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H_∞ Filtering with Inequality Constraints for Aircraft Turbofan Engine Health Estimation

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***Abstract*— H_∞ filters (also called minimax filters) can estimate the state variables of a dynamic system. However, in the application of state estimators, some known signal information is often either ignored or dealt with heuristically. The contribution of this paper is three-fold: first, this paper extends previous work on equality-constrained H_∞ filtering and derives some new theoretical results; second, this paper shows how the inequality-constrained H_∞ filtering problem can be reduced to a standard quadratic programming problem; third, this paper shows how inequality-constrained H_∞ filtering can be applied to aircraft engine health estimation. The incorporation of state constraints significantly increases the computational effort of the filter but also improves its estimation accuracy. The improved estimation accuracy is shown in this paper both theoretically and experimentally. We also show that the Kalman filter performs better for aircraft engine health estimation under nominal conditions, but (in agreement with theory) the H_∞ filter performs better in the presence of unmodeled errors with respect to worst case estimation error.**

I. INTRODUCTION

In the application of state estimators there is often known model or signal information that is either ignored or dealt with heuristically. This paper presents a way to generalize a one-step-ahead H_∞ filter in such a way that known inequality constraints among the state variables are satisfied by the state estimate. This paper has its roots in the unconstrained H_∞ filtering theory presented in [1], which was extended in [2] to include state equality constraints. The present paper generalizes the results of [2] to show how the inequality-constrained H_∞ filter can be reduced to a standard H_∞ filter combined with a quadratic programming problem. It is shown in this paper that the constrained estimate has a lower error covariance bound than the unconstrained estimate.

Numerous efforts have been pursued to incorporate constraints into H_∞ control and estimation problems. For instance, H_∞ control can be achieved subject to constraints on the system time response [3], state variables [4], controller poles [5], state integrals [6], and control variables [7]. H_∞ filters can be designed with poles that are constrained to a specific region [8]. FIR and IIR filters can be designed such that the H_∞ norm of the error transfer function is minimized while constraining the filter output to lie within a prescribed envelope [9], [10]. To our knowledge the first effort to incorporate state inequality constraints into H_∞ filtering problems is given in [2]. This paper extends those results theoretically and also applies them to aircraft engine health parameter estimation.

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This paper applies inequality-constrained H_∞ filtering to estimate aircraft engine component efficiencies and flow capacities, which are referred to as health parameters. The health parameters that we try to estimate can be modeled as slowly varying biases. The state vector of the dynamic model is augmented to include the health parameters [11] which are then estimated with the recursive H_∞ filter. In addition, we have some *a priori* knowledge of the engine's health parameters: we know that they never improve. Engine health always degrades over time (unless maintenance is performed) and we can incorporate this information into state constraints to improve our health parameter estimation. This is similar to constrained Kalman filtering [11].

This paper makes three new contributions to the state estimation literature.

- First we extend previous work on equality-constrained H_∞ filtering to obtain new theoretical results. We show that the equality-constrained H_∞ filter is equivalent to the unconstrained filter projected onto the constraint surface. We then show that the covariance bound of the constrained filter is smaller than that of the unconstrained filter.
- Second we use the projection theorem that we derived to show how the inequality-constrained filter can be posed as a quadratic programming problem. This makes inequality-constrained H_∞ filtering more straightforward and practical than previous approaches (because we can use the tools that are available for solving quadratic programming problems).
- Third we show how inequality-constrained H_∞ filtering can be applied to aircraft engine health estimation. This is the part of the paper where the newly introduced theory meets practical application.

Section II begins with a discussion of existing H_∞ filtering theory, both unconstrained and equality-constrained. We then present some new results related to equality-constrained H_∞ filtering theory and generalize those results to inequality constraints. This new filter is unbiased and has an error covariance bound that is smaller than that of the unconstrained filter. Section III discusses turbofan health parameter estimation and Section IV presents simulation results. We also show in Section IV that the Kalman filter provides health estimation better than the H_∞ filter under nominal conditions, but the H_∞ filter performs better in the presence of unmodeled errors with respect to worst case estimation errors. Section V presents concluding remarks and discussion.

II. H_∞ FILTERING

The literature contains a number of different formulations of the H_∞ filtering problem [12], [13], [14]. Section II-A reviews the *a priori* H_∞ filtering formulation that is developed in [1]. We then continue in Section II-B by reviewing the recently developed theory of H_∞ state estimation with state equality constraints [2]. Section II-C then presents some new analysis related to the equality-constrained H_∞ filter, and finally Section II-D extends those results to H_∞ filtering with inequality constraints.

A. Unconstrained H_∞ Filtering

This subsection reviews existing results on H_∞ filtering. Consider the discrete linear time-invariant system given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k + \delta_k \\ y_k &= Cx_k + m_k \end{aligned} \tag{1}$$

where k is the time index, x is the state vector, y is the measurement, $\{w_k\}$ and $\{m_k\}$ are white noise sequences, and $\{\delta_k\}$ is a (possibly worst case) noise sequence generated by an adversary. We assume that $\{w_k\}$ and $\{m_k\}$ are uncorrelated unity-variance white noise sequences. In general, A, B, and C can be time-varying matrices, but we will omit the time subscript on these matrices for ease of notation. The problem is to find an estimate $\hat{x}_{k+1}$ of x_{k+1} given the measurements $\{y_0, y_1, \cdots, y_k\}$. We will restrict the state estimator to have an observer structure so that it results in an unbiased estimate [15].

$$\hat{x}_{k+1} = A\hat{x}_k + K_k(y_k - C\hat{x}_k) \tag{2}$$

where K_k is the estimator gain. The noise δ_k in (1) is introduced by an adversary that has the goal of maximizing the estimation error. We will assume that our adversary's input to the system is given as follows [1], [2].

$$\delta_k = L_k(G_k(x_k - \hat{x}_k) + n_k) \tag{3}$$

where L_k is a gain to be determined, G_k is a given matrix, and $\{n_k\}$ is a noise sequence. We will assume that $\{w_k\}$, $\{m_k\}$, and $\{n_k\}$ are uncorrelated unity-variance white noise sequences that are uncorrelated with x_0. The matrix G_k can be considered as a tuning parameter or weighting matrix that can be adjusted on the basis of our *a priori* knowledge about the adversary's noise input [2].

The estimation error is defined as

$$e_k = x_k - \hat{x}_k \tag{4}$$

Following the development of [1], [2] we define the following variables.

$$\begin{aligned} e_{1,0} &= x_0 - \hat{x}_0 \\ e_{1,k+1} &= (A - K_kC + L_kG_k)e_{1,k} + Bw_k - K_km_k \\ e_{2,0} &= 0 \\ e_{2,k+1} &= (A - K_kC + L_kG_k)e_{2,k} + L_kn_k \end{aligned} \tag{5}$$

Note that $e_k = e_{1,k} + e_{2,k}$. We define the cost function as

$$J = \text{Trace} \sum_{k=0}^{N} W_k E(e_{1,k}e_{1,k}^T - e_{2,k}e_{2,k}^T) \tag{6}$$

where W_k is any positive definite weighting matrix. The differential game is for the filter designer to find a gain sequence $\{K_k\}$ that minimizes J, and for the adversary to find a gain sequence $\{L_k\}$ that maximizes J. As such, J is considered a function of $\{K_k\}$ and $\{L_k\}$, which we denote in shorthand notation as K and L.

The cost function in (6) is not intuitive, but is used here because the solution of the problem results in a state estimator that bounds the infinity norm of the transfer function from the random noise terms to the state estimation error [1]. That is, if (6) has a saddle point, then the following bound holds for the induced l_2 norm of the estimator.

$$\sup_{w_k, m_k} \frac{\sum_{k=0}^{N} ||G_ke_k||_2^2}{\sum_{k=0}^{N} (||w_k||_2^2 + ||m_k||_2^2)} < 1 \tag{7}$$

If G_k is too "large" then the differential game will not have a saddle point. Note that the induced l_2 norm above reduces to the system H_∞ norm when the system is time invariant.

J is, in general, a function of K and L, so we write it as $J(K, L)$. It is shown in [1] that J can be written as

$$J(K, L) = \text{Trace} \sum_{k=0}^{N} W_kQ_k \tag{8}$$

where Q_k is described as follows.

$$\begin{aligned} Q_0 &= E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \\ F_k &= A - K_kC + L_kG_k \\ Q_{k+1} &= F_kQ_kF_k^T + BB^T + K_kK_k^T - L_kL_k^T \end{aligned} \tag{9}$$

The differential game is for the filter designer to find a gain sequence $\{K_k^*\}$ that minimizes J, and for the adversary to find a gain sequence $\{L_k^*\}$ that maximizes J. That is,

$$J(K^*, L) \leq J(K^*, L^*) \leq J(K, L^*) \text{ for all } K, L \tag{10}$$

In the development below we use the notation $A > B$ (where A and B are matrices with equivalent dimensions) to indicate that $(A - B)$ is positive definite. The notation $A \geq B$ indicates that $(A - B)$ is positive semidefinite. The following theorem derives from [1].

Theorem 1: The following gain matrices satisfy (10).

$$\begin{aligned} K_k^* &= A\hat{\Sigma}_kC^T \\ L_k^* &= A\hat{\Sigma}_kG_k^T \end{aligned} \tag{11}$$

where $\hat{\Sigma}_k$ and $\hat{Q}_k$ are the positive definite solutions to the following set of equations.

$$\begin{aligned} \hat{Q}_0 &= E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \\ \hat{Q}_k(I - C^TC\hat{\Sigma}_k) &= (I - \hat{Q}_kG_k^TG_k)\hat{\Sigma}_k \\ \hat{Q}_{k+1} &= A\hat{\Sigma}_kA^T + BB^T \end{aligned} \tag{12}$$

For the gain matrices given in (11), Q_k in (9) is equal to $\hat{Q}_k$ in (12). In addition, it can be shown that

$$E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \leq \hat{Q}_k \tag{13}$$

That is, $\hat{Q}_k$ bounds the error covariance of the estimator.
Proof: See [1].

The conditions required for the validity of the above theorem are as follows: $\hat{Q}_k > 0$, $\hat{\Sigma}_k > 0$, $(I - G_k\hat{Q}_k G_k^T) > 0$, and $(I + C\hat{Q}_k C^T) > 0$.

B. H_∞ Filtering with State Equality Constraints

This section reviews existing results on equality-constrained H_∞ filtering. Suppose that in addition to everything in the preceding subsection, we know that the states satisfy the following constraint.

$$D_k x_k = d_k \tag{14}$$

We assume that the D_k matrix is full rank and normalized so that $D_k D_k^T = I$. We define the following matrix for notational convenience.

$$V_k = D_k^T D_k \tag{15}$$

We assume that both the noisy system and the noise-free system satisfy the above state constraint; therefore $\{L_k\}$ must be such that the noise-free system satisfies the state constraint (14). We use the notation $\tilde{x}_k$ to denote the constrained H_∞ estimate, which should satisfy the state constraint; that is, $D_k\tilde{x}_k = d_k$. The following theorem derives from [2].

Theorem 2: The following gain matrices satisfy the constrained saddle point of (10).

$$\begin{aligned} K_k^* &= (I - V_{k+1})A\tilde{\Sigma}_k C^T \\ L_k^* &= (I - V_{k+1})A\tilde{\Sigma}_k G_k^T \end{aligned} \tag{16}$$

where $\tilde{\Sigma}_k$ and $\tilde{Q}_k$ are the positive definite solutions to the following set of equations.

$$\begin{aligned} \tilde{Q}_0 &= E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \\ \tilde{Q}_k(I - C^T C\tilde{\Sigma}_k) &= (I - \tilde{Q}_k G_k^T G_k)\tilde{\Sigma}_k \\ \tilde{Q}_{k+1} &= (I - V_{k+1})A\tilde{\Sigma}_k A^T(I - V_{k+1}) + \\ & BB^T \end{aligned} \tag{17}$$

For the gain matrices given in (16), Q_k in (9) is equal to $\tilde{Q}_k$ in (17). In addition, it can be shown that

$$E[(x_k - \tilde{x}_k)(x_k - \tilde{x}_k)^T] \leq \tilde{Q}_k \tag{18}$$

That is, $\tilde{Q}_k$ bounds the error covariance of the estimator.
Proof: See [2].

The conditions required for the validity of this theorem are the same as those following Theorem 1.

C. One-Step-Ahead H_∞ Filtering with State Equality Constraints

In this section we consider one-step-ahead H_∞ filters. That is, we consider each time to be the initial time ($k = 0$). We assume that at the initial time the unconstrained state estimate is equal to the constrained state estimate ($\hat{x}_0 = \tilde{x}_0$). We then use (2), (11), and (12) to compute the unconstrained state estimate at the next time step, and the bound on the its error covariance. We use (2), (16), and (17) to compute the constrained state estimate at the next time step, and the bound on its error covariance.

Denote the unconstrained optimal gains given in (11) as $\hat{K}_k$ and $\hat{L}_k$. Denote the one-step-ahead state estimate obtained by using $\hat{K}_k$ as $\hat{x}_k$. Denote the constrained optimal gains given in (16) as $\tilde{K}_k$ and $\tilde{L}_k$. Denote the one-step-ahead state estimate obtained by using $\tilde{K}_k$ as $\tilde{x}_k$.

In this section we show that the constrained one-step-ahead H_∞ state estimate is equal to the unconstrained one-step-ahead state estimate projected onto the constraint surface. We also show that the error covariance bound of the constrained one-step-ahead H_∞ state estimate is less than or equal to the error covariance bound of the unconstrained one-step-ahead H_∞ state estimate.

Theorem 3: The constrained one-step-ahead H_∞ state estimate is equal to the unconstrained one-step-ahead state estimate projected onto the constraint surface.

Proof: We assume, as stated above, that at the present time the two state estimates are equal and satisfy the state constraint; that is, $\tilde{x}_k = \hat{x}_k$. Then $\tilde{Q}_k = \hat{Q}_k$ and $\tilde{\Sigma}_k = \hat{\Sigma}_k$. So the constrained state estimate at the next time step is seen from (2), (11), and (16) to be as follows.

$$\begin{aligned} \tilde{x}_{k+1} &= A\tilde{x}_k + \tilde{K}_k(y_k - C\tilde{x}_k) \\ &= A\hat{x}_k + (I - V_{k+1})\hat{K}_k(y_k - C\hat{x}_k) \end{aligned} \tag{19}$$

Recall that the noise-free system satisfies the state constraint. Therefore, since the present state estimate satisfies the state constraint (that is, $D_k\hat{x}_k = d_k$) we can see that $D_{k+1}A\hat{x}_k = d_{k+1}$ [2]. It therefore follows that $D_{k+1}^T D_{k+1} A\hat{x}_k = D_{k+1}^T d_{k+1}$, which can be written as $V_{k+1}A\hat{x}_k = D_{k+1}^T d_{k+1}$. We therefore add $D_{k+1}^T d_{k+1}$ and subtract $V_{k+1}A\hat{x}_k$ on the right side of (19) to obtain

$$\begin{aligned} \tilde{x}_{k+1} &= (I - V_{k+1})(A\hat{x}_k + \hat{K}_k(y_k - C\hat{x}_k)) + D_{k+1}^T d_{k+1} \\ &= (I - V_{k+1})\hat{x}_{k+1} + D_{k+1}^T d_{k+1} \\ &= \hat{x}_{k+1} - D_{k+1}^T(D_{k+1}\hat{x}_{k+1} - d_{k+1}) \end{aligned} \tag{20}$$

But this equation is just equal to $\hat{x}_{k+1}$ projected onto the constraint surface. That is, $\tilde{x}_{k+1}$ given in (20) is the solution to the following problem.

$$\begin{aligned} &\min_{\tilde{x}_{k+1}} (\tilde{x}_{k+1} - \hat{x}_{k+1})^T(\tilde{x}_{k+1} - \hat{x}_{k+1}) \\ &\text{such that } D_{k+1}\tilde{x}_{k+1} = d_{k+1} \end{aligned} \tag{21}$$

So the one-step-ahead H_∞ state estimate subject to state equality constraints is equal to the one-step-ahead unconstrained estimate projected onto the constraint surface.

QED

Theorem 4: Consider the error covariance bound $\hat{Q}_{k+1}$ of the one-step-ahead unconstrained estimator given in (13), and the error covariance bound $\tilde{Q}_{k+1}$ of the one-step-ahead constrained estimator given in (18). The constrained bound is less than or equal to the unconstrained bound. That is, $\text{Trace}(\hat{Q}_{k+1} - \tilde{Q}_{k+1}) \geq 0$.

Proof: We again assume, as stated above, that at the present time the two state estimates are equal; that is, $\tilde{x}_k = \hat{x}_k$. Therefore $\hat{Q}_k = \tilde{Q}_k$ and $\hat{\Sigma}_k = \tilde{\Sigma}_k$ and we obtain the following from (12) and (17).

$$\hat{Q}_{k+1} - \tilde{Q}_{k+1} = A\hat{\Sigma}_k A^T - (I - V_{k+1})A\hat{\Sigma}_k A^T (I - V_{k+1}) \tag{22}$$

Recall that $\text{Trace}(FG) = \text{Trace}(GF)$ for any square matrices F and G of equal dimensions. We therefore obtain

$$\begin{aligned} &\text{Trace}(\hat{Q}_{k+1} - \tilde{Q}_{k+1}) \\ &= \text{Trace}(A\hat{\Sigma}_k A^T) - \text{Trace}(A\hat{\Sigma}_k A^T (I - V_{k+1})^2) \\ &= \text{Trace}(A\hat{\Sigma}_k A^T) - \text{Trace}(A\hat{\Sigma}_k A^T (I - V_{k+1})) \end{aligned} \tag{23}$$

where we have used the fact in the second equality that $(I - V_{k+1})^2 = (I - V_{k+1})$ (because $D_{k+1}D_{k+1}^T = I$). Now recall that $V_{k+1} = D_{k+1}^T D_{k+1}$. But $D_{k+1}D_{k+1}^T = I$ with all eigenvalues equal to 1. Therefore each eigenvalue of V_{k+1} is equal to either 0 or 1, from which we see that each eigenvalue of $(I - V_{k+1})$ is also equal to 0 or 1. Using these facts, we obtain from [16, p. 433] that

$$\text{Trace}(A\hat{\Sigma}_k A^T (I - V_{k+1})) \leq \text{Trace}(A\hat{\Sigma}_k A^T) \tag{24}$$

We combine this result with (23) to obtain $\text{Trace}(\hat{Q}_{k+1} - \tilde{Q}_{k+1}) \geq 0$, which is the desired result.

QED

D. H_∞ Filtering with State Inequality Constraints

This section shows how the equality-constrained H_∞ filtering results of the previous section can be extended to inequality-constrained H_∞ filtering. Consider the dynamic system of (1) where we are given the constraint

$$D_k x_k \leq d_k \tag{25}$$

where D_k is a known $s \times n$ matrix, s is the number of constraints, n is the number of states, and $s \leq n$. As before, we assume that D_k is full rank, i.e., that D_k has rank s. We want to find a state estimate $\tilde{x}_k$ that satisfies saddle point (10) subject to constraint (25). If the unconstrained state estimate $\hat{x}_k$ satisfies the constraint (25), our problem is solved. If the unconstrained state estimate does not satisfy the constraint, then, inspired by (21), we can find $\tilde{x}_k$ by solving

$$\min_{\tilde{x}_k} (\tilde{x}_k - \hat{x}_k)^T (\tilde{x}_k - \hat{x}_k) \text{ such that } D_k \tilde{x}_k \leq d_k \tag{26}$$

The problem defined by (26) is a quadratic programming problem [17], [18]. One way to solve a quadratic programming problem is with an active set method. This uses the fact that it is only those constraints that are active at the problem solution that are significant in the optimality conditions. Assume that t of the s inequality constraints are active at the solution of (26), and denote by $\hat{D}_k$ and $\hat{d}_k$ the t rows of D_k and t elements of d_k corresponding to the active constraints. If the set of active constraints were known *a priori* then the solution of (26) would be a solution of the equality-constrained problem

$$\min_{\tilde{x}_k} (\tilde{x}_k - \hat{x}_k)^T (\tilde{x}_k - \hat{x}_k) \text{ such that } \hat{D}_k \tilde{x}_k = \hat{d}_k \tag{27}$$

The inequality constrained problem (26) is equivalent to the equality-constrained problem (27). But the equality-constrained problem was discussed in Sections II-B and II-C, so those results apply to the inequality-constrained problem. For instance, we can use the results of Section II-B to see that the inequality-constrained state estimate is unbiased. We can use the results of Section II-C to see that the error covariance bound of the inequality-constrained state estimate is less than or equal to that of the unconstrained state estimate.

III. TURBOFAN ENGINE HEALTH MONITORING

A good overview of turbofan engine technology can be found in [19]. A single inlet supplies airflow to the fan. Air leaving the fan separates into two streams: one through the engine core, and the other through the bypass duct. The fan is driven by the low pressure turbine. The air passing through the engine core moves through the compressor, which is driven by the high pressure turbine. Fuel is injected in the main combustor and burned to produce hot gas for driving the turbines. The two air streams combine in the augmentor duct, where additional fuel is added to further increase the temperature. The air leaves the augmentor through the nozzle, which has a variable cross section area.

The performance of gas turbine engines deteriorates over time. This deterioration reduces the fuel economy of the turbine. Airlines periodically collect performance data in order to evaluate the health of the engine and its components. The health evaluation is then used to determine maintenance schedules. The data used to perform health evaluations are typically collected during flight and are later transferred to ground-based computers for post flight analysis.

The turbofan simulation model that we used in this paper is based on a gas turbine engine simulation software package called DIGTEM (Digital Turbofan Engine Model) [20]. DIGTEM is written in Fortran and includes 16 states x, 6 controls u, 8 health parameters p, and 12 measurements y. The elements in these vectors (along with their nominal values and the measurement signal-to-noise ratios) are summarized in [21]. The system equations can be linearized about the nominal operating point by using the first order approximation of the Taylor series expansion. We then discretize the system and augment the health parameter vector onto the state vector [21]. This gives us a linear system in δx and δp, which are the deviations of the states and health parameters from their nominal values. The state vector is then redefined as $x = \begin{bmatrix} \delta x^T & \delta p^T \end{bmatrix}^T$. The linearized system equation then becomes

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} A_1 & A_2 \\ 0 & I \end{bmatrix} x_k + w_k \\ \delta y_k &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x_k + v_k \end{aligned} \tag{28}$$

The first 16 elements of x comprise the original state, and the last 8 elements of x comprise the health parameter vector. Now we can use a linear filter to estimate x_k. The application of Kalman filtering to this problem was discussed in [11]. However, H_∞ filtering might be more advantageous in cases where the noise statistics are not well known, the system matrices are not well known, or the user is more interested in minimizing worst-case estimation error rather than root-mean-square estimation error [14]. This will be demonstrated in the simulation results in Section IV.

It is known that health parameters do not improve over time. That is, $p(1)$, $p(2)$, $p(3)$, $p(4)$, $p(6)$, and $p(8)$ are monotonically nonincreasing and are always less than or equal to zero. Similarly, $p(5)$ and $p(7)$ are monotonically nondecreasing and are always greater than or equal to zero. In addition, we know that the health parameters vary slowly with time. We can use this knowledge of the health parameter behavior to enforce constraints on their estimates. For example, since $\tilde{x}(17)$ is the constrained estimate of $p(1)$, we can enforce the following constraints on $\tilde{x}(17)$.

$$\begin{aligned} \tilde{x}(17) &\leq 0 \\ \tilde{x}_{k+1}(17) &\leq \tilde{x}_k(17) + \gamma_{17}^+ \\ \tilde{x}_{k+1}(17) &\geq \tilde{x}_k(17) - \gamma_{17}^- \end{aligned} \tag{29}$$

where γ_{17}^+ and γ_{17}^- are factors chosen by the user that allow the state estimate to vary only within prescribed limits. Typically we choose $\gamma_{17}^- > \gamma_{17}^+$ so that the state estimate can change more in the negative direction than in the positive direction. This is in keeping with our *a priori* knowledge that this particular state is monotonically nonincreasing. Ideally we would have $\gamma_{17}^+ = 0$ since $x(17)$ never increases. However, since the state variable estimate varies around the true value of the state variable, we choose $\gamma_{17}^+ > 0$. This allows some time-varying increase in the state variable estimate to compensate for a state variable estimate that is smaller than the true state variable value.

IV. SIMULATION RESULTS

We simulated the methods discussed in this paper using MATLAB. We simulated 3 seconds of steady state engine data measured at 5 Hz during each flight. Each of these routine services was performed at the single operating point and signal-to-noise ratios shown in [21]. For the constrained filter we chose the γ variables in (29) such that the maximum allowable rate of change in $\tilde{x}$ was 9% per 500 flights in the direction of expected change, and 3% per 500 flights in the opposite direction. These numbers are in line with turbofan engine performance data collected by NASA and reported in the literature [22].

We simulated a degradation over 500 flights of the following values for the health parameters: -1% for fan airflow, -2% for fan efficiency, -3% for compressor airflow, -2% for compressor efficiency, $+3\%$ for high pressure turbine airflow, -2% for high pressure turbine enthalpy change, $+2\%$ for low pressure turbine airflow, and -1% for low pressure turbine enthalpy change. Table I shows the performance of the filters averaged over eight simulations like this (each simulation being subject to a different random noise history). The Kalman filters perform better than the H_∞ filters in this scenario, and the constrained filters perform better than the unconstrained filters.

Health Parameter	Unconstr Kalman	Constr Kalman	Unconstr H_∞	Constr H_∞
Fan Airflow	0.192	0.154	0.274	0.232
Fan Efficiency	0.164	0.145	0.366	0.261
Compressor Flow	0.194	0.183	0.251	0.220
Compressor Eff.	0.124	0.101	0.168	0.123
HPT Flow	0.167	0.155	0.170	0.145
HPT ΔEnthalpy	0.126	0.113	0.159	0.125
LPT Flow	0.146	0.123	0.169	0.137
LPT ΔEnthalpy	0.249	0.220	0.304	0.235
Average	0.170	0.149	0.233	0.185

TABLE I

AVERAGE RMS FILTER ESTIMATION ERRORS (PERCENT) WHEN THE MEASUREMENT BIAS IS ZERO. HPT STANDS FOR HIGH PRESSURE TURBINE, AND LPT STANDS FOR LOW PRESSURE TURBINE.

However, one of the attractions of H_∞ filtering is its performance in the presence of worst case noise inputs. Another attraction is its robustness to nonideal conditions such as unmodeled system dynamics or noise [1], [14], [23]. With this in mind, we considered measurement biases of one-half of a standard deviation and repeated the simulations. We ran 24 simulations with measurement biases. Each simulation imposed a bias of either plus one-half or minus one-half of a standard deviation on one of the 12 measurements. In each simulation, all eight of the health parameters degraded linearly over 500 flights by an amount between -3% and $+3\%$ (as detailed above). The worst-case filter performance (over the 24 simulations) is shown in Table II, where we see that the performance of the H_∞ filter is better than that of the Kalman filter. This is consistent with the theoretical properties of the H_∞ filter relative to optimal worst-case performance and robustness to unmodeled noise. A filter designer may therefore desire to use an H_∞ filter rather than a Kalman filter if worst-case performance in the presence of modeling errors is more important than RMS performance or performance under nominal conditions. Table II also shows that the constrained H_∞ filter performs better than the unconstrained H_∞ filter.

The filters with inequality constraints require about four times the computational effort of the unconstrained filters. This is because of the additional quadratic programming problem that is required for the constrained filters. However, computational effort is not a critical issue for the particular application of turbofan health estimation since it is performed on ground-based computers after each flight.

V. CONCLUSION AND DISCUSSION

We have presented a method for incorporating linear state inequality constraints in an H_∞ filter. Theoretical results prove the benefits of the incorporation of inequality con-

Health Parameter	Unconstr Kalman	Constr Kalman	Unconstr H_∞	Constr H_∞
Fan Airflow	1.489	1.214	2.006	0.578
Fan Efficiency	2.716	2.551	0.988	0.864
Compressor Flow	1.405	1.276	1.871	1.580
Compressor Eff.	1.200	1.119	1.228	0.865
HPT Airflow	1.512	1.318	1.778	1.684
HPT ΔEnthalpy	0.990	0.998	0.539	0.544
LPT Flow	1.000	0.853	1.469	0.865
LPT ΔEnthalpy	2.436	2.218	0.702	0.574
Maximum	2.716	2.551	2.006	1.684

TABLE II

MAXIMUM RMS FILTER ESTIMATION ERRORS (PERCENT) WHEN THE MEASUREMENTS ARE BIASED $\pm\sigma/2$.

straints, and simulation results demonstrate the effectiveness of this method for turbofan engine health estimation.

The constrained filters require a larger computational effort than the unconstrained filters. (This is true for both the Kalman and H_∞ filters.) This is due to the addition of the quadratic programming problem that must be solved in the constrained filters. The engineer must perform a tradeoff between computational effort and estimation accuracy.

Comparison between the Kalman and H_∞ filters showed that the Kalman filters perform better under nominal conditions or if the objective is the minimization of root-mean-square error. However, if the system has measurement biases and the objective is the minimization of worst-case error, then the H_∞ filters perform significantly better than the Kalman filters. The results shown in this paper verify the robustness of the H_∞ filter relative to the Kalman filter. If the user has high confidence in the sensors, then the Kalman filter may be the best choice for turbofan health estimation. But if the user wants to minimize the worst case estimation error in the presence of possible sensor biases, then the H_∞ filter may be the best choice. Incorporation of state constraints will further improve filter performance.

Although this paper has considered only linear state constraints, nonlinear constraints can be linearized as in [24].

This paper opens several avenues for further research. The stability of the constrained H_∞ filter is an open issue. Possible applications of this work include neural network and fuzzy logic training in cases where finite precision arithmetic during implementation is a consideration. H_∞ filtering can result in neural net or fuzzy logic parameters that are robust to discretization errors. Training with constrained filters can be used to enforce sum normal fuzzy membership functions [25], or improve neural network robustness [26]. Constrained H_∞ filtering could be extended to systems with uncertain parameters. Combining the constrained H_∞ and Kalman filters to obtain a mixed constrained H_2/H_∞ filter could be important. A nonlinear constrained H_∞ filter could be designed using the moving horizon approach [27], [28].

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