Chapter 1

Complex Aperture Theory – Volume Apertures – General Results

Let us begin by explaining the meaning of the word "aperture" as it shall be used in this and subsequent chapters. In the field of optics, a rectangular or circular hole in an opaque screen is referred to as a rectangular or circular *aperture*. The *complex aperture function* is equal to the amplitude and phase of the light distribution within the aperture. It determines the *Fresnel* and *Fraunhofer diffraction patterns* of the aperture. The complex aperture function can describe the light distribution within, for example, a single rectangular hole, or an array of rectangular holes in an opaque screen. In electromagnetics, the meaning of the word "aperture" has been extended to connote either a single electromagnetic antenna or an array of electromagnetic antennas.

Similarly, in acoustics, the word "aperture" is used to refer to either a *single electroacoustic transducer* or an *array of electroacoustic transducers*. When used in the *active mode* as a *transmitter*, an electroacoustic transducer converts electrical signals (voltages and currents) into acoustic signals (sound waves). When used in the *passive mode* as a *receiver*, an electroacoustic transducer converts acoustic signals into electrical signals.

General definitions of the near-field and far-field beam patterns of transmit and receive volume apertures, along with other important general equations, shall be derived in this chapter. Starting with an exact solution of a linear wave equation, it will be shown that the definitions of the near-field and far-field beam patterns are analogous to Fresnel and Fraunhofer diffraction integrals in optics. The reason to start with volume apertures is because linear and planar apertures, and linear, planar, and volume arrays are just special cases of a volume aperture. As a consequence, we will be able to use these general definitions and the other general results obtained in this chapter throughout the entire book.

1.1 Coupling Transmitted and Received Electrical Signals to the Fluid Medium

1.1.1 Transmit Coupling Equation

Consider a closed-surface, transmit aperture enclosing a volume V as shown in Fig. 1.1-1, where the subscripts T and M refer to the transmit aperture and fluid medium, respectively. Imagine that the closed surface is very thin

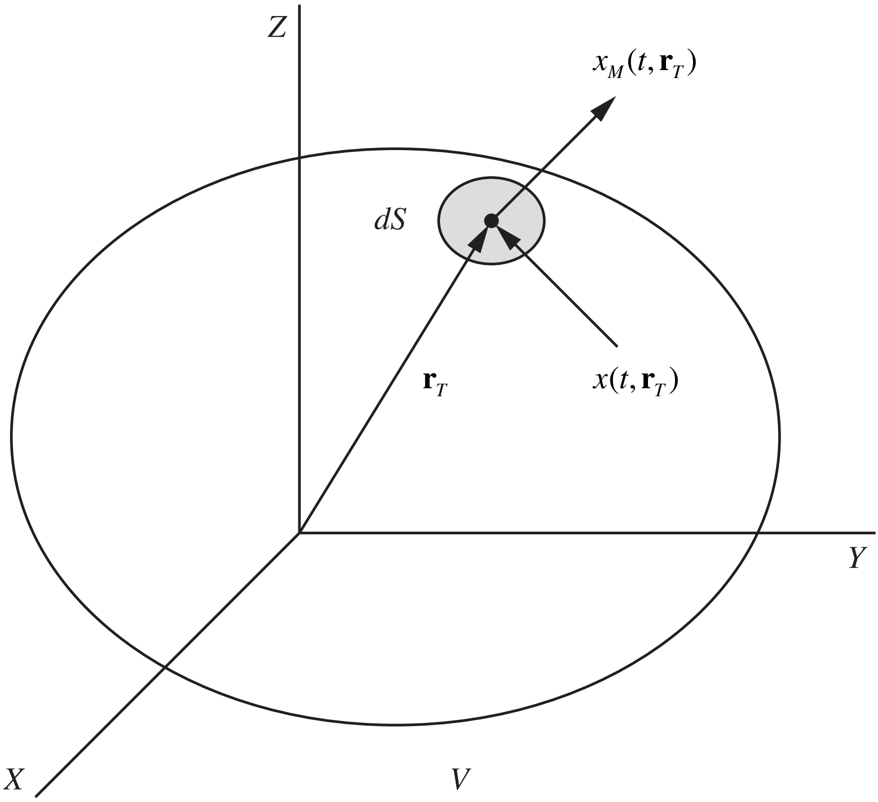

Figure 1.1-1 Closed-surface, transmit aperture enclosing a volume V .

electroacoustic transducer material and that the enclosed volume V is *empty*. This physical situation is known as a *volume aperture*. A real-world example of a volume aperture is a spherical array of electroacoustic transducers, that is, an array of electroacoustic transducers lying on the surface of a hollow sphere. Imagine applying an input electrical signal $x(t, \mathbf{r}_T)$ to the transmit aperture at time t and location $\mathbf{r}_T = (x_T, y_T, z_T)$, where $\mathbf{r}_T$ is the position vector to a point on the surface of the transmit aperture. Therefore, $x(t, \mathbf{r}_T)$ is shorthand notation for $x(t, x_T, y_T, z_T)$. Assume that $x(t, \mathbf{r}_T)$ is a voltage signal with units of volts (V). If we treat the infinitesimal surface area element dS of the transmit aperture located at $\mathbf{r}_T$ as a *linear*, *time-invariant filter* with *impulse response* $\alpha_T(t, \mathbf{r}_T)$ with units of $\left(\sec^{-1}/\mathrm{V}\right)/\sec$, then the output acoustic signal from the transmit aperture $x_M(t, \mathbf{r}_T)$ with units of $\sec^{-1}$, which is also the input acoustic signal to the fluid medium, can be expressed as a *time-domain convolution integral* as follows:

$$x_M(t,\mathbf{r}_T) = \int_{-\infty}^{\infty} x(\tau,\mathbf{r}_T)\alpha_T(t-\tau,\mathbf{r}_T)d\tau \,, \tag{1.1-1}$$

or

$$x_M(t,\mathbf{r}_T) = x(t,\mathbf{r}_T) \underset{t}{*} \alpha_T(t,\mathbf{r}_T)\,, \tag{1.1-2}$$

where the asterisk denotes convolution with respect to time t . The input acoustic signal to the fluid medium $x_M(t,\mathbf{r}_T)$ is also known as the *source distribution*. It represents the volume flow rate (source strength in cubic meters per second) per unit volume of fluid at time t and location $\mathbf{r}_T$.

Taking the Fourier transform of (1.1-2) with respect to t yields

$$X_M(f,\mathbf{r}_T) = X(f,\mathbf{r}_T)A_T(f,\mathbf{r}_T)\,, \tag{1.1-3}$$

where

$$X_M(f,\mathbf{r}_T) = F_t\{x_M(t,\mathbf{r}_T)\} = \int_{-\infty}^{\infty} x_M(t,\mathbf{r}_T)\exp(-j2\pi ft)dt \tag{1.1-4}$$

is the complex frequency spectrum of the input acoustic signal to the fluid medium (source distribution) at location $\mathbf{r}_T$ of the transmit aperture with units of $\mathrm{sec}^{-1}/\mathrm{Hz}$,

$$X(f,\mathbf{r}_T) = F_t\{x(t,\mathbf{r}_T)\} = \int_{-\infty}^{\infty} x(t,\mathbf{r}_T)\exp(-j2\pi ft)dt \tag{1.1-5}$$

is the complex frequency spectrum of the input electrical signal at location $\mathbf{r}_T$ of the transmit aperture with units of V/Hz , and

$$A_T(f,\mathbf{r}_T) = F_t\{\alpha_T(t,\mathbf{r}_T)\} = \int_{-\infty}^{\infty} \alpha_T(t,\mathbf{r}_T)\exp(-j2\pi ft)dt \tag{1.1-6}$$

is the *complex frequency response of the transmit aperture* at $\mathbf{r}_T$ (a.k.a. the *complex transmit aperture function*) with units of $\mathrm{sec}^{-1}/\mathrm{V}$, where f represents *input (transmitted)* frequencies in hertz. Since

$$x_M(t,\mathbf{r}_T) = F_f^{-1}\{X_M(f,\mathbf{r}_T)\} = \int_{-\infty}^{\infty} X_M(f,\mathbf{r}_T)\exp(+j2\pi ft)df \,, \tag{1.1-7}$$

substituting (1.1-3) into (1.1-7) yields

$$\boxed{x_M(t,\mathbf{r}_T) = \int_{-\infty}^{\infty} X(f,\mathbf{r}_T)A_T(f,\mathbf{r}_T)\exp(+j2\pi ft)df} \tag{1.1-8}$$

Compared to (1.1-1), (1.1-8) is a more useful representation of the input acoustic signal (source distribution) for our purposes. Equation (1.1-8) is the *transmit coupling equation*. It models the *coupling* of the input electrical signal to the fluid medium, that is, the production of the input acoustic signal (source distribution) due to the input electrical signal via the transmit aperture whose performance is described, in part, by the complex transmit aperture function.

1.1.2 Receive Coupling Equation

Now consider a closed-surface, receive aperture enclosing a volume V as shown in Fig. 1.1-2, where the subscripts R and M refer to the receive aperture and fluid medium, respectively. As with the transmit aperture, imagine that the closed surface is very thin electroacoustic transducer material and that the enclosed volume V is *empty*. By following the same reasoning used in the development of (1.1-1) through (1.1-8), the output electrical signal from the receive aperture $y(t,\mathbf{r}_R)$ at time t and location $\mathbf{r}_R=(x_R,y_R,z_R)$, with units of $\mathrm{V/m^3}$, can be expressed as a time-domain convolution integral as follows:

$$y(t,\mathbf{r}_R)=\int_{-\infty}^{\infty} y_M(\tau,\mathbf{r}_R)\alpha_R(t-\tau,\mathbf{r}_R)\,d\tau\ , \tag{1.1-9}$$

or

$$y(t,\mathbf{r}_R)=y_M(t,\mathbf{r}_R)\underset{t}{*}\alpha_R(t,\mathbf{r}_R)\ , \tag{1.1-10}$$

where $\mathbf{r}_R$ is the position vector to a point on the surface of the receive aperture; $y_M(t,\mathbf{r}_R)$, with units of $\mathrm{m^2/sec}$, is the output acoustic signal (velocity potential) from the fluid medium, which is also the input acoustic signal to the receive aperture (i.e., it is the acoustic field incident upon the receive aperture) at time t and location $\mathbf{r}_R$; and $\alpha_R(t,\mathbf{r}_R)$, with units of $\left(\left(\mathrm{V}/\left(\mathrm{m^2/sec}\right)\right)/\mathrm{m^3}\right)/\mathrm{sec}$, is the *impulse response* of the infinitesimal surface area element dS of the receive aperture located at $\mathbf{r}_R$, which is treated as a *linear*, *time-invariant filter*. Note that $y(t,\mathbf{r}_R)$ is shorthand notation for $y(t,x_R,y_R,z_R)$.

Taking the Fourier transform of (1.1-10) with respect to t yields

$$Y(\eta,\mathbf{r}_R)=Y_M(\eta,\mathbf{r}_R)A_R(\eta,\mathbf{r}_R)\ , \tag{1.1-11}$$

where

$$Y(\eta,\mathbf{r}_R)=F_t\{y(t,\mathbf{r}_R)\}=\int_{-\infty}^{\infty} y(t,\mathbf{r}_R)\exp(-j2\pi\eta t)\,dt \tag{1.1-12}$$

is the complex frequency spectrum of the output electrical signal at location $\mathbf{r}_R$ of

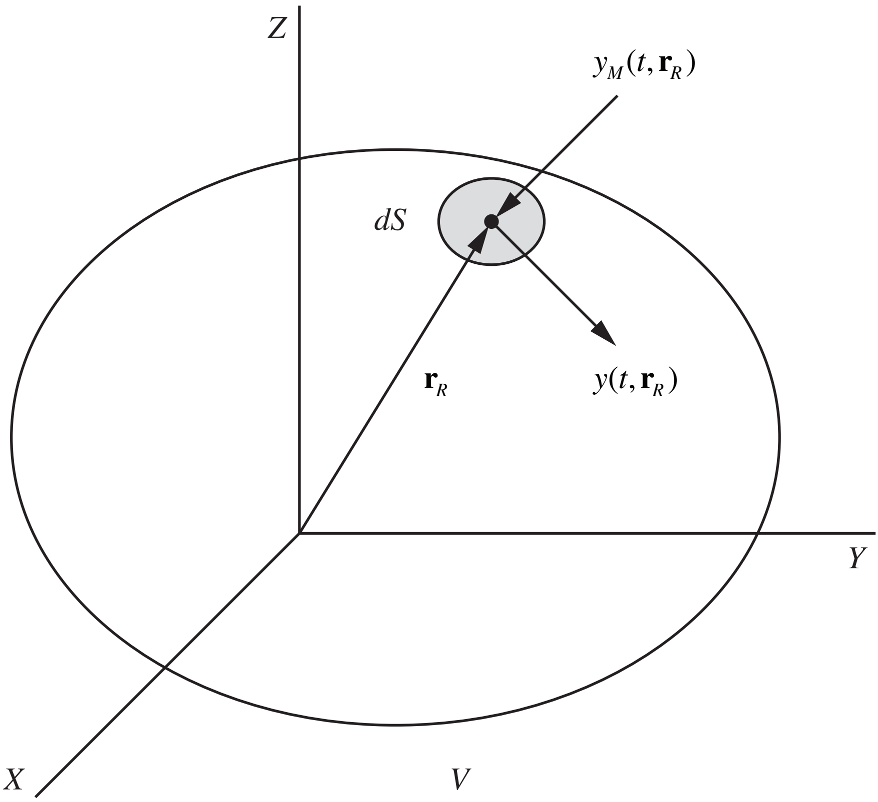

Figure 1.1-2 Closed-surface, receive aperture enclosing a volume V .

the receive aperture with units of $\left(\mathrm{V/Hz}\right)/\mathrm{m}^3$,

$$Y_M(\eta, \mathbf{r}_R) = F_t\{y_M(t, \mathbf{r}_R)\} = \int_{-\infty}^{\infty} y_M(t, \mathbf{r}_R) \exp(-j2\pi\eta t)\, dt \qquad (1.1\text{-}13)$$

is the complex frequency spectrum of the acoustic field incident upon the receive aperture at location $\mathbf{r}_R$ with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$, and

$$A_R(\eta, \mathbf{r}_R) = F_t\{\alpha_R(t, \mathbf{r}_R)\} = \int_{-\infty}^{\infty} \alpha_R(t, \mathbf{r}_R) \exp(-j2\pi\eta t)\, dt \qquad (1.1\text{-}14)$$

is the *complex frequency response of the receive aperture* at $\mathbf{r}_R$ (a.k.a. the *complex receive aperture function*) with units of $\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^3$, where η represents *output (received)* frequencies in hertz and, in general, $\eta \neq f$. For example, if there is any relative motion between transmit and receive apertures, the received frequencies will not be equal to the transmitted frequencies because of Doppler shift and bandwidth compression or expansion. Taking the inverse Fourier transform of (1.1-11) with respect to η yields

$$\boxed{y(t,\mathbf{r}_R)=\int_{-\infty}^{\infty} Y_M(\eta,\mathbf{r}_R)A_R(\eta,\mathbf{r}_R)\exp(+j2\pi\eta t)\,d\eta} \tag{1.1-15}$$

Note that (1.1-15) is analogous to (1.1-8). Equation (1.1-15) is the *receive coupling equation*. It models the *coupling* of the fluid medium to the output electrical signal, that is, the production of the output electrical signal due to the input acoustic signal (incident acoustic field) via the receive aperture whose performance is described, in part, by the complex receive aperture function.

1.2 The Near-Field Beam Pattern of a Volume Aperture

1.2.1 Transmit Aperture

The *exact* solution of the following linear, three-dimensional, lossless, inhomogeneous, wave equation

$$\nabla^2\varphi(t,\mathbf{r})-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(t,\mathbf{r})=x_M(t,\mathbf{r}) \tag{1.2-1}$$

for free-space propagation in an ideal (nonviscous), homogeneous, fluid medium is given by

$$\varphi(t,\mathbf{r})=-\frac{1}{4\pi}\int_{V_0}\frac{x_M\left(t-\left[\left|\mathbf{r}-\mathbf{r}_0\right|/c\right],\mathbf{r}_0\right)}{\left|\mathbf{r}-\mathbf{r}_0\right|}dV_0\ , \tag{1.2-2}$$

where $\varphi(t,\mathbf{r})$ is the scalar velocity potential in squared meters per second, c is the constant speed of sound in the fluid medium in meters per second, and $x_M(t,\mathbf{r})$ is the source distribution in inverse seconds. The wave equation given by (1.2-1) is referred to as being inhomogeneous because the right-hand side is nonzero. If the transmit coupling equation for the source distribution given by (1.1-8) is substituted into (1.2-2), then

$$\boxed{\varphi(t,\mathbf{r})=\int_{-\infty}^{\infty}\int_{V_0}X(f,\mathbf{r}_0)A_T(f,\mathbf{r}_0)g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)dV_0\exp(+j2\pi ft)\,df} \tag{1.2-3}$$

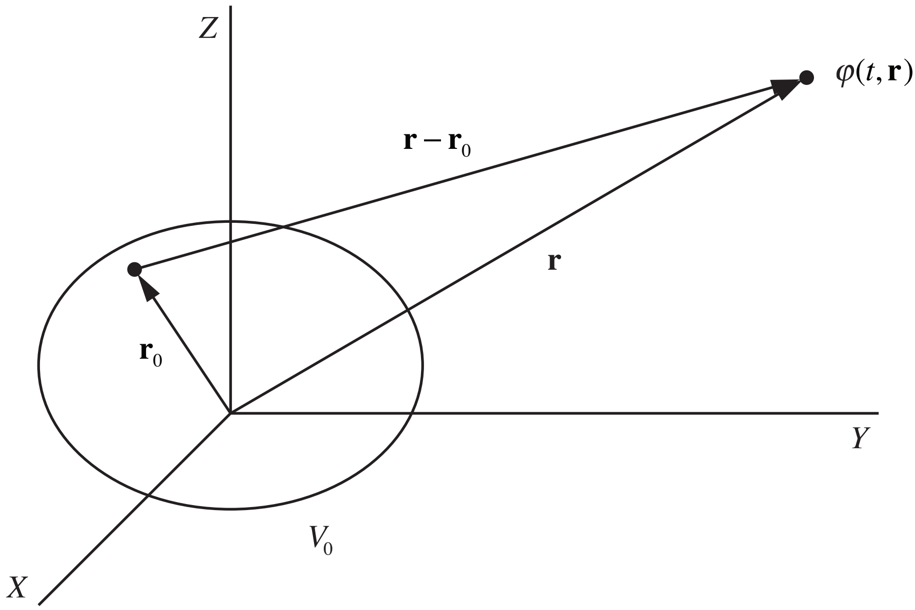

Figure 1.2-1 Closed-surface source distribution enclosing a volume V_0 .

where

$$g_f(\mathbf{r}\,|\,\mathbf{r}_0) \triangleq -\frac{\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}-\mathbf{r}_0\right|} \tag{1.2-4}$$

is the time-independent, free-space, Green's function (a.k.a. the spatial impulse response at frequency f hertz) of an unbounded, ideal (nonviscous), homogeneous, fluid medium with units of inverse meters,

$$k = 2\pi f/c = 2\pi/\lambda \tag{1.2-5}$$

is the wavenumber in radians per meter where $c = f\lambda$, λ is the wavelength in meters,

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{1.2-6}$$

is the position vector to a field point, and

$$\mathbf{r}_0 = x_0\hat{x} + y_0\hat{y} + z_0\hat{z} \tag{1.2-7}$$

is the position vector to a source point (see Fig. 1.2-1). The Green's function given by (1.2-4) is the *complex frequency response* of an unbounded, ideal (nonviscous), homogeneous, fluid medium at frequency f hertz and location $\mathbf{r}=(x,y,z)$ due to the application of a unit-amplitude impulse (i.e., a unit-amplitude, omnidirectional point-source) at location $\mathbf{r}_0=(x_0,y_0,z_0)$. The physical significance of (1.2-3) can be stated as follows: for each frequency component contained in the complex frequency spectrum of the input electrical signal, "sum" the contributions due to each source point contained in the source distribution in order to compute the total acoustic field $\varphi(t,\mathbf{r})$.

The acoustic pressure $p(t,\mathbf{r})$ in pascals (Pa) and the acoustic fluid-velocity-vector (a.k.a. the acoustic particle-velocity-vector) $\mathbf{u}(t,\mathbf{r})$ in meters per second can be obtained from the scalar velocity potential $\varphi(t,\mathbf{r})$ in squared meters per second as follows:

$$p(t,\mathbf{r})=-\rho_0(\mathbf{r})\frac{\partial}{\partial t}\varphi(t,\mathbf{r})=-\rho_0\frac{\partial}{\partial t}\varphi(t,\mathbf{r}), \tag{1.2-8}$$

where ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter and

$$\mathbf{u}(t,\mathbf{r})=\nabla\varphi(t,\mathbf{r}), \tag{1.2-9}$$

where

$$\nabla=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z} \tag{1.2-10}$$

is the gradient expressed in the rectangular coordinates (x,y,z). Note that $1\,\text{Pa}=1\,\text{N}/\text{m}^2$, where $1\,\text{N}=1\,\text{kg-m}/\text{sec}^2$. Therefore, $1\,\text{Pa}=1\,\text{kg}/(\text{m-sec}^2)$. Also note that the ambient density ρ_0 and the speed of sound c are constants because we are dealing with a homogeneous fluid medium.

The solution of the wave equation given by (1.2-3) is an *exact* expression. However, by *approximating* the time-independent, free-space, Green's function using *Fresnel* and *Fraunhofer expansions*, both the near-field and far-field beam patterns (a.k.a. directivity functions) of a transmit volume aperture can be derived, respectively, along with corresponding near-field and far-field expressions for the velocity potential and acoustic pressure. We begin the analysis by noting that the range between source and field points, $\left|\mathbf{r}-\mathbf{r}_0\right|$, appears both in the amplitude and phase of the Green's function given by (1.2-4). Since

$$\left|\mathbf{r}-\mathbf{r}_0\right|=\sqrt{(\mathbf{r}-\mathbf{r}_0)\bullet(\mathbf{r}-\mathbf{r}_0)}, \tag{1.2-11}$$

it can be shown that

$$\left|\mathbf{r}-\mathbf{r}_0\right|=r\sqrt{1+b}, \tag{1.2-12}$$

where the parameter

$$b = \left(\frac{r_0}{r}\right)^2 - 2\frac{\hat{r} \bullet \mathbf{r}_0}{r} \tag{1.2-13}$$

is dimensionless,

$$r_0 = |\mathbf{r}_0| = \sqrt{x_0^2 + y_0^2 + z_0^2} \tag{1.2-14}$$

is the range to a source point in meters,

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \tag{1.2-15}$$

is the range to a field point in meters, and

$$\hat{r} = u\hat{x} + v\hat{y} + w\hat{z} \tag{1.2-16}$$

is the dimensionless *unit vector* in the direction of $\mathbf{r}$, that is,

$$\mathbf{r} = r\hat{r}\,, \tag{1.2-17}$$

where

$$u = \cos\alpha = \sin\theta\cos\psi\,, \tag{1.2-18}$$

$$v = \cos\beta = \sin\theta\sin\psi\,, \tag{1.2-19}$$

and

$$w = \cos\gamma = \cos\theta \tag{1.2-20}$$

are dimensionless *direction cosines* with respect to the X, Y, and Z axes, respectively, expressed in terms of the spherical angles θ and ψ (see Fig. 1.2-2). Note that u, v, and w take on values between -1 and 1 since $0° \le \theta \le 180°$ and $0° \le \psi \le 360°$, and the sum of their squares is equal to one:

$$u^2 + v^2 + w^2 = 1\,. \tag{1.2-21}$$

The Green's function given by (1.2-4) shall be approximated by approximating the range $|\mathbf{r} - \mathbf{r}_0|$.

In order to approximate $|\mathbf{r} - \mathbf{r}_0|$, we shall use a *binomial expansion* of $\sqrt{1+b}$ in (1.2-12) which yields

$$\boxed{|\mathbf{r} - \mathbf{r}_0| = r\sqrt{1+b} \approx r\left(1 + \frac{b}{2} - \frac{b^2}{8} + \cdots\right), \qquad |b| < 1} \tag{1.2-22}$$

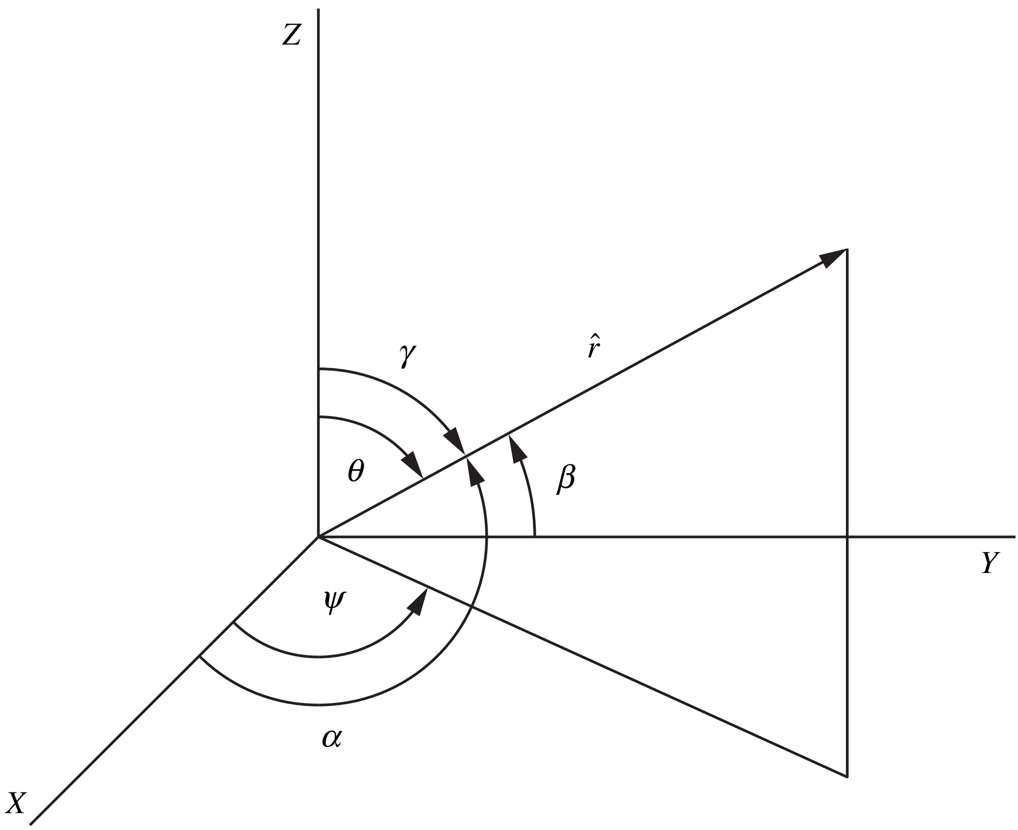

Figure 1.2-2 Unit vector $\hat{r}$ and spherical angles θ and ψ. Note that angle ψ is measured in a counter-clockwise direction.

Note that (1.2-22) is valid only if $|b| < 1$, where b is given by (1.2-13). In Appendix 1A it is shown that $|b| < 1$ whenever

$$\boxed{r > \begin{cases} r_0\left(\sqrt{1+\cos^2\phi}+\cos\phi\right), & 0° \le \phi \le 90° \\ r_0\left(\sqrt{1+\cos^2\phi}-\cos\phi\right), & 90° \le \phi \le 180° \end{cases}} \tag{1.2-23}$$

where ϕ is the angle between $\hat{r}$ and $\mathbf{r}_0$.

Let us first approximate the spherical spreading amplitude factor $1/|\mathbf{r}-\mathbf{r}_0|$. Assuming that $|b| < 1$ and using only the *first term* of the binomial expansion in (1.2-22), $|\mathbf{r}-\mathbf{r}_0| \approx r$. Therefore,

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}_0\right|} \approx \frac{1}{r}, \qquad |b|<1 . \tag{1.2-24}$$

Although using only the first term of the binomial expansion is satisfactory for approximating the amplitude factor, it is *not* satisfactory for approximating the phase factor $\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right)$ because small changes in the range $\left|\mathbf{r}-\mathbf{r}_0\right|$ can lead to large changes in phase.

In order to approximate $\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right)$, we shall use the *first three terms* of the binomial expansion as shown in (1.2-22). If all terms involving r_0 raised to powers greater than two are neglected, then (1.2-22) reduces to

$$\left|\mathbf{r}-\mathbf{r}_0\right| \approx r - \hat{r}\bullet\mathbf{r}_0 + \frac{r_0^2 - (\hat{r}\bullet\mathbf{r}_0)^2}{2r}, \qquad |b|<1 . \tag{1.2-25}$$

Therefore, by substituting (1.2-24) and (1.2-25) into (1.2-4), we obtain

$$g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) \approx -\frac{\exp(-jkr)}{4\pi r}\exp(+jk\hat{r}\bullet\mathbf{r}_0)\exp\left[-jk\frac{r_0^2-(\hat{r}\bullet\mathbf{r}_0)^2}{2r}\right], \qquad |b|<1 , \tag{1.2-26}$$

which is a *2nd-order*, *near-field approximation (expansion)* of the time-independent, free-space, Green's function involving *all* terms up to 2nd power in r_0.

The Fresnel approximation of $g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)$ can also be obtained from (1.2-22) by using only the *first two terms* of the binomial expansion instead of the first three, which is equivalent to neglecting the dot product term $(\hat{r}\bullet\mathbf{r}_0)^2$ in (1.2-25) and (1.2-26). Therefore, the *Fresnel approximation (Fresnel expansion)* of the time-independent, free-space, Green's function is given by

$$\boxed{g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) \approx -\frac{\exp(-jkr)}{4\pi r}\exp(+jk\hat{r}\bullet\mathbf{r}_0)\exp\left(-jk\frac{r_0^2}{2r}\right), \qquad |b|<1} \tag{1.2-27}$$

The Fresnel approximation of the Green's function given by (1.2-27) is *not* a true 2nd-order approximation like (1.2-26) is because it is based on neglecting the additional 2nd-order term $(\hat{r}\bullet\mathbf{r}_0)^2$.

Neglecting $(\hat{r}\bullet\mathbf{r}_0)^2$ in (1.2-26) in order to obtain (1.2-27) requires that

$$r_0^2 >> (\hat{r} \bullet \mathbf{r}_0)^2 , \tag{1.2-28}$$

or, equivalently, that

$$r_0^2 \geq K(\hat{r} \bullet \mathbf{r}_0)^2 , \tag{1.2-29}$$

where the constant $K > 1$ controls the accuracy of the Fresnel approximation. And by taking the positive square root of (1.2-29), we obtain

$$r_0 \geq \sqrt{K} \left| \hat{r} \bullet \mathbf{r}_0 \right| , \tag{1.2-30}$$

where the magnitude of the dot product is used because the range r_0 must be positive. Since we already designated ϕ as the angle between $\hat{r}$ and $\mathbf{r}_0$, where $\hat{r}$ is the unit vector in the direction of the position vector $\mathbf{r}$ to a field point, and $\mathbf{r}_0$ is the position vector to a source point, (1.2-30) reduces to

$$1 \geq \sqrt{K} \left| \cos\phi \right|, \qquad 0° < \phi < 180° . \tag{1.2-31}$$

From (1.2-31) it can be seen that the angle ϕ cannot be equal to $0°$ and $180°$ because $K > 1$. When $0° < \phi < 90°$, $\cos\phi > 0$ and $\left|\cos\phi\right| = \cos\phi$. When $90° < \phi < 180°$, $\cos\phi < 0$ and $\left|\cos\phi\right| = -\cos\phi = \cos(180° - \phi)$. Therefore, (1.2-31) can be rewritten as (see Fig. 1.2-3)

$$\phi_{\min} \leq \phi \leq \phi_{\max} , \tag{1.2-32}$$

where

$$\phi_{\min} = \cos^{-1}\left(1/\sqrt{K}\right), \qquad 0° < \phi_{\min} < 90° , \tag{1.2-33}$$

and

$$\phi_{\max} = 180° - \phi_{\min} , \qquad 90° < \phi_{\max} < 180° . \tag{1.2-34}$$

Since a factor of 10 is a typical choice to represent "much greater than," if we let $K = 10$, then (1.2-32) becomes

$$\boxed{72° \leq \phi \leq 108°} \tag{1.2-35}$$

Equation (1.2-35) is the *Fresnel angle criterion*. For example, for the problem illustrated in Fig. 1.2-3, (1.2-35) indicates that the Fresnel approximation of the Green's function is very accurate inside an angular region no greater than $18°$ from the Y axis. This is analogous to the *paraxial assumption* that is made in the derivation of the Fresnel diffraction integral in optics. However, outside this angular region (which is equivalent to $K < 10$), the accuracy of the Fresnel

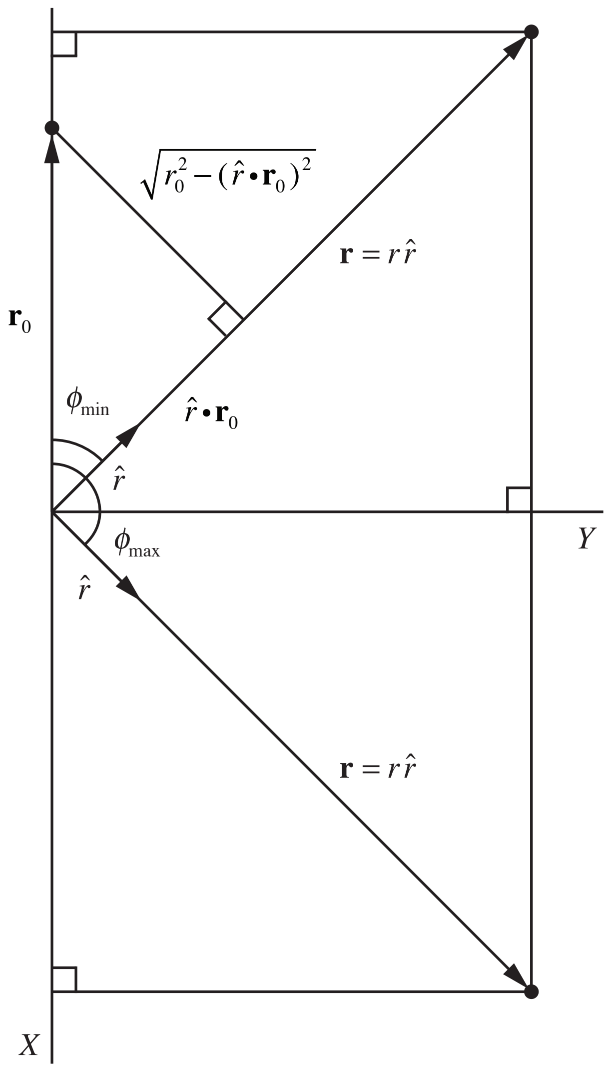

Figure 1.2-3 Illustration of the angles $\phi_{\min}$ and $\phi_{\max}$ for a source point on the negative X axis and two field points in the XY plane.

approximation decreases. Even if a source distribution is an arbitrarily shaped closed-surface, the inequality constraint given by (1.2-35) can easily be tested since $\phi = \cos^{-1}(\hat{r} \bullet \hat{r}_0)$, where the unit vector $\hat{r}_0 = \mathbf{r}_0 / r_0$.

Since we have just shown that in order for the Fresnel approximation of the Green's function to be accurate (1.2-35) must be satisfied, setting the range to a source point r_0 equal to its *maximum* value (call it R_0) and substituting either $\phi = 72°$ or $\phi = 108°$ into (1.2-23) yields

$$\boxed{r > 1.356 R_0} \tag{1.2-36}$$

Equation (1.2-36), which is based on $K = 10$, stipulates the *minimum* range that a field point has to be from an aperture in order to guarantee that $|b| < 1$ in order for the binomial expansion of $\sqrt{1+b}$ and, hence, the Fresnel approximation of the Green's function to be valid.

Next, by using the Fresnel approximation of the Green's function given by (1.2-27), we shall derive the Fraunhofer range criterion, the Fresnel range criterion, and the range to the near-field/far-field boundary. We begin by examining the quadratic phase factor

$$\exp\left(-jk\frac{r_0^2}{2r}\right)$$

that appears on the right-hand side of (1.2-27). This quadratic phase factor accurately models *wavefront curvature* in the Fresnel region of the aperture, and since $k = 2\pi/\lambda$, it can be rewritten as

$$\exp\left(-j\pi\frac{r_0^2}{\lambda r}\right).$$

We shall consider this quadratic phase factor to be *insignificant* if

$$\pi\frac{r_0^2}{\lambda r} < 1, \tag{1.2-37}$$

or

$$r > \pi r_0^2/\lambda. \tag{1.2-38}$$

In order to obtain a conservative criterion based on (1.2-38), consider the worst case, which is when the numerator is equal to its maximum value. This corresponds to setting r_0 equal to its maximum value R_0. Therefore, if

$$\boxed{r > \pi R_0^2/\lambda > 2.414 R_0} \tag{1.2-39}$$

then we shall consider the quadratic phase factor to be *insignificant* for all values of ϕ, that is, $0° \le \phi \le 180°$, since it will be approximately equal to $\exp(j0) = 1$, a constant, which indicates no phase variation as a function of r_0 and r. If (1.2-39) is satisfied, then $|b| < 1$ for $0° \le \phi \le 180°$. The factor $2.414 R_0$ was obtained by setting $r_0 = R_0$ and $\phi = 0°$ or $\phi = 180°$ in (1.2-23). Equation (1.2-39) is the *Fraunhofer (far-field) range criterion*. The Fraunhofer region of an aperture is commonly referred to as the far-field region. However, if

$$\boxed{1.356R_0 < r < \pi R_0^2/\lambda} \tag{1.2-40}$$

then we shall consider the quadratic phase factor to be *significant*, that is, a nonnegligible function of r_0 and r. Equation (1.2-40) is the *Fresnel range criterion*, where the minimum range requirement given by (1.2-36) has been incorporated into (1.2-40) (see Fig. 1.2-4). The ratio $\pi R_0^2/\lambda$, which appears in both (1.2-39) and (1.2-40), is designated as the *range to the near-field/far-field boundary*, that is, $r_{\text{NF/FF}} = \pi R_0^2/\lambda$, and depends on the ratio between an aperture's size and the wavelength. The numerator πR_0^2 is the *cross-sectional area* (in squared meters) of a transmit aperture with radius R_0 meters. The Fresnel region of an aperture, although commonly referred to as the near-field region, is actually only a *subset* of the near-field. For example, if $r < r_{\text{NF/FF}}$ but the Fresnel angle criterion given by (1.2-35) is *not* satisfied, then a field point is considered to be in the near-field region of a transmit aperture, *not* the Fresnel region.

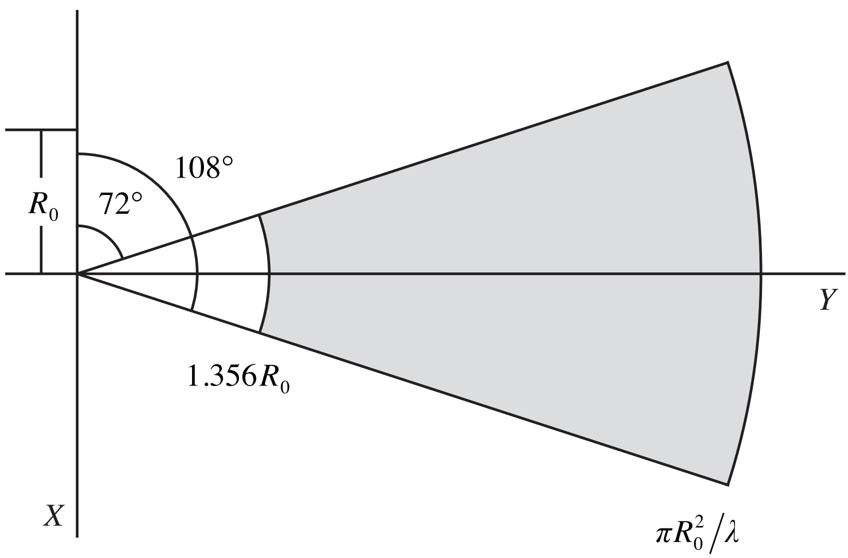

Figure 1.2-4 Illustration of the Fresnel criteria. The shaded region is the Fresnel region in the XY plane for source points lying along the X axis where $\max r_0 = R_0$ meters. Recall that the angles $72°$ and $108°$, and the factor 1.356, are based on $K = 10$.

Since we have a Fresnel approximation of the Green's function given by (1.2-27), we can proceed with the derivation of the near-field beam pattern. Substituting (1.2-27) into the exact solution of the wave equation given by (1.2-3) yields the following approximate solution:

$$\varphi(t,\mathbf{r}) \approx -\frac{1}{4\pi r}\int_{-\infty}^{\infty}\int_{V_0} X(f,\mathbf{r}_0)A_T(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\exp(+jk\hat{r}\bullet\mathbf{r}_0)\,dV_0 \times \exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \tag{1.2-41}$$

for $t \geq r/c$, where the expression $t-(r/c)$ is known as the *retarded time*. Retarded time is a measure of the amount of time that has *elapsed* since the acoustic field transmitted by a source *first appears* at a receiver located r meters from the source at time $t = r/c$ seconds. Expand the exponent of $\exp(+jk\hat{r}\bullet\mathbf{r}_0)$ next.

With the use of (1.2-5), (1.2-7), and (1.2-16), we can write that

$$\exp(+jk\hat{r}\bullet\mathbf{r}_0) = \exp\left[+j2\pi(f_X x_0 + f_Y y_0 + f_Z z_0)\right], \tag{1.2-42}$$

where

$$f_X = u/\lambda = \sin\theta\cos\psi/\lambda\,, \tag{1.2-43}$$

$$f_Y = v/\lambda = \sin\theta\sin\psi/\lambda\,, \tag{1.2-44}$$

and

$$f_Z = w/\lambda = \cos\theta/\lambda \tag{1.2-45}$$

are *spatial frequencies* in the X, Y, and Z directions, respectively, with units of cycles per meter, and u, v, and w are the dimensionless direction cosines with respect to the X, Y, and Z axes, respectively, given by (1.2-18) through (1.2-20). Note that the *propagation-vector components* k_X, k_Y, and k_Z, with units of radians per meter, are related to the spatial frequencies f_X, f_Y, and f_Z, as follows: $k_X = 2\pi f_X = ku$, $k_Y = 2\pi f_Y = kv$, and $k_Z = 2\pi f_Z = kw$, where $k = 2\pi/\lambda$ is the wavenumber. If we substitute (1.2-42) into (1.2-41) and designate the volume integral by I, then

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\mathbf{r}_0)A_T(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\times \exp\left[+j2\pi(f_X x_0 + f_Y y_0 + f_Z z_0)\right]dx_0\,dy_0\,dz_0\,. \tag{1.2-46}$$

Equation (1.2-46) can be interpreted as being a *three-dimensional spatial Fourier transform*, that is,

$$I = F_{\mathbf{r}_0}\left\{X_M(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\right\} = F_{\mathbf{r}_0}\left\{X(f,\mathbf{r}_0)\left[A_T(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\right]\right\}, \tag{1.2-47}$$

where $F_{\mathbf{r}_0}\{\bullet\}$ is shorthand notation for a three-dimensional spatial Fourier transform with respect to x_0, y_0, and z_0; that is, $F_{\mathbf{r}_0}\{\bullet\} = F_{x_0}F_{y_0}F_{z_0}\{\bullet\}$, and $X_M(f,\mathbf{r}_0) = X(f,\mathbf{r}_0)A_T(f,\mathbf{r}_0)$ [see (1.1-3)]. By inspecting the right-hand side of (1.2-47), it can be seen that the complex transmit aperture function and quadratic phase factor have intentionally been grouped together, separate from the complex frequency spectrum of the transmitted electrical signal.

It is well known from Fourier transform theory that the time-domain Fourier transform of the product of two functions of time is equal to a convolution integral in the frequency domain. Similarly, the three-dimensional spatial Fourier transform of the product of two functions of three spatial variables is equal to a three-dimensional convolution integral in the spatial-frequency domain. Therefore, (1.2-47) can be expressed as follows:

$$I = \mathcal{X}_M(f,r,\boldsymbol{\alpha}) = \mathrm{X}(f,\boldsymbol{\alpha}) \underset{\boldsymbol{\alpha}}{*} \mathcal{D}_T(f,r,\boldsymbol{\alpha}), \tag{1.2-48}$$

or

$$\mathcal{X}_M(f,r,\boldsymbol{\alpha}) = \int_{-\infty}^{\infty} \mathrm{X}(f,\boldsymbol{\nu})\mathcal{D}_T(f,r,\boldsymbol{\alpha}-\boldsymbol{\nu})\,d\boldsymbol{\nu}, \tag{1.2-49}$$

where

$$\begin{aligned}\mathcal{X}_M(f,r,\boldsymbol{\alpha}) &= F_{\mathbf{r}_0}\left\{X_M(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\right\} \\ &= \int_{-\infty}^{\infty} X_M(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_0)\,d\mathbf{r}_0\end{aligned} \tag{1.2-50}$$

is the near-field, complex frequency-and-angular spectrum of the input acoustic signal to the fluid medium (source distribution),

$$\mathrm{X}(f,\boldsymbol{\alpha}) = F_{\mathbf{r}_0}\{X(f,\mathbf{r}_0)\} = \int_{-\infty}^{\infty} X(f,\mathbf{r}_0)\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_0)\,d\mathbf{r}_0 \tag{1.2-51}$$

is the complex frequency-and-angular spectrum of the transmitted electrical signal, and

$$\boxed{\begin{aligned}\mathcal{D}_T(f,r,\boldsymbol{\alpha}) &= F_{\mathbf{r}_0}\left\{A_T(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\right\} \\ &= \int_{-\infty}^{\infty} A_T(f,\mathbf{r}_0)\exp\left(-j\frac{k}{2r}r_0^2\right)\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_0)\,d\mathbf{r}_0\end{aligned}} \tag{1.2-52}$$

is the *near-field beam pattern* (a.k.a. the *near-field directivity function*) of the transmit aperture, where

$$\boldsymbol{\alpha} = (f_X, f_Y, f_Z) \tag{1.2-53}$$

and

$$\mathbf{r}_0 = (x_0, y_0, z_0)\,. \tag{1.2-54}$$

As was discussed in Subsection 1.1.1, the complex transmit aperture function $A_T(f,\mathbf{r}_0)$ is the complex frequency response of the transmit aperture at $\mathbf{r}_0$. Since the units of $A_T(f,\mathbf{r}_0)$ are $\sec^{-1}/\text{V}$, the units of $\mathcal{D}_T(f,r,\boldsymbol{\alpha})$ are $\left(\text{m}^3/\sec\right)/\text{V}$. Equations (1.2-50) and (1.2-52) are most accurate in the Fresnel region of a transmit aperture where both the Fresnel angle criterion given by (1.2-35) and the Fresnel range criterion given by (1.2-40) are satisfied. They are less accurate in the near-field region outside the Fresnel region where $r < r_{\text{NF/FF}}$, but the Fresnel angle criterion given by (1.2-35) is *not* satisfied. The right-hand side of (1.2-49) is shorthand notation for a three-dimensional convolution integral because $\boldsymbol{\nu} = (\nu_X, \nu_Y, \nu_Z)$ is a three-dimensional vector whose components are spatial frequencies in the X, Y, and Z directions, respectively, and $d\boldsymbol{\nu} = d\nu_X\, d\nu_Y\, d\nu_Z$. The right-hand sides of (1.2-50) through (1.2-52) are also shorthand notation for triple integrals because $d\mathbf{r}_0 = dx_0\, dy_0\, dz_0$. Since the spatial frequencies f_X, f_Y, and f_Z are related to the direction cosines u, v, and w, respectively, which are related to the spherical angles θ and ψ [see (1.2-43) through (1.2-45)], the near-field beam pattern given by (1.2-52) can ultimately be expressed as a function of frequency f, the range r to a field point, and the spherical angles θ and ψ; that is, $\mathcal{D}_T(f,r,\boldsymbol{\alpha}) \rightarrow \mathcal{D}_T(f,r,\theta,\psi)$. The phrase "angular spectrum" is used because spatial frequencies can be expressed in terms of spherical angles. Just as the frequency spectrum of a signal contains information as to which frequency components contain most of the signal's energy, the angular spectrum contains information as to the directions in which most of the signal's energy is propagating. The form of (1.2-52) is also known as a *Fresnel diffraction integral* in optics or as a *three-dimensional spatial Fresnel transform*. The important functions and their units at a transmit volume aperture are summarized in Table 1B-1 in Appendix 1B.

If $\mathcal{X}_M(f,r,\boldsymbol{\alpha})$ is substituted into (1.2-41) in place of the volume integral, then the approximate solution of the wave equation can be expressed as

$$\varphi(t,r,\theta,\psi) \approx -\frac{1}{4\pi r}\int_{-\infty}^{\infty} \mathcal{X}_M(f,r,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \qquad (1.2\text{-}55)$$

for $t \geq r/c$, and if (1.2-55) is then substituted into (1.2-8), we obtain the following equation for the acoustic pressure radiated by a transmit aperture:

$$p(t,r,\theta,\psi) \approx j\frac{\rho_0}{2r}\int_{-\infty}^{\infty} f\,\mathcal{X}_M(f,r,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \qquad (1.2\text{-}56)$$

for $t \geq r/c$, where $\mathcal{X}_M(f,r,\boldsymbol{\alpha})$ is given by (1.2-49) and $\boldsymbol{\alpha}=(f_X,f_Y,f_Z)$. The function $\mathcal{X}_M(f,r,\boldsymbol{\alpha})$ can ultimately be expressed as a function of frequency f, range r, and the spherical angles θ and ψ; that is, $\mathcal{X}_M(f,r,\boldsymbol{\alpha}) \rightarrow \mathcal{X}_M(f,r,\theta,\psi)$. Although (1.2-55) and (1.2-56) are most accurate in the Fresnel region of a transmit aperture and less accurate in the near-field region outside the Fresnel region, they are very accurate everywhere in the Fraunhofer (far-field) region as well. Recall that the quadratic phase factor is insignificant in the far-field. Near-field equations can be used to evaluate acoustic fields in the far-field. However, far-field equations can only be used to evaluate acoustic fields in the far-field – they are *not* valid in the near-field.

Example 1.2-1

An *identical* input electrical signal is applied at all locations $\mathbf{r}_0$ of a transmit aperture, that is,

$$x(t,\mathbf{r}_0)=x(t), \qquad (1.2\text{-}57)$$

so that

$$X(f,\mathbf{r}_0)=X(f), \qquad (1.2\text{-}58)$$

where $X(f)$ is the complex frequency spectrum (in volts per hertz) of $x(t)$. If we decide that any electronics and digital signal processing used for purposes of amplitude shading, beam steering, etc., are to be considered as part of the aperture, then stating that an identical input electrical signal is applied at all locations of a transmit aperture is *realistic* and *true* in most practical cases.

Substituting (1.2-58) into (1.2-51) yields

$$\mathsf{X}(f,\boldsymbol{\alpha})=X(f)\int_{-\infty}^{\infty}\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_0)\,d\mathbf{r}_0 = X(f)F_{\mathbf{r}_0}\{1\}, \qquad (1.2\text{-}59)$$

and since

$$F_{\mathbf{r}_0}\{1\} = \delta(\boldsymbol{\alpha}), \tag{1.2-60}$$

the complex frequency-and-angular spectrum of the transmitted electrical signal given by (1.2-59) reduces to

$$\boxed{\mathsf{X}(f,\boldsymbol{\alpha}) = X(f)\delta(\boldsymbol{\alpha})} \tag{1.2-61}$$

where the impulse function $\delta(\boldsymbol{\alpha})$ is nonzero only when $\boldsymbol{\alpha} = \mathbf{0}$. Substituting (1.2-61) into (1.2-49) yields

$$\mathcal{X}_M(f,r,\boldsymbol{\alpha}) = X(f)\int_{-\infty}^{\infty} \delta(\boldsymbol{\nu})\mathcal{D}_T(f,r,\boldsymbol{\alpha}-\boldsymbol{\nu})\,d\boldsymbol{\nu}, \tag{1.2-62}$$

and by making use of the sifting property of impulse functions, the near-field, complex frequency-and-angular spectrum of the input acoustic signal to the fluid medium given by (1.2-62) reduces to

$$\mathcal{X}_M(f,r,\boldsymbol{\alpha}) = X(f)\mathcal{D}_T(f,r,\boldsymbol{\alpha}). \tag{1.2-63}$$

And by substituting (1.2-63) into (1.2-55) and (1.2-56), we obtain the following equations for the near-field velocity potential and acoustic pressure:

$$\varphi(t,r,\theta,\psi) \approx -\frac{1}{4\pi r}\int_{-\infty}^{\infty} X(f)\mathcal{D}_T(f,r,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \tag{1.2-64}$$

and

$$p(t,r,\theta,\psi) \approx j\frac{\rho_0}{2r}\int_{-\infty}^{\infty} fX(f)\mathcal{D}_T(f,r,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \tag{1.2-65}$$

for $t \geq r/c$, where the near-field beam pattern $\mathcal{D}_T(f,r,\boldsymbol{\alpha})$ can ultimately be expressed as a function of frequency f, the range r to a field point, and the spherical angles θ and ψ; that is, $\mathcal{D}_T(f,r,\boldsymbol{\alpha}) \rightarrow \mathcal{D}_T(f,r,\theta,\psi)$. Because an "*identical* input electrical signal" is used, the three-dimensional convolution integral equation for $\mathcal{X}_M(f,r,\boldsymbol{\alpha})$ given by (1.2-49) reduced to the simple product given by (1.2-63).

Equations (1.2-64) and (1.2-65) can be simplified further if the identical input electrical signal is *time-harmonic*, that is, if

$$x(t,\mathbf{r}_0) = x(t) = A_x \exp(+j2\pi f_0 t), \tag{1.2-66}$$

where A_x is a complex amplitude with units of volts (V), then

$$X(f,\mathbf{r}_0) = X(f) = A_x \delta(f - f_0), \tag{1.2-67}$$

where the impulse function $\delta(f - f_0)$ has units of inverse hertz. Substituting (1.2-67) into (1.2-64) and (1.2-65) yields

$$\varphi(t,r,\theta,\psi) \approx -\frac{A_x}{4\pi r}\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)\exp\left[+j2\pi f_0\left(t-\frac{r}{c}\right)\right] \tag{1.2-68}$$

and

$$p(t,r,\theta,\psi) \approx jA_x\frac{\rho_0 f_0}{2r}\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)\exp\left[+j2\pi f_0\left(t-\frac{r}{c}\right)\right], \tag{1.2-69}$$

respectively, for $t \ge r/c$, where $\boldsymbol{\alpha}_0 = (f_{X_0}, f_{Y_0}, f_{Z_0})$; $f_{X_0} = u/\lambda_0$, $f_{Y_0} = v/\lambda_0$, and $f_{Z_0} = w/\lambda_0$; $c = f_0\lambda_0$; and u, v, and w are the direction cosines given by (1.2-18) through (1.2-20), respectively. The near-field beam pattern $\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)$ can ultimately be expressed as a function of frequency f_0, the range r to a field point, and the spherical angles θ and ψ; that is, $\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0) \to \mathcal{D}_T(f_0,r,\theta,\psi)$.

We conclude this example by checking the units on the right-hand sides of both (1.2-68) and (1.2-69). Since A_x has units of V, the near-field beam pattern $\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)$ has units of $\left(\text{m}^3/\text{sec}\right)/\text{V}$, and r has units of m; the product $A_x\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)$ has units of m^3/sec, which are the units of source strength, and $A_x\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)/r$ has units of m^2/sec, which are the correct units for the scalar velocity potential $\varphi(t,r,\theta,\psi)$ given by (1.2-68). And since ρ_0 has units of kg/m^3, f_0 has units of Hz, and $A_x\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)/r$ has units of m^2/sec; $A_x\rho_0 f_0\mathcal{D}_T(f_0,r,\boldsymbol{\alpha}_0)/r$ has units of $\text{kg}/(\text{m-sec}^2) = \text{N}/\text{m}^2 = \text{Pa}$, which are the correct units for the acoustic pressure $p(t,r,\theta,\psi)$ given by (1.2-69). ■

1.2.2 Receive Aperture

In this subsection we shall derive the equation for the near-field beam pattern of a receive volume aperture by evaluating the receive coupling equation given by (1.1-15) for the special case of a sound-source (target) radiating sound in an unbounded, ideal (nonviscous), homogeneous, fluid medium. The sound-source (target) is modeled as an omnidirectional point-source located at $\mathbf{r}_S = (x_S, y_S, z_S)$ with arbitrary time dependence $s_0(t)$ (see Fig. 1.2-5). The source distribution for this case, with units of inverse seconds, is given by

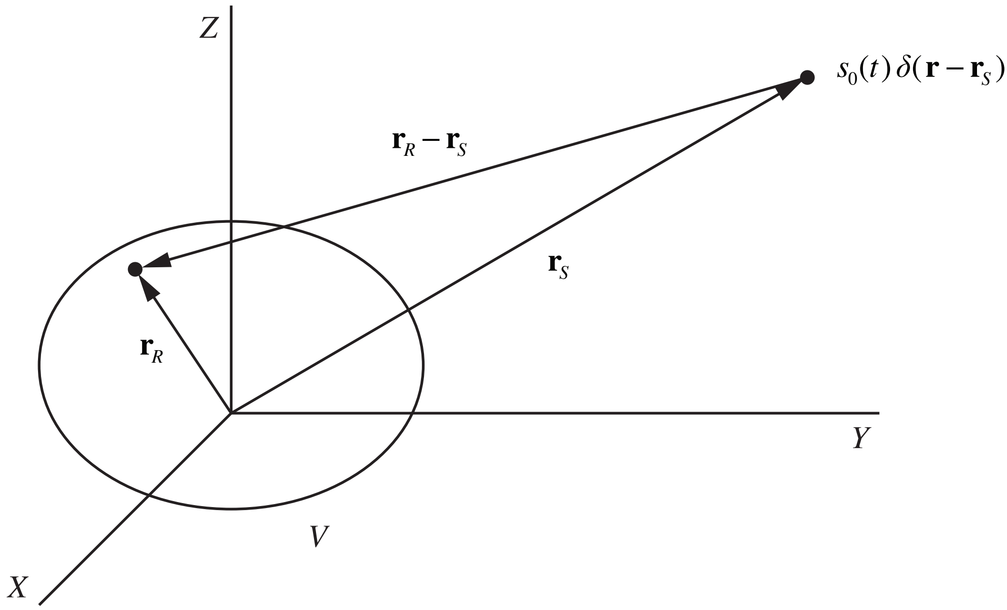

Figure 1.2-5 Omnidirectional point-source with arbitrary time dependence ensonifying a closed-surface, receive aperture enclosing a volume V .

$$x_M(t,\mathbf{r}) = s_0(t)\delta(\mathbf{r}-\mathbf{r}_S)\,, \tag{1.2-70}$$

where $s_0(t)$, $t \geq 0$, is the target's *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}_S)$, with units of inverse cubic meters, represents a unit-amplitude, omnidirectional point-source at $\mathbf{r}_S$. Substituting (1.2-70) into the exact solution of the linear wave equation given by (1.2-2) yields

$$\varphi(t,\mathbf{r}) = -\frac{1}{4\pi}\int_{V_0} \frac{s_0\Big(t-\big[\left|\mathbf{r}-\mathbf{r}_0\right|/c\big]\Big)\delta(\mathbf{r}_0-\mathbf{r}_S)}{\left|\mathbf{r}-\mathbf{r}_0\right|}\,dV_0\,, \tag{1.2-71}$$

and by using the sifting property of impulse functions and letting $y_M(t,\mathbf{r}) = \varphi(t,\mathbf{r})$, we obtain

$$y_M(t,\mathbf{r}) = -\frac{1}{4\pi R}s_0(t-\tau)\,, \tag{1.2-72}$$

for $t \geq \tau$, where

$$R = \left|\mathbf{r}-\mathbf{r}_S\right| \tag{1.2-73}$$

is the range in meters between the sound-source (target) and field point,

$$\tau = R/c \tag{1.2-74}$$

is the travel time or time delay in seconds, and c is the speed of sound in meters per second. Equation (1.2-72) is the velocity potential in squared meters per second of a *spherical wave with arbitrary time dependence*.

In order to evaluate the receive coupling equation given by (1.1-15), which is rewritten below for convenience,

$$y(t,\mathbf{r}_R) = \int_{-\infty}^{\infty} Y_M(\eta,\mathbf{r}_R) A_R(\eta,\mathbf{r}_R) \exp(+j2\pi\eta t)\, d\eta \,, \tag{1.1-15}$$

we need to evaluate (1.2-72) at $\mathbf{r} = \mathbf{r}_R$ and then take the Fourier transform with respect to t according to (1.1-13). Doing so yields

$$Y_M(\eta,\mathbf{r}_R) = S_0(\eta)\, g_\eta\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right), \tag{1.2-75}$$

where $Y_M(\eta,\mathbf{r}_R)$ is the complex frequency spectrum of the acoustic field incident upon the receive aperture at $\mathbf{r}_R = (x_R, y_R, z_R)$ with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$, $S_0(\eta)$ is the complex frequency spectrum of the target's source strength in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{Hz}$,

$$g_\eta\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_S\right|\right)}{4\pi\left|\mathbf{r}_R - \mathbf{r}_S\right|}, \tag{1.2-76}$$

is the time-independent, free-space, Green's function of an unbounded, ideal (nonviscous), homogeneous, fluid medium with units of m^{-1},

$$k = 2\pi\eta/c = 2\pi/\lambda \tag{1.2-77}$$

is the wavenumber in $\mathrm{rad/m}$ where $c = \eta\lambda$,

$$\mathbf{r}_R = x_R\hat{x} + y_R\hat{y} + z_R\hat{z} \tag{1.2-78}$$

is the position vector to an aperture point, and

$$\mathbf{r}_S = x_S\hat{x} + y_S\hat{y} + z_S\hat{z} \tag{1.2-79}$$

is the position vector to the sound-source (target) (see Fig. 1.2-5). Substituting (1.2-75) into the receive coupling equation given by (1.1-15) yields

$$\boxed{y(t,\mathbf{r}_R) = \int_{-\infty}^{\infty} S_0(\eta) A_R(\eta,\mathbf{r}_R)\, g_\eta\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right) \exp(+j2\pi\eta t)\, d\eta} \tag{1.2-80}$$

where $y(t, \mathbf{r}_R)$ is the output electrical signal from the receive aperture at time t and location $\mathbf{r}_R$ with units of $\mathrm{V/m^3}$.

The next step is to obtain the Fresnel approximation of the Green's function given by (1.2-76) and substitute it into (1.2-80). If we follow the same procedure that was used in Subsection 1.2.1, then

$$\left|\mathbf{r}_R - \mathbf{r}_S\right| = \left|\mathbf{r}_S - \mathbf{r}_R\right| = r_S\sqrt{1+b}\,, \tag{1.2-81}$$

where now the dimensionless parameter b is given by

$$b = \left(\frac{r_R}{r_S}\right)^2 - 2\frac{\hat{r}_S \bullet \mathbf{r}_R}{r_S}\,, \tag{1.2-82}$$

$$r_R = \left|\mathbf{r}_R\right| = \sqrt{x_R^2 + y_R^2 + z_R^2} \tag{1.2-83}$$

is the range to an aperture point in meters,

$$r_S = \left|\mathbf{r}_S\right| = \sqrt{x_S^2 + y_S^2 + z_S^2} \tag{1.2-84}$$

is the range to the sound-source (target) in meters, and

$$\hat{r}_S = u_S\,\hat{x} + v_S\,\hat{y} + w_S\,\hat{z} \tag{1.2-85}$$

is the dimensionless *unit vector* in the direction of $\mathbf{r}_S$, that is,

$$\mathbf{r}_S = r_S\,\hat{r}_S\,, \tag{1.2-86}$$

where

$$u_S = \cos\alpha_S = \sin\theta_S\cos\psi_S\,, \tag{1.2-87}$$

$$v_S = \cos\beta_S = \sin\theta_S\sin\psi_S\,, \tag{1.2-88}$$

and

$$w_S = \cos\gamma_S = \cos\theta_S \tag{1.2-89}$$

are dimensionless *direction cosines* with respect to the X, Y, and Z axes, respectively, expressed in terms of the spherical angles θ_S and ψ_S (see Fig. 1.2-6). Therefore, the Fresnel approximation of the Green's function is given by

$$\boxed{g_\eta\!\left(\mathbf{r}_R \,\middle|\, \mathbf{r}_S\right) \approx -\frac{\exp(-jkr_S)}{4\pi r_S}\exp(+jk\hat{r}_S \bullet \mathbf{r}_R)\exp\!\left(-jk\frac{r_R^2}{2r_S}\right), \qquad \left|b\right| < 1}$$

(1.2-90)

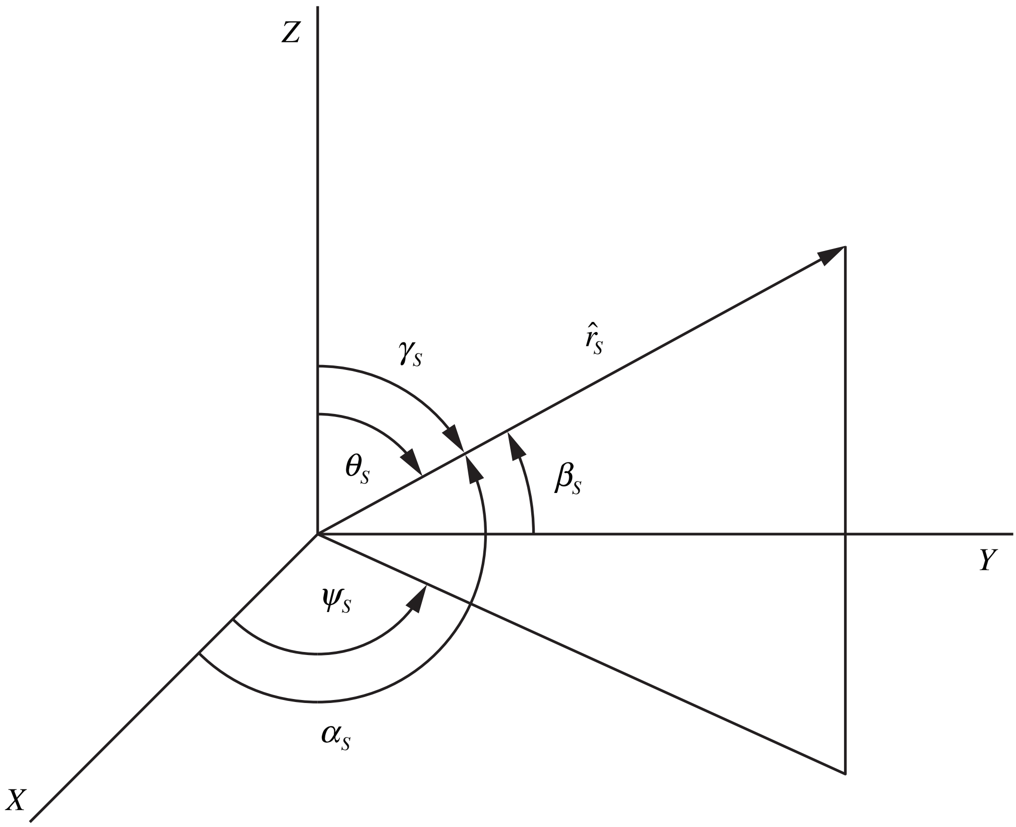

Figure 1.2-6 Unit vector $\hat{r}_S$ and spherical angles θ_S and ψ_S. Note that angle ψ_S is measured in a counter-clockwise direction.

where b is given by (1.2-82).

If we now let ϕ be the angle between $\hat{r}_S$ and $\mathbf{r}_R$, and if we follow the same procedure that was used in Subsection 1.2.1, then a sound-source (target) will be in the Fresnel region of a receive aperture if

$$\boxed{72° \le \phi \le 108°} \tag{1.2-91}$$

and

$$\boxed{1.356 R_R < r_S < \pi R_R^2 / \lambda} \tag{1.2-92}$$

where R_R is the maximum radial extent of the receive aperture, that is, $\max r_R = R_R$. Equation (1.2-91) is the *Fresnel angle criterion*. The Fresnel approximation of the Green's function will be very accurate in the angular region defined by (1.2-91). The Fresnel angle criterion given by (1.2-91) can easily be tested since $\phi = \cos^{-1}(\hat{r}_S \bullet \hat{r}_R)$, where the unit vector $\hat{r}_R = \mathbf{r}_R / r_R$. Equation (1.2-92) is the *Fresnel range criterion*. If $r_S > 1.356 R_R$, then $|b| < 1$ for ϕ satisfying (1.2-91), where b is given by (1.2-82). As a result, the binomial expansion of $\sqrt{1+b}$ in (1.2-81) and, hence, the Fresnel approximation of the Green's function given by (1.2-90) is valid. The range to the near-field/far-field boundary $r_{\text{NF/FF}} = \pi R_R^2 / \lambda$ depends on the ratio between an aperture's size and the wavelength, where πR_R^2 is the *cross-sectional area* (in squared meters) of a receive aperture with radius R_R meters. If $r_S < r_{\text{NF/FF}}$ but the Fresnel angle criterion given by (1.2-91) is *not* satisfied, then a sound-source is considered to be in the near-field region of a receive aperture, *not* the Fresnel region.

If

$$\boxed{r_S > \pi R_R^2 / \lambda > 2.414 R_R} \tag{1.2-93}$$

then a sound-source (target) will be in the Fraunhofer (far-field) region of a receive aperture. Equation (1.2-93) is the *Fraunhofer (far-field) range criterion*. If (1.2-93) is satisfied, then $|b| < 1$ for all values of ϕ, that is, $0° \le \phi \le 180°$. The factor $2.414 R_R$ was obtained by setting $r = r_S$, $r_0 = r_R = R_R$, and $\phi = 0°$ or $\phi = 180°$ in (1.2-23).

Now that we have the Fresnel approximation of the Green's function, substituting (1.2-90) into (1.2-80) yields

$$y(t, \mathbf{r}_R) \approx -\frac{1}{4\pi r_S} \int_{-\infty}^{\infty} S_0(\eta) A_R(\eta, \mathbf{r}_R) \exp\left(-j \frac{k}{2 r_S} r_R^2\right) \exp(+j2\pi \boldsymbol{\beta} \bullet \mathbf{r}_R) \times \exp\left[+j2\pi\eta\left(t - \frac{r_S}{c}\right)\right] d\eta \tag{1.2-94}$$

for $t \ge r_S / c$, where

$$\boldsymbol{\beta} = (\beta_X, \beta_Y, \beta_Z), \tag{1.2-95}$$

and

$$\beta_X = u_S / \lambda = \sin\theta_S \cos\psi_S / \lambda, \tag{1.2-96}$$

$$\beta_Y = v_S / \lambda = \sin\theta_S \sin\psi_S / \lambda, \tag{1.2-97}$$

and

$$\beta_Z = w_S / \lambda = \cos\theta_S / \lambda \tag{1.2-98}$$

are *spatial frequencies* in the X, Y, and Z directions, respectively, with units of cycles per meter, and u_S, v_S, and w_S are the dimensionless direction cosines with respect to the X, Y, and Z axes, respectively, given by (1.2-87) through (1.2-89). The *total* output electrical signal from the receive aperture (with units of volts) is given by

$$\boxed{y(t) = \int_{-\infty}^{\infty} y(t, \mathbf{r}_R)\, d\mathbf{r}_R} \tag{1.2-99}$$

and by substituting (1.2-94) into (1.2-99), we obtain

$$\boxed{y(t) \approx -\frac{1}{4\pi r_S} \int_{-\infty}^{\infty} S_0(\eta)\, \mathcal{D}_R(\eta, r_S, \boldsymbol{\beta}) \exp\left[+j2\pi\eta\left(t - \frac{r_S}{c}\right)\right] d\eta} \tag{1.2-100}$$

for $t \geq r_S/c$, where

$$\boxed{\begin{aligned} \mathcal{D}_R(\eta, r_S, \boldsymbol{\beta}) &= F_{\mathbf{r}_R}\left\{ A_R(\eta, \mathbf{r}_R) \exp\left(-j\frac{k}{2r_S} r_R^2\right)\right\} \\ &= \int_{-\infty}^{\infty} A_R(\eta, \mathbf{r}_R) \exp\left(-j\frac{k}{2r_S} r_R^2\right) \exp(+j2\pi\boldsymbol{\beta} \bullet \mathbf{r}_R)\, d\mathbf{r}_R \end{aligned}}$$

(1.2-101)

is the *near-field beam pattern* (a.k.a. the *directivity function*) of the receive aperture, where $F_{\mathbf{r}_R}\{\bullet\}$ is shorthand notation for a three-dimensional spatial Fourier transform with respect to x_R, y_R, and z_R; that is, $F_{\mathbf{r}_R}\{\bullet\} = F_{x_R} F_{y_R} F_{z_R}\{\bullet\}$, $\mathbf{r}_R = (x_R, y_R, z_R)$, and $d\mathbf{r}_R = dx_R\, dy_R\, dz_R$. As was discussed in Subsection 1.1.2, the complex receive aperture function $A_R(\eta, \mathbf{r}_R)$ is the complex frequency response of the receive aperture at $\mathbf{r}_R$. Since the units of $A_R(\eta, \mathbf{r}_R)$ are $\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^3$, the units of $\mathcal{D}_R(\eta, r_S, \boldsymbol{\beta})$ are $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$. Equations (1.2-100) and (1.2-101) are most accurate when the sound-source is in the Fresnel region of a receive aperture where both the Fresnel angle criterion given by (1.2-91) and the Fresnel range criterion given by (1.2-92) are satisfied. They are less accurate when the sound-source is in the near-field region outside the Fresnel region where $r_S < r_{\mathrm{NF/FF}}$, but the Fresnel angle criterion given by (1.2-91) is *not* satisfied. Equation (1.2-100) shows that if a sound-source (target) is in the Fresnel region of a receive aperture, then the total output electrical signal depends on the near-field beam pattern of the receive aperture, *not* the far-field beam pattern. Since the spatial frequencies β_X,

β_Y, and β_Z are related to the direction cosines u_S, v_S, and w_S, respectively, which are related to the spherical angles θ_S and ψ_S [see (1.2-96) through (1.2-98)], the near-field beam pattern given by (1.2-101) can ultimately be expressed as a function of frequency η, the range r_S to a sound-source (target), and the spherical angles θ_S and ψ_S; that is, $\mathcal{D}_R(\eta, r_S, \boldsymbol{\beta}) \rightarrow \mathcal{D}_R(\eta, r_S, \theta_S, \psi_S)$. The form of (1.2-101) is also known as a *Fresnel diffraction integral* in optics or as a *three-dimensional spatial Fresnel transform*. The important functions and their units at a receive volume aperture are summarized in Table 1B-2 in Appendix 1B.

1.3 The Far-Field Beam Pattern of a Volume Aperture

1.3.1 Transmit Aperture

If a field point is in the far-field region of a transmit aperture, that is, if (1.2-39) is satisfied, then the *Fraunhofer approximation (Fraunhofer expansion)* of the time-independent, free-space, Green's function given by (1.2-4) can be obtained from its Fresnel approximation given by (1.2-27) by neglecting the quadratic phase factor. Doing so yields the following Fraunhofer (far-field) approximation:

$$\boxed{g_f\left(\mathbf{r} \,|\, \mathbf{r}_0\right) \approx -\frac{\exp(-jkr)}{4\pi r} \exp(+jk\hat{r} \bullet \mathbf{r}_0), \qquad |b| < 1} \tag{1.3-1}$$

where b is given by (1.2-13). Recall that if (1.2-39) is satisfied, that is, if $r > \pi R_0^2 / \lambda > 2.414 R_0$, then $|b| < 1$ for all values of ϕ, that is, $0° \leq \phi \leq 180°$, where ϕ is the angle between $\hat{r}$ and $\mathbf{r}_0$ (see Fig. 1.3-1).

Similarly, the far-field beam pattern can be obtained from the near-field beam pattern given by (1.2-52) by neglecting the quadratic phase factor. Therefore, the *far-field beam pattern* (a.k.a. the *far-field directivity function*) of the transmit aperture is given by

$$\boxed{D_T(f, \boldsymbol{\alpha}) = F_{\mathbf{r}_0}\left\{A_T(f, \mathbf{r}_0)\right\} = \int_{-\infty}^{\infty} A_T(f, \mathbf{r}_0) \exp(+j2\pi \boldsymbol{\alpha} \bullet \mathbf{r}_0)\, d\mathbf{r}_0} \tag{1.3-2}$$

where $\boldsymbol{\alpha} = (f_X, f_Y, f_Z)$, $F_{\mathbf{r}_0}\{\bullet\}$ is shorthand notation for a three-dimensional spatial Fourier transform with respect to x_0, y_0, and z_0; that is, $F_{\mathbf{r}_0}\{\bullet\} = F_{x_0} F_{y_0} F_{z_0}\{\bullet\}$, $\mathbf{r}_0 = (x_0, y_0, z_0)$, and $d\mathbf{r}_0 = dx_0\, dy_0\, dz_0$. Since the units of $A_T(f, \mathbf{r}_0)$ are sec^{-1}/V, the units of $D_T(f, \boldsymbol{\alpha})$ are $\left(\text{m}^3/\text{sec}\right)/\text{V}$, which are the same units for the near-

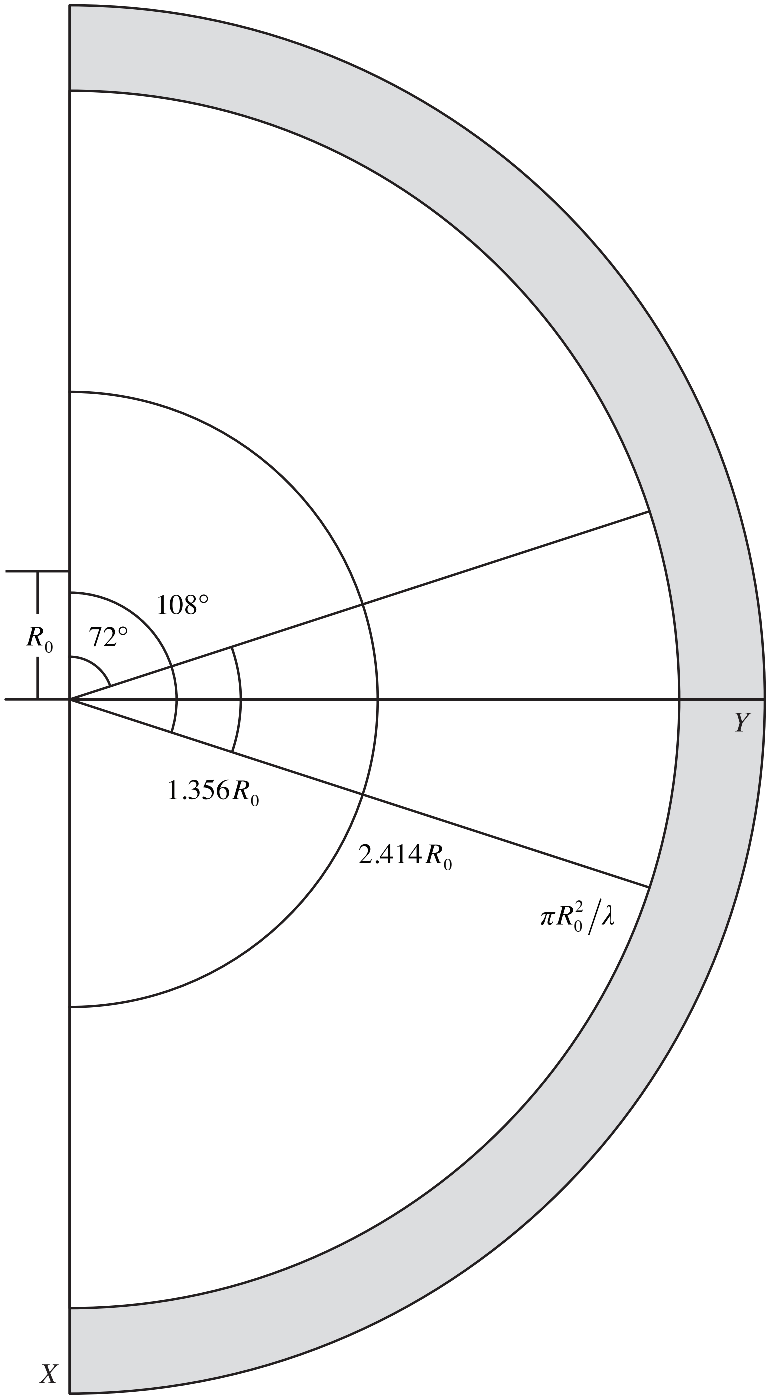

Figure 1.3-1 Illustration of the far-field criterion. The shaded region is the beginning of the Fraunhofer (far-field) region in the XY plane for source points lying along the X axis where $\max r_0 = R_0$ meters. Recall that the angles $72°$ and $108°$, and the factor 1.356, are based on $K = 10$.

field beam pattern. It also follows from (1.3-2) that

$$\boxed{A_T(f,\mathbf{r}_0)=F_{\boldsymbol{\alpha}}^{-1}\left\{D_T(f,\boldsymbol{\alpha})\right\}=\int_{-\infty}^{\infty}D_T(f,\boldsymbol{\alpha})\exp(-j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_0)\,d\boldsymbol{\alpha}} \tag{1.3-3}$$

where $F_{\boldsymbol{\alpha}}^{-1}\{\bullet\}$ is shorthand notation for a three-dimensional inverse spatial Fourier transform with respect to spatial frequencies f_X, f_Y, and f_Z; that is, $F_{\boldsymbol{\alpha}}^{-1}\{\bullet\}=F_{f_X}^{-1}F_{f_Y}^{-1}F_{f_Z}^{-1}\{\bullet\}$, and $d\boldsymbol{\alpha}=df_X\,df_Y\,df_Z$. From (1.3-2) and (1.3-3) it can be seen that *an aperture function and its far-field beam pattern form a spatial Fourier transform pair*. Note that the far-field beam pattern is *not* a function of the range r to a field point as the near-field beam pattern is. Since the spatial frequencies f_X, f_Y, and f_Z are related to the direction cosines u, v, and w, respectively, which are related to the spherical angles θ and ψ [see (1.2-43) through (1.2-45)], the far-field beam pattern given by (1.3-2) can ultimately be expressed as a function of frequency f and the spherical angles θ and ψ, that is, $D_T(f,\boldsymbol{\alpha})\rightarrow D_T(f,\theta,\psi)$. The form of (1.3-2) is also known as a *Fraunhofer diffraction integral* in optics or as a *three-dimensional spatial Fourier transform*. The important functions and their units at a transmit volume aperture are summarized in Table 1B-1 in Appendix 1B.

The far-field, complex frequency-and-angular spectrum of the input acoustic signal to the fluid medium (source distribution) can be obtained from the near-field version given by (1.2-49) by replacing the near-field beam pattern with the far-field beam pattern. Doing so yields

$$\mathsf{X}_M(f,\boldsymbol{\alpha})=\int_{-\infty}^{\infty}\mathsf{X}(f,\boldsymbol{\nu})D_T(f,\boldsymbol{\alpha}-\boldsymbol{\nu})\,d\boldsymbol{\nu}\,. \tag{1.3-4}$$

Equation (1.3-4) can also be obtained by taking the three-dimensional spatial Fourier transform of (1.1-3). Replacing $\mathcal{X}_M(f,r,\boldsymbol{\alpha})$ with $\mathsf{X}_M(f,\boldsymbol{\alpha})$ in (1.2-55) and (1.2-56) yields the following equations for the velocity potential and acoustic pressure valid in the far-field region of a transmit aperture:

$$\varphi(t,r,\theta,\psi)\approx-\frac{1}{4\pi r}\int_{-\infty}^{\infty}\mathsf{X}_M(f,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \tag{1.3-5}$$

and

$$p(t,r,\theta,\psi)\approx j\frac{\rho_0}{2r}\int_{-\infty}^{\infty}f\,\mathsf{X}_M(f,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df \tag{1.3-6}$$

for $t\geq r/c$, where $\mathsf{X}_M(f,\boldsymbol{\alpha})$ is given by (1.3-4) and $\boldsymbol{\alpha}=(f_X,f_Y,f_Z)$. The function

$\mathsf{X}_M(f,\boldsymbol{\alpha})$ can ultimately be expressed as a function of frequency f and the spherical angles θ and ψ, that is, $\mathsf{X}_M(f,\boldsymbol{\alpha}) \rightarrow \mathsf{X}_M(f,\theta,\psi)$.

Example 1.3-1

This example is a continuation of Example 1.2-1. Recall from Example 1.2-1 that when an *identical* input electrical signal is applied at all locations $\mathbf{r}_0$ of a transmit aperture, the complex frequency-and-angular spectrum of the transmitted electrical signal reduced to [see (1.2-61)]

$$\mathsf{X}(f,\boldsymbol{\alpha}) = X(f)\delta(\boldsymbol{\alpha}) . \tag{1.3-7}$$

Substituting (1.3-7) into (1.3-4) yields

$$\mathsf{X}_M(f,\boldsymbol{\alpha}) = X(f)\int_{-\infty}^{\infty}\delta(\boldsymbol{\nu})D_T(f,\boldsymbol{\alpha}-\boldsymbol{\nu})d\boldsymbol{\nu} , \tag{1.3-8}$$

and by making use of the sifting property of impulse functions, the far-field, complex frequency-and-angular spectrum of the input acoustic signal to the fluid medium given by (1.3-8) reduces to

$$\mathsf{X}_M(f,\boldsymbol{\alpha}) = X(f)D_T(f,\boldsymbol{\alpha}) . \tag{1.3-9}$$

And by substituting (1.3-9) into (1.3-5) and (1.3-6), we obtain the following equations for the velocity potential and acoustic pressure valid in the far-field region of a transmit aperture:

$$\boxed{\varphi(t,r,\theta,\psi) \approx -\frac{1}{4\pi r}\int_{-\infty}^{\infty} X(f)D_T(f,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df} \tag{1.3-10}$$

and

$$\boxed{p(t,r,\theta,\psi) \approx j\frac{\rho_0}{2r}\int_{-\infty}^{\infty} fX(f)D_T(f,\boldsymbol{\alpha})\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]df} \tag{1.3-11}$$

for $t \geq r/c$, where the far-field beam pattern $D_T(f,\boldsymbol{\alpha})$ can ultimately be expressed as a function of frequency f and the spherical angles θ and ψ, that is,

$D_T(f,\boldsymbol{\alpha}) \to D_T(f,\theta,\psi)$. Because an "*identical* input electrical signal" is used, the three-dimensional convolution integral equation for $\mathsf{X}_M(f,\boldsymbol{\alpha})$ given by (1.3-4) reduced to the simple product given by (1.3-9).

If the identical input electrical signal is time-harmonic, then it was shown in Example 1.2-1 that the complex frequency spectrum $X(f)$ reduces to [see (1.2-67)]

$$X(f) = A_x \delta(f - f_0), \tag{1.3-12}$$

where the amplitude factor A_x is complex with units of volts, and the impulse function $\delta(f - f_0)$ has units of inverse hertz. Substituting (1.3-12) into (1.3-10) and (1.3-11) yields

$$\boxed{\varphi(t,r,\theta,\psi) \approx -\frac{A_x}{4\pi r} D_T(f_0,\boldsymbol{\alpha}_0)\exp\left[+j2\pi f_0\left(t - \frac{r}{c}\right)\right]} \tag{1.3-13}$$

and

$$\boxed{p(t,r,\theta,\psi) \approx jA_x \frac{\rho_0 f_0}{2r} D_T(f_0,\boldsymbol{\alpha}_0)\exp\left[+j2\pi f_0\left(t - \frac{r}{c}\right)\right]} \tag{1.3-14}$$

respectively, for $t \geq r/c$, where $\boldsymbol{\alpha}_0 = (f_{X_0}, f_{Y_0}, f_{Z_0})$; $f_{X_0} = u/\lambda_0$, $f_{Y_0} = v/\lambda_0$, and $f_{Z_0} = w/\lambda_0$; $\lambda_0 = c/f_0$; and u, v, and w are the direction cosines given by (1.2-18) through (1.2-20), respectively. The far-field beam pattern $D_T(f_0,\boldsymbol{\alpha}_0)$ can ultimately be expressed as a function of frequency f_0 and the spherical angles θ and ψ, that is, $D_T(f_0,\boldsymbol{\alpha}_0) \to D_T(f_0,\theta,\psi)$.

We conclude this example by checking the units on the right-hand sides of both (1.3-13) and (1.3-14). Since A_x has units of V, the far-field beam pattern $D_T(f_0,\boldsymbol{\alpha}_0)$ has units of $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$, and r has units of m; the product $A_x D_T(f_0,\boldsymbol{\alpha}_0)$ has units of $\mathrm{m}^3/\mathrm{sec}$, which are the units of source strength, and $A_x D_T(f_0,\boldsymbol{\alpha}_0)/r$ has units of $\mathrm{m}^2/\mathrm{sec}$, which are the correct units for the scalar velocity potential $\varphi(t,r,\theta,\psi)$ given by (1.3-13). And since ρ_0 has units of kg/m^3, f_0 has units of Hz, and $A_x D_T(f_0,\boldsymbol{\alpha}_0)/r$ has units of $\mathrm{m}^2/\mathrm{sec}$; $A_x \rho_0 f_0 D_T(f_0,\boldsymbol{\alpha}_0)/r$ has units of $\mathrm{kg}/(\mathrm{m\text{-}sec}^2) = \mathrm{N}/\mathrm{m}^2 = \mathrm{Pa}$, which are the correct units for the acoustic pressure $p(t,r,\theta,\psi)$ given by (1.3-14). Equations (1.3-13) and (1.3-14) will be used in Section 4.1 to derive the equation for the directivity of a transmit volume aperture. ■

1.3.2 Receive Aperture

If a sound-source (target) is in the far-field region of a receive aperture, that is, if (1.2-93) is satisfied, then the *Fraunhofer approximation (Fraunhofer expansion)* of the time-independent, free-space, Green's function given by (1.2-76) can be obtained from its Fresnel approximation given by (1.2-90) by neglecting the quadratic phase factor. Doing so yields the following Fraunhofer (far-field) approximation:

$$\boxed{g_\eta\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right) \approx -\frac{\exp(-jkr_S)}{4\pi r_S}\exp(+jk\hat{r}_S \bullet \mathbf{r}_R), \qquad |b|<1} \tag{1.3-15}$$

where b is given by (1.2-82). Recall that if (1.2-93) is satisfied, that is, if $r_S > \pi R_R^2/\lambda > 2.414 R_R$, then $|b|<1$ for all values of ϕ, that is, $0° \le \phi \le 180°$, where ϕ is the angle between $\hat{r}_S$ and $\mathbf{r}_R$.

Similarly, the far-field beam pattern can be obtained from the near-field beam pattern given by (1.2-101) by neglecting the quadratic phase factor. Therefore, the *far-field beam pattern* (a.k.a. the *directivity function*) of the receive aperture is given by

$$\boxed{D_R(\eta,\boldsymbol{\beta}) = F_{\mathbf{r}_R}\left\{A_R(\eta,\mathbf{r}_R)\right\} = \int_{-\infty}^{\infty} A_R(\eta,\mathbf{r}_R)\exp(+j2\pi\boldsymbol{\beta}\bullet\mathbf{r}_R)\,d\mathbf{r}_R} \tag{1.3-16}$$

where $\boldsymbol{\beta} = (\beta_X, \beta_Y, \beta_Z)$, $F_{\mathbf{r}_R}\{\bullet\}$ is shorthand notation for a three-dimensional spatial Fourier transform with respect to x_R, y_R, and z_R; that is, $F_{\mathbf{r}_R}\{\bullet\} = F_{x_R}F_{y_R}F_{z_R}\{\bullet\}$, $\mathbf{r}_R = (x_R, y_R, z_R)$, and $d\mathbf{r}_R = dx_R\,dy_R\,dz_R$. Since the units of $A_R(\eta,\mathbf{r}_R)$ are $\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^3$, the units of $D_R(\eta,\boldsymbol{\beta})$ are $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$, which are the same units for the near-field beam pattern. It also follows from (1.3-16) that

$$\boxed{A_R(\eta,\mathbf{r}_R) = F_{\boldsymbol{\beta}}^{-1}\left\{D_R(\eta,\boldsymbol{\beta})\right\} = \int_{-\infty}^{\infty} D_R(\eta,\boldsymbol{\beta})\exp(-j2\pi\boldsymbol{\beta}\bullet\mathbf{r}_R)\,d\boldsymbol{\beta}} \tag{1.3-17}$$

where $F_{\boldsymbol{\beta}}^{-1}\{\bullet\}$ is shorthand notation for a three-dimensional inverse spatial Fourier transform with respect to spatial frequencies β_X, β_Y, and β_Z; that is, $F_{\boldsymbol{\beta}}^{-1}\{\bullet\} = F_{\beta_X}^{-1}F_{\beta_Y}^{-1}F_{\beta_Z}^{-1}\{\bullet\}$, and $d\boldsymbol{\beta} = d\beta_X\,d\beta_Y\,d\beta_Z$. From (1.3-16) and (1.3-17) it can be seen that *an aperture function and its far-field beam pattern form a spatial Fourier transform pair*. Note that the far-field beam pattern is *not* a function of the range r_S to a sound-source (target) as the near-field beam pattern is. Since the spatial frequencies β_X, β_Y, and β_Z are related to the direction cosines u_S, v_S, and w_S, respectively, which are related to the spherical angles θ_S and ψ_S [see (1.2-96) through (1.2-98)], the far-field beam pattern given by (1.3-16) can ultimately be expressed as a function of frequency η and the spherical angles θ_S and ψ_S, that is, $D_R(\eta, \boldsymbol{\beta}) \to D_R(\eta, \theta_S, \psi_S)$. The form of (1.3-16) is also known as a *Fraunhofer diffraction integral* in optics or as a *three-dimensional spatial Fourier transform*. The important functions and their units at a receive volume aperture are summarized in Table 1B-2 in Appendix 1B.

Example 1.3-2

In this example we shall derive equations for two more functions that involve the far-field beam pattern of a receive aperture. We begin with the derivation of the equation for the complex frequency-and-angular spectrum of the output electrical signal from a receive aperture. Recall that the complex frequency spectrum of the output electrical signal from a receive aperture at $\mathbf{r}_R$ is given by (1.1-11), which is rewritten below for convenience:

$$Y(\eta, \mathbf{r}_R) = Y_M(\eta, \mathbf{r}_R) A_R(\eta, \mathbf{r}_R), \tag{1.1-11}$$

where $Y_M(\eta, \mathbf{r}_R)$ is the complex frequency spectrum of the acoustic field incident upon the receive aperture at $\mathbf{r}_R$. Taking the spatial Fourier transform of (1.1-11) with respect to $\mathbf{r}_R$ yields

$$\mathrm{Y}(\eta, \boldsymbol{\beta}) = \mathrm{Y}_M(\eta, \boldsymbol{\beta}) \underset{\boldsymbol{\beta}}{*} D_R(\eta, \boldsymbol{\beta}), \tag{1.3-18}$$

or

$$\mathrm{Y}(\eta, \boldsymbol{\beta}) = \int_{-\infty}^{\infty} \mathrm{Y}_M(\eta, \boldsymbol{\gamma}) D_R(\eta, \boldsymbol{\beta} - \boldsymbol{\gamma}) \, d\boldsymbol{\gamma}, \tag{1.3-19}$$

where

$$\mathrm{Y}(\eta, \boldsymbol{\beta}) = F_{\mathbf{r}_R}\{Y(\eta, \mathbf{r}_R)\} = \int_{-\infty}^{\infty} Y(\eta, \mathbf{r}_R) \exp(+j2\pi \boldsymbol{\beta} \bullet \mathbf{r}_R) \, d\mathbf{r}_R \tag{1.3-20}$$

is the complex frequency-and-angular spectrum of the output electrical signal from a receive aperture,

$$\mathrm{Y}_M(\eta, \boldsymbol{\beta}) = F_{\mathbf{r}_R}\{Y_M(\eta, \mathbf{r}_R)\} = \int_{-\infty}^{\infty} Y_M(\eta, \mathbf{r}_R) \exp(+j2\pi \boldsymbol{\beta} \bullet \mathbf{r}_R) \, d\mathbf{r}_R \tag{1.3-21}$$

is the complex frequency-and-angular spectrum of the acoustic field incident upon the receive aperture, $D_R(\eta, \boldsymbol{\beta})$ is the far-field beam pattern of the receive aperture given by (1.3-16), $\boldsymbol{\gamma} = (\gamma_X, \gamma_Y, \gamma_Z)$ is a three-dimensional vector whose components are spatial frequencies in the X, Y, and Z directions, respectively, and $d\boldsymbol{\gamma} = d\gamma_X \, d\gamma_Y \, d\gamma_Z$. The right-hand side of (1.3-19) is shorthand notation for a three-dimensional convolution integral. Equation (1.3-19) is analogous to the far-field, complex frequency-and-angular spectrum of the input acoustic signal to the fluid medium given by (1.3-4).

The second function involves the total output electrical signal from a receive aperture. If a sound-source (target) is in the far-field region of a receive aperture, then the total output electrical signal from the receive aperture can be obtained from (1.2-100) by replacing the near-field beam pattern $\mathcal{D}_R(\eta, r_S, \boldsymbol{\beta})$ with the far-field beam pattern $D_R(\eta, \boldsymbol{\beta})$. Doing so yields

$$\boxed{y(t) \approx -\frac{1}{4\pi r_S} \int_{-\infty}^{\infty} S_0(\eta) \, D_R(\eta, \boldsymbol{\beta}) \exp\left[+j2\pi\eta\left(t - \frac{r_S}{c}\right)\right] d\eta} \tag{1.3-22}$$

for $t \geq r_S/c$, where $S_0(\eta)$ is the complex frequency spectrum of the target's source strength and $D_R(\eta, \boldsymbol{\beta}) \rightarrow D_R(\eta, \theta_S, \psi_S)$. ■

Problems

Section 1.2

1-1 Verify (1.2-12) and (1.2-13).

1-2 Using Fig. P1-2, show that

$$u = \cos\alpha = \sin\theta\cos\psi ,$$

$$v = \cos\beta = \sin\theta\sin\psi ,$$

and

$$w = \cos\gamma = \cos\theta ,$$

where u, v, and w are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively.

1-3 Verify (1.2-25).

1-4 Consider the derivation of the Fresnel expansion (Fresnel approximation) of the time-independent, free-space, Green's function $g_f(\mathbf{r} \,|\, \mathbf{r}_0)$. Compute the values for both $\phi_{\min}$ and $\phi_{\max}$ in degrees, and the corresponding *minimum* range that a field point has to be from a transmit aperture for

(a) $K = 2$

(b) $K = 5$

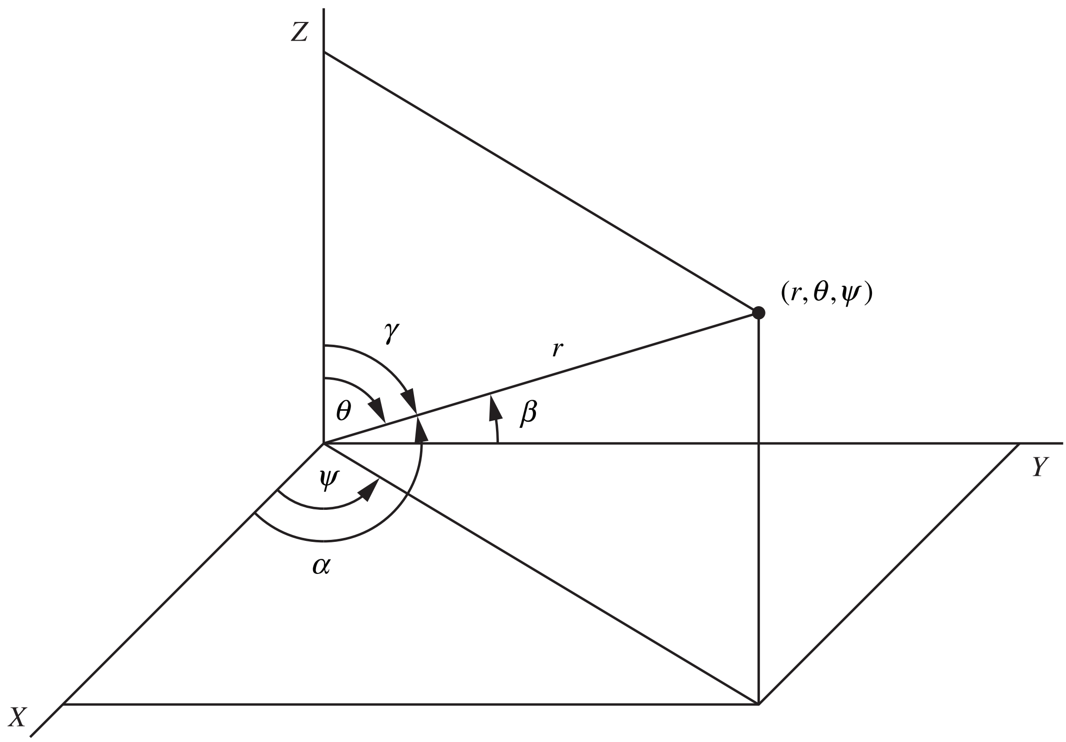

Figure P1-2

Appendix 1A

In this appendix we shall derive a criterion that will guarantee that $|b| < 1$ where [see (1.2-13)]

$$b = \left(\frac{r_0}{r}\right)^2 - 2\frac{\hat{r} \bullet \mathbf{r}_0}{r}, \tag{1A-1}$$

or, expanding the dot product in (1A-1),

$$b = \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos\phi, \tag{1A-2}$$

where ϕ is the angle between $\hat{r}$ and $\mathbf{r}_0$. Therefore, with the use of (1A-2), $|b|<1$ can be expressed as

$$\left|\frac{r_0^2}{r^2}-2\frac{r_0}{r}\cos\phi\right|<1\,. \tag{1A-3}$$

If we restrict ourselves for the moment to $90°\le\phi\le180°$, then the absolute value sign on the left-hand side of (1A-3) can be removed, that is,

$$\frac{r_0^2}{r^2}-2\frac{r_0}{r}\cos\phi<1, \qquad 90°\le\phi\le180°\,, \tag{1A-4}$$

or

$$r^2+2\,r_0\,r\cos\phi>r_0^2, \qquad 90°\le\phi\le180°\,. \tag{1A-5}$$

Completing the square on the left-hand side of (1A-5) yields

$$r^2+2\,r_0\,r\cos\phi+r_0^2\cos^2\phi>r_0^2+r_0^2\cos^2\phi, \qquad 90°\le\phi\le180°\,, \tag{1A-6}$$

$$(r+r_0\cos\phi)^2>r_0^2(1+\cos^2\phi), \qquad 90°\le\phi\le180°\,, \tag{1A-7}$$

and, finally,

$$r>r_0\left(\sqrt{1+\cos^2\phi}-\cos\phi\right), \qquad 90°\le\phi\le180°\,. \tag{1A-8}$$

For $0°\le\phi\le90°$, we substitute the trigonometric identity

$$\cos(180°-\phi)=-\cos\phi \tag{1A-9}$$

into (1A-8) which yields

$$r>r_0\left[\sqrt{1+\cos^2(180°-\phi)}+\cos(180°-\phi)\right], \qquad 90°\le\phi\le180°\,. \tag{1A-10}$$

When $\phi\le180°$, $180°-\phi\ge0°$, and when $\phi\ge90°$, $180°-\phi\le90°$. Therefore, (1A-10) is equivalent to

$$r>r_0\left(\sqrt{1+\cos^2\phi}+\cos\phi\right), \qquad 0°\le\phi\le90°\,. \tag{1A-11}$$

Combining (1A-8) and (1A-11) yields

$$\boxed{r > \begin{cases} r_0\left(\sqrt{1+\cos^2\phi}+\cos\phi\right), & 0° \le \phi \le 90° \\ r_0\left(\sqrt{1+\cos^2\phi}-\cos\phi\right), & 90° \le \phi \le 180° \end{cases}} \tag{1A-12}$$

If (1A-12) is satisfied, then $|b| < 1$.

Appendix 1B Important Functions and their Units at a Transmit and Receive Volume Aperture

Table 1B-1 Important Functions and their Units at a Transmit Volume Aperture

Function	Description	Units
$x(t, \mathbf{r}_T)$	input electrical signal to the transmit aperture at time t and location $\mathbf{r}_T = (x_T, y_T, z_T)$	V
$X(f, \mathbf{r}_T)$	complex frequency spectrum of the input electrical signal at location $\mathbf{r}_T = (x_T, y_T, z_T)$ of the transmit aperture	$\mathrm{V/Hz}$
$\alpha_T(t, \mathbf{r}_T)$	impulse response of the transmit aperture at time t and location $\mathbf{r}_T = (x_T, y_T, z_T)$	$\left(\mathrm{sec}^{-1}/\mathrm{V}\right)/\mathrm{sec}$
$A_T(f, \mathbf{r}_T)$	complex frequency response of the transmit aperture at $\mathbf{r}_T = (x_T, y_T, z_T)$, also known as the complex transmit aperture function	$\mathrm{sec}^{-1}/\mathrm{V}$
$\mathcal{D}_T(f, r, \boldsymbol{\alpha})$	near-field beam pattern (directivity function) of the transmit aperture	$\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$
$D_T(f, \boldsymbol{\alpha})$	far-field beam pattern (directivity function) of the transmit aperture	$\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$
$x_M(t, \mathbf{r}_T)$	output acoustic signal from the transmit aperture at time t and location $\mathbf{r}_T = (x_T, y_T, z_T)$, which is also the input acoustic signal to the fluid medium or source distribution	sec^{-1}
$X_M(f, \mathbf{r}_T)$	complex frequency spectrum of the input acoustic signal to the fluid medium (source distribution) at location $\mathbf{r}_T = (x_T, y_T, z_T)$ of the transmit aperture	$\mathrm{sec}^{-1}/\mathrm{Hz}$

Comments

1. the units of source strength (volume flow rate) are $\mathrm{m}^3/\mathrm{sec}$

2. $\mathrm{sec}^{-1} = \left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{m}^3$ source strength per unit volume

3. $\mathrm{sec}^{-1}/\mathrm{V} = \left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{m}^3\right)/\mathrm{V}$ source strength per unit volume, per volt