0.0151327	0.0273362	0.0607388	0.1589150	0.4198524	0.8267536
3.3	0.0075766	0.0114879	0.0213228	0.0491588	0.1351350	0.3801714	0.7993388
3.4	0.0055883	0.0086401	0.0164801	0.0394319	0.1139476	0.3417567	0.7694987
3.5	0.0040830	0.0064378	0.0126201	0.0313457	0.0952653	0.3049531	0.7373555
3.6	0.0029549	0.0047521	0.0095750	0.0246927	0.0789620	0.2700573	0.7030894
3.7	0.0021183	0.0034749	0.0071973	0.0192750	0.0648811	0.2373114	0.6669370
3.8	0.0015042	0.0025171	0.0053597	0.0149086	0.0528448	0.2068995	0.6291874
3.9	0.0010580	0.0018061	0.0039539	0.0114254	0.0426617	0.1789451	0.5901752
4	0.0007370	0.0012837	0.0028895	0.0086753	0.0341348	0.1535135	0.5502721

	$a = 1/2$	$a = \sqrt{2}/2$	$a = 1$	$a = \sqrt{2}$	$a = 2$	$a = 2\sqrt{2}$	$a = 4$
b	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$	$Q(a,b)$
4	0.0007370	0.0012837	0.0028895	0.0086753	0.0341348	0.1535135	0.5502721
4.1	0.0005086	0.0009038	0.0020918	0.0065262	0.0270679	0.1306137	0.5098758
4.2	0.0003476	0.0006302	0.0015000	0.0048638	0.0212707	0.1102048	0.4693988
4.3	0.0002353	0.0004353	0.0010655	0.0035910	0.0165638	0.0922015	0.4292554
4.4	0.0001577	0.0002978	0.0007497	0.0026265	0.0127809	0.0764824	0.3898493
4.5	0.0001047	0.0002018	0.0005224	0.0019030	0.0097718	0.0628974	0.3515611
4.6	0.0000689	0.0001354	0.0003606	0.0013658	0.0074024	0.0512764	0.3147382
4.7	0.0000449	0.0000900	0.0002465	0.0009710	0.0055557	0.0414366	0.2796846
4.8	0.0000289	0.0000593	0.0001669	0.0006838	0.0041311	0.0331895	0.2466545
4.9	0.0000185	0.0000386	0.0001120	0.0004770	0.0030432	0.0263476	0.2158473
5	0.0000117	0.0000249	0.0000744	0.0003295	0.0022208	0.0207290	0.1874047
5.1	0.0000073	0.0000159	0.0000489	0.0002255	0.0016055	0.0161618	0.1614116
5.2	0.0000045	0.0000101	0.0000319	0.0001528	0.0011498	0.0124868	0.1378976
5.3	0.0000028	0.0000063	0.0000206	0.0001026	0.0008156	0.0095596	0.1168417
5.4	0.0000017	0.0000039	0.0000131	0.0000682	0.0005731	0.0072517	0.0981776
5.5	0.0000010	0.0000024	0.0000083	0.0000449	0.0003989	0.0054504	0.0818009

Appendix 12D How to Compute Values for σ_0/σ_1

In this appendix we shall derive an equation for the ratio σ_0/σ_1. Substituting (12.3-25) and (12.4-29) into (12.4-23) yields

$$\sigma_1^2 = \frac{\mathrm{Var}\left(\tilde{l}_{Trgt}\right)}{2} + \sigma_0^2 , \tag{12D-1}$$

from which we can conclude that

$$\boxed{\sigma_1^2 > \sigma_0^2} \tag{12D-2}$$

Since [see (12.3-49)]

$$\alpha_0^2 = \gamma_0/\sigma_0^2 \tag{12D-3}$$

and [see (12.4-58)]

$$\alpha_1^2 = \gamma_1/\sigma_1^2 , \tag{12D-4}$$

$$\frac{\alpha_1^2}{\alpha_0^2} = \frac{\gamma_1}{\gamma_0}\frac{\sigma_0^2}{\sigma_1^2} , \tag{12D-5}$$

or

$$\boxed{\frac{\sigma_0^2}{\sigma_1^2} = \frac{\gamma_0}{\gamma_1}\frac{\alpha_1^2}{\alpha_0^2} < 1} \tag{12D-6}$$

From (12D-6),

$$\frac{\gamma_0}{\gamma_1}\frac{\alpha_1^2}{\alpha_0^2} < 1 , \tag{12D-7}$$

or

$$\boxed{\gamma_0/\gamma_1 < \alpha_0^2/\alpha_1^2} \tag{12D-8}$$

We shall establish the relationship between γ_0 and γ_1 next.

Substituting (12.3-7) into (12.3-29) yields

$$\gamma_0 = \left|E\left\{\tilde{l}\,\middle|\,\mathrm{H}_0\right\}\right|^2 = \left|E\left\{\tilde{l}_{Rev}\right\}\right|^2 , \tag{12D-9}$$

and substituting (12.4-12) into (12.4-33) yields

$$\gamma_1 = \left| E\left\{ \tilde{l} \middle| \mathrm{H}_1 \right\} \right|^2 = \left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + \left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\}, \tag{12D-10}$$

where use was made of the identity

$$|A+B|^2 = |A|^2 + 2\,\mathrm{Re}\left\{ AB^* \right\} + |B|^2 . \tag{12D-11}$$

Note that $\gamma_0 > 0$ and $\gamma_1 > 0$ because the target and reverberation scattering functions are *nonzero-mean* in this analysis. Substituting (12D-9) into (12D-10) yields

$$\gamma_1 = \gamma_0 + \left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\}, \tag{12D-12}$$

and since it can be shown that (see Appendix 12E)

$$\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\} > 0 \tag{12D-13}$$

whether $\alpha_1^2/2 > \alpha_0^2/2$ or $\alpha_0^2/2 > \alpha_1^2/2$,

$$\boxed{\gamma_1 > \gamma_0} \tag{12D-14}$$

Therefore, from (12D-6),

$$\boxed{\frac{\sigma_0}{\sigma_1} = \sqrt{\frac{\gamma_0}{\gamma_1}} \frac{\alpha_1}{\alpha_0} < 1} \tag{12D-15}$$

where [see (12D-8)],

$$\boxed{\sqrt{\gamma_0/\gamma_1} < \alpha_0/\alpha_1} \tag{12D-16}$$

and [see (12D-14)],

$$\boxed{\sqrt{\gamma_0/\gamma_1} < 1} \tag{12D-17}$$

Appendix 12E

In Appendix 12D it was shown that [see (12D-12)]

$$\gamma_1 = \gamma_0 + \left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\}, \tag{12E-1}$$

where $\gamma_0 > 0$ and $\gamma_1 > 0$ because the target and reverberation scattering functions are *nonzero-mean* in Appendix 12D. In this appendix we shall show that

$$\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\} > 0 \tag{12E-2}$$

whether $\alpha_1^2/2 > \alpha_0^2/2$ or $\alpha_0^2/2 > \alpha_1^2/2$, and as a result [see (12D-14)],

$$\boxed{\gamma_1 > \gamma_0} \tag{12E-3}$$

Substituting (12D-9) into (12.3-49) yields

$$\frac{\alpha_0^2}{2} = \frac{\left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2}{2\sigma_0^2}, \tag{12E-4}$$

and by substituting (12D-10) into (12.4-58), we obtain

$$\frac{\alpha_1^2}{2} = \frac{\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + \left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\}}{2\sigma_1^2}. \tag{12E-5}$$

If $\alpha_1^2/2 > \alpha_0^2/2$, then

$$\frac{\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + \left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\}}{2\sigma_1^2} > \frac{\left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2}{2\sigma_0^2}, \tag{12E-6}$$

or

$$\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\} > \left(\frac{\sigma_1^2}{\sigma_0^2} - 1 \right) \left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2. \tag{12E-7}$$

Since $\sigma_1^2/\sigma_0^2 > 1$ [see (12D-2)],

$$\left| E\left\{ \tilde{l}_{Trgt} \right\} \right|^2 + 2\,\mathrm{Re}\left\{ E\left\{ \tilde{l}_{Trgt} \right\} E\left\{ \tilde{l}^*_{Rev} \right\} \right\} > \left(\frac{\sigma_1^2}{\sigma_0^2} - 1 \right) \left| E\left\{ \tilde{l}_{Rev} \right\} \right|^2 > 0, \tag{12E-8}$$

or