$$D(f,\theta,\psi)=\int_0^{2\pi}\int_0^a A(f,r_A,\phi_A)\exp\left[+j\frac{2\pi r_A}{\lambda}\sin\theta\cos(\psi-\phi_A)\right]r_A\,dr_A\,d\phi_A\,, \tag{3C-12}$$

where

$$A(f,r_A,\phi_A)=a(f,r_A,\phi_A)\exp[+j\theta(f,r_A,\phi_A)] \tag{3C-13}$$

is the complex frequency response (complex aperture function) of the circular aperture, where r_A and ϕ_A are the polar coordinates of an aperture point [see (3.3-10)]. If $a(f,r_A,\phi_A)\geq 0$, then

$$\boxed{D_{\max}=\max\left|D(f,\theta,\psi)\right|=\int_0^{2\pi}\int_0^a a(f,r_A,\phi_A)r_A\,dr_A\,d\phi_A} \tag{3C-14}$$

or

$$\boxed{D_{\max}=\max\left|\mathcal{S}(f)\right|=\int_0^{2\pi}\int_0^a a(f,r_A,\phi_A)r_A\,dr_A\,d\phi_A} \tag{3C-15}$$

Note that if (3C-12) is evaluated at broadside (i.e., at $\theta=0°$), and if the phase response $\theta(f,r_A,\phi_A)=0$ and the amplitude response $a(f,r_A,\phi_A)\geq 0$, then

$$\boxed{D_{\max}=\max\left|D(f,\theta,\psi)\right|=\left|D(f,0°,\psi)\right|} \tag{3C-16}$$

Chapter 4

Time-Average Radiated Acoustic Power

4.1 Directivity and Directivity Index

In this section we shall derive an equation for the *directivity* of an aperture, where the aperture is characterized by its far-field beam pattern. Recall that the term "aperture" can refer to either a single electroacoustic transducer or an array of electroacoustic transducers. The *directivity index* is simply the decibel equivalent of the directivity.

The directivity of an aperture, when used in the active mode as a transmitter, is a measure of its ability to concentrate the available acoustic power into a preferred direction. When used in the passive mode as a receiver, the directivity of an aperture is a measure of its ability to distinguish between several sound-sources located at different positions in the fluid medium. Therefore, directivity is basically a measure of the beamwidth and sidelobe levels of a far-field beam pattern. In the analysis that follows, the aperture is being used as a transmitter. Note that if the transducer or transducers that make up an aperture are *reversible*, that is, if they can be used as either transmitters or receivers, then the transmit and receive far-field beam patterns of the aperture are *identical*.

In order to derive an equation for the directivity of an aperture when used as a transmitter (a sound-source), we must derive equations for both the time-average intensity vector of the radiated acoustic field, and the time-average radiated acoustic power. However, in order to derive equations for the intensity and power, we must first derive equations for both the radiated acoustic pressure and the radiated acoustic fluid-velocity-vector due to the sound-source. We begin by using the following results obtained in Example 1.3-1: when an identical, time-harmonic, input electrical signal with complex amplitude A_x volts (V) is applied at all points on the surface of a *closed-surface*, transmit volume aperture, the scalar velocity potential in squared meters per second and the acoustic pressure in pascals (Pa) of the radiated acoustic field in the *far-field* region of the aperture are given by

$$\varphi(t,r,\theta,\psi) \approx -\frac{A_x}{4\pi r} D(f,\theta,\psi) \exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right] \tag{4.1-1}$$

and

$$p(t,r,\theta,\psi) \approx jk\rho_0 c \frac{A_x}{4\pi r} D(f,\theta,\psi) \exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right], \tag{4.1-2}$$

respectively, for $t \ge r/c$, where $D(f,\theta,\psi)$ is the unnormalized, far-field beam pattern of the aperture with units of $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$,

$$k = 2\pi f/c = 2\pi/\lambda \tag{4.1-3}$$

is the wavenumber in radians per meter, ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter, and c is the constant speed of sound in the fluid medium in meters per second. Note that $1\,\mathrm{Pa} = 1\,\mathrm{N}/\mathrm{m}^2$, where $1\,\mathrm{N} = 1\,\mathrm{kg\text{-}m}/\mathrm{sec}^2$. Therefore, $1\,\mathrm{Pa} = 1\,\mathrm{kg}/(\mathrm{m\text{-}sec}^2)$. Also note that the ambient density ρ_0 and the speed of sound c are constants because we are dealing with a homogeneous fluid medium. Since we already have an equation for the radiated acoustic pressure as given by (4.1-2), we need to derive an equation for the radiated acoustic fluid-velocity-vector next.

Recall from Subsection 1.2.1 that the acoustic fluid-velocity-vector (a.k.a. the acoustic particle-velocity-vector) $\mathbf{u}(t,r,\theta,\psi)$ in meters per second can be obtained from the scalar velocity potential $\varphi(t,r,\theta,\psi)$ as follows:

$$\mathbf{u}(t,r,\theta,\psi) = \nabla\varphi(t,r,\theta,\psi)\,, \tag{4.1-4}$$

where

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\psi}\hat{\psi} \tag{4.1-5}$$

is the gradient expressed in the spherical coordinates (r,θ,ψ), and $\hat{r}$, $\hat{\theta}$, and $\hat{\psi}$ are *unit vectors* in the r, θ, and ψ directions, respectively. As will be shown later, we only need to derive an equation for the *radial* component of the acoustic fluid-velocity-vector because only the radial component contributes to the time-average radiated acoustic power. By referring to (4.1-4) and (4.1-5), the radial component of the acoustic fluid-velocity-vector is given by

$$\mathbf{u}_r(t,r,\theta,\psi) = \frac{\partial}{\partial r}\varphi(t,r,\theta,\psi)\hat{r}\,, \tag{4.1-6}$$

and by substituting (4.1-1) into (4.1-6), we obtain

$$\mathbf{u}_r(t,r,\theta,\psi) \approx -\frac{A_x}{4\pi}D(f,\theta,\psi)\frac{\partial}{\partial r}\left[\frac{\exp(-jkr)}{r}\right]\exp(+j2\pi ft)\hat{r}\,, \tag{4.1-7}$$

where k is given by (4.1-3). Since

$$\frac{\partial}{\partial r}\frac{\exp(-jkr)}{r} = -k^2\left[\frac{1}{(kr)^2} + j\frac{1}{kr}\right]\exp(-jkr)\,, \tag{4.1-8}$$

and since the field point is in the far-field region of the aperture where it is assumed that $kr >> 1$, (4.1-8) reduces to

$$\frac{\partial}{\partial r}\frac{\exp(-jkr)}{r} \approx -jk\frac{\exp(-jkr)}{r}, \qquad kr >> 1\,. \tag{4.1-9}$$

Therefore, substituting (4.1-9) into (4.1-7) yields

$$\mathbf{u}_r(t,r,\theta,\psi) \approx jk\frac{A_x}{4\pi r}D(f,\theta,\psi)\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right]\hat{r} \tag{4.1-10}$$

for $t \geq r/c$, or

$$\mathbf{u}_r(t,r,\theta,\psi) = u_r(t,r,\theta,\psi)\hat{r}\,, \tag{4.1-11}$$

where

$$u_r(t,r,\theta,\psi) \approx jk\frac{A_x}{4\pi r}D(f,\theta,\psi)\exp\left[+j2\pi f\left(t-\frac{r}{c}\right)\right] \tag{4.1-12}$$

is the complex acoustic *fluid-speed* (a.k.a. the complex acoustic *particle-speed*) in meters per second in the radial direction. Note that the acoustic pressure given by (4.1-2) can be expressed as

$$p(t,r,\theta,\psi) = \rho_0 c\, u_r(t,r,\theta,\psi)\,, \tag{4.1-13}$$

where the factor $\rho_0 c$ is referred to as the characteristic impedance of the fluid medium in rayls ($1\,\text{rayl} = 1\,\text{Pa-sec/m}$). Equation (4.1-13) is a plane-wave relationship between acoustic pressure and acoustic fluid-speed. Therefore, the radiated acoustic field behaves like a plane wave in the radial direction in the far-field region of the aperture.

Now that we have equations for both the acoustic pressure and the acoustic fluid-velocity-vector in the radial direction, we can derive an equation for the time-average intensity vector in the radial direction. We begin by rewriting the acoustic pressure and acoustic fluid-velocity-vector given by (4.1-2) and (4.1-10) as follows:

$$p(t,r,\theta,\psi) = p_f(r,\theta,\psi)\exp(+j2\pi ft) \tag{4.1-14}$$

and

$$\mathbf{u}_r(t,r,\theta,\psi) = \mathbf{u}_{f,r}(r,\theta,\psi)\exp(+j2\pi ft)\,, \tag{4.1-15}$$

where

$$p_f(r,\theta,\psi) \approx jk\rho_0 c \frac{A_x}{4\pi r} D(f,\theta,\psi)\exp(-jkr) \tag{4.1-16}$$

and

$$\mathbf{u}_{f,r}(r,\theta,\psi) \approx jk \frac{A_x}{4\pi r} D(f,\theta,\psi)\exp(-jkr)\hat{r} \,. \tag{4.1-17}$$

From (4.1-14) and (4.1-15) it is clear that the acoustic pressure and acoustic fluid-velocity-vector are time-harmonic acoustic fields. A time-harmonic acoustic field is equal to the product between a spatial-dependent part and the time-dependent part $\exp(+j2\pi ft)$. As a result, it depends on a single frequency f and is a periodic function of time. Either $\cos(2\pi ft)$ or $\sin(2\pi ft)$ can be used instead of $\exp(+j2\pi ft)$, but it is more customary to use $\exp(+j2\pi ft)$. For the special case of time-harmonic acoustic fields where

$$p(t,\mathbf{r}) = p_f(\mathbf{r})\exp(+j2\pi ft) \tag{4.1-18}$$

and

$$\mathbf{u}(t,\mathbf{r}) = \mathbf{u}_f(\mathbf{r})\exp(+j2\pi ft) \,, \tag{4.1-19}$$

the time-average intensity vector in watts per squared meter is given by

$$\boxed{\mathbf{I}_{\text{avg}}(\mathbf{r}) = \frac{1}{2}\text{Re}\left\{ p_f(\mathbf{r})\mathbf{u}_f^*(\mathbf{r}) \right\}} \tag{4.1-20}$$

where the asterisk denotes complex conjugate. Therefore, in our problem, the time-average intensity vector in the radial direction can be computed by using the following equation:

$$\mathbf{I}_{\text{avg},r}(r,\theta,\psi) = \frac{1}{2}\text{Re}\left\{ p_f(r,\theta,\psi)\mathbf{u}_{f,r}^*(r,\theta,\psi) \right\} . \tag{4.1-21}$$

Substituting (4.1-16) and (4.1-17) into (4.1-21) yields

$$\mathbf{I}_{\text{avg},r}(r,\theta,\psi) \approx \frac{k^2\rho_0 c}{32\pi^2 r^2} \left|A_x\right|^2 \left|D(f,\theta,\psi)\right|^2 \hat{r} \,. \tag{4.1-22}$$

We shall now use (4.1-22) to derive an equation for the time-average radiated acoustic power.

The time-average radiated acoustic power in watts can be obtained by evaluating the following closed-surface integral:

$$\boxed{P_{\text{avg}} = \oint_S \mathbf{I}_{\text{avg}}(\mathbf{r}) \bullet \mathbf{dS}} \tag{4.1-23}$$

where S is *any closed-surface* enclosing the sound-source (transmit aperture),

$$\mathbf{dS} = dS\,\hat{n}\,, \tag{4.1-24}$$

dS is an infinitesimal element of surface area, and $\hat{n}$ is a unit vector normal to the surface S, pointing in the conventional outward direction away from the enclosed volume and, hence, the source. Note that $1\text{ W} = 1\text{ J/sec}$, where $1\text{ J} = 1\text{ N-m}$ and $1\text{ N} = 1\text{ kg-m/sec}^2$. Therefore, $1\text{ W} = 1\text{ kg-m}^2/\text{sec}^3$. If we enclose the transmit aperture with a *sphere* of radius r meters, then

$$\mathbf{dS} = r^2 \sin\theta\, d\theta\, d\psi\, \hat{r}\,. \tag{4.1-25}$$

Since $\mathbf{dS}$ only has a radial component,

$$\mathbf{I}_{\text{avg}}(\mathbf{r}) \bullet \mathbf{dS} = I_{\text{avg},r}(r,\theta,\psi)\,dS \approx \frac{k^2 \rho_0 c}{32\pi^2} \left|A_x\right|^2 \left|D(f,\theta,\psi)\right|^2 \sin\theta\, d\theta\, d\psi\,, \tag{4.1-26}$$

where $I_{\text{avg},r}(r,\theta,\psi) = \left|\mathbf{I}_{\text{avg},r}(r,\theta,\psi)\right|$ [see (4.1-22)]. As can be seen from (4.1-26), even if we had computed all three components of the radiated acoustic fluid-velocity-vector $\mathbf{u}(t,r,\theta,\psi)$ instead of just the radial component $\mathbf{u}_r(t,r,\theta,\psi)$ so that the time-average intensity vector $\mathbf{I}_{\text{avg}}(\mathbf{r})$ would have had three components, only the magnitude of the *radial component* of $\mathbf{I}_{\text{avg}}(\mathbf{r})$ would remain after the dot product between $\mathbf{I}_{\text{avg}}(\mathbf{r})$ and $\mathbf{dS}$. And finally, by substituting (4.1-26) into (4.1-23), we obtain

$$\boxed{P_{\text{avg}} \approx \frac{k^2 \rho_0 c}{32\pi^2} \left|A_x\right|^2 \int_0^{2\pi} \int_0^{\pi} \left|D(f,\theta,\psi)\right|^2 \sin\theta\, d\theta\, d\psi} \tag{4.1-27}$$

which is the time-average acoustic power radiated by a transmit volume aperture at frequency f hertz. Equation (4.1-27) is a general result that is also applicable to linear and planar apertures since they are just special cases of a volume aperture as was discussed in Chapters 2 and 3. Therefore, (4.1-27) can be used to compute the time-average acoustic power radiated at frequency f hertz by a single electroacoustic transducer (e.g., a continuous line source or rectangular piston) or by a linear, planar, or volume array of electroacoustic transducers. All that is needed is the far-field beam pattern $D(f,\theta,\psi)$ of the aperture. We are now in a position to compute the directivity of a transmit aperture.

The *directivity* D of a transmit aperture is defined as follows:

$$\boxed{\mathrm{D} \triangleq I_{\max} / I_{\mathrm{ref}}} \tag{4.1-28}$$

where $I_{\max}$ is the maximum value of the magnitude of the far-field, time-average intensity vector in the radial direction of the acoustic field radiated by the aperture, and I_{ref} is a reference intensity. By referring to (4.1-22),

$$I_{\max} = \frac{k^2 \rho_0 c}{32\pi^2 r^2} \left| A_x \right|^2 D_{\max}^2 , \tag{4.1-29}$$

where $D_{\max}$ is the maximum value of the magnitude of the unnormalized, far-field beam pattern, that is, $D_{\max} = \max \left| D(f,\theta,\psi) \right|$, and is referred to as the normalization factor (see Subsection 2.2.1).

The reference intensity I_{ref} is defined as the time-average intensity of the acoustic field radiated by an omnidirectional, spherical point-source that radiates the same time-average acoustic power as the aperture, that is,

$$I_{\mathrm{ref}} \triangleq \frac{P_{\mathrm{avg}}}{4\pi r^2} , \tag{4.1-30}$$

where P_{avg} is given by (4.1-27). Substituting (4.1-27) into (4.1-30) yields

$$I_{\mathrm{ref}} = \frac{k^2 \rho_0 c}{32\pi^2 r^2} \left| A_x \right|^2 D_{\max}^2 \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left| D_N(f,\theta,\psi) \right|^2 \sin\theta \, d\theta \, d\psi , \tag{4.1-31}$$

where

$$D_N(f,\theta,\psi) = D(f,\theta,\psi) / D_{\max} \tag{4.1-32}$$

is the normalized, far-field beam pattern of the aperture.

Substituting (4.1-29) and (4.1-31) into (4.1-28) yields the following general expression for the directivity of a transmit volume aperture at frequency f hertz:

$$\boxed{\mathrm{D} = \frac{4\pi}{\displaystyle\int_0^{2\pi} \int_0^{\pi} \left| D_N(f,\theta,\psi) \right|^2 \sin\theta \, d\theta \, d\psi}} \tag{4.1-33}$$

Note that the directivity D is dimensionless. The *directivity index* DI is the decibel equivalent of the directivity and is defined as follows:

$$\boxed{\mathrm{DI} \triangleq 10\log_{10}\mathrm{D}\ \mathrm{dB}} \tag{4.1-34}$$

Values of the directivity range from a minimum of $\mathrm{D}=1$ ($\mathrm{DI}=0\ \mathrm{dB}$) for an *unbaffled*, omnidirectional source where $|D_N(f,\theta,\psi)|=1$ for $0\le\theta\le\pi$ and $0\le\psi\le 2\pi$, to large numbers for highly directional sources. If $|D_N(f,\theta,\psi)|=1$ for $0\le\theta\le\pi$ and $0\le\psi\le 2\pi$, then the denominator in (4.1-33) reduces to

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\psi &= 2\pi\int_0^{\pi}\sin\theta\, d\theta \\ &= -2\pi\cos\theta\Big|_0^{\pi} \\ &= 4\pi,\end{aligned} \tag{4.1-35}$$

and as a result, the directivity $\mathrm{D}=1$ ($\mathrm{DI}=0\ \mathrm{dB}$). However, for an omnidirectional source mounted on a very large baffle in the XY plane, the range of values for θ are from 0 to $\pi/2$. Therefore, the denominator in (4.1-33) reduces to

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\, d\theta\, d\psi &= 2\pi\int_0^{\pi/2}\sin\theta\, d\theta \\ &= -2\pi\cos\theta\Big|_0^{\pi/2} \\ &= 2\pi,\end{aligned} \tag{4.1-36}$$

and as a result, the directivity $\mathrm{D}=2$ ($\mathrm{DI}=3\ \mathrm{dB}$). An example of an omnidirectional source is a spherical sound-source vibrating in the monopole mode of vibration. Also, as a general principle, if the size of an aperture is small compared to a wavelength, then the unnormalized, far-field beam pattern of the aperture reduces to a *constant*, that is, the aperture is omnidirectional – the magnitude of the normalized, far-field beam pattern of the aperture is equal to 1 for all angles (e.g., see Fig. 2.2-3).

The directivity of an aperture can be increased by decreasing the beamwidth of the far-field beam pattern of the aperture. The beamwidth can be decreased by keeping the size of the aperture constant while increasing the frequency, or by keeping the frequency constant while increasing the size of the aperture. And finally, it should be emphasized that both the directivity and the directivity index of either a single electroacoustic transducer or an array of electroacoustic transducers can be computed from (4.1-33) and (4.1-34), respectively, as long as the normalized, far-field beam pattern is known.

4.2 The Source Level of a Directional Sound-Source

In this section we shall derive three main equations. The first equation expresses the source level (SL) of a transmit volume aperture as a function of the time-average acoustic power P_{avg} radiated by the aperture and the directivity index (DI) of the aperture. The second equation expresses the SL of a transmit volume aperture as a function of P_0, where P_0 is the maximum value of the magnitude of the acoustic pressure measured at a distance (range) of one meter from the aperture along its acoustic axis. The third equation expresses the time-average acoustic power P_{avg} radiated by a transmit volume aperture as a function of the SL and directivity (D) of the aperture.

We begin the derivation of the first main equation by rewriting (4.1-27) as follows:

$$P_{\text{avg}} = \frac{k^2 \rho_0 c}{8\pi} |S_0|^2 \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} |D_N(f,\theta,\psi)|^2 \sin\theta \, d\theta \, d\psi \, , \tag{4.2-1}$$

where

$$S_0 = A_x D_{\max} \tag{4.2-2}$$

is the *maximum* value of *source strength* in cubic meters per second at frequency f hertz, and $D_N(f,\theta,\psi)$ is the normalized, far-field beam pattern of the aperture [see (4.1-32)]. Substituting (4.1-33) into (4.2-1) yields

$$P_{\text{avg}} = \frac{k^2 \rho_0 c}{8\pi} \frac{|S_0|^2}{\mathrm{D}} \, , \tag{4.2-3}$$

and by solving for $|S_0|^2$, we obtain

$$|S_0|^2 = \mathrm{D} \frac{8\pi}{k^2 \rho_0 c} P_{\text{avg}} \, , \tag{4.2-4}$$

where D is the directivity of the transmit volume aperture. Note that if $\mathrm{D} = 1$, then (4.2-3) reduces to the time-average acoustic power radiated by an omnidirectional, spherical sound-source vibrating in the monopole mode of vibration with $ka \ll 1$, where a is the radius of the sphere in meters.

The *source level* (SL) is defined as follows:

$$\boxed{\mathrm{SL} \triangleq 20 \log_{10}\left(\frac{\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\text{ m}} \right\}}{P_{\text{ref}}} \right) \text{dB re } P_{\text{ref}}} \tag{4.2-5}$$

where $\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\,\text{m}} \right\}$ is the maximum value of the root-mean-square (rms) acoustic pressure measured at a distance (range) of one meter from the source along its acoustic axis, that is, in the direction of the maximum response of the source, and P_{ref} is the rms reference pressure. Since we are dealing with time-harmonic acoustic fields,

$$\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\,\text{m}} \right\} = \frac{\sqrt{2}}{2} \max\left\{ \left| p_f(r,\theta,\psi) \right|\big|_{r=1\,\text{m}} \right\}. \tag{4.2-6}$$

Computing the magnitude of the spatial-dependent part of the radiated acoustic pressure given by (4.1-16) yields

$$\left| p_f(r,\theta,\psi) \right| = k\rho_0 c \frac{|A_x|}{4\pi r} \left| D(f,\theta,\psi) \right|, \tag{4.2-7}$$

and as a result,

$$\max\left\{ \left| p_f(r,\theta,\psi) \right|\big|_{r=1\,\text{m}} \right\} = k\rho_0 c \frac{|A_x|}{4\pi r} D_{\max} \bigg|_{r=1\,\text{m}}, \tag{4.2-8}$$

or

$$\max\left\{ \left| p_f(r,\theta,\psi) \right|\big|_{r=1\,\text{m}} \right\} = k\rho_0 c \frac{|S_0|}{4\pi r} \bigg|_{r=1\,\text{m}}, \tag{4.2-9}$$

where $D_{\max} = \max\left| D(f,\theta,\psi) \right|$ and S_0 is the maximum value of source strength at frequency f given by (4.2-2). And by substituting (4.2-9) into (4.2-6), we obtain

$$\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\,\text{m}} \right\} = \frac{\sqrt{2}}{2} k\rho_0 c \frac{|S_0|}{4\pi r} \bigg|_{r=1\,\text{m}}. \tag{4.2-10}$$

With the use of (4.2-10), we shall now compute the SL according to (4.2-5). Dividing (4.2-10) by P_{ref} and then squaring yields

$$\left[\frac{\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\,\text{m}} \right\}}{P_{\text{ref}}} \right]^2 = k^2 \frac{(\rho_0 c)^2}{P_{\text{ref}}^2} \frac{|S_0|^2}{32\pi^2 r^2} \bigg|_{r=1\,\text{m}}, \tag{4.2-11}$$

and by substituting (4.2-4) into (4.2-11), we obtain

$$\left[\frac{\max\left\{ p_{\text{rms}}(\mathbf{r})\big|_{r=1\,\text{m}} \right\}}{P_{\text{ref}}} \right]^2 = \mathrm{D} \frac{\rho_0 c}{P_{\text{ref}}^2} \frac{P_{\text{avg}}}{4\pi r^2} \bigg|_{r=1\,\text{m}}, \tag{4.2-12}$$

where both sides of (4.2-12) are dimensionless. Therefore,

$$10\log_{10}\left[\frac{\max\left\{p_{\text{rms}}(\mathbf{r})\big|_{r=1\text{ m}}\right\}}{P_{\text{ref}}}\right]^2 = 10\log_{10}\left[\mathrm{D}\frac{\rho_0 c}{P_{\text{ref}}^2}\frac{P_{\text{avg}}}{4\pi r^2}\bigg|_{r=1\text{ m}}\right]$$
$$= 10\log_{10}\mathrm{D} + 10\log_{10}(\rho_0 c) + 10\log_{10}P_{\text{avg}} - 10\log_{10}(4\pi r^2 P_{\text{ref}}^2)\big|_{r=1\text{ m}},$$
(4.2-13)

or

$$\boxed{\mathrm{SL} = 10\log_{10}P_{\text{avg}} + \mathrm{DI} + 10\log_{10}(\rho_0 c) - 10\log_{10}(4\pi P_{\text{ref}}^2)\ \text{dB re}\ P_{\text{ref}}}$$
(4.2-14)

where P_{avg} is the time-average acoustic power radiated by a transmit aperture, and DI is the directivity index of the aperture given by (4.1-34). As can be seen from (4.2-14), the SL of a directional sound-source (transmit aperture) is increased by the value of the DI because the available time-average acoustic power is radiated in preferred directions by the main-lobe of the transmit aperture's far-field beam pattern. The SL of a time-harmonic, omnidirectional sound-source as a function of P_{avg} can be obtained from (4.2-14) by setting $\mathrm{DI} = 0\ \text{dB}$. For problems in underwater acoustics where $P_{\text{ref}} = 1\ \mu\text{Pa (rms)}$, (4.2-14) reduces to

$$\boxed{\mathrm{SL} = 10\log_{10}P_{\text{avg}} + \mathrm{DI} + 10\log_{10}(\rho_0 c) + 109.01\ \text{dB re}\ 1\ \mu\text{Pa (rms)}}$$
(4.2-15)

and if $\rho_0 = 1026\ \text{kg/m}^3$ and $c = 1500\ \text{m/sec}$, which are typical values for seawater, then (4.2-15) reduces to

$$\mathrm{SL} = 10\log_{10}P_{\text{avg}} + \mathrm{DI} + 170.88\ \text{dB re}\ 1\ \mu\text{Pa (rms)}. \tag{4.2-16}$$

The second main equation to be derived expresses the source level (SL) of a transmit volume aperture as a function of the maximum value P_0 of the magnitude of the acoustic pressure measured at a distance (range) of one meter from the aperture along its acoustic axis. From (4.2-9),

$$P_0 = \max\left\{\left|p_f(r,\theta,\psi)\right|\big|_{r=1\text{ m}}\right\} = k\rho_0 c\frac{|S_0|}{4\pi r}\bigg|_{r=1\text{ m}}. \tag{4.2-17}$$

Substituting (4.2-17) into (4.2-10) yields

$$\max\left\{ p_{\text{rms}}(\mathbf{r})\Big|_{r=1\text{ m}} \right\} = \frac{\sqrt{2}}{2} P_0 \,, \tag{4.2-18}$$

and by substituting (4.2-18) into (4.2-5), we obtain

$$\boxed{\text{SL} = 20\log_{10}\left(\frac{\sqrt{2}\,P_0/2}{P_{\text{ref}}} \right) \text{dB re } P_{\text{ref}}} \tag{4.2-19}$$

Two additional useful equations shall be derived next. First, solving for P_0 from (4.2-19) yields

$$\boxed{P_0 = \sqrt{2}\,P_{\text{ref}}\,10^{(\text{SL}/20)}} \tag{4.2-20}$$

and second, substituting wavenumber $k = 2\pi f/c$ into (4.2-17) and solving for $|S_0|$ yields

$$\boxed{|S_0| = \frac{2r}{f\rho_0} P_0 \Big|_{r=1\text{ m}}} \tag{4.2-21}$$

where $|S_0|$ is the magnitude of the maximum value of source strength in cubic meters per second at frequency f hertz [see (4.2-2)]. Therefore, for a given value of SL, P_0 can be computed using (4.2-20), and by substituting that value of P_0 into (4.2-21), the corresponding value for $|S_0|$ can be obtained.

The third and final main equation to be derived expresses the time-average acoustic power P_{avg} radiated by a transmit volume aperture as a function of the source level (SL) and directivity (D) of the aperture. From (4.2-13),

$$\text{SL} = 10\log_{10}\left[\text{D}\frac{\rho_0 c}{P_{\text{ref}}^2} \frac{P_{\text{avg}}}{4\pi r^2} \Big|_{r=1\text{ m}} \right], \tag{4.2-22}$$

and by solving for P_{avg}, we obtain

$$\boxed{P_{\text{avg}} = 4\pi r^2 \frac{P_{\text{ref}}^2}{\text{D}\,\rho_0 c} 10^{(\text{SL}/10)} \Big|_{r=1\text{ m}}} \tag{4.2-23}$$

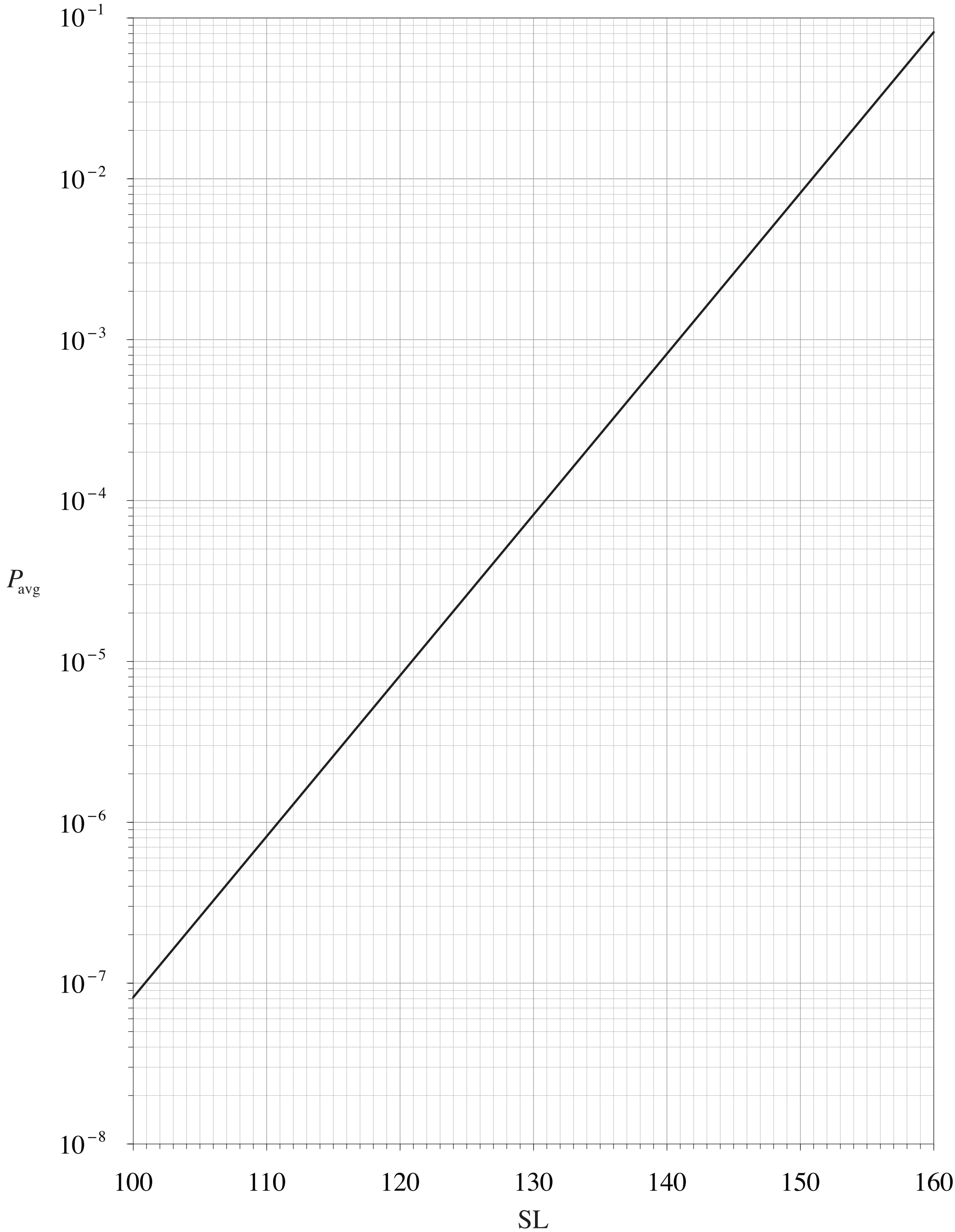

Figure 4.2-1 Time-average acoustic power P_{avg} in watts radiated by a time-harmonic, omnidirectional sound-source versus source level (SL) from 100 to 160 dB re 1 μPa (rms).

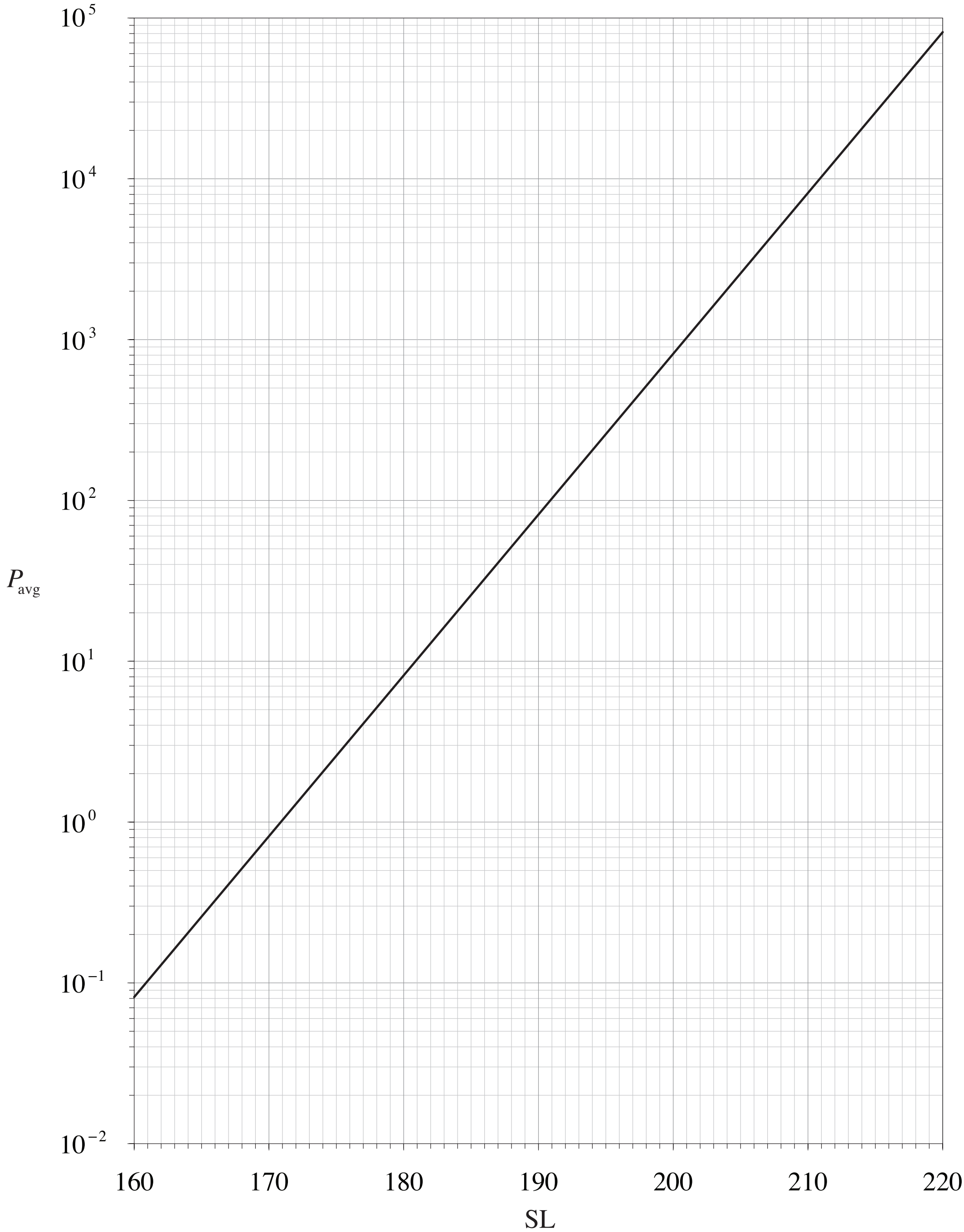

Figure 4.2-2 Time-average acoustic power P_{avg} in watts radiated by a time-harmonic, omnidirectional sound-source versus source level (SL) from 160 to 220 dB re 1 μPa (rms).

Figures 4.2-1 and 4.2-2 are plots of (4.2-23) for $\rho_0 = 1026\ \mathrm{kg/m^3}$, $c = 1500\ \mathrm{m/sec}$, $\mathrm{D} = 1$ (time-harmonic, omnidirectional sound-source), and $P_{\mathrm{ref}} = 1\ \mu\mathrm{Pa\ (rms)}$ for SL values from 100 to $160\ \mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$, and from 160 to $220\ \mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$, respectively. And finally, in order to get a feeling for low and high values for P_0, if we use the same values for ρ_0, c, D, and P_{ref}, and if $P_0 = 1\ \mathrm{Pa}$, then from (4.2-19), $\mathrm{SL} = 116.99\ \mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$, and from (4.2-23), $P_{\mathrm{avg}} = 4.08\ \mu\mathrm{W}$; and if $P_0 = 1\ \mathrm{atm} = 1.01325 \times 10^5\ \mathrm{Pa}$, then $\mathrm{SL} = 217.1$ $\mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$ and $P_{\mathrm{avg}} = 41.89\ \mathrm{kW}$.

Problems

Section 4.1

4-1 Find the *real*, time-harmonic, radiated acoustic pressure in the far-field region of the following transmit apertures:

(a) a linear aperture lying along the Z axis with complex frequency response modeled by the cosine amplitude window

(b) a rectangular piston lying in the XY plane

4-2 Verify (4.1-8).

4-3 In Chapter 5, a side-looking sonar (SLS) lying in the YZ plane is modeled as a single rectangular piston with dimensions $L_Y = 0.1\ \mathrm{m}$ and $L_Z = 2.5\ \mathrm{m}$. If $f = 30\ \mathrm{kHz}$, $\rho_0 = 1026\ \mathrm{kg/m^3}$, and $c = 1500\ \mathrm{m/sec}$, then find the maximum value of the magnitude of the far-field, time-average intensity vector in the radial direction of the acoustic field radiated by the SLS at a range $r = 139\ \mathrm{m}$.

Section 4.2

4-4 If the source level of a directional transmit aperture (a sound-source) is $204\ \mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$, and the directivity index of the aperture is $3\ \mathrm{dB}$, then what is the time-average radiated power in watts? Use $\rho_0 = 1026\ \mathrm{kg/m^3}$, $c = 1500\ \mathrm{m/sec}$, and $P_{\mathrm{ref}} = 1\ \mu\mathrm{Pa\ (rms)}$.

4-5 If the source level of a transmit aperture (a sound-source) is $180\ \mathrm{dB\ re\ 1\ \mu Pa\ (rms)}$ at $25\ \mathrm{kHz}$, then using $\rho_0 = 1026\ \mathrm{kg/m^3}$ and $P_{\mathrm{ref}} = 1\ \mu\mathrm{Pa\ (rms)}$, find