$$\left|E\left\{\tilde{l}_{Trgt}\right\}\right|^2 + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Trgt}\right\}E\left\{\tilde{l}^*_{Rev}\right\}\right\} > 0\,, \tag{12E-9}$$

which is the desired result [see (12E-2)].

Similarly, if $\alpha_0^2/2 > \alpha_1^2/2$, then

$$\frac{\left|E\left\{\tilde{l}_{Rev}\right\}\right|^2}{2\sigma_0^2} > \frac{\left|E\left\{\tilde{l}_{Trgt}\right\}\right|^2 + \left|E\left\{\tilde{l}_{Rev}\right\}\right|^2 + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Trgt}\right\}E\left\{\tilde{l}^*_{Rev}\right\}\right\}}{2\sigma_1^2}\,, \tag{12E-10}$$

or

$$\frac{\sigma_1^2}{\sigma_0^2} > 1 + \frac{\left|E\left\{\tilde{l}_{Trgt}\right\}\right|^2 + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Trgt}\right\}E\left\{\tilde{l}^*_{Rev}\right\}\right\}}{\left|E\left\{\tilde{l}_{Rev}\right\}\right|^2}\,. \tag{12E-11}$$

Since $\sigma_1^2/\sigma_0^2 > 1$ [see (12D-2)],

$$\left|E\left\{\tilde{l}_{Trgt}\right\}\right|^2 + 2\,\mathrm{Re}\left\{E\left\{\tilde{l}_{Trgt}\right\}E\left\{\tilde{l}^*_{Rev}\right\}\right\} > 0\,, \tag{12E-12}$$

which is the desired result [see (12E-2)].

Chapter 13

The Auto-Ambiguity Function and Signal Design

Transmitted electrical signals used in *active* sonar (and radar) systems are amplitude-and-angle-modulated carriers. The *auto-ambiguity function* of a transmitted electrical signal contains information about the range and Doppler resolving capabilities of the signal. As such, it is a very important function that is used to design transmit signals to satisfy range and Doppler resolution requirements and to analyze the performance of active sonar (and radar) systems.

In this chapter we shall derive and discuss the normalized, auto-ambiguity functions of two very common transmitted electrical signals used in active sonar systems, namely, the *rectangular-envelope*, *CW* (continuous-wave) *pulse* and the *rectangular-envelope*, *LFM* (linear-frequency-modulated) *pulse*. We shall then use the information obtained from the ambiguity functions to design rectangular-envelope, CW and LFM pulses to satisfy range and Doppler resolution requirements. Since the auto-ambiguity function depends on the complex envelope of a transmitted signal (see Section 12.2), different signals have different ambiguity functions and, in general, different range and Doppler resolving capabilities.

13.1 The Rectangular-Envelope CW Pulse

A CW (continuous-wave) pulse is nothing more than a finite duration, *double-sideband*, *suppressed-carrier* (DSBSC) waveform given by

$$\boxed{x(t) = a(t)\cos(2\pi f_c t)\operatorname{rect}\left[(t-0.5T)/T\right]} \tag{13.1-1}$$

where $a(t)$ is a real amplitude-modulating function, $\cos(2\pi f_c t)$ is the carrier waveform, f_c is the carrier frequency in hertz, and

$$\operatorname{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{13.1-2}$$

is the time-shifted rectangle function (see Fig. 13.1-1), where T is the pulse length in seconds. An example of a CW pulse is shown in Fig. 13.1-2. The complex envelope and envelope of (13.1-1) are given by

$$\boxed{\tilde{x}(t) = a(t)\operatorname{rect}\left[(t-0.5T)/T\right]} \tag{13.1-3}$$

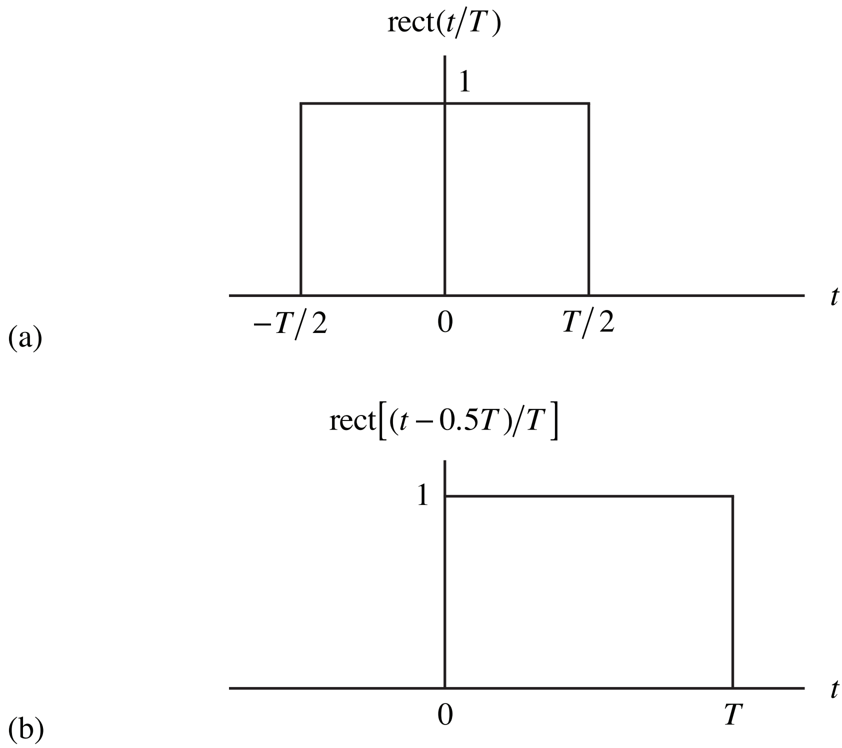

Figure 13.1-1 (a) Rectangle function and (b) time-shifted rectangle function.

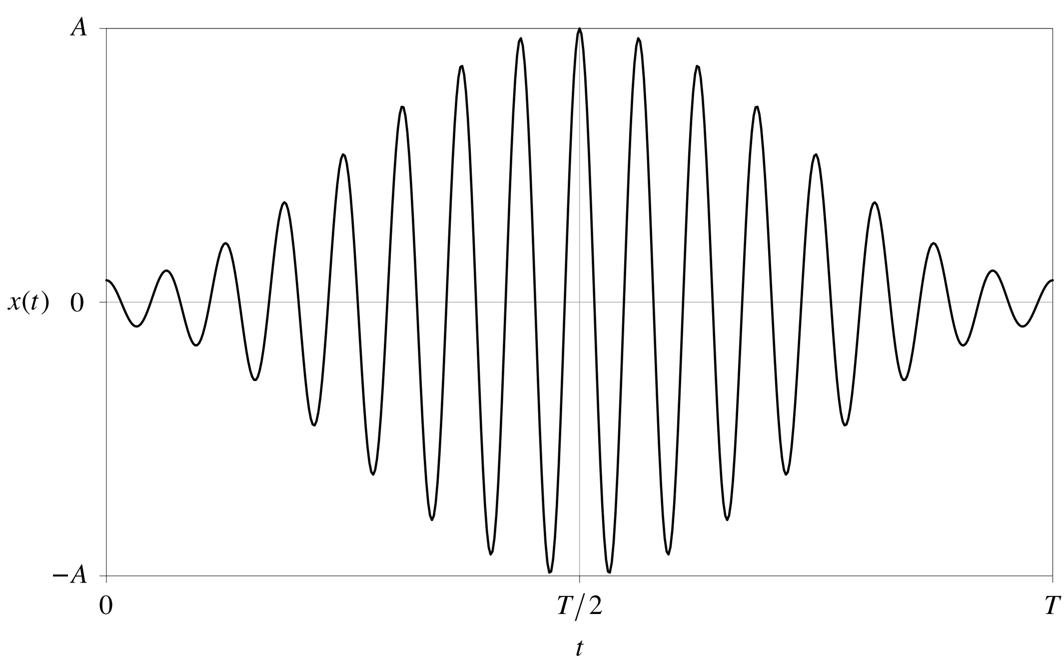

Figure 13.1-2 A CW pulse with pulse length T seconds.

and

$$\boxed{\mathcal{E}(t) = |a(t)|\operatorname{rect}[(t-0.5T)/T]} \tag{13.1-4}$$

respectively (see Section 11.2). In addition, the energies of $\tilde{x}(t)$ and $x(t)$, and the time-average power of $x(t)$ are given by

$$\boxed{E_{\tilde{x}} = \int_0^T a^2(t)\,dt} \tag{13.1-5}$$

$$\boxed{E_x = \frac{1}{2}E_{\tilde{x}} = \frac{1}{2}\int_0^T a^2(t)\,dt} \tag{13.1-6}$$

and

$$\boxed{P_{\text{avg},x} = \frac{E_x}{T} = \frac{1}{2T}\int_0^T a^2(t)\,dt} \tag{13.1-7}$$

respectively (see Section 11.2). If $\tilde{x}(t)$ and $x(t)$ are voltage signals, then $E_{\tilde{x}}$ and E_x have units of *joules-ohms*. However, if $\tilde{x}(t)$ and $x(t)$ are current signals, then $E_{\tilde{x}}$ and E_x have units of *joules per ohm* (see Subsection 11.1.1). If $x(t)$ has units of joules-ohms or joules per ohm, then $P_{\text{avg},x}$ will have units of watts-ohms or watts per ohm, respectively.

If the real amplitude-modulating function $a(t)$ is set equal to a *positive constant* A, that is, if $a(t) = A > 0$, then the CW pulse given by (13.1-1) reduces to the *rectangular-envelope*, *CW pulse*

$$\boxed{x(t) = A\cos(2\pi f_c t)\operatorname{rect}[(t-0.5T)/T]} \tag{13.1-8}$$

An example of a rectangular-envelope, CW pulse is shown in Fig. 13.1-3. The complex envelope and envelope of (13.1-8) are given by

$$\boxed{\tilde{x}(t) = A\operatorname{rect}[(t-0.5T)/T]} \tag{13.1-9}$$

and

$$\boxed{\mathcal{E}(t) = A\operatorname{rect}[(t-0.5T)/T]} \tag{13.1-10}$$

respectively [see (13.1-3) and (13.1-4)]. Since the envelope given by (13.1-10) is a rectangle, hence the terminology *rectangular-envelope*, CW pulse. The energy

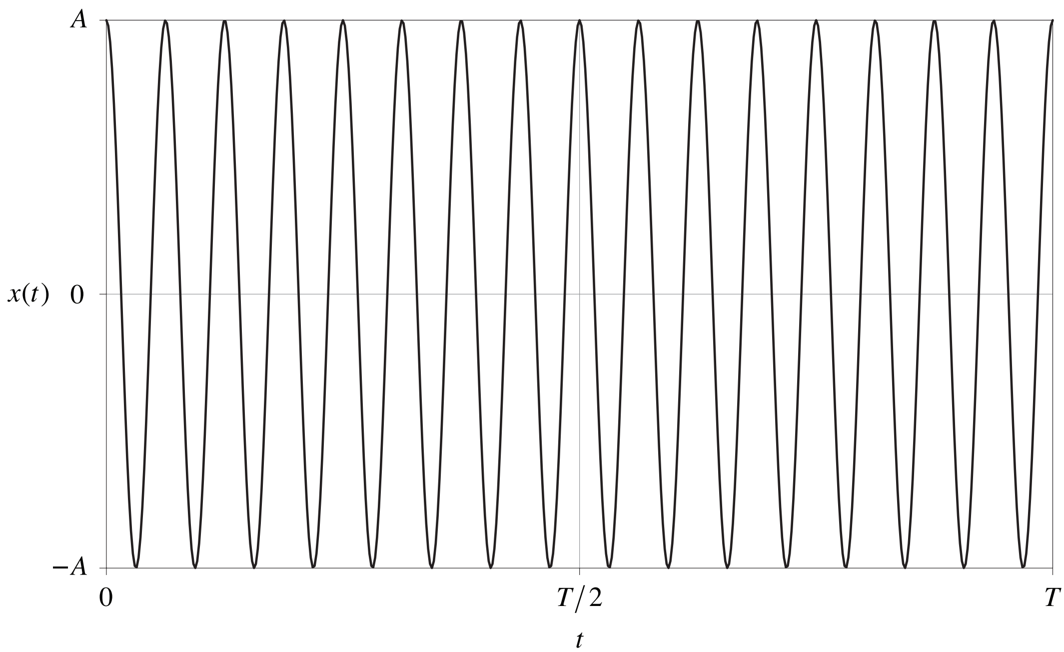

Figure 13.1-3 A rectangular-envelope, CW pulse with pulse length T seconds.

of $\tilde{x}(t)$ and $x(t)$, and the time-average power of $x(t)$ for a rectangular-envelope, CW pulse are given by

$$\boxed{E_{\tilde{x}} = A^2 T} \tag{13.1-11}$$

$$\boxed{E_x = \frac{1}{2} E_{\tilde{x}} = \frac{A^2}{2} T} \tag{13.1-12}$$

and

$$\boxed{P_{\text{avg},x} = \frac{E_x}{T} = \frac{A^2}{2}} \tag{13.1-13}$$

respectively [see (13.1-5) through (13.1-7)]. The unnormalized and normalized, auto-ambiguity functions of a rectangular-envelope, CW pulse shall be derived next.

The *unnormalized*, auto-ambiguity function of a transmitted electrical signal $x(t)$, with complex envelope $\tilde{x}(t)$, is given by (see Section 12.2)

$$\boxed{\begin{aligned} \mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) &= \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t - e_{\tau,Trgt})\exp(-j2\pi e_{D,Trgt} t)\, dt \\ &= F\left\{\tilde{x}(t)\tilde{x}^*(t - e_{\tau,Trgt})\right\} \\ &= \left\langle \tilde{x}(t), \tilde{x}(t - e_{\tau,Trgt})\exp(+j2\pi e_{D,Trgt} t)\right\rangle \end{aligned}} \tag{13.1-14}$$

where

$$\boxed{e_{\tau,Trgt} = \hat{\tau}_{Trgt} - \tau_{Trgt}} \tag{13.1-15}$$

is the *round-trip time-delay estimation error of the target* in seconds, where $\hat{\tau}_{Trgt}$ is the estimate of τ_{Trgt}, and

$$\boxed{e_{D,Trgt} = \hat{\eta}_{D,Trgt} - \eta_{D,Trgt}} \tag{13.1-16}$$

is the *Doppler-shift estimation error of the target* in hertz, where $\hat{\eta}_{D,Trgt}$ is the estimate of $\eta_{D,Trgt}$. Two other very important functions related to the auto-ambiguity function are the range and Doppler profiles. By setting $e_{D,Trgt} = 0$ in (13.1-14), we obtain the *unnormalized*, *round-trip time-delay* (*range*) *profile*:

$$\boxed{\begin{aligned} \mathcal{X}(e_{\tau,Trgt}, 0) &= \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t - e_{\tau,Trgt})\, dt \\ &= \tilde{x}(e_{\tau,Trgt}) * \tilde{x}^*(-e_{\tau,Trgt}) \\ &= F^{-1}\left\{\tilde{X}(f)\tilde{X}^*(f)\right\} \end{aligned}} \tag{13.1-17}$$

where

$$\tilde{x}(e_{\tau,Trgt}) \leftrightarrow \tilde{X}(f) \tag{13.1-18}$$

and

$$\tilde{x}^*(-e_{\tau,Trgt}) \leftrightarrow \tilde{X}^*(f)\,. \tag{13.1-19}$$

The range profile $\mathcal{X}(e_{\tau,Trgt}, 0)$ given by (13.1-17) is the *time-domain autocorrelation function* of $\tilde{x}(t)$. Furthermore, $\mathcal{X}(e_{\tau,Trgt}, 0) = F^{-1}\left\{\left|\tilde{X}(f)\right|^2\right\}$. Similarly, by setting $e_{\tau,Trgt} = 0$ in (13.1-14), we obtain the *unnormalized*, *Doppler profile*:

$$\boxed{\begin{aligned} \mathcal{X}(0, e_{D,Trgt}) &= \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t)\exp(-j2\pi e_{D,Trgt}t)\,dt \\ &= F\left\{\tilde{x}(t)\tilde{x}^*(t)\right\} \\ &= \tilde{X}(e_{D,Trgt}) * \tilde{X}^*(-e_{D,Trgt}) \\ &= \int_{-\infty}^{\infty} \tilde{X}(f)\tilde{X}^*(f - e_{D,Trgt})\,df \end{aligned}} \tag{13.1-20}$$

where

$$\tilde{x}(t) \leftrightarrow \tilde{X}(e_{D,Trgt}) \tag{13.1-21}$$

and

$$\tilde{x}^*(t) \leftrightarrow \tilde{X}^*(-e_{D,Trgt})\,. \tag{13.1-22}$$

The Doppler profile $\mathcal{X}(0, e_{D,Trgt})$ given by (13.1-20) is the *frequency-domain autocorrelation function* of $\tilde{X}(f)$. Furthermore, $\mathcal{X}(0, e_{D,Trgt}) = F\left\{\left|\tilde{x}(t)\right|^2\right\} = F\left\{a^2(t)\right\}$. As we shall show later, both the range and Doppler profiles contain very important information that is pertinent to signal design.

In order to derive the unnormalized, auto-ambiguity function of a rectangular-envelope, CW pulse, we start by substituting its complex envelope given by (13.1-9) into (13.1-14). Doing so yields

$$\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{t - 0.5T}{T}\right) \operatorname{rect}\left[\frac{t - (0.5T + e_{\tau,Trgt})}{T}\right] \times \exp(-j2\pi e_{D,Trgt}t)\,dt, \tag{13.1-23}$$

or

$$\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 F\left\{\operatorname{rect}\left(\frac{t - 0.5T}{T}\right) \operatorname{rect}\left[\frac{t - (0.5T + e_{\tau,Trgt})}{T}\right]\right\}. \tag{13.1-24}$$

By referring to Fig. 13.1-4, it can be seen that for $0 \le e_{\tau,Trgt} \le T$, (13.1-23) reduces to

$$\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 \int_{e_{\tau,Trgt}}^{T} \exp(-j2\pi e_{D,Trgt}t)\,dt, \qquad 0 \le e_{\tau,Trgt} \le T\,, \tag{13.1-25}$$

which simplifies to

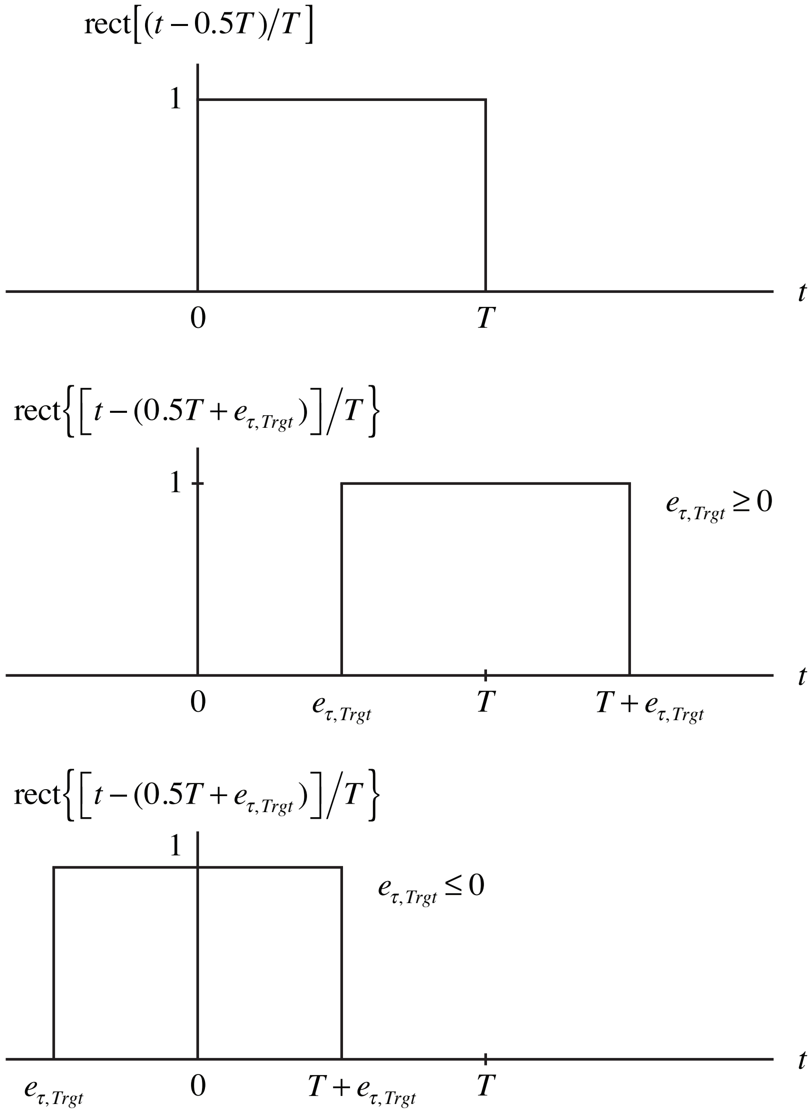

Figure 13.1-4 Time-shifted rectangle functions.

$$\begin{aligned}\mathcal{X}(e_{\tau,Trgt},e_{D,Trgt}) = A^2(T-e_{\tau,Trgt})\text{sinc}\left[e_{D,Trgt}(T-e_{\tau,Trgt})\right]\times \\ \exp\left[-j\pi e_{D,Trgt}(T+e_{\tau,Trgt})\right], \qquad 0\leq e_{\tau,Trgt}\leq T.\end{aligned} \tag{13.1-26}$$

Similarly, for $-T\leq e_{\tau,Trgt}\leq 0$, (13.1-23) reduces to

$$\mathcal{X}(e_{\tau,Trgt},e_{D,Trgt}) = A^2\int_0^{T+e_{\tau,Trgt}}\exp(-j2\pi e_{D,Trgt}t)\,dt, \qquad -T\leq e_{\tau,Trgt}\leq 0, \tag{13.1-27}$$

which simplifies to

$$\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) = A^2(T + e_{\tau,Trgt})\operatorname{sinc}\left[e_{D,Trgt}(T + e_{\tau,Trgt})\right]\times$$
$$\exp\left[-j\pi e_{D,Trgt}(T + e_{\tau,Trgt})\right], \qquad -T \le e_{\tau,Trgt} \le 0. \tag{13.1-28}$$

Equations (13.1-26) and (13.1-28) can be combined into the following single equation:

$$\mathcal{X}(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 T\left[1 - \frac{\left|e_{\tau,Trgt}\right|}{T}\right]\operatorname{sinc}\left\{e_{D,Trgt}\left[T - \left|e_{\tau,Trgt}\right|\right]\right\}\times$$
$$\exp\left[-j\pi e_{D,Trgt}(T + e_{\tau,Trgt})\right], \qquad -T \le e_{\tau,Trgt} \le T. \tag{13.1-29}$$

Equation (13.1-29) is the *unnormalized*, auto-ambiguity function of a rectangular-envelope, CW pulse.

In order to normalize an unnormalized, auto-ambiguity function, we need to divide it by the energy of the complex envelope of the transmitted electrical signal (see Section 12.2). Therefore, dividing (13.1-29) by (13.1-11) yields

$$\boxed{\begin{aligned}\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt}) = \left[1 - \frac{\left|e_{\tau,Trgt}\right|}{T}\right]&\operatorname{sinc}\left\{e_{D,Trgt}\left[T - \left|e_{\tau,Trgt}\right|\right]\right\}\times\\ &\exp\left[-j\pi e_{D,Trgt}(T + e_{\tau,Trgt})\right], \qquad -T \le e_{\tau,Trgt} \le T\end{aligned}} \tag{13.1-30}$$

which is the *normalized*, *auto-ambiguity function of a rectangular-envelope*, *CW pulse*. By setting $e_{D,Trgt} = 0$ in (13.1-30) and taking the magnitude of the resulting expression, we obtain the magnitude of the *normalized*, *round-trip time-delay* (*range*) *profile*:

$$\boxed{\left|\mathcal{X}_N(e_{\tau,Trgt}, 0)\right| = 1 - \frac{\left|e_{\tau,Trgt}\right|}{T}, \qquad -T \le e_{\tau,Trgt} \le T} \tag{13.1-31}$$

since $\operatorname{sinc}(0) = 1$. Equation (13.1-31) is the equation of a triangle (see Fig. 13.1-5). Similarly, by setting $e_{\tau,Trgt} = 0$ in (13.1-30) and taking the magnitude of the resulting expression, we obtain the magnitude of the *normalized*, *Doppler profile* (see Fig. 13.1-6):

$$\boxed{\left|\mathcal{X}_N(0, e_{D,Trgt})\right| = \left|\mathrm{sinc}(e_{D,Trgt}\,T)\right|} \tag{13.1-32}$$

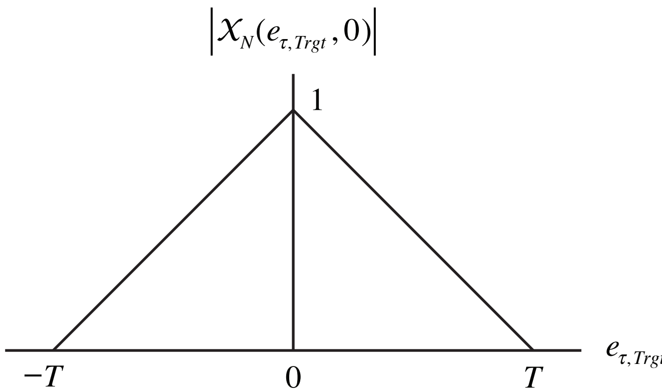

Figure 13.1-5 Magnitude of the normalized, round-trip time-delay (range) profile of a rectangular-envelope, CW pulse with pulse length T seconds.

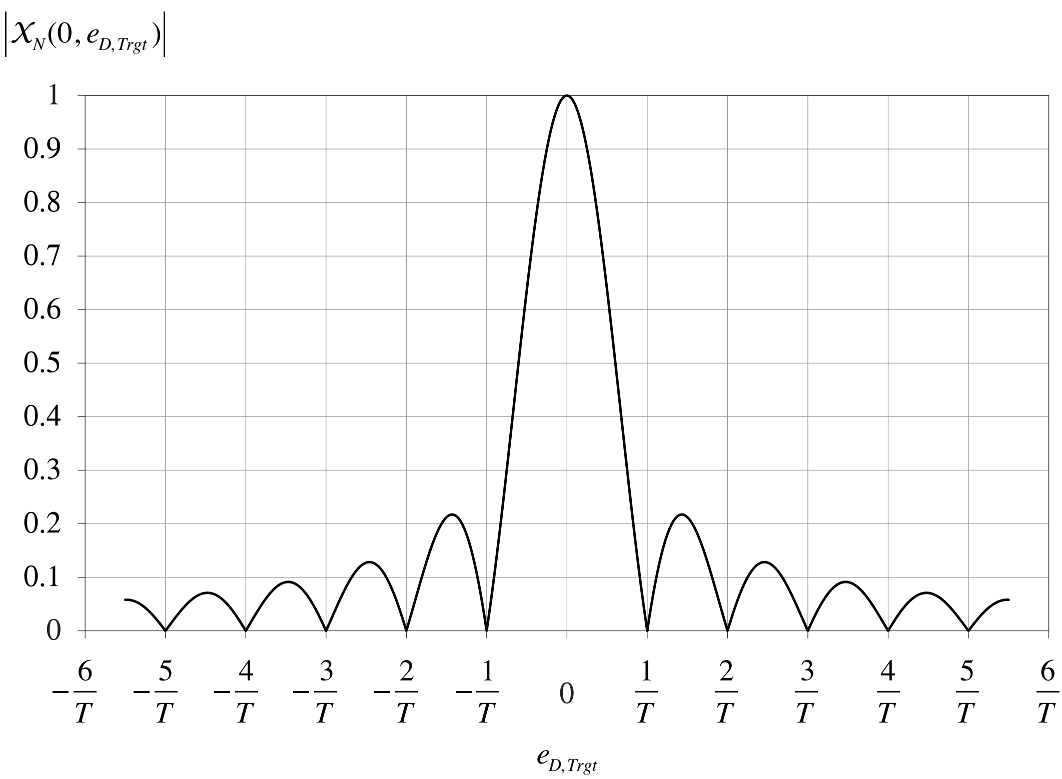

Figure 13.1-6 Magnitude of the normalized, Doppler profile of a rectangular-envelope, CW pulse with pulse length T seconds.

By inspecting the range profile given by (13.1-31) and referring to Fig. 13.1-5, it can be seen that the width of the mainlobe of the ambiguity function along the $e_{\tau,Trgt}$ axis is *directly proportional to the pulse length* T since the range profile decreases to zero at $e_{\tau,Trgt} = \pm T$ seconds. However, by inspecting the Doppler profile given by (13.1-32) and referring to Fig. 13.1-6, it can be seen that the width of the mainlobe of the ambiguity function along the $e_{D,Trgt}$ axis is *inversely proportional to the pulse length* T since the locations of the *first* zero crossings of the Doppler profile are at $e_{D,Trgt} = \pm 1/T$ hertz. A single parameter, the pulse length T, controls both the round-trip time-delay (range) and Doppler resolving capabilities of a rectangular-envelope, CW pulse. One can make the width of the mainlobe of the ambiguity function arbitrarily small in either direction by varying T, but *not* in both directions simultaneously. As a result, *a short duration rectangular-envelope, CW pulse has good range resolution but poor Doppler resolution, and a long duration rectangular-envelope, CW pulse has poor range resolution but good Doppler resolution.*

Besides wanting, in general, an ambiguity function to have as narrow a mainlobe about the origin as possible, it is also desirable that an ambiguity function have sidelobe levels as low as possible. The sidelobe levels of an ambiguity function can be reduced by using an amplitude-modulating function $a(t)$ other than rectangular (i.e., a constant), analogous to using different amplitude weights to reduce the sidelobe levels of the far-field beam pattern of an array. In fact, one can use time-domain versions of the same continuous amplitude windows discussed in Section 2.2 for $a(t)$. However, as with far-field beam patterns, there is a trade-off, since the width of the mainlobe of an ambiguity function generally increases as the sidelobe levels decrease.

Example 13.1-1 Design of a Rectangular-Envelope, CW Pulse

A rectangular-envelope, CW pulse cannot provide good range and Doppler resolution simultaneously. Therefore, we shall first design a rectangular-envelope, CW pulse to satisfy a range-resolution requirement and then compute the resulting Doppler resolution. In general, in signal-design problems, satisfying a range-resolution requirement is of primary concern. We will then reverse the problem, that is, we will design a rectangular-envelope, CW pulse to satisfy a Doppler-resolution requirement and then compute the resulting range resolution. We will also estimate the bandwidth of the resulting pulses in both cases and solve for the amplitude of a rectangular-envelope, CW pulse that is needed to satisfy a time-average-power requirement or constraint. Being able to estimate the bandwidth of a transmitted electrical signal is important because one needs to know if the sensitivity functions of the electroacoustic transducers in a sonar array have sufficient bandwidth so as not to filter out any important frequency components in the input electrical signals to the transducers.

We begin the design procedure by introducing the *range estimation error of the target* in meters given by

$$\boxed{e_{r,Trgt} = \hat{r}_{Trgt} - r_{Trgt}} \tag{13.1-33}$$

where $\hat{r}_{Trgt}$ is the estimate of the range to the target r_{Trgt}. The round-trip time-delay estimation error $e_{\tau,Trgt}$ is related to the range estimation error $e_{r,Trgt}$ by

$$e_{\tau,Trgt} = 2\,e_{r,Trgt}/c\,, \tag{13.1-34}$$

where c is the constant speed of sound in meters per second. As can be seen from (13.1-34), an error in estimating the range to a target will result in an error in estimating the round-trip time delay, and vice-versa. Since estimation errors can be either positive or negative in value, we take the magnitude of (13.1-34) to obtain

$$\left|e_{\tau,Trgt}\right| = 2\left|e_{r,Trgt}\right|/c\,. \tag{13.1-35}$$

Solving for $\left|e_{r,Trgt}\right|$ yields

$$\left|e_{r,Trgt}\right| = c\left|e_{\tau,Trgt}\right|/2\,. \tag{13.1-36}$$

In signal design, the widths of the mainlobe of an auto-ambiguity function along the $e_{\tau,Trgt}$ and $e_{D,Trgt}$ axes are used as measures of the range and Doppler resolutions, respectively, of a transmitted electrical signal. For a rectangular-envelope, CW pulse, the round-trip time-delay (range) profile decreases to zero at [see (13.1-31)]

$$e_{\tau,Trgt} = \pm T\,, \tag{13.1-37}$$

and the locations of the first zero crossings of the Doppler profile are at [see (13.1-32)]

$$e_{D,Trgt} = \pm 1/T\,, \tag{13.1-38}$$

where T is the pulse length in seconds. Therefore, from (13.1-37),

$$\left|e_{\tau,Trgt}\right| = T\,, \tag{13.1-39}$$

and from (13.1-38),

$$\left|e_{D,Trgt}\right| = 1/T\,. \tag{13.1-40}$$

Let us now design a rectangular-envelope, CW pulse to satisfy a range-resolution requirement.

Specifying a value for range resolution is equivalent to specifying a desired value for the magnitude of the range estimation error of the target $\left|e_{r,Trgt}\right|$ that corresponds to the location of the first zero crossing along the positive $e_{\tau,Trgt}$ axis. If (13.1-39) is substituted into (13.1-35), then

$$\boxed{T = 2\left|e_{r,Trgt}\right|/c} \tag{13.1-41}$$

Equation (13.1-41) is used to compute the pulse length T in seconds that is required for a rectangular-envelope, CW pulse to provide a range resolution equal to $\left|e_{r,Trgt}\right|$ meters. The resulting Doppler resolution is obtained by substituting the value of T from (13.1-41) into (13.1-40). For example, if $\left|e_{r,Trgt}\right| = 1\text{ m}$ ($e_{r,Trgt} = \pm 1\text{ m}$) and $c = 1500\text{ m/sec}$, then $T = 1.333\text{ msec}$ and $\left|e_{D,Trgt}\right| = 750\text{ Hz}$ ($e_{D,Trgt} = \pm 750\text{ Hz}$). Also, if $\left|e_{r,Trgt}\right| = 1\text{ m}$, then $\left|e_{\tau,Trgt}\right| = 1.333\text{ msec}$ [see (13.1-39)]. The magnitude of the normalized, auto-ambiguity function $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ of this signal is shown in Fig. 13.1-7. As can be seen from the figure, if the round-trip time-delay estimation error $e_{\tau,Trgt}$ is zero or very small, $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ remains "big" even if the Doppler estimation error $e_{D,Trgt}$ is "big". For example, from Fig. 13.1-7, $\left|\chi_N(0, \pm 400\text{ Hz})\right| \approx 0.7$. This waveform is an example of a *Doppler-tolerant* waveform, that is, a waveform that is tolerant of Doppler estimation errors. In other words, if a very good estimate of the range to a target is made so that $e_{\tau,Trgt}$ is zero or very small, then $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ remains "big" even if $e_{D,Trgt}$ is "big". As a result, the target can still be detected. Recall that the numerator of the signal-to-interference ratio (SIR) is directly proportional to the magnitude-squared of the normalized, auto-ambiguity function (see Section 12.2). Therefore, as $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right| \to 0$, the $\text{SIR} \to 0$ and, as a result, the target won't be detected.

Before we estimate the bandwidth of a rectangular-envelope, CW pulse, let us state the following two relationships that pertain to all amplitude-and-angle-modulated carriers:

$$\boxed{f_c > \text{BW}_{\tilde{x}}} \tag{13.1-42}$$

and

$$\boxed{\text{BW}_x = 2\,\text{BW}_{\tilde{x}}} \tag{13.1-43}$$

where f_c is the carrier frequency in hertz, $\text{BW}_{\tilde{x}}$ is the bandwidth in hertz of the

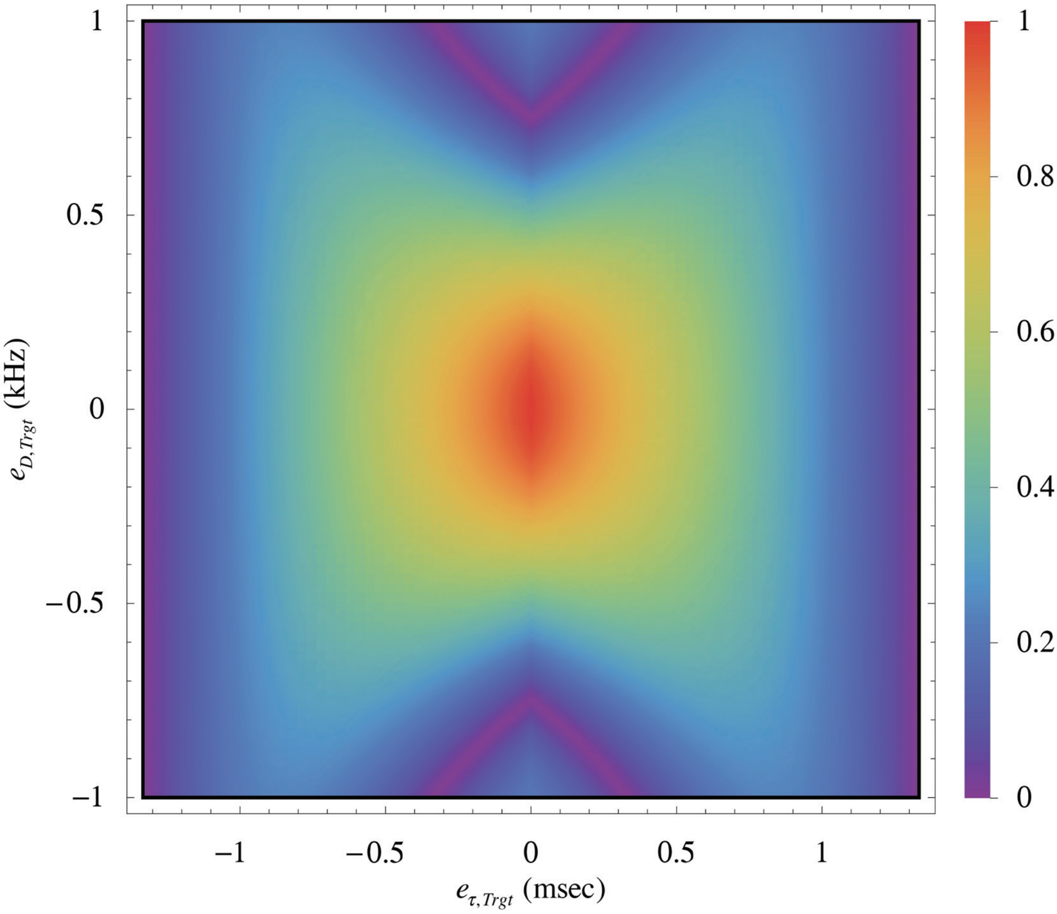

Figure 13.1-7 Magnitude of the normalized, auto-ambiguity function of a rectangular-envelope, CW pulse with pulse length $T = 1.333$ msec .

lowpass (baseband) complex envelope $\tilde{x}(t)$, and BW_x is the bandwidth in hertz of the bandpass, amplitude-and-angle-modulated carrier $x(t)$. As can be seen from (13.1-42), $\mathrm{BW}_{\tilde{x}}$ is the lower bound for determining a value for the carrier frequency f_c. The inequality in (13.1-42) must be satisfied in order to avoid signal distortion.

The complex envelope of a rectangular-envelope, CW pulse is given by (13.1-9) with unnormalized and normalized Fourier transforms

$$\tilde{X}(f) = AT\,\mathrm{sinc}(fT)\exp(-j\pi fT) \tag{13.1-44}$$

and

$$\tilde{X}_N(f) = \mathrm{sinc}(fT)\exp(-j\pi fT), \tag{13.1-45}$$

respectively. Since

$$\max\left|\tilde{X}_N(f)\right| = 1, \qquad f = 0, \tag{13.1-46}$$

and

$$\left|\tilde{X}_N(f)\right| < 0.1, \qquad f > 3/T, \tag{13.1-47}$$

a conservative estimate of the bandwidth in hertz of the complex envelope of a rectangular-envelope, CW pulse is given by (see Fig. 13.1-8)

$$\boxed{\mathrm{BW}_{\tilde{x}} \approx 5/T} \tag{13.1-48}$$

Substituting (13.1-48) into (13.1-42) and (13.1-43) yields

$$\boxed{f_c > 5/T} \tag{13.1-49}$$

and

$$\boxed{\mathrm{BW}_x \approx 10/T} \tag{13.1-50}$$

for a rectangular-envelope, CW pulse. For example, for the previously designed rectangular-envelope, CW pulse with pulse length $T = 1.333$ msec; $\mathrm{BW}_{\tilde{x}} \approx 3750$ Hz, $f_c > 3750$ Hz, and $\mathrm{BW}_x \approx 7500$ Hz.

Next we shall design a rectangular-envelope, CW pulse to satisfy a Doppler-resolution requirement. Specifying a value for Doppler resolution is equivalent to specifying a desired value for the magnitude of the Doppler estimation error of the target $\left|e_{D,Trgt}\right|$ that corresponds to the location of the first zero crossing along the positive $e_{D,Trgt}$ axis. Solving for T from (13.1-40) yields

$$\boxed{T = 1\Big/\left|e_{D,Trgt}\right|} \tag{13.1-51}$$

and by substituting (13.1-39) into (13.1-36), we obtain

$$\left|e_{r,Trgt}\right| = cT/2. \tag{13.1-52}$$

Equation (13.1-51) is used to compute the pulse length T in seconds that is required for a rectangular-envelope, CW pulse to provide a Doppler resolution equal to $\left|e_{D,Trgt}\right|$ hertz. The resulting range resolution is obtained by substituting the value of T from (13.1-51) into (13.1-52). For example, if $\left|e_{D,Trgt}\right| = 1$ Hz ($e_{D,Trgt} = \pm 1$ Hz) and $c = 1500$ m/sec, then $T = 1$ sec, $\left|e_{r,Trgt}\right| = 750$ m

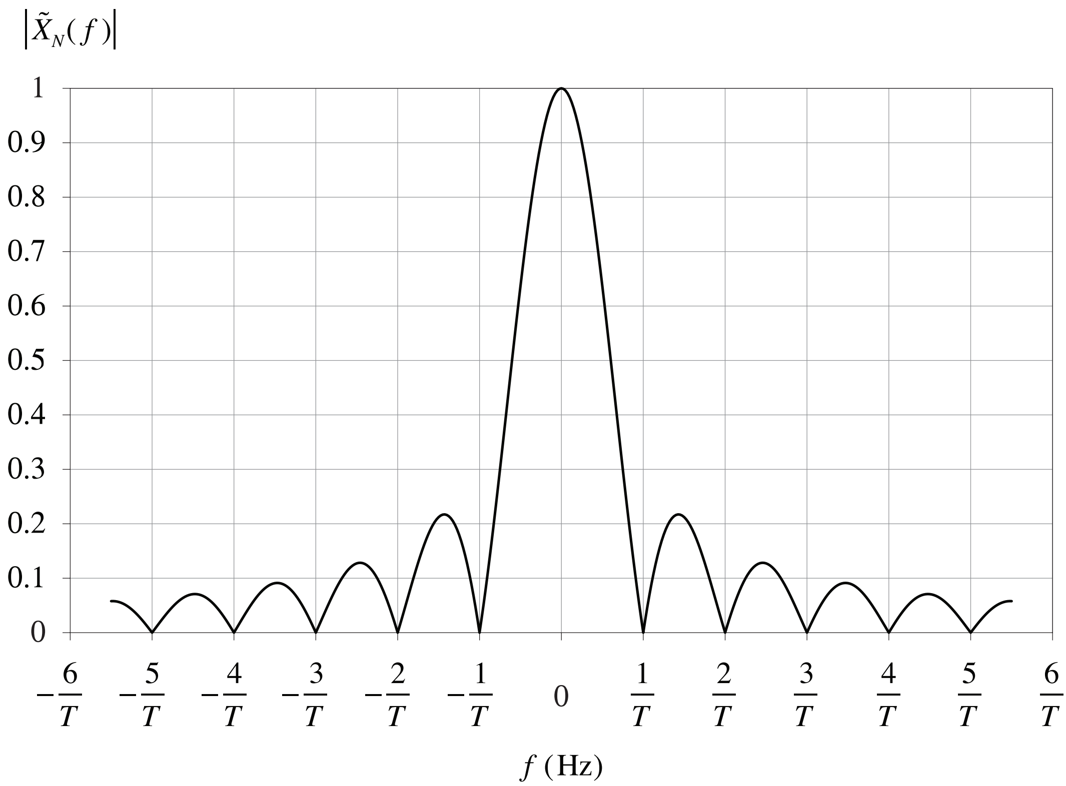

Figure 13.1-8 Normalized magnitude spectrum of the complex envelope of a rectangular-envelope, CW pulse with pulse length T seconds.

($e_{r,Trgt} = \pm 750$ m), $\text{BW}_{\tilde{x}} \approx 5$ Hz, $f_c > 5$ Hz, and $\text{BW}_x \approx 10$ Hz. Also, if $|e_{r,Trgt}| = 750$ m, then $|e_{\tau,Trgt}| = 1$ sec [see (13.1-39)]. The magnitude of the normalized, auto-ambiguity function $|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})|$ of this signal is shown in Fig. 13.1-9. As was previously mentioned, since the numerator of the SIR is directly proportional to the magnitude-squared of the normalized, auto-ambiguity function (see Section 12.2), as $|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})| \to 0$, the SIR $\to 0$ and, as a result, the target won't be detected.

Our last calculation is to solve for the amplitude A of a rectangular-envelope, CW pulse that will satisfy a time-average-power requirement or constraint. Since the time-average power $P_{\text{avg},x}$ of a rectangular-envelope, CW pulse is given by [see (13.1-13)]

$$P_{\text{avg},x} = A^2/2, \tag{13.1-53}$$

solving for the amplitude A yields

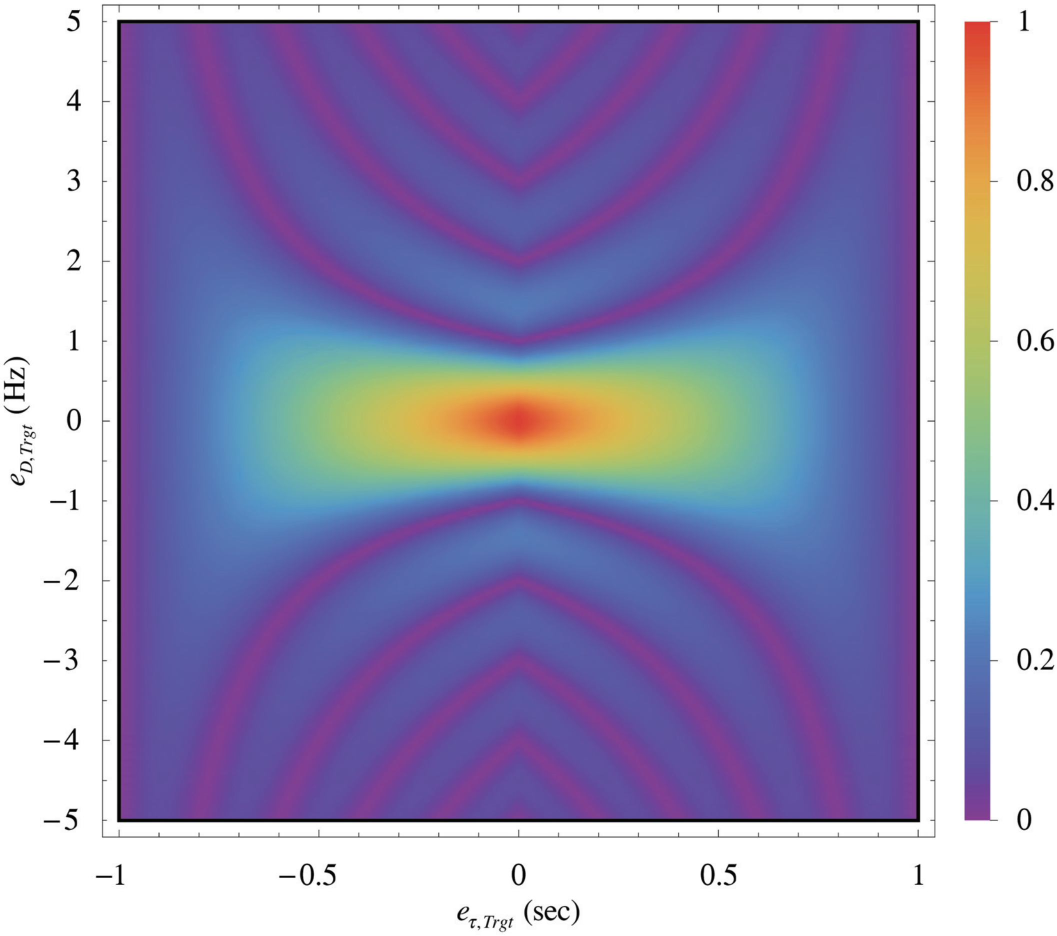

Figure 13.1-9 Magnitude of the normalized, auto-ambiguity function of a rectangular-envelope, CW pulse with pulse length $T = 1\,\text{sec}$.

$$\boxed{A = \sqrt{2P_{\text{avg},x}}} \tag{13.1-54}$$

If $x(t)$ is a voltage signal, then A has units of volts and $P_{\text{avg},x}$ has units of watts-ohms. Once the pulse length T has been calculated using either (13.1-41) or (13.1-51), and the amplitude A has been calculated using (13.1-54), the energies $E_{\tilde{x}}$ and E_x of the rectangular-envelope, CW pulse can be calculated using (13.1-11) and (13.1-12), respectively. If $x(t)$ is a voltage signal, then $E_{\tilde{x}}$ and E_x have units of joules-ohms.

Finally, we know that the range resolution of a rectangular-envelope, CW pulse improves as pulse length T decreases. However, as T decreases, signal bandwidth BW_x increases [see (13.1-50)] and signal energy E_x decreases [see

(13.1-12)]. In order to increase E_x, signal amplitude A must be increased which increases the time-average power $P_{\text{avg},x}$ of the signal [see (13.1-53)]. ■

13.2 The Rectangular-Envelope LFM Pulse

It was shown in Section 13.1 that a rectangular-envelope, CW pulse cannot provide good range and Doppler resolution simultaneously. In order to obtain good range and Doppler resolution *simultaneously*, a more complicated signal must be used that contains several parameters that can be varied. A rectangular-envelope, LFM (linear-frequency-modulated) pulse is such a signal and shall be discussed next.

If we begin with a finite duration, amplitude-and-angle-modulated carrier

$$x(t) = a(t)\cos\left[2\pi f_c t + \theta(t)\right]\text{rect}\left[(t-0.5T)/T\right], \tag{13.2-1}$$

and if we let the real angle-modulating function (phase deviation) be given by

$$\theta(t) = D_p\left[t-(T/2)\right]^2 \tag{13.2-2}$$

so that the frequency deviation

$$\frac{d}{dt}\theta(t) = 2D_p\left[t-(T/2)\right] \tag{13.2-3}$$

is equal to a *linear* function of time, then the signal

$$\boxed{x(t) = a(t)\cos\left\{2\pi f_c t + D_p\left[t-(T/2)\right]^2\right\}\text{rect}\left[(t-0.5T)/T\right]} \tag{13.2-4}$$

is known as a *linear-frequency-modulated* (LFM) *pulse*, where $a(t)$ is a real amplitude-modulating function, $\cos(2\pi f_c t)$ is the carrier waveform, f_c is the carrier frequency in hertz, D_p is the phase-deviation constant with units of radians per second-squared, T is the pulse length in seconds, and the time-shifted rectangle function is given by (13.1-2). An example of a LFM pulse is shown in Fig. 13.2-1. The complex envelope and envelope of (13.2-4) are given by

$$\boxed{\tilde{x}(t) = a(t)\exp\left\{+jD_p\left[t-(T/2)\right]^2\right\}\text{rect}\left[(t-0.5T)/T\right]} \tag{13.2-5}$$

and

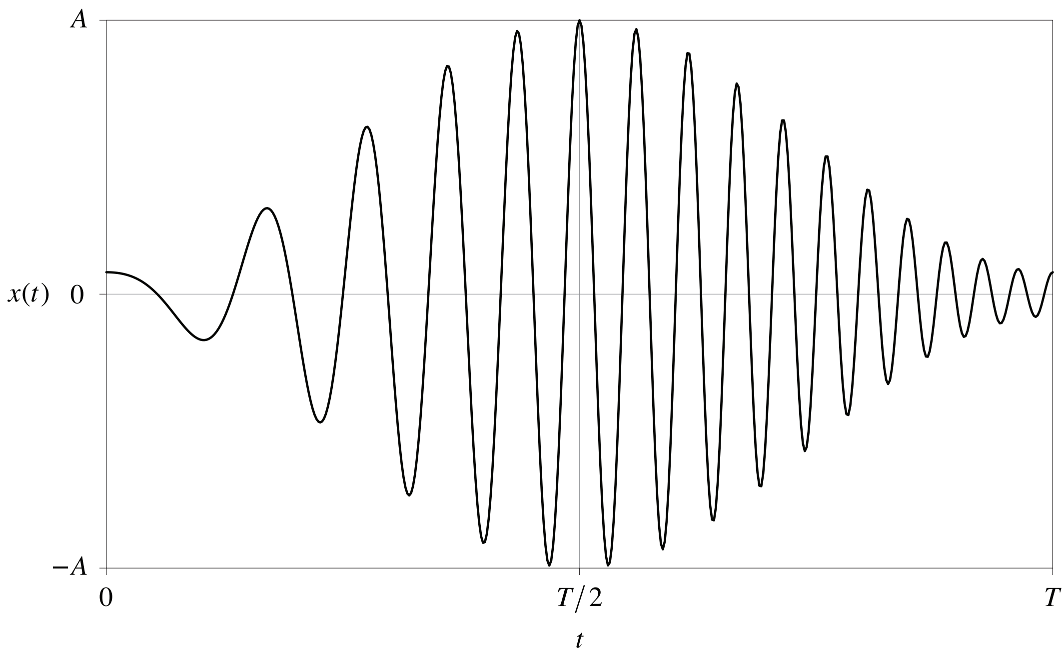

Figure 13.2-1 A LFM pulse with pulse length T seconds.

$$\boxed{\mathcal{E}(t) = |a(t)|\operatorname{rect}[(t-0.5T)/T]} \tag{13.2-6}$$

respectively (see Section 11.2). The energies of $\tilde{x}(t)$ and $x(t)$, and the time-average power of $x(t)$ for a LFM pulse are identical to those for a CW pulse and are given by (13.1-5) through (13.1-7), respectively.

If the real amplitude-modulating function $a(t)$ is set equal to a *positive constant* A, that is, if $a(t) = A > 0$, then the LFM pulse given by (13.2-4) reduces to the *rectangular-envelope*, *LFM pulse*

$$\boxed{x(t) = A\cos\left\{2\pi f_c t + D_p\left[t-(T/2)\right]^2\right\}\operatorname{rect}[(t-0.5T)/T]} \tag{13.2-7}$$

An example of a rectangular-envelope, LFM pulse is shown in Fig. 13.2-2. The complex envelope and envelope of (13.2-7) are given by

$$\boxed{\tilde{x}(t) = A\exp\left\{+jD_p\left[t-(T/2)\right]^2\right\}\operatorname{rect}[(t-0.5T)/T]} \tag{13.2-8}$$

and

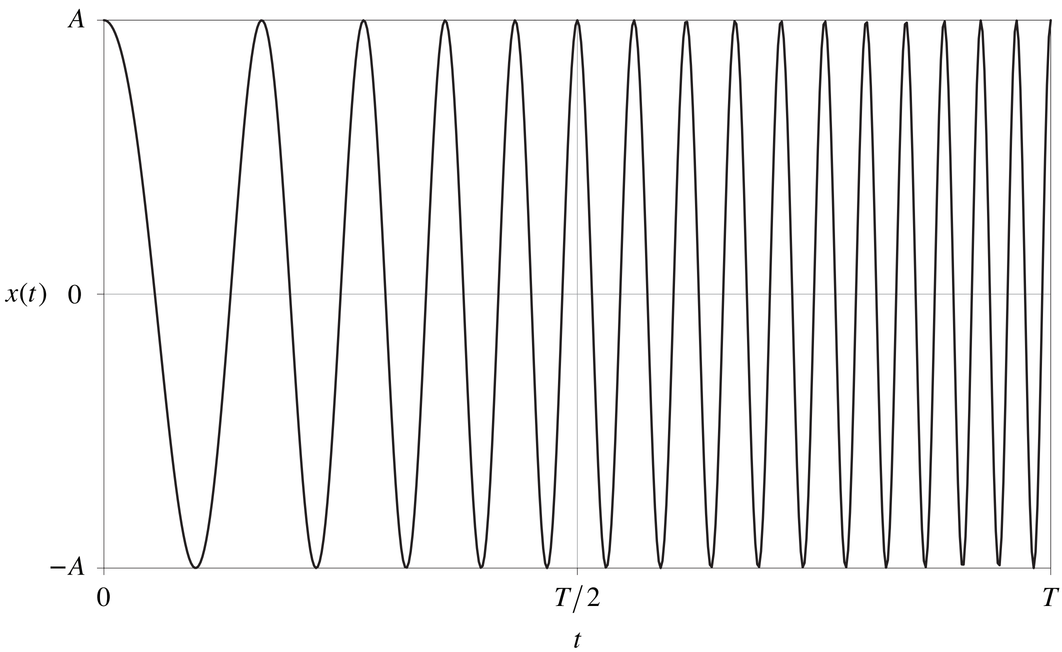

Figure 13.2-2 A rectangular-envelope, LFM pulse with pulse length T seconds.

$$\boxed{\mathcal{E}(t) = A\,\text{rect}\big[(t-0.5T)/T\big]} \tag{13.2-9}$$

respectively [see (13.2-5) and (13.2-6)]. Since the envelope given by (13.2-9) is a rectangle, hence the terminology *rectangular-envelope*, LFM pulse. The energies of $\tilde{x}(t)$ and $x(t)$, and the time-average power of $x(t)$ for a rectangular-envelope, LFM pulse are identical to those for a rectangular-envelope, CW pulse and are given by (13.1-11) through (13.1-13), respectively.

The instantaneous phase in radians and instantaneous frequency in hertz of both the LFM pulse given by (13.2-4) and the rectangular-envelope, LFM pulse given by (13.2-7) are given by

$$\begin{aligned}\theta_i(t) &\triangleq 2\pi f_c t + \theta(t) \\ &= 2\pi f_c t + D_p\big[t-(T/2)\big]^2, \qquad 0 \le t \le T\end{aligned} \tag{13.2-10}$$

and

$$\begin{aligned}f_i(t) &\triangleq \frac{1}{2\pi}\frac{d}{dt}\theta_i(t) \\ &= f_c + \frac{1}{\pi}D_p\big[t-(T/2)\big], \qquad 0 \le t \le T,\end{aligned} \tag{13.2-11}$$

respectively (see Fig. 13.2-3). From (13.2-11) or Fig. 13.2-3,

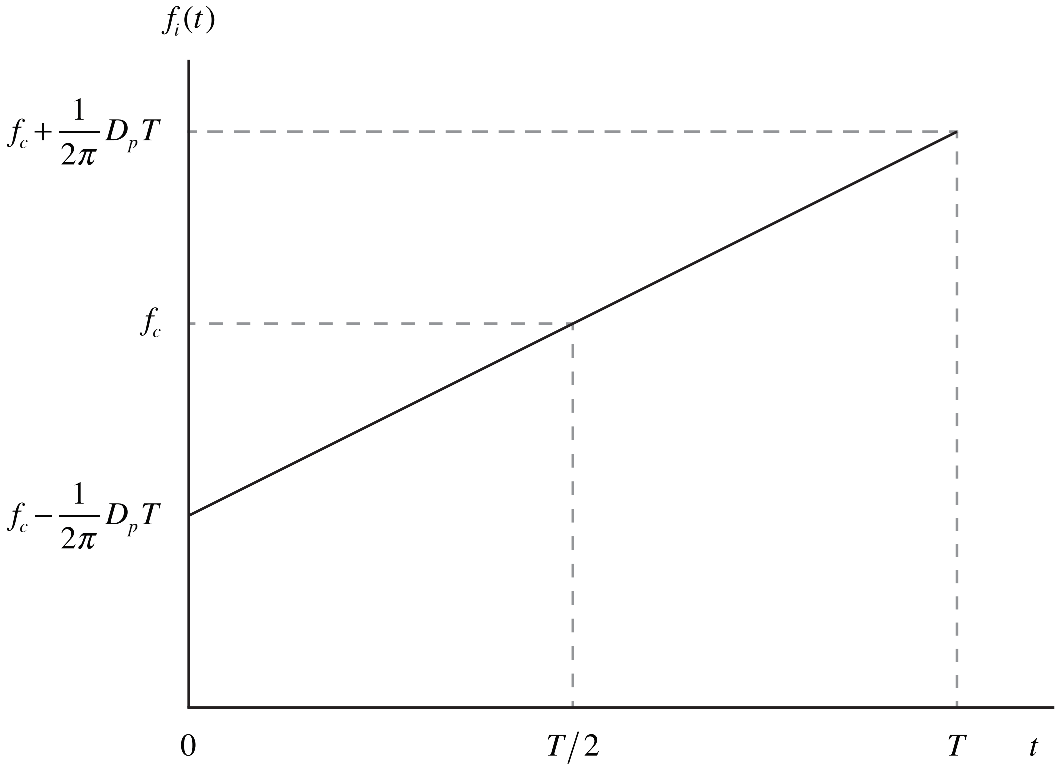

Figure 13.2-3 Instantaneous frequency $f_i(t)$ in hertz versus time t in seconds of a LFM pulse with pulse length T seconds and phase-deviation constant $D_p > 0$.

$$f_i(T) - f_i(0) = f_c + \frac{1}{2\pi} D_p T - \left[f_c - \frac{1}{2\pi} D_p T \right] = \frac{1}{\pi} D_p T \ . \qquad (13.2\text{-}12)$$

Therefore, the *swept-bandwidth* $\mathrm{BW}_{\mathrm{swept}}$ in hertz is defined as

$$\boxed{\mathrm{BW}_{\mathrm{swept}} \triangleq \frac{1}{\pi} \left| D_p \right| T} \qquad (13.2\text{-}13)$$

where the absolute value of the phase-deviation constant D_p is used in (13.2-13) in order to keep the swept-bandwidth positive since D_p can be negative. A LFM pulse is also known as a *chirp pulse*. When the phase-deviation constant $D_p > 0$, a LFM pulse is called an "up-chirp" because the instantaneous frequency $f_i(t)$ *increases* with time; and when $D_p < 0$, it is called a "down-chirp" because $f_i(t)$ *decreases* with time.

In order to derive the unnormalized, auto-ambiguity function of a rectangular-envelope, LFM pulse, we start by substituting its complex envelope given by (13.2-8) into (13.1-14). Doing so yields

$$\begin{aligned}\chi(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 \exp\left(-jD_p e_{\tau,Trgt}^2\right)\exp\left(-jD_p T\, e_{\tau,Trgt}\right)\times \\ \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{t-0.5T}{T}\right)\operatorname{rect}\left[\frac{t-(0.5T+e_{\tau,Trgt})}{T}\right]\times \\ \exp\left[-j2\pi\left(e_{D,Trgt}-\frac{1}{\pi}D_p e_{\tau,Trgt}\right)t\right]dt,\end{aligned} \tag{13.2-14}$$

or

$$\begin{aligned}\chi(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 \exp\left(-jD_p e_{\tau,Trgt}^2\right)\exp\left(-jD_p T\, e_{\tau,Trgt}\right)\times \\ F\left\{\operatorname{rect}\left(\frac{t-0.5T}{T}\right)\operatorname{rect}\left[\frac{t-(0.5T+e_{\tau,Trgt})}{T}\right]\exp\left(+j2D_p e_{\tau,Trgt}\, t\right)\right\}.\end{aligned} \tag{13.2-15}$$

If we designate (13.1-23) for the rectangular-envelope, CW pulse as $\chi_{\text{RECW}}(e_{\tau,Trgt}, e_{D,Trgt})$, then (13.2-14) can be expressed as

$$\begin{aligned}\chi(e_{\tau,Trgt}, e_{D,Trgt}) = \chi_{\text{RECW}}\left(e_{\tau,Trgt}, e_{D,Trgt}-\frac{1}{\pi}D_p e_{\tau,Trgt}\right)\exp\left(-jD_p e_{\tau,Trgt}^2\right)\times \\ \exp\left(-jD_p T\, e_{\tau,Trgt}\right).\end{aligned} \tag{13.2-16}$$

Therefore, by replacing $e_{D,Trgt}$ with $e_{D,Trgt}-\left(D_p e_{\tau,Trgt}/\pi\right)$ in (13.1-29) for the unnormalized, auto-ambiguity function of a rectangular-envelope, CW pulse, and then substituting the resulting expression into (13.2-16) yields

$$\begin{aligned}\chi(e_{\tau,Trgt}, e_{D,Trgt}) = A^2 T\left[1-\frac{\left|e_{\tau,Trgt}\right|}{T}\right]\operatorname{sinc}\left\{\left[e_{D,Trgt}-\frac{1}{\pi}D_p e_{\tau,Trgt}\right]\left[T-\left|e_{\tau,Trgt}\right|\right]\right\}\times \\ \exp\left[-j\pi e_{D,Trgt}(T+e_{\tau,Trgt})\right], \qquad -T \le e_{\tau,Trgt} \le T.\end{aligned} \tag{13.2-17}$$

Equation (13.2-17) is the *unnormalized*, auto-ambiguity function of a rectangular-envelope, LFM pulse.

Since the energy of the complex envelope of a rectangular-envelope, LFM pulse is $A^2 T$, dividing (13.2-17) by $A^2 T$ yields

$$\boxed{\begin{aligned}\chi_N(e_{\tau,Trgt}, e_{D,Trgt}) = \left[1 - \frac{\left|e_{\tau,Trgt}\right|}{T}\right] \operatorname{sinc}\left\{\left[e_{D,Trgt} - \frac{1}{\pi} D_p e_{\tau,Trgt}\right]\left[T - \left|e_{\tau,Trgt}\right|\right]\right\} \times \\ \exp\left[-j\pi e_{D,Trgt}(T + e_{\tau,Trgt})\right], \qquad -T \le e_{\tau,Trgt} \le T\end{aligned}}$$

(13.2-18)

which is the *normalized*, *auto-ambiguity function of a rectangular-envelope*, *LFM pulse*. The magnitude of (13.2-18) has $|\operatorname{sinc}(x)|$ type profiles along every line with a constant $e_{\tau,Trgt}$ value, that is, along every line parallel to and including the $e_{D,Trgt}$ axis. For a given value of $e_{\tau,Trgt}$, the maximum value of the corresponding profile is $1 - \left(\left|e_{\tau,Trgt}\right| / T\right)$, and this maximum value is located at $e_{D,Trgt} = \left(D_p e_{\tau,Trgt} / \pi\right)$, which is the equation of a straight line in the $e_{\tau,Trgt}\, e_{D,Trgt}$ plane that passes through the origin. Therefore, when $D_p > 0$ ("up-chirp"), $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ will be concentrated mainly in the first and third quadrants of the $e_{\tau,Trgt}\, e_{D,Trgt}$ plane; and when $D_p < 0$ ("down-chirp"), $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ will be concentrated mainly in the second and fourth quadrants. In order to cover all four quadrants, one needs to transmit an up-chirp pulse followed by a down-chirp pulse or vice-versa.

By setting $e_{D,Trgt} = 0$ in (13.2-18) and taking the magnitude of the resulting expression, we obtain the magnitude of the *normalized*, *round-trip time-delay* (*range*) *profile*:

$$\boxed{\left|\chi_N(e_{\tau,Trgt}, 0)\right| = \left[1 - \frac{\left|e_{\tau,Trgt}\right|}{T}\right] \left|\operatorname{sinc}\left\{\frac{1}{\pi} D_p e_{\tau,Trgt}\left[T - \left|e_{\tau,Trgt}\right|\right]\right\}\right|, \qquad -T \le e_{\tau,Trgt} \le T}$$

(13.2-19)

since $\operatorname{sinc}(-x) = \operatorname{sinc}(x)$. Similarly, by setting $e_{\tau,Trgt} = 0$ in (13.2-18) and taking the magnitude of the resulting expression, we obtain the magnitude of the *normalized*, *Doppler profile*:

$$\boxed{\left|\chi_N(0, e_{D,Trgt})\right| = \left|\operatorname{sinc}(e_{D,Trgt} T)\right|} \qquad (13.2\text{-}20)$$

which is identical to the magnitude of the normalized, Doppler profile of a rectangular-envelope, CW pulse given by (13.1-32) (see Fig. 13.1-6).

Since $\operatorname{sinc}(\pm 1) = 0$, the range profile given by (13.2-19) has its *first* zero crossings along the $e_{\tau,Trgt}$ axis when the magnitude of the argument of the sinc function is equal to 1, that is,

$$\left| \frac{1}{\pi} D_p e_{\tau,Trgt} \left[T - \left| e_{\tau,Trgt} \right| \right] \right| = 1, \qquad \left| e_{\tau,Trgt} \right| \leq T , \tag{13.2-21}$$

or

$$\left| e_{\tau,Trgt} \right|^2 - T \left| e_{\tau,Trgt} \right| + \frac{\pi}{\left| D_p \right|} = 0 . \tag{13.2-22}$$

Using the quadratic formula to solve (13.2-22) yields

$$\left| e_{\tau,Trgt} \right| = \frac{T}{2} \left[1 \pm \sqrt{1 - \frac{4\pi}{\left| D_p \right| T^2}} \right], \tag{13.2-23}$$

and by choosing the minus sign (since we want the locations of the *first* zero crossings),

$$\boxed{\left| e_{\tau,Trgt} \right| = \frac{T}{2} \left[1 - \sqrt{1 - \frac{4\pi}{\left| D_p \right| T^2}} \right]} \tag{13.2-24}$$

where the condition

$$\frac{4\pi}{\left| D_p \right| T^2} \leq 1 , \tag{13.2-25}$$

or

$$\boxed{T \times \mathrm{BW}_{\mathrm{swept}} \geq 4} \tag{13.2-26}$$

must be satisfied at all times in order to ensure that $\left| e_{\tau,Trgt} \right|$ is real and not complex, where the swept-bandwidth $\mathrm{BW}_{\mathrm{swept}}$ is given by (13.2-13). However, if we impose the more stringent condition

$$\frac{4\pi}{\left| D_p \right| T^2} \leq 0.1 , \tag{13.2-27}$$

or

$$\boxed{T \times \mathrm{BW}_{\mathrm{swept}} \geq 40} \tag{13.2-28}$$

then we can use the first two terms in a *binomial expansion* of the square root in (13.2-24). Doing so yields

$$\left|e_{\tau,Trgt}\right| \approx \frac{T}{2}\left[1-\left(1-\frac{1}{2}\frac{4\pi}{\left|D_p\right|T^2}\right)\right], \tag{13.2-29}$$

$$\left|e_{\tau,Trgt}\right| \approx \frac{\pi}{\left|D_p\right|T}, \tag{13.2-30}$$

or

$$\boxed{e_{\tau,Trgt} \approx \pm 1/\mathrm{BW}_{\mathrm{swept}}} \tag{13.2-31}$$

Equation (13.2-31) gives the locations of the *first* zero crossings of the range profile when (13.2-28) is satisfied. As a result, the width of the mainlobe of the ambiguity function along the $e_{\tau,Trgt}$ axis is *inversely proportional to the swept-bandwidth* $\mathrm{BW}_{\mathrm{swept}}$ when (13.2-28) is satisfied.

By inspecting the Doppler profile given by (13.2-20) and referring to Fig. 13.1-6, it can be seen that the width of the mainlobe of the ambiguity function along the $e_{D,Trgt}$ axis is *inversely proportional to the pulse length* T since the locations of the *first* zero crossings of the Doppler profile are at

$$\boxed{e_{D,Trgt} = \pm 1/T} \tag{13.2-32}$$

By inspecting (13.2-30) and (13.2-32), it can also be seen that the width of the mainlobe of the ambiguity function along *both* the $e_{\tau,Trgt}$ and $e_{D,Trgt}$ axes is *inversely proportional to the pulse length* T.

Now that we have two parameters that we can vary, namely, the phase-deviation constant D_p and the pulse length T, we can *simultaneously* control both the range and Doppler resolving capabilities of a rectangular-envelope, LFM pulse subject to the constraint given by (13.2-28).

Example 13.2-1 Design of a Rectangular-Envelope, LFM Pulse

As was mentioned in Example 13.1-1, the widths of the mainlobe of the ambiguity function along the $e_{\tau,Trgt}$ and $e_{D,Trgt}$ axes are used as measures of the range and Doppler resolutions, respectively, of a transmitted electrical signal. The locations of the first zero crossings of the round-trip time-delay (range) profile of a rectangular-envelope, LFM pulse are given by (13.2-31) provided that (13.2-28) is satisfied, and the locations of the first zero crossings of the Doppler profile are given by (13.2-32). Therefore, from (13.2-31),

$$\left|e_{\tau,Trgt}\right| \approx 1/\mathrm{BW}_{\mathrm{swept}}\ , \tag{13.2-33}$$

and from (13.2-32),

$$\left|e_{D,Trgt}\right| = 1/T\ . \tag{13.2-34}$$

Let us now design a rectangular-envelope, LFM pulse to satisfy a range-resolution requirement. In general, in signal-design problems, satisfying a range-resolution requirement is of primary concern.

Specifying a value for range resolution is equivalent to specifying a desired value for the magnitude of the range estimation error of the target $\left|e_{r,Trgt}\right|$ that corresponds to the location of the first zero crossing along the positive $e_{\tau,Trgt}$ axis. If (13.2-33) is substituted into (13.1-35), then

$$\boxed{\mathrm{BW}_{\mathrm{swept}} \approx \frac{c}{2\left|e_{r,Trgt}\right|}} \tag{13.2-35}$$

Equation (13.2-35) is used to compute the swept-bandwidth $\mathrm{BW}_{\mathrm{swept}}$ in hertz that is required for a rectangular-envelope, LFM pulse to provide a range resolution equal to $\left|e_{r,Trgt}\right|$ meters. The pulse length T in seconds must then satisfy the condition [see (13.2-28)]

$$\boxed{T \geq 40/\mathrm{BW}_{\mathrm{swept}}} \tag{13.2-36}$$

The resulting Doppler resolution is obtained by substituting the value of T from (13.2-36) into (13.2-34). Since we have a choice as to how big to make T, as T increases, Doppler resolution gets better, that is, $\left|e_{D,Trgt}\right|$ gets smaller, without changing the range resolution $\left|e_{r,Trgt}\right|$. Furthermore, as pulse length T increases, signal bandwidth BW_x decreases and signal energy E_x increases [see (13.1-12)], but the waveform becomes less Doppler-tolerant because the Doppler resolution gets better. Therefore, if the minimum value of pulse length is used in order to maintain some Doppler tolerance, signal energy can be increased by increasing the signal amplitude A, which increases the time-average power $P_{\mathrm{avg},x}$ [see (13.1-13)]. And by solving for $\left|D_p\right|$ from (13.2-13), we obtain

$$\left|D_p\right| = \pi \, \mathrm{BW}_{\text{swept}} / T \,, \tag{13.2-37}$$

or

$$\boxed{D_p = \pm \pi \, \mathrm{BW}_{\text{swept}} / T} \tag{13.2-38}$$

Equation (13.2-38) is used to compute the phase-deviation constant D_p in radians per second-squared that is required for a rectangular-envelope, LFM pulse to provide a range resolution equal to $\left|e_{r,Trgt}\right|$ meters, and a Doppler resolution equal to $\left|e_{D,Trgt}\right|$ hertz. For example, if $\left|e_{r,Trgt}\right| = 1\text{ m}$ ($e_{r,Trgt} = \pm 1\text{ m}$) and $c = 1500\text{ m/sec}$, then $\mathrm{BW}_{\text{swept}} = 750\text{ Hz}$ and $T \geq 53.333\text{ msec}$. Also, if $\left|e_{r,Trgt}\right| = 1\text{ m}$, then $\left|e_{\tau,Trgt}\right| = 1.333\text{ msec}$ [see (13.1-35)]. If we let $T = 54\text{ msec}$, then $\left|e_{D,Trgt}\right| = 18.519\text{ Hz}$ ($e_{D,Trgt} = \pm 18.519\text{ Hz}$) and $D_p = \pm 43{,}633.231\text{ rad/sec}^2$. The magnitude of the normalized, auto-ambiguity function $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right|$ of this signal is shown in Fig. 13.2-4. Figure 13.2-5 is a plot of the magnitude of the normalized, round-trip time-delay (range) profile of the auto-ambiguity function shown in Fig. 13.2-4. Since the numerator of the signal-to-interference ratio (SIR) is directly proportional to the magnitude-squared of the normalized, auto-ambiguity function (see Section 12.2), as $\left|\chi_N(e_{\tau,Trgt}, e_{D,Trgt})\right| \to 0$, the SIR $\to 0$ and, as a result, the target won't be detected.

Unlike the complex envelope of a rectangular-envelope, CW pulse; an exact, closed-form, analytical expression for the Fourier transform of the complex envelope of a rectangular-envelope, LFM pulse does not exist – the Fourier transform must be computed numerically. However, a conservative estimate of the bandwidth in hertz of the complex envelope of a rectangular-envelope, LFM pulse when (13.2-28) is satisfied is given by

$$\boxed{\mathrm{BW}_{\tilde{x}} \approx \mathrm{BW}_{\text{swept}} + \frac{5}{T}} \tag{13.2-39}$$

Substituting (13.2-39) into (13.1-42) and (13.1-43) yields

$$\boxed{f_c > \mathrm{BW}_{\text{swept}} + \frac{5}{T}} \tag{13.2-40}$$

and

$$\boxed{\mathrm{BW}_x \approx 2\,\mathrm{BW}_{\text{swept}} + \frac{10}{T}} \tag{13.2-41}$$

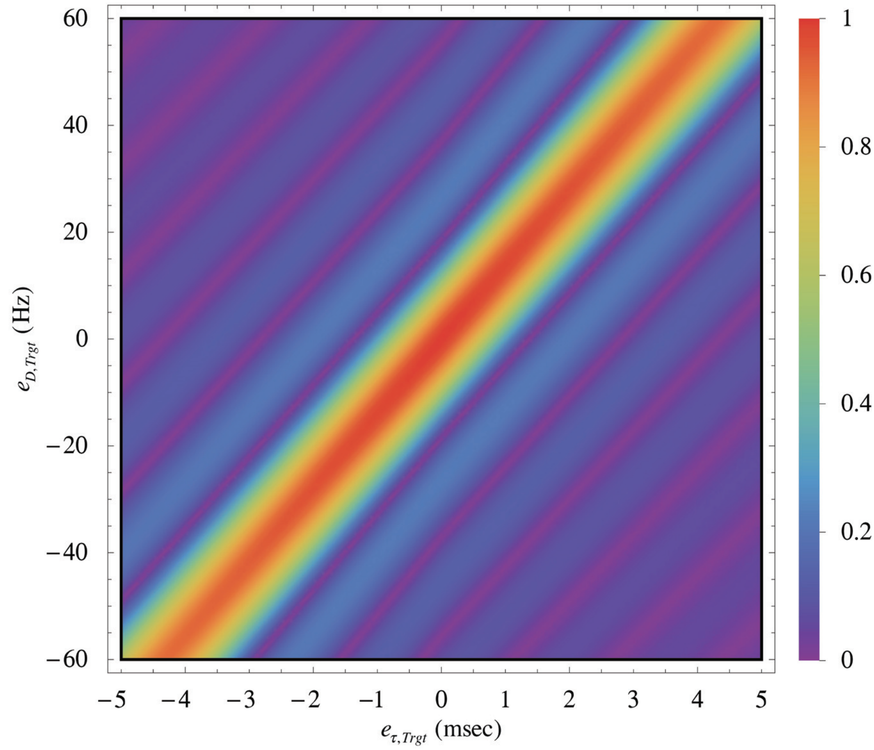

Figure 13.2-4 Magnitude of the normalized, auto-ambiguity function of a rectangular-envelope, up-chirp, LFM pulse with pulse length $T = 54$ msec and swept bandwidth $\mathrm{BW}_{\mathrm{swept}} = 750$ Hz .

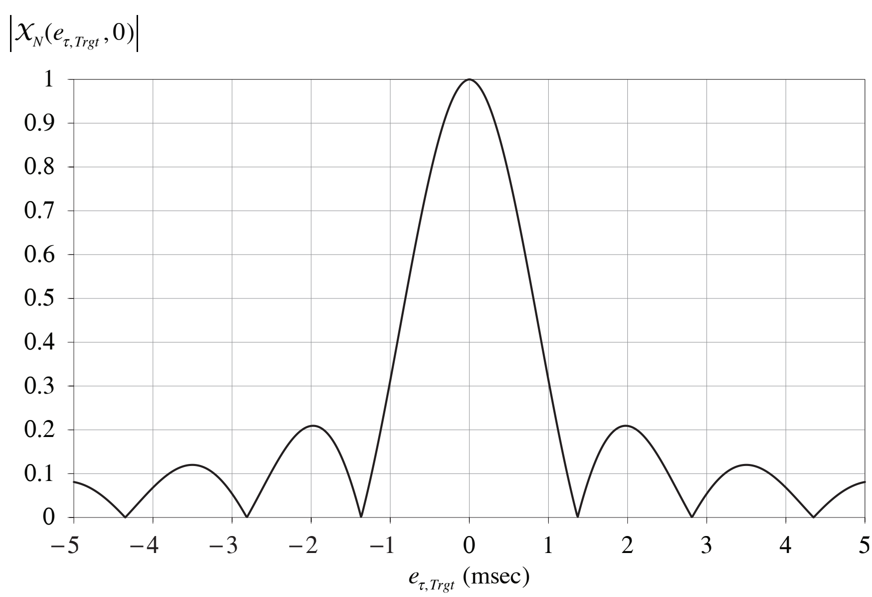

Figure 13.2-5 Magnitude of the normalized, round-trip time-delay (range) profile of the auto-ambiguity function shown in Fig. 13.2-4.

for a rectangular-envelope, LFM pulse. For example, for the previously designed rectangular-envelope, LFM pulse with pulse length $T = 54\ \text{msec}$ and swept-bandwidth $\text{BW}_{\text{swept}} = 750\ \text{Hz}$; $\text{BW}_{\tilde{x}} \approx 842.6\ \text{Hz}$, $f_c > 842.6\ \text{Hz}$, and $\text{BW}_x \approx 1685.2\ \text{Hz}$ (see Fig. 13.2-6).

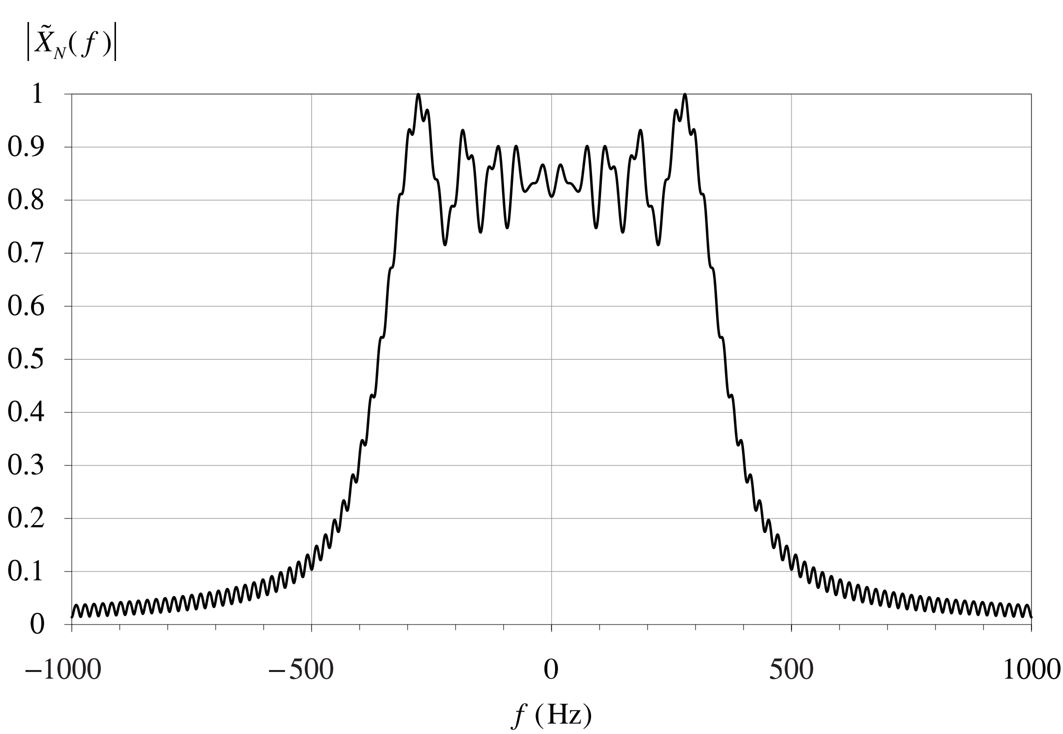

Figure 13.2-6 Normalized magnitude spectrum of the complex envelope of a rectangular-envelope, LFM pulse with pulse length $T = 54\ \text{msec}$ and swept bandwidth $\text{BW}_{\text{swept}} = 750\ \text{Hz}$.

Next we shall design a rectangular-envelope, LFM pulse to satisfy a Doppler-resolution requirement. Specifying a value for Doppler resolution is equivalent to specifying a desired value for the magnitude of the Doppler estimation error of the target $|e_{D,Trgt}|$ that corresponds to the location of the first zero crossing along the positive $e_{D,Trgt}$ axis. Solving for T from (13.2-34) yields

$$\boxed{T = 1\big/\left|e_{D,Trgt}\right|} \qquad (13.2\text{-}42)$$

Equation (13.2-42) is used to compute the pulse length T in seconds that is required for a rectangular-envelope, LFM pulse to provide a Doppler resolution equal to $|e_{D,Trgt}|$ hertz. The swept-bandwidth $\mathrm{BW}_{\mathrm{swept}}$ must then satisfy the condition [see (13.2-28)]

$$\boxed{\mathrm{BW}_{\mathrm{swept}} \geq 40/T} \tag{13.2-43}$$

And by solving for $|e_{r,Trgt}|$ from (13.2-35), we obtain

$$|e_{r,Trgt}| \approx \frac{c}{2\,\mathrm{BW}_{\mathrm{swept}}}. \tag{13.2-44}$$

The resulting range resolution is obtained by substituting the value of $\mathrm{BW}_{\mathrm{swept}}$ from (13.2-43) into (13.2-44). Since we have a choice as to how big to make $\mathrm{BW}_{\mathrm{swept}}$, as $\mathrm{BW}_{\mathrm{swept}}$ increases, range resolution gets better, that is, $|e_{r,Trgt}|$ gets smaller, without changing the Doppler resolution $|e_{D,Trgt}|$. However, as swept-bandwidth $\mathrm{BW}_{\mathrm{swept}}$ increases, signal bandwidth BW_x increases [see (13.2-41)]. After T and $\mathrm{BW}_{\mathrm{swept}}$ have been determined, the required value for the phase-deviation constant D_p in radians per second-squared is computed from (13.2-38). For example, if $|e_{D,Trgt}| = 1\text{ Hz}$ ($e_{D,Trgt} = \pm 1\text{ Hz}$) and $c = 1500\text{ m/sec}$, then $T = 1\text{ sec}$ and $\mathrm{BW}_{\mathrm{swept}} \geq 40\text{ Hz}$. If we let $\mathrm{BW}_{\mathrm{swept}} = 40\text{ Hz}$, then $|e_{r,Trgt}| \approx 18.75\text{ m}$ ($e_{r,Trgt} \approx \pm 18.75\text{ m}$), $D_p = \pm 125.664\text{ rad/sec}^2$, $\mathrm{BW}_{\tilde{x}} \approx 45\text{ Hz}$, $f_c > 45\text{ Hz}$, and $\mathrm{BW}_x \approx 90\text{ Hz}$ (see Fig. 13.2-7). Also, if $|e_{r,Trgt}| \approx 18.75\text{ m}$, then $|e_{\tau,Trgt}| \approx 25\text{ msec}$ [see (13.1-35)]. The magnitude of the normalized, auto-ambiguity function $|\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt})|$ of this signal is shown in Fig. 13.2-8. Figure 13.2-9 is a plot of the magnitude of the normalized, round-trip time-delay (range) profile of the auto-ambiguity function shown in Fig. 13.2-8. As was previously mentioned, since the numerator of the SIR is directly proportional to the magnitude-squared of the normalized, auto-ambiguity function (see Section 12.2), as $|\mathcal{X}_N(e_{\tau,Trgt}, e_{D,Trgt})| \to 0$, the SIR $\to 0$ and, as a result, the target won't be detected.

Finally, since the energy and time-average power of a rectangular-envelope, LFM and CW pulse are equal, (13.1-54) can be used to solve for the amplitude A of a rectangular-envelope, LFM pulse that will satisfy a time-average-power requirement. Tables 13.2-1 and 13.2-2 summarize the four different signal designs discussed in Examples 13.1-1 and 13.2-1. From Table 13.2-1 it can be seen that the rectangular-envelope, LFM pulse has more energy than the rectangular-envelope, CW pulse because for the same value of signal amplitude A, the *minimum* value of pulse length used for the rectangular-envelope, LFM pulse is $T = 54\text{ msec}$, which is much greater than the *required* value of $T = 1.333\text{ msec}$ for the rectangular-envelope, CW pulse. ∎

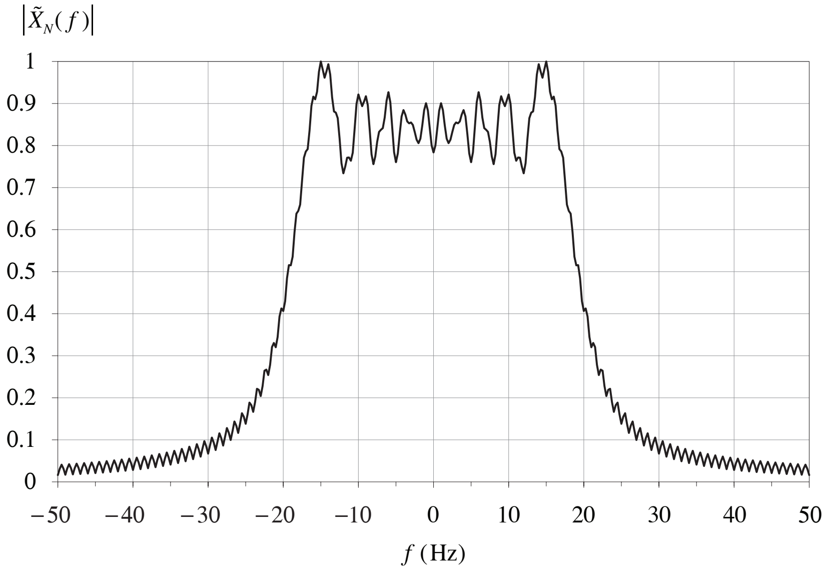

Figure 13.2-7 Normalized magnitude spectrum of the complex envelope of a rectangular-envelope, LFM pulse with pulse length $T = 1\,\text{sec}$ and swept bandwidth $\text{BW}_{\text{swept}} = 40\ \text{Hz}$.

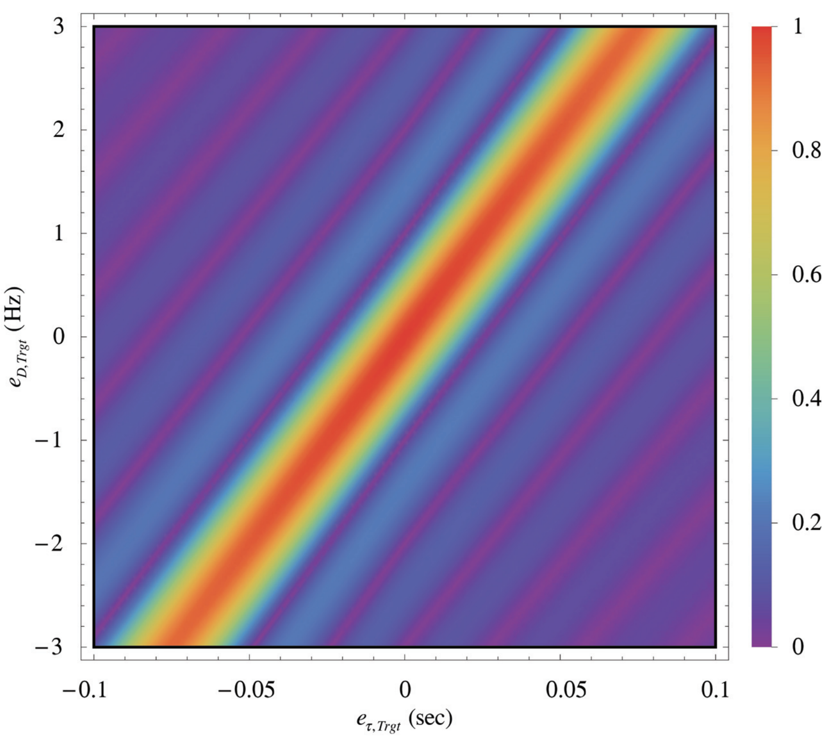

Figure 13.2-8 Magnitude of the normalized, auto-ambiguity function of a rectangular-envelope, up-chirp, LFM pulse with pulse length $T = 1\,\text{sec}$ and swept bandwidth $\text{BW}_{\text{swept}} = 40\ \text{Hz}$.

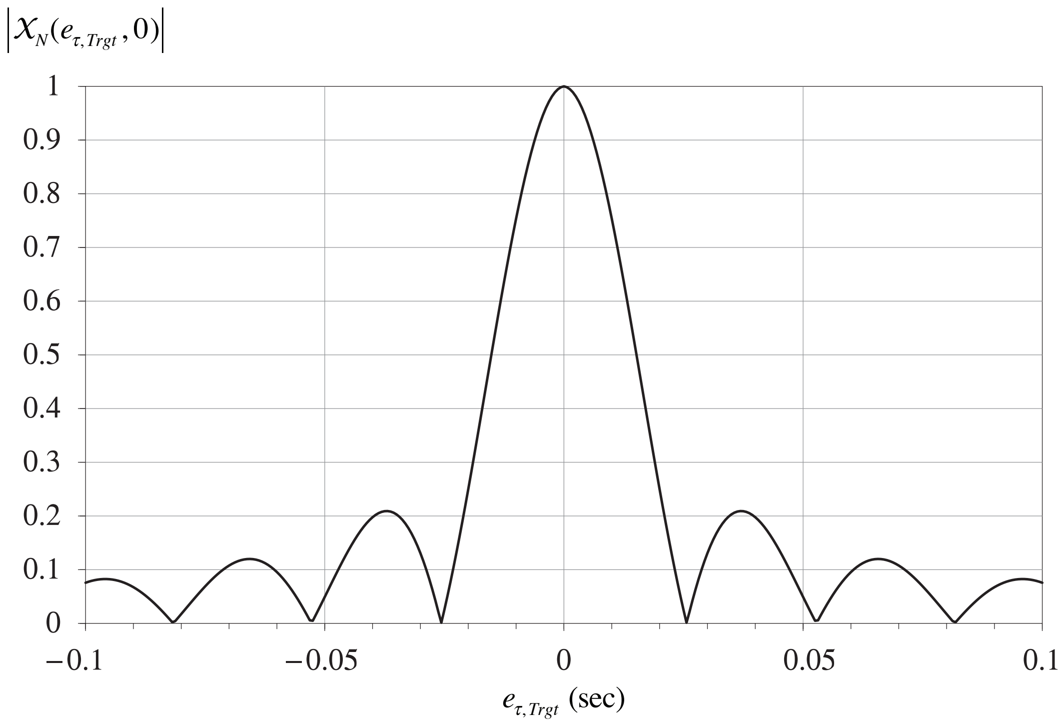

Figure 13.2-9 Magnitude of the normalized, round-trip time-delay (range) profile of the auto-ambiguity function shown in Fig. 13.2-8.

Table 13.2-1 Parameter Values for Rectangular-Envelope, CW and LFM Pulses Designed for a Range Resolution of 1 m

	T	BW_{swept}	BW_x	$\left\|e_{r,Trgt}\right\|$	$\left\|e_{D,Trgt}\right\|$
RE CW Pulse	1.333 msec	NA	7500 Hz	1 m	750 Hz
RE LFM Pulse	54 msec	750 Hz	1685.2 Hz	1 m	18.519 Hz

Table 13.2-2 Parameter Values for Rectangular-Envelope, CW and LFM Pulses Designed for a Doppler Resolution of 1 Hz

	T	BW_{swept}	BW_x	$\left\|e_{r,Trgt}\right\|$	$\left\|e_{D,Trgt}\right\|$
RE CW Pulse	1 sec	NA	10 Hz	750 m	1 Hz
RE LFM Pulse	1 sec	40 Hz	90 Hz	18.75 m	1 Hz

Problems

Section 13.1

13-1 Design a rectangular-envelope, CW pulse $x(t)$ with a range resolution of ± 0.15 m. In addition, $x(t)$ must have a time-average power of 200 W-Ω. Use $c = 1500$ m/sec.

(a) Find the required pulse length in seconds.

(b) Find the required amplitude in volts.

(c) What is the energy of $x(t)$ in joules-ohms?

(d) What is the Doppler resolution of $x(t)$ in hertz?

(e) What is the bandwidth of $x(t)$ in hertz?

(f) What is the minimum allowed value for the carrier frequency in hertz?

13-2 A rectangular-envelope, CW pulse $x(t)$ has an amplitude of 15 V and a pulse length of 100 msec. If $c = 1500$ m/sec, then

(a) what is the range resolution of $x(t)$ in meters?

(b) what is the Doppler resolution of $x(t)$ in hertz?

(c) what is the energy of $x(t)$ in joules-ohms?

(d) what is the time-average power of $x(t)$ in watts-ohms?

(e) what is the bandwidth of $x(t)$ in hertz?

(f) what is the minimum allowed value for the carrier frequency in hertz?

Section 13.2

13-3 Design a rectangular-envelope, LFM pulse $x(t)$ with a range resolution of ± 0.15 m. In addition, $x(t)$ must have a time-average power of 450 W-Ω. Use $c = 1500$ m/sec.

(a) Find the required swept-bandwidth in hertz.

(b) Find the required amplitude in volts.

(c) Find the minimum allowed pulse length in seconds.