Chapter 10

Bistatic Scattering

10.1 Target Strength

In this section we shall analyze the *bistatic scattering* problem shown in Fig. 10.1-1 for the purpose of deriving an equation that defines the *target strength* of an object. Objects of interest can be, for example, gas bubbles in water, fish, marine mammals (dolphins and whales), submarines, mines, etc. As will be shown later, the target strength of an object can be shown to be directly proportional to either the *differential scattering cross-section* or *scattering function* of an object, or to the ratio between the scattered and incident time-average intensities. Target strength is one of those important parameters that determines the signal-to-noise ratio (SNR) at a receiver and whether or not an object (target) can be detected.

A bistatic scattering problem is one in which the transmitter and receiver are *not* at the same location. Referring to Fig. 10.1-1, the transmitter (sound-source) is located at $\mathbf{r}_T = (x_T, y_T, z_T)$, the object (target or scatterer) is located at $\mathbf{r}_S = (x_S, y_S, z_S)$, where the subscript S denotes scatterer, and the receiver is located at $\mathbf{r}_R = (x_R, y_R, z_R)$. For our purposes, the transmitter is a time-harmonic, omnidirectional point-source, and the receiver is an omnidirectional point-element. The corresponding equation for the source distribution $x_M(t, \mathbf{r})$, with units of inverse seconds, is given by (see Appendix 6C)

$$x_M(t, \mathbf{r}) = S_0 \delta(\mathbf{r} - \mathbf{r}_T) \exp(+j2\pi f t), \tag{10.1-1}$$

where S_0 is the source strength in cubic meters per second at frequency f hertz, the impulse function $\delta(\mathbf{r} - \mathbf{r}_T)$, with units of inverse cubic meters, represents a unit-amplitude, omnidirectional point-source at $\mathbf{r}_T$, and f is the source frequency. An exact solution of the linear wave equation

$$\nabla^2 \varphi(t, \mathbf{r}) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi(t, \mathbf{r}) = x_M(t, \mathbf{r}), \tag{10.1-2}$$

where $\varphi(t, \mathbf{r})$ is the scalar velocity potential in squared meters per second, c is the constant speed of sound in the fluid medium in meters per second, and $x_M(t, \mathbf{r})$ is the source distribution in inverse seconds given by (10.1-1), is discussed in Appendix 10A, along with other equations that we shall need that describe the radiated acoustic field.

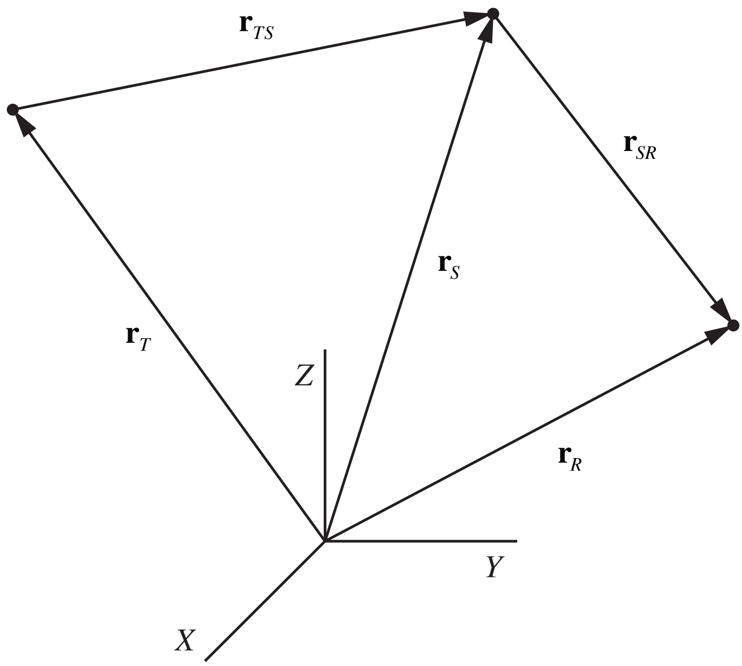

Figure 10.1-1 Bistatic scattering geometry. The transmitter (sound-source) is located at $\mathbf{r}_T = (x_T, y_T, z_T)$, the object (target or scatterer) is located at $\mathbf{r}_S = (x_S, y_S, z_S)$, and the receiver is located at $\mathbf{r}_R = (x_R, y_R, z_R)$.

We begin the analysis of the bistatic scattering problem by computing the acoustic pressure incident upon the object due to a time-harmonic, omnidirectional point-source. The time-harmonic acoustic pressure (in pascals) incident upon the object at $\mathbf{r}_S$ is given by (see Appendix 10A)

$$p_i(t, \mathbf{r}_S) = p_{f,i}(\mathbf{r}_S)\exp(+j2\pi f t), \tag{10.1-3}$$

where

$$p_{f,i}(\mathbf{r}_S) = -jk\rho_0 c\,\varphi_{f,i}(\mathbf{r}_S), \tag{10.1-4}$$

$$\varphi_{f,i}(\mathbf{r}_S) = S_0\, g_f\left(\mathbf{r}_S \,\middle|\, \mathbf{r}_T\right), \tag{10.1-5}$$

$$S_0 = A_x \mathcal{S}_T(f), \tag{10.1-6}$$

$$\begin{aligned} g_f\left(\mathbf{r}_S \,\middle|\, \mathbf{r}_T\right) &= -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}_S - \mathbf{r}_T\right|\right]}{4\pi\left|\mathbf{r}_S - \mathbf{r}_T\right|}\exp\left(-jk\left|\mathbf{r}_S - \mathbf{r}_T\right|\right) \\ &= -\frac{\exp[-\alpha(f) r_{TS}]}{4\pi r_{TS}}\exp(-jkr_{TS}), \end{aligned} \tag{10.1-7}$$

$$\mathbf{r}_{TS} = \mathbf{r}_S - \mathbf{r}_T = (x_S - x_T)\hat{x} + (y_S - y_T)\hat{y} + (z_S - z_T)\hat{z}\,, \tag{10.1-8}$$

$$r_{TS} = \left|\mathbf{r}_{TS}\right| = \sqrt{(x_S - x_T)^2 + (y_S - y_T)^2 + (z_S - z_T)^2}\,, \tag{10.1-9}$$

and

$$\mathbf{r}_{TS} = r_{TS}\hat{r}_{TS}\,, \tag{10.1-10}$$

where $k = 2\pi f/c = 2\pi/\lambda$ is the real wavenumber in radians per meter, $c = f\lambda$ is the constant speed of sound in the fluid medium in meters per second, λ is the wavelength in meters, ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter, S_0 is the source strength in cubic meters per second at frequency f hertz, A_x is the complex amplitude in volts of the time-harmonic, input electrical signal applied to the omnidirectional point-source, $\mathcal{S}_T(f)$ is the complex, transmitter sensitivity function of the omnidirectional point-source in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$ (see Table 6.1-2 and Appendix 6B), $\alpha(f)$ is the real, nonnegative, frequency-dependent, attenuation coefficient of seawater in nepers per meter, and $\hat{r}_{TS}$ is the dimensionless unit vector in the direction of the vector $\mathbf{r}_{TS}$.

The time-average intensity vector (in watts per squared meter) incident upon the object at $\mathbf{r}_S$ is given by (see Appendix 10A)

$$\mathbf{I}_{\mathrm{avg},i}(\mathbf{r}_S) = I_{\mathrm{avg},i}(\mathbf{r}_S)\hat{r}_{TS}\,, \tag{10.1-11}$$

where

$$I_{\mathrm{avg},i}(\mathbf{r}_S) = \frac{1}{2}k^2\rho_0 c\left(\frac{|S_0|}{4\pi r_{TS}}\right)^2 \exp[-2\alpha(f)r_{TS}]\,. \tag{10.1-12}$$

If the origin of a Cartesian coordinate system (with the same orientation as the one shown in Fig. 10.1-1) is placed at $\mathbf{r}_T$, then the spherical angles θ_{TS} and ψ_{TS} that describe the direction of the unit vector $\hat{r}_{TS}$ are measured as shown in Fig. 1.2-2 and are given by (see Appendix 10A)

$$\boxed{\theta_{TS} = \cos^{-1}\left(\frac{z_S - z_T}{r_{TS}}\right)} \tag{10.1-13}$$

and

$$\boxed{\psi_{TS} = \tan^{-1}\left(\frac{y_S - y_T}{x_S - x_T}\right)} \tag{10.1-14}$$

Equations (10.1-13) and (10.1-14) are the *angles of incidence* at the object.

If we enclose the omnidirectional point-source at $\mathbf{r}_T$ with a sphere of radius r_{TS} meters, then the time-average radiated power (in watts) is given by (see Appendix 10A)

$$P_{\text{avg},i} = \frac{1}{2} k^2 \rho_0 c \frac{|S_0|^2}{4\pi} \exp[-2\alpha(f) r_{TS}] = 4\pi r_{TS}^2 I_{\text{avg},i}(\mathbf{r}_S)\,, \tag{10.1-15}$$

where the source strength S_0 is given by (10.1-6), or

$$P_{\text{avg},i} = 4\pi R^2 \frac{P_0^2}{2\rho_0 c} \exp\left[-2\alpha(f)(r_{TS} - R)\right]\Big|_{R=1\text{ m}}\,, \tag{10.1-16}$$

where P_0 is the magnitude of the time-harmonic, radiated acoustic pressure measured at a range $R = 1$ m from the omnidirectional point-source at $\mathbf{r}_T$. The source level (SL) and P_0 are related as follows (see Section 4.2):

$$\text{SL} = 20\log_{10}\left(\frac{\sqrt{2}P_0/2}{P_{\text{ref}}}\right) \text{dB re } P_{\text{ref}}\,, \tag{10.1-17}$$

or

$$P_0 = \sqrt{2}\, P_{\text{ref}}\, 10^{(\text{SL}/20)}\,, \tag{10.1-18}$$

where $P_{\text{ref}} = 1\,\mu\text{Pa (rms)}$ is the root-mean-square (rms) reference pressure used in underwater acoustics.

In order to compute the acoustic field scattered by the object centered at $\mathbf{r}_S$, we shall treat the object as a discrete point-scatterer that acts like a time-harmonic point-source with source strength S_0' cubic meters per second at frequency f hertz, where

$$S_0' = \varphi_{f,i}(\mathbf{r}_S) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})\,, \tag{10.1-19}$$

$\varphi_{f,i}(\mathbf{r}_S)$ is the spatial-dependent part of the time-harmonic, scalar velocity potential (in squared meters per second) incident upon the object at $\mathbf{r}_S$ [see (10.1-5)], and $S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})$ is the *scattering function* (a.k.a. the scattering amplitude[1] or complex acoustical scattering length[2]) of the object (scatterer) with

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[1] A. Ishimaru, *Wave Propagation and Scattering in Random Media*, Vol. 1, Academic Press, New York, 1978, pp. 9-12, 39-40.

[2] H. Medwin and C. S. Clay, *Fundamentals of Acoustical Oceanography*, Academic Press, Boston, 1998, pp. 235-241.

units of meters, where θ_{TS} and ψ_{TS} are the angles of incidence at the object given by (10.1-13) and (10.1-14), respectively, and θ_{SR} and ψ_{SR} are the angles of scatter at the receiver to be given later. The scattering function is a complex function (magnitude and phase) and is, in general, a function of frequency and the directions of wave propagation from the source to the object, and from the object to the receiver. For a given object and fixed scattering geometry, the magnitude of the scattering function can be relatively large for certain resonant or characteristic frequencies of the object, and relatively small for other frequencies. The resonant or characteristic frequencies of an object can be used to identify or classify an object. The magnitude of the scattering function versus frequency is referred to as the *acoustic color* of an object. Substituting (10.1-5) into (10.1-19) yields

$$S_0' = S_0\, g_f\left(\mathbf{r}_S \,|\, \mathbf{r}_T\right) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \,, \tag{10.1-20}$$

where S_0 is given by (10.1-6) and $g_f\left(\mathbf{r}_S \,|\, \mathbf{r}_T\right)$ is given by (10.1-7). For a fixed frequency and geometry, S_0' is a constant.

The time-harmonic acoustic pressure (in pascals) of the scattered acoustic field incident upon the receiver at $\mathbf{r}_R$ is given by (see Appendix 10A)

$$p_s(t, \mathbf{r}_R) = p_{f,s}(\mathbf{r}_R) \exp(+j2\pi f t) \,, \tag{10.1-21}$$

where

$$p_{f,s}(\mathbf{r}_R) = -jk\rho_0 c\, \varphi_{f,s}(\mathbf{r}_R) \,, \tag{10.1-22}$$

$$\varphi_{f,s}(\mathbf{r}_R) = S_0'\, g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right), \tag{10.1-23}$$

$$\begin{aligned} g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right) &= -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}_R - \mathbf{r}_S\right|\right]}{4\pi\left|\mathbf{r}_R - \mathbf{r}_S\right|} \exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_S\right|\right) \\ &= -\frac{\exp[-\alpha(f) r_{SR}]}{4\pi r_{SR}} \exp(-jk r_{SR}), \end{aligned} \tag{10.1-24}$$

$$\mathbf{r}_{SR} = \mathbf{r}_R - \mathbf{r}_S = (x_R - x_S)\hat{x} + (y_R - y_S)\hat{y} + (z_R - z_S)\hat{z} \,, \tag{10.1-25}$$

$$r_{SR} = \left|\mathbf{r}_{SR}\right| = \sqrt{(x_R - x_S)^2 + (y_R - y_S)^2 + (z_R - z_S)^2} \,, \tag{10.1-26}$$

and

$$\mathbf{r}_{SR} = r_{SR}\, \hat{r}_{SR} \,, \tag{10.1-27}$$

where $\hat{r}_{SR}$ is the dimensionless unit vector in the direction of the vector $\mathbf{r}_{SR}$.

The time-average intensity vector (in watts per squared meter) of the scattered acoustic field incident upon the receiver at $\mathbf{r}_R$ is given by (see Appendix 10A)

$$\mathbf{I}_{\text{avg},s}(\mathbf{r}_R) = I_{\text{avg},s}(\mathbf{r}_R)\hat{r}_{SR}\,, \tag{10.1-28}$$

where

$$I_{\text{avg},s}(\mathbf{r}_R) = \frac{1}{2}k^2\rho_0 c\left(\frac{|S_0'|}{4\pi r_{SR}}\right)^2 \exp[-2\alpha(f)r_{SR}]\,. \tag{10.1-29}$$

Substituting (10.1-20) into (10.1-29) yields

$$I_{\text{avg},s}(\mathbf{r}_R) = I_{\text{avg},i}(\mathbf{r}_S)\left[\frac{|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})|}{4\pi r_{SR}}\right]^2 \exp[-2\alpha(f)r_{SR}]\,, \tag{10.1-30}$$

where $I_{\text{avg},i}(\mathbf{r}_S)$ is given by (10.1-12). If the origin of a Cartesian coordinate system (with the same orientation as the one shown in Fig. 10.1-1) is placed at $\mathbf{r}_S$, then the spherical angles θ_{SR} and ψ_{SR} that describe the direction of the unit vector $\hat{r}_{SR}$ are measured as shown in Fig. 1.2-2 and are given by (see Appendix 10A)

$$\boxed{\theta_{SR} = \cos^{-1}\left(\frac{z_R - z_S}{r_{SR}}\right)} \tag{10.1-31}$$

and

$$\boxed{\psi_{SR} = \tan^{-1}\left(\frac{y_R - y_S}{x_R - x_S}\right)} \tag{10.1-32}$$

Equations (10.1-31) and (10.1-32) are the *angles of scatter* at the receiver.

If we enclose the object centered at $\mathbf{r}_S$ with a sphere of radius r_{SR} meters, then the *differential* time-average scattered power (in watts) incident upon the receiver at $\mathbf{r}_R$ is given by (see Appendix 10A)

$$dP_{\text{avg},s} = \mathbf{I}_{\text{avg},s}(\mathbf{r}_R)\bullet\mathbf{dS}\,, \tag{10.1-33}$$

where

$$\mathbf{dS} = dS\,\hat{r}_{SR} = r_{SR}^2\sin\theta_{SR}\,d\theta_{SR}\,d\psi_{SR}\,\hat{r}_{SR}\,. \tag{10.1-34}$$

Substituting (10.1-28), (10.1-30), and (10.1-34) into (10.1-33) yields

$$dP_{\text{avg},s} = I_{\text{avg},i}(\mathbf{r}_S)\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp[-2\alpha(f)r_{SR}]\sin\theta_{SR}\,d\theta_{SR}\,d\psi_{SR}\,, \tag{10.1-35}$$

or

$$dP_{\text{avg},s} = I_{\text{avg},i}(\mathbf{r}_S)\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp[-2\alpha(f)r_{SR}]\,d\Omega\,, \tag{10.1-36}$$

where

$$\boxed{\begin{aligned}\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) &= \left[\frac{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|}{4\pi}\right]^2 \\ &= r_{SR}^2\frac{I_{\text{avg},s}(\mathbf{r}_R)}{I_{\text{avg},i}(\mathbf{r}_S)}\exp[+2\alpha(f)r_{SR}]\end{aligned}} \tag{10.1-37}$$

is the *differential scattering cross-section* (in squared meters) of the object and

$$d\Omega = dS/r_{SR}^2 = \sin\theta_{SR}\,d\theta_{SR}\,d\psi_{SR} \tag{10.1-38}$$

is the differential of solid angle. The right-hand side of (10.1-37) was obtained by using (10.1-30). The factor r_{SR}^2 on the right-hand side of (10.1-37) is the surface area on a sphere of radius r_{SR} meters subtended by a solid angle of one steradian. Note that the differential scattering cross-section is directly proportional to the magnitude-squared of the scattering function[1, 2]. The *total* time-average scattered power (in watts) is obtained by integrating either (10.1-35) or (10.1-36). Doing so yields

$$\boxed{P_{\text{avg},s} = I_{\text{avg},i}(\mathbf{r}_S)\sigma_{\text{s}}(f,\theta_{TS},\psi_{TS})\exp[-2\alpha(f)r_{SR}]} \tag{10.1-39}$$

where

$$\boxed{\begin{aligned}\sigma_{\text{s}}(f,\theta_{TS},\psi_{TS}) &= \int_0^{2\pi}\int_0^{\pi}\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\sin\theta_{SR}\,d\theta_{SR}\,d\psi_{SR} \\ &= \int_0^{4\pi}\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\,d\Omega\end{aligned}} \tag{10.1-40}$$

is the *scattering cross-section*[1] (in squared meters) of the object, also known as the *total scattering cross-section*[2, 3]. For a given object and fixed scattering

[3] C. S. Clay and H. Medwin, *Acoustical Oceanography*, Wiley, New York, 1977, pp. 180-184.

geometry, the scattering cross-section may be less than, equal to, or greater than the actual geometrical cross-sectional area of the object depending on the frequency.

If the scatter from the object is *omnidirectional*, that is, if the scattering function and, hence, the differential scattering cross-section are *not* functions of the angles of incidence and scatter, then $S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = S_S(f)$ and $\sigma_\mathrm{d}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \sigma_\mathrm{d}(f)$. Therefore, $\sigma_\mathrm{s}(f,\theta_{TS},\psi_{TS}) = \sigma_\mathrm{s}(f)$, where

$$\sigma_\mathrm{s}(f) = 4\pi\sigma_\mathrm{d}(f) \tag{10.1-41}$$

and

$$\sigma_\mathrm{d}(f) = \left[\frac{|S_S(f)|}{4\pi}\right]^2 . \tag{10.1-42}$$

Substituting (10.1-41) and (10.1-42) into (10.1-39) yields

$$P_{\mathrm{avg},s} = 4\pi I_{\mathrm{avg},i}(\mathbf{r}_S)\left[\frac{|S_S(f)|}{4\pi}\right]^2 \exp[-2\alpha(f)r_{SR}] , \tag{10.1-43}$$

or

$$P_{\mathrm{avg},s} = 4\pi r_{SR}^2 I_{\mathrm{avg},s}(\mathbf{r}_R) , \tag{10.1-44}$$

where $I_{\mathrm{avg},s}(\mathbf{r}_R)$ is given by (10.1-30). Equation (10.1-44) is the total time-average scattered power (in watts) from an omnidirectional scatterer and is analogous to the time-average power radiated by an omnidirectional point-source.

Another popular scattering cross-section is the *bistatic scattering cross-section* (a.k.a. the *bistatic radar cross-section*) given by[1]

$$\sigma_\mathrm{bi}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = 4\pi\sigma_\mathrm{d}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) . \tag{10.1-45}$$

Substituting (10.1-37) into (10.1-45) yields

$$\boxed{\begin{aligned}\sigma_\mathrm{bi}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) &= \frac{|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})|^2}{4\pi} \\ &= 4\pi r_{SR}^2 \frac{I_{\mathrm{avg},s}(\mathbf{r}_R)}{I_{\mathrm{avg},i}(\mathbf{r}_S)}\exp[+2\alpha(f)r_{SR}]\end{aligned}} \tag{10.1-46}$$

where $4\pi r_{SR}^2$ is the surface area of a sphere with radius r_{SR} meters. If we let

$$\sigma_{\text{bi}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \frac{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|^2}{4\pi} = \pi a_{\text{eff}}^2 , \quad (10.1\text{-}47)$$

where πa_{eff}^2 is an *effective cross-sectional area* (in squared meters) of the object, and a_{eff} is an *effective radius* (in meters), then

$$\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right| = 2\pi a_{\text{eff}} , \quad (10.1\text{-}48)$$

that is, the magnitude of the scattering function can be interpreted as an *effective circumference* (in meters) of the object based on the bistatic scattering cross-section. Therefore,

$$\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \frac{\sigma_{\text{bi}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})}{4\pi} = \frac{a_{\text{eff}}^2}{4} . \quad (10.1\text{-}49)$$

We are now in a position to define target strength.

We shall define *target strength* (TS) as follows:[2, 3]

$$\boxed{\text{TS} \triangleq 10\log_{10}\left[\frac{\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})}{A_{\text{ref}}}\right] \text{dB re } A_{\text{ref}}} \quad (10.1\text{-}50)$$

and by substituting (10.1-37) into (10.1-50), we obtain

$$\boxed{\text{TS} = 10\log_{10}\left[\frac{r_{SR}^2}{A_{\text{ref}}}\frac{I_{\text{avg},s}(\mathbf{r}_R)}{I_{\text{avg},i}(\mathbf{r}_S)}\exp[+2\alpha(f)r_{SR}]\right] \text{dB re } A_{\text{ref}}} \quad (10.1\text{-}51)$$

where A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\,\text{m}^2$. Since acoustic measurements are usually taken in terms of acoustic pressure, what we need to do next is to rewrite (10.1-37) and (10.1-51) so that $\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})$ and TS can be related to source level (SL) and the scattered acoustic pressure incident upon the receiver. One of the most common parameters used to describe a sound-source is SL.

Since

$$I_{\text{avg},i}(\mathbf{r}_S) = \frac{\left|p_{f,i}(\mathbf{r}_S)\right|^2}{2\rho_0 c} \quad (10.1\text{-}52)$$

and

$$I_{\text{avg},s}(\mathbf{r}_R) = \frac{\left|p_{f,s}(\mathbf{r}_R)\right|^2}{2\rho_0 c}, \tag{10.1-53}$$

$$\frac{I_{\text{avg},s}(\mathbf{r}_R)}{I_{\text{avg},i}(\mathbf{r}_S)} = \frac{\left|p_{f,s}(\mathbf{r}_R)\right|^2}{\left|p_{f,i}(\mathbf{r}_S)\right|^2}, \tag{10.1-54}$$

where

$$\left|p_{f,i}(\mathbf{r}_S)\right|^2 = k^2(\rho_0 c)^2 \left(\frac{|S_0|}{4\pi r_{TS}}\right)^2 \exp[-2\alpha(f) r_{TS}] \tag{10.1-55}$$

and the source strength S_0 is given by (10.1-6). Equation (10.1-55) relates the incident acoustic pressure to source strength. However, the incident acoustic pressure can also be related to source level (SL). Since (see Appendix 10A)

$$|S_0| = \frac{4\pi R}{k\rho_0 c} \exp[+\alpha(f)R] P_0 \bigg|_{R=1\text{ m}}, \tag{10.1-56}$$

substituting (10.1-56) into (10.1-55) yields

$$\left|p_{f,i}(\mathbf{r}_S)\right|^2 = \left[P_0 \frac{R}{r_{TS}} \exp\left[-\alpha(f)(r_{TS} - R)\right]\right]^2 \Bigg|_{R=1\text{ m}}, \tag{10.1-57}$$

where the acoustic pressure magnitude P_0 and SL are related by (10.1-18). By substituting (10.1-57) into (10.1-54), we obtain

$$\frac{I_{\text{avg},s}(\mathbf{r}_R)}{I_{\text{avg},i}(\mathbf{r}_S)} = \left[\frac{\left|p_{f,s}(\mathbf{r}_R)\right|}{P_0} \frac{r_{TS}}{R} \exp\left[+\alpha(f)(r_{TS} - R)\right]\right]^2 \Bigg|_{R=1\text{ m}}. \tag{10.1-58}$$

Therefore, substituting (10.1-58) into (10.1-37) and (10.1-51) yields the differential scattering cross-section

$$\boxed{\begin{aligned}\sigma_{\text{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) &= \left[\frac{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|}{4\pi}\right]^2 \\ &= r_{SR}^2 \left[\frac{\left|p_{f,s}(\mathbf{r}_R)\right|}{P_0} \frac{r_{TS}}{R} \exp\left[+\alpha(f)(r_{TS} + r_{SR} - R)\right]\right]^2 \Bigg|_{R=1\text{ m}}\end{aligned}}$$

(10.1-59)

and target strength

$$\boxed{\mathrm{TS} = 10\log_{10}\left[\frac{r_{SR}^2}{A_{\mathrm{ref}}}\left[\frac{\left|p_{f,s}(\mathbf{r}_R)\right|}{P_0}\frac{r_{TS}}{R}\exp\left[+\alpha(f)(r_{TS}+r_{SR}-R)\right]\right]^2\Bigg|_{R=1\ \mathrm{m}}\right] \text{dB re } A_{\mathrm{ref}}}$$

(10.1-60)

of the object, respectively, where P_0 and SL are related by (10.1-18), and $A_{\mathrm{ref}} = 1\ \mathrm{m}^2$.

Since $p_{f,s}(\mathbf{r}_R)$ is complex, it can be expressed in polar form as follows:

$$p_{f,s}(\mathbf{r}_R) = \left|p_{f,s}(\mathbf{r}_R)\right|\exp\left[+j\angle p_{f,s}(\mathbf{r}_R)\right]. \tag{10.1-61}$$

Substituting (10.1-61) into (10.1-21) yields

$$p_s(t,\mathbf{r}_R) = \left|p_{f,s}(\mathbf{r}_R)\right|\exp\left\{+j\left[2\pi ft + \angle p_{f,s}(\mathbf{r}_R)\right]\right\}, \tag{10.1-62}$$

or, equivalently,

$$\mathrm{Re}\left\{p_s(t,\mathbf{r}_R)\right\} = \left|p_{f,s}(\mathbf{r}_R)\right|\cos\left[2\pi ft + \angle p_{f,s}(\mathbf{r}_R)\right]. \tag{10.1-63}$$

Therefore, $\left|p_{f,s}(\mathbf{r}_R)\right|$ and $\angle p_{f,s}(\mathbf{r}_R)$ are the amplitude (in pascals) and phase (in radians) of the time-harmonic, scattered acoustic pressure incident upon the receiver at $\mathbf{r}_R$.

The amplitude factor $\left|p_{f,s}(\mathbf{r}_R)\right|$ can be related to the output electrical signal from an omnidirectional point-element (the receiver) at $\mathbf{r}_R$ as follows. Since the scattered acoustic pressure incident upon the omnidirectional point-element is time-harmonic, the output electrical signal in volts is time-harmonic, that is,

$$y_s(t,\mathbf{r}_R) = y_{f,s}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.1-64}$$

where

$$y_{f,s}(\mathbf{r}_R) = \left|y_{f,s}(\mathbf{r}_R)\right|\exp\left[+j\angle y_{f,s}(\mathbf{r}_R)\right]. \tag{10.1-65}$$

Substituting (10.1-65) into (10.1-64) yields

$$y_s(t,\mathbf{r}_R) = \left|y_{f,s}(\mathbf{r}_R)\right|\exp\left\{+j\left[2\pi ft + \angle y_{f,s}(\mathbf{r}_R)\right]\right\}, \tag{10.1-66}$$

or, equivalently,

$$\operatorname{Re}\{y_s(t,\mathbf{r}_R)\} = \left|y_{f,s}(\mathbf{r}_R)\right| \cos\left[2\pi f t + \angle y_{f,s}(\mathbf{r}_R)\right], \tag{10.1-67}$$

where $\left|y_{f,s}(\mathbf{r}_R)\right|$ and $\angle y_{f,s}(\mathbf{r}_R)$ are the amplitude (in volts) and phase (in radians) of the time-harmonic, output electrical signal. Furthermore, since $F_t\{\varphi_s(t,\mathbf{r}_R)\} = \varphi_{f,s}(\mathbf{r}_R)\delta(\eta - f)$ and $F_t\{y_s(t,\mathbf{r}_R)\} = y_{f,s}(\mathbf{r}_R)\delta(\eta - f)$, in the frequency domain at frequency $\eta = f$ Hz [see (7.3-27)],

$$y_{f,s}(\mathbf{r}_R) = \varphi_{f,s}(\mathbf{r}_R)\mathcal{S}_R(f), \tag{10.1-68}$$

where $\varphi_{f,s}(\mathbf{r}_R)$ is the spatial-dependent part of the time-harmonic velocity potential (in $\mathrm{m}^2/\mathrm{sec}$) of the scattered acoustic field incident upon the omnidirectional point-element (the receiver) at $\mathbf{r}_R$, and $\mathcal{S}_R(f)$ is the complex, receiver sensitivity function with units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$ (see Table 6.1-2 and Appendix 6B). Since [see (10.1-22)]

$$p_{f,s}(\mathbf{r}_R) = -j2\pi f \rho_0 \varphi_{f,s}(\mathbf{r}_R), \tag{10.1-69}$$

$$\left|p_{f,s}(\mathbf{r}_R)\right| = 2\pi f \rho_0 \left|\varphi_{f,s}(\mathbf{r}_R)\right|, \tag{10.1-70}$$

and from (10.1-68),

$$\left|\varphi_{f,s}(\mathbf{r}_R)\right| = \frac{\left|y_{f,s}(\mathbf{r}_R)\right|}{\left|\mathcal{S}_R(f)\right|}. \tag{10.1-71}$$

Substituting (10.1-71) into (10.1-70) yields

$$\left|p_{f,s}(\mathbf{r}_R)\right| = 2\pi f \rho_0 \frac{\left|y_{f,s}(\mathbf{r}_R)\right|}{\left|\mathcal{S}_R(f)\right|}, \tag{10.1-72}$$

and because (see Appendix 6B)

$$\left|\mathcal{S}_R(f)\right| = 2\pi f \rho_0 \mathrm{RS}(f), \tag{10.1-73}$$

where $\mathrm{RS}(f)$ is the receiver sensitivity in $\mathrm{V/Pa}$, substituting (10.1-73) into (10.1-72) yields

$$\boxed{\left|p_{f,s}(\mathbf{r}_R)\right| = \frac{\left|y_{f,s}(\mathbf{r}_R)\right|}{\mathrm{RS}(f)}} \tag{10.1-74}$$

Therefore, by substituting (10.1-74) into (10.1-59) and (10.1-60), the differential scattering cross-section and target strength of an object can be computed using the measured amplitude $\left|y_{f,s}(\mathbf{r}_R)\right|$ of the time-harmonic, output electrical signal [see (10.1-67)].

In the case of a monostatic (backscatter) geometry, where the transmitter (sound-source) and receiver are at the same location,

$$\mathbf{r}_R = \mathbf{r}_T \ , \tag{10.1-75}$$

$$\mathbf{r}_{SR} = -\mathbf{r}_{TS} \, , \tag{10.1-76}$$

$$r_{SR} = r_{TS} \, , \tag{10.1-77}$$

and

$$\hat{r}_{SR} = -\hat{r}_{TS} \, . \tag{10.1-78}$$

Therefore (see Figs. 10.1-2 and 10.1-3),

$$\theta_{SR} = 180° - \theta_{TS} \tag{10.1-79}$$

and

$$\psi_{SR} = 180° + \psi_{TS} \, . \tag{10.1-80}$$

In addition, the bistatic scattering cross-section $\sigma_{\mathrm{bi}}(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})$ given by (10.1-45) is replaced by the *backscattering cross-section* (a.k.a. the *radar cross-section*[1]) $\sigma_{\mathrm{bs}}(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})$, where θ_{SR} and ψ_{SR} are given by (10.1-79) and (10.1-80), respectively.

10.2 Computing the Scattering Function of an Object

By referring to (10.1-59) and (10.1-60), it can be seen that the differential scattering cross-section and target strength of an object can be computed without knowing its scattering function. In this section we shall derive an equation for the complex scattering function that will enable us to compute its value using the scattered acoustic pressure incident upon a receiver.

We begin the derivation by expanding the equation for the spatial-dependent part of the time-harmonic, scattered acoustic pressure given by (10.1-22). Substituting (10.1-23) and (10.1-19) into (10.1-22) yields

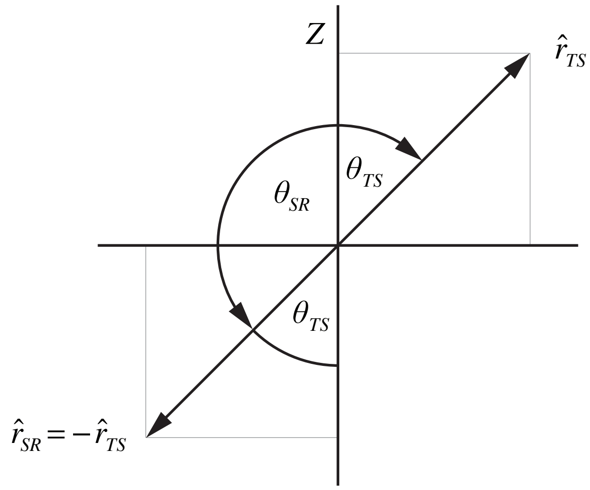

Figure 10.1-2 Measurement of the vertical angles θ_{TS} and θ_{SR} for a backscatter geometry. The vertical angles are measured from the positive Z axis to the unit vectors $\hat{r}_{TS}$ and $\hat{r}_{SR} = -\hat{r}_{TS}$ in the plane containing the unit vectors and the Z axis.

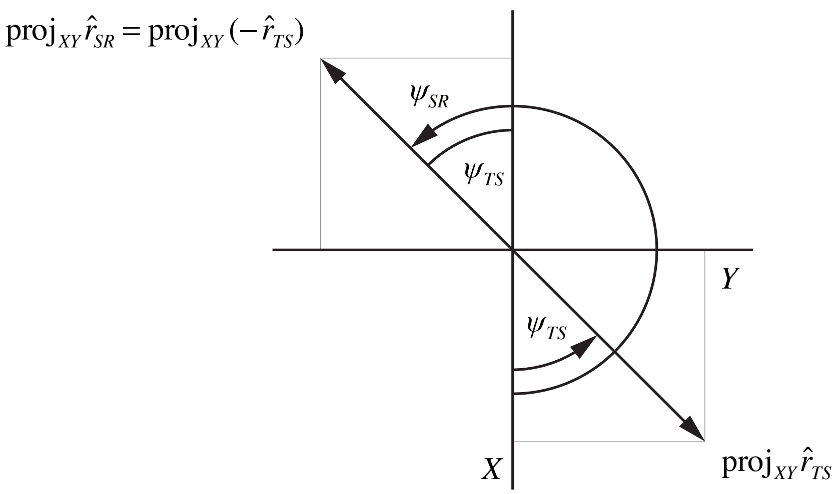

Figure 10.1-3 Measurement of the azimuthal (bearing) angles ψ_{TS} and ψ_{SR} for a backscatter geometry. The bearing angles are measured in a counter-clockwise direction from the positive X axis to the orthogonal projections $\mathrm{proj}_{XY}\hat{r}_{TS}$ and $\mathrm{proj}_{XY}\hat{r}_{SR} = \mathrm{proj}_{XY}(-\hat{r}_{TS})$ of the unit vectors $\hat{r}_{TS}$ and $\hat{r}_{SR} = -\hat{r}_{TS}$ in the XY plane.

$$p_{f,s}(\mathbf{r}_R) = p_{f,i}(\mathbf{r}_S) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right), \tag{10.2-1}$$

where [see (10.1-4) and (10.1-5)]

$$p_{f,i}(\mathbf{r}_S) = -jk\rho_0 c\, S_0\, g_f\left(\mathbf{r}_S \,|\, \mathbf{r}_T\right) \tag{10.2-2}$$

is the spatial-dependent part of the time-harmonic, incident acoustic pressure. Solving for the scattering function using (10.2-1) yields

$$S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) = \frac{p_{f,s}(\mathbf{r}_R)}{p_{f,i}(\mathbf{r}_S)\, g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right)}, \tag{10.2-3}$$

and by substituting (10.2-2) into (10.2-3), we obtain

$$S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) = j \frac{p_{f,s}(\mathbf{r}_R)}{k\rho_0 c\, S_0\, g_f\left(\mathbf{r}_S \,|\, \mathbf{r}_T\right) g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right)}. \tag{10.2-4}$$

Further substituting (10.1-7) and (10.1-24) into (10.2-4) yields

$$\begin{aligned} S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) = j \frac{(4\pi)^2 r_{TS} r_{SR}}{k\rho_0 c\, S_0} p_{f,s}(\mathbf{r}_R) \exp\left[+\alpha(f)(r_{TS} + r_{SR})\right] \times \\ \exp\left[+jk(r_{TS} + r_{SR})\right], \end{aligned} \tag{10.2-5}$$

or, since wavenumber $k = 2\pi f / c$,

$$\boxed{S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) = j \frac{8\pi r_{TS} r_{SR}}{f \rho_0 S_0} p_{f,s}(\mathbf{r}_R) \exp\left[+\alpha(f)(r_{TS} + r_{SR})\right] \exp(+j2\pi f \tau)} \tag{10.2-6}$$

where the source strength S_0 is given by (10.1-6), $p_{f,s}(\mathbf{r}_R)$ is the spatial-dependent part of the time-harmonic, scattered acoustic pressure incident upon the receiver at $\mathbf{r}_R$ [see (10.1-61) through (10.1-63)], and

$$\tau = (r_{TS} + r_{SR}) / c \tag{10.2-7}$$

is the *bistatic* time delay in seconds.

The acoustic pressure $p_{f,s}(\mathbf{r}_R)$ can be related to the time-harmonic, output electrical signal from an omnidirectional point-element (the receiver) at $\mathbf{r}_R$ as follows. With the use of (10.1-65), (10.1-68), and (10.1-69),

$$\boxed{p_{f,s}(\mathbf{r}_R) = -jk\rho_0 c \frac{\left|y_{f,s}(\mathbf{r}_R)\right|}{\mathcal{S}_R(f)} \exp\left[+j\angle y_{f,s}(\mathbf{r}_R)\right]} \tag{10.2-8}$$

where $\left|y_{f,s}(\mathbf{r}_R)\right|$ and $\angle y_{f,s}(\mathbf{r}_R)$ are the amplitude (in volts) and phase (in radians) of the time-harmonic, output electrical signal $y_s(t,\mathbf{r}_R)$ [see (10.1-67)], and $\mathcal{S}_R(f)$ is the complex, receiver sensitivity function with units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$ (see Table 6.1-2 and Appendix 6B). Substituting (10.2-8) into (10.2-5) yields

$$\boxed{\begin{aligned} S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \frac{(4\pi)^2 r_{TS} r_{SR}}{S_0 \mathcal{S}_R(f)} \left|y_{f,s}(\mathbf{r}_R)\right| \exp\left[+\alpha(f)(r_{TS}+r_{SR})\right] \times \\ \exp\left\{+j\left[2\pi f \tau + \angle y_{f,s}(\mathbf{r}_R)\right]\right\} \end{aligned}}$$

(10.2-9)

where τ is given by (10.2-7). Note that the scattering function $S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})$ has units of meters.

10.3 Direct Path

In this section we shall complete the analysis of the bistatic scattering problem by analyzing the *direct path* between the transmitter (sound-source) located at $\mathbf{r}_T = (x_T, y_T, z_T)$ and the receiver located at $\mathbf{r}_R = (x_R, y_R, z_R)$, where the transmitter is a time-harmonic, omnidirectional point-source, and the receiver is an omnidirectional point-element (see Fig. 10.3-1). The equations that describe the direct path are very similar to the equations that describe the acoustic field incident upon the object. The time-harmonic acoustic pressure (in pascals) incident upon the receiver at $\mathbf{r}_R$ is given by (see Appendix 10A)

$$p_d(t,\mathbf{r}_R) = p_{f,d}(\mathbf{r}_R)\exp(+j2\pi f t)\,, \tag{10.3-1}$$

where

$$p_{f,d}(\mathbf{r}_R) = -jk\rho_0 c\,\varphi_{f,d}(\mathbf{r}_R)\,, \tag{10.3-2}$$

$$\varphi_{f,d}(\mathbf{r}_R) = S_0\, g_f\left(\mathbf{r}_R \middle| \mathbf{r}_T\right), \tag{10.3-3}$$

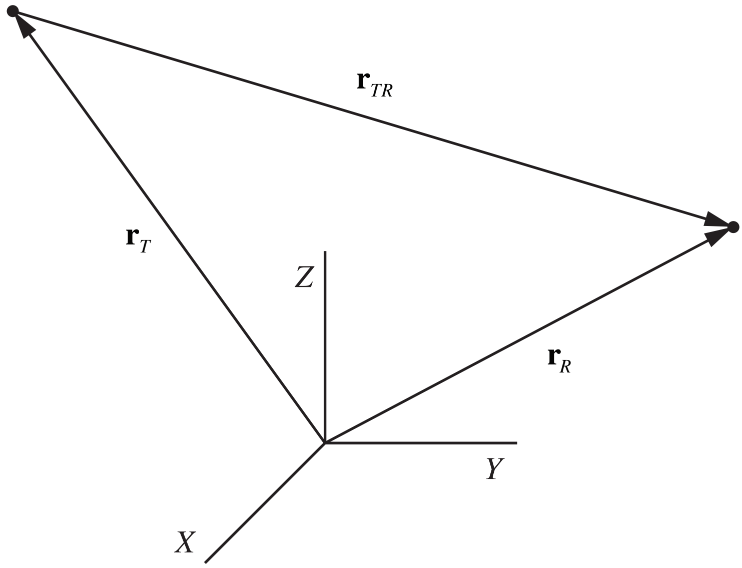

Figure 10.3-1 Direct path between the transmitter (sound-source) located at $\mathbf{r}_T = (x_T, y_T, z_T)$ and the receiver located at $\mathbf{r}_R = (x_R, y_R, z_R)$.

$$S_0 = A_x \mathcal{S}_T(f), \tag{10.3-4}$$

$$\begin{aligned} g_f\left(\mathbf{r}_R \mid \mathbf{r}_T\right) &= -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}_R - \mathbf{r}_T\right|\right]}{4\pi\left|\mathbf{r}_R - \mathbf{r}_T\right|}\exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_T\right|\right) \\ &= -\frac{\exp[-\alpha(f)r_{TR}]}{4\pi r_{TR}}\exp(-jkr_{TR}), \end{aligned} \tag{10.3-5}$$

$$\mathbf{r}_{TR} = \mathbf{r}_R - \mathbf{r}_T = (x_R - x_T)\hat{x} + (y_R - y_T)\hat{y} + (z_R - z_T)\hat{z}, \tag{10.3-6}$$

$$r_{TR} = \left|\mathbf{r}_{TR}\right| = \sqrt{(x_R - x_T)^2 + (y_R - y_T)^2 + (z_R - z_T)^2}, \tag{10.3-7}$$

and

$$\mathbf{r}_{TR} = r_{TR}\hat{r}_{TR}, \tag{10.3-8}$$

where $\hat{r}_{TR}$ is the dimensionless unit vector in the direction of the vector $\mathbf{r}_{TR}$.

The time-average intensity vector (in watts per squared meter) incident upon the receiver at $\mathbf{r}_R$ is given by (see Appendix 10A)

$$\mathbf{I}_{\text{avg},d}(\mathbf{r}_R) = I_{\text{avg},d}(\mathbf{r}_R)\hat{r}_{TR}, \tag{10.3-9}$$

where

$$I_{\text{avg},d}(\mathbf{r}_R) = \frac{1}{2} k^2 \rho_0 c \left(\frac{|S_0|}{4\pi r_{TR}} \right)^2 \exp[-2\alpha(f) r_{TR}] . \tag{10.3-10}$$

If the origin of a Cartesian coordinate system (with the same orientation as the one shown in Fig. 10.1-1) is placed at $\mathbf{r}_T$, then the spherical angles θ_{TR} and ψ_{TR} that describe the direction of the unit vector $\hat{r}_{TR}$ are measured as shown in Fig. 1.2-2 and are given by (see Appendix 10A)

$$\boxed{\theta_{TR} = \cos^{-1}\left(\frac{z_R - z_T}{r_{TR}} \right)} \tag{10.3-11}$$

and

$$\boxed{\psi_{TR} = \tan^{-1}\left(\frac{y_R - y_T}{x_R - x_T} \right)} \tag{10.3-12}$$

Equations (10.3-11) and (10.3-12) are the *angles of incidence* at the receiver.

And finally, if we enclose the omnidirectional point-source at $\mathbf{r}_T$ with a sphere of radius r_{TR} meters, then the time-average radiated power (in watts) is given by (see Appendix 10A)

$$P_{\text{avg},d} = \frac{1}{2} k^2 \rho_0 c \frac{|S_0|^2}{4\pi} \exp[-2\alpha(f) r_{TR}] = 4\pi r_{TR}^2 I_{\text{avg},d}(\mathbf{r}_R) , \tag{10.3-13}$$

where the source strength S_0 is given by (10.1-6), or

$$P_{\text{avg},d} = 4\pi R^2 \frac{P_0^2}{2\rho_0 c} \exp\big[-2\alpha(f)(r_{TR} - R)\big]\Big|_{R=1\,\text{m}} , \tag{10.3-14}$$

where P_0 is the magnitude of the time-harmonic, radiated acoustic pressure measured at a range $R = 1\,\text{m}$ from the omnidirectional point-source at $\mathbf{r}_T$. The acoustic pressure magnitude P_0 and source level (SL) are related by (10.1-18).

10.4 Sonar Equations

10.4.1 Scattered Path

In this subsection we shall derive equations for the sound-pressure level

(SPL), transmission loss (TL), and acoustic signal-to-noise ratio at the location of the receiver for the bistatic scattering problem shown in Fig. 10.1-1. The receiver is an omnidirectional point-element at $\mathbf{r}_R$. In order to derive equations for the SPL and TL, we first have to find the root-mean-square (rms) scattered acoustic pressure.

The time-harmonic, scattered acoustic pressure (in pascals) incident upon the receiver at $\mathbf{r}_R$ is given by

$$p_s(t,\mathbf{r}_R) = p_{f,s}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.4-1}$$

where, by substituting (10.2-2) into (10.2-1),

$$p_{f,s}(\mathbf{r}_R) = -jk\rho_0 c S_0 g_f\left(\mathbf{r}_S \middle| \mathbf{r}_T\right) S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) g_f\left(\mathbf{r}_R \middle| \mathbf{r}_S\right). \tag{10.4-2}$$

Further substituting (10.1-7) and (10.1-24) into (10.4-2) yields

$$p_{f,s}(\mathbf{r}_R) = -j\frac{k\rho_0 c S_0}{(4\pi)^2 r_{TS} r_{SR}} S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\times$$
$$\exp\left[-jk(r_{TS}+r_{SR})\right]. \tag{10.4-3}$$

Since the scattered acoustic pressure $p_s(t,\mathbf{r}_R)$ is a time-harmonic acoustic field, its rms value is given by

$$p_{\text{rms},s}(\mathbf{r}_R) = \frac{\sqrt{2}}{2}\left|p_{f,s}(\mathbf{r}_R)\right|. \tag{10.4-4}$$

Computing the magnitude of (10.4-3) yields

$$\left|p_{f,s}(\mathbf{r}_R)\right| = \frac{k\rho_0 c\left|S_0\right|}{(4\pi)^2 r_{TS} r_{SR}}\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|\exp\left[-\alpha(f)(r_{TS}+r_{SR})\right], \tag{10.4-5}$$

and by substituting (10.1-56) into (10.4-5), we obtain

$$\left|p_{f,s}(\mathbf{r}_R)\right| = P_0\frac{R}{r_{TS} r_{SR}}\frac{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|}{4\pi}\exp\left[-\alpha(f)(r_{TS}+r_{SR}-R)\right]\Bigg|_{R=1\text{ m}}. \tag{10.4-6}$$

Therefore, substituting (10.4-6) into (10.4-4) yields the rms scattered acoustic pressure

$$p_{\mathrm{rms},s}(\mathbf{r}_R) = \frac{\sqrt{2}}{2} P_0 \frac{R}{r_{TS} r_{SR}} \frac{\left| S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) \right|}{4\pi} \exp\left[-\alpha(f)(r_{TS}+r_{SR}-R)\right] \Bigg|_{R=1\ \mathrm{m}} . \tag{10.4-7}$$

Now that we have an equation for the rms scattered acoustic pressure, we can begin the SPL calculation by 1) dividing both sides of (10.4-7) by the rms reference pressure P_{ref}, 2) squaring both sides of the resulting equation, and 3) multiplying the right-hand side of the resulting equation by $A_{\mathrm{ref}} / A_{\mathrm{ref}} = 1$, where A_{ref} is a reference cross-sectional area. Doing steps one through three yields

$$\left[\frac{p_{\mathrm{rms},s}(\mathbf{r}_R)}{P_{\mathrm{ref}}}\right]^2 = \left(\frac{\sqrt{2}P_0/2}{P_{\mathrm{ref}}}\right)^2 \left(\frac{R\sqrt{A_{\mathrm{ref}}}}{r_{TS} r_{SR}}\right)^2 \frac{\sigma_{\mathrm{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})}{A_{\mathrm{ref}}} \times \left\{\exp\left[-\alpha(f)(r_{TS}+r_{SR}-R)\right]\right\}^2 \Big|_{R=1\ \mathrm{m}} , \tag{10.4-8}$$

where

$$\sigma_{\mathrm{d}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \left[\frac{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|}{4\pi}\right]^2 \tag{10.4-9}$$

is the differential scattering cross-section in squared meters [see (10.1-37)]. Note that both sides of (10.4-8) are *dimensionless*. Taking $10\log_{10}$ of (10.4-8) and setting $R = 1\ \mathrm{m}$ and $A_{\mathrm{ref}} = 1\ \mathrm{m}^2$ in the radicand yields

$$\mathrm{SPL} = \mathrm{SL} - 20\log_{10}(r_{TS} r_{SR}) + \mathrm{TS} + 20\log_{10}\left\{\exp\left[-\alpha(f)(r_{TS}+r_{SR}-1)\right]\right\}, \tag{10.4-10}$$

where

$$\mathrm{SPL} = 20\log_{10}\left[\frac{p_{\mathrm{rms},s}(\mathbf{r}_R)}{P_{\mathrm{ref}}}\right] \mathrm{dB\ re}\ P_{\mathrm{ref}} \tag{10.4-11}$$

is the sound-pressure level at the receiver,

$$\mathrm{SL} = 20\log_{10}\left(\frac{\sqrt{2}P_0/2}{P_{\mathrm{ref}}}\right) \mathrm{dB\ re}\ P_{\mathrm{ref}} \tag{10.4-12}$$

is the source level [see (10.1-17)], where P_0 is the magnitude of the time-harmonic, radiated acoustic pressure measured at a range $R = 1\,\text{m}$ from the omnidirectional point-source at $\mathbf{r}_T$,

$$\text{TS} = 10\log_{10}\left[\frac{\sigma_\text{d}(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})}{A_\text{ref}}\right] \text{dB re } A_\text{ref} \tag{10.4-13}$$

is the target strength [see (10.1-50)], $\alpha(f)$ is the real, nonnegative, frequency-dependent, attenuation coefficient of seawater in nepers per meter, $P_\text{ref} = 1\,\mu\text{Pa (rms)}$ is the rms reference pressure used in underwater acoustics, and $A_\text{ref} = 1\,\text{m}^2$ is the reference cross-sectional area. Since

$$\begin{aligned} 20\log_{10}\left\{\exp\left[-\alpha(f)(r_{TS} + r_{SR} - 1)\right]\right\} &= -20\,\alpha(f)(r_{TS} + r_{SR} - 1)\log_{10} e \\ &= -8.686\,\alpha(f)(r_{TS} + r_{SR} - 1) \\ &= -\alpha'(f)(r_{TS} + r_{SR} - 1), \end{aligned} \tag{10.4-14}$$

where

$$\boxed{\alpha'(f) = 8.686\,\alpha(f)} \tag{10.4-15}$$

is the attenuation coefficient in decibels per meter (see Section 7.2), substituting (10.4-14) into (10.4-10) and rearranging terms yields the following equation for the sound-pressure level at the location of the receiver for the bistatic scattering problem shown in Fig. 10.1-1:

$$\boxed{\text{SPL} = \text{SL} - 20\log_{10}(r_{TS}\, r_{SR}) - \alpha'(f)(r_{TS} + r_{SR} - 1) + \text{TS}} \tag{10.4-16}$$

Equation (10.4-16) is an example of what is commonly referred to as an *active sonar equation* because of the intentional transmission of sound and the appearance of the TS term. Now that we have an equation for the SPL, we can compute the transmission loss (TL) given by

$$\text{TL} = \text{SL} - \text{SPL}\,. \tag{10.4-17}$$

Substituting (10.4-16) into (10.4-17) yields

$$\boxed{\text{TL} = 20\log_{10}(r_{TS}\, r_{SR}) + \alpha'(f)(r_{TS} + r_{SR} - 1) - \text{TS}} \tag{10.4-18}$$

For the case of a monostatic (backscatter) geometry, where the transmitter (sound-source) and receiver are at the same location, $r_{SR} = r_{TS}$ [see (10.1-77)]. Therefore, (10.4-16) and (10.4-18) reduce as follows:

$$\boxed{\text{SPL} = \text{SL} - 40\log_{10}(r_{TS}) - \alpha'(f)(2r_{TS} - 1) + \text{TS}} \tag{10.4-19}$$

and

$$\boxed{\text{TL} = 40\log_{10}(r_{TS}) + \alpha'(f)(2r_{TS} - 1) - \text{TS}} \tag{10.4-20}$$

Acoustic Signal-to-Noise Ratio

In the development of the sonar equations in this subsection, ambient noise in the ocean was not taken into account. In order to develop sonar equations that include ambient noise, we shall use the following definition for the *acoustic signal-to-noise ratio* (SNR_a) as a starting point:

$$\boxed{\text{SNR}_\text{a} \triangleq \frac{\left\langle \left| p_s(t, \mathbf{r}_R) \right|^2 \right\rangle}{E\left\{ \left| p_{n_a}(t, \mathbf{r}_R) \right|^2 \right\}}} \tag{10.4-21}$$

where $\left\langle \left| p_s(t, \mathbf{r}_R) \right|^2 \right\rangle$ is the time-average second moment (time-average power) of the *deterministic* (nonrandom), scattered acoustic pressure $p_s(t, \mathbf{r}_R)$ incident upon the receiver at $\mathbf{r}_R$, and $E\left\{ \left| p_{n_a}(t, \mathbf{r}_R) \right|^2 \right\}$ is the statistical second moment (statistical average power) of the acoustic pressure due to ambient noise $p_{n_a}(t, \mathbf{r}_R)$ incident upon the receiver at $\mathbf{r}_R$, where $E\{\bullet\}$ is the ensemble-average (expectation) operator. The case where $p_s(t, \mathbf{r}_R)$ is a *random process* shall be considered later in this subsection. For an *aperiodic*, *finite duration* signal defined in the time interval $[0, T]$,

$$\left\langle \left| p_s(t, \mathbf{r}_R) \right|^2 \right\rangle = \frac{1}{T} \int_0^T \left| p_s(t, \mathbf{r}_R) \right|^2 dt = p_{\text{rms}, s}^2(\mathbf{r}_R), \tag{10.4-22}$$

and for a *periodic* signal with *fundamental period* T_0 seconds,

$$\left\langle \left| p_s(t, \mathbf{r}_R) \right|^2 \right\rangle = \frac{1}{T_0} \int_0^{T_0} \left| p_s(t, \mathbf{r}_R) \right|^2 dt = p_{\text{rms}, s}^2(\mathbf{r}_R), \tag{10.4-23}$$

where $p_{\mathrm{rms},s}(\mathbf{r}_R)$ is the root-mean-square (rms) value of the scattered acoustic pressure $p_s(t,\mathbf{r}_R)$.

Although we know from (10.4-22) and (10.4-23) that the numerator of (10.4-21) is equal to $p^2_{\mathrm{rms},s}(\mathbf{r}_R)$, we shall compute $\left\langle \left| p_s(t,\mathbf{r}_R)\right|^2 \right\rangle$ using our model for $p_s(t,\mathbf{r}_R)$ as a check. In order to evaluate the numerator, we shall treat $p_s(t,\mathbf{r}_R)$ as being a *real* function by rewriting (10.4-1) as

$$p_s(t,\mathbf{r}_R) = \left| p_{f,s}(\mathbf{r}_R)\right| \cos\left[2\pi f t + \angle p_{f,s}(\mathbf{r}_R)\right], \tag{10.4-24}$$

where $\left| p_{f,s}(\mathbf{r}_R)\right|$ and $\angle p_{f,s}(\mathbf{r}_R)$ are the magnitude and phase of $p_{f,s}(\mathbf{r}_R)$, respectively. Since $p_s(t,\mathbf{r}_R)$ is a periodic function of time with fundamental period $T_0 = 1/f$ seconds, the time-average second moment (time-average power) of $p_s(t,\mathbf{r}_R)$ is given by [see (10.4-23)]

$$\begin{aligned}
\left\langle \left| p_s(t,\mathbf{r}_R)\right|^2 \right\rangle &= \frac{1}{T_0}\int_0^{T_0} \left| p_s(t,\mathbf{r}_R)\right|^2 dt \\
&= \frac{1}{T_0}\left| p_{f,s}(\mathbf{r}_R)\right|^2 \int_0^{T_0} \cos^2\left[2\pi f t + \angle p_{f,s}(\mathbf{r}_R)\right] dt \\
&= \frac{1}{2T_0}\left| p_{f,s}(\mathbf{r}_R)\right|^2 \int_0^{T_0} \left\{1 + \cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\} dt \\
&= \frac{1}{2}\left| p_{f,s}(\mathbf{r}_R)\right|^2 ,
\end{aligned} \tag{10.4-25}$$

or

$$\left\langle \left| p_s(t,\mathbf{r}_R)\right|^2 \right\rangle = p^2_{\mathrm{rms},s}(\mathbf{r}_R) \tag{10.4-26}$$

as expected, where $p_{\mathrm{rms},s}(\mathbf{r}_R)$ is given by (10.4-7).

In order to evaluate the denominator of (10.4-21), we shall assume that the random process $p_{n_a}(t,\mathbf{r}_R)$ is *wide-sense stationary* (WSS) *in time*. Therefore, the autocorrelation function of $p_{n_a}(t,\mathbf{r}_R)$ at $\mathbf{r}_R$, given by

$$R_{p_{n_a}}(t,t',\mathbf{r}_R,\mathbf{r}_R) = E\left\{ p_{n_a}(t,\mathbf{r}_R)\, p^*_{n_a}(t',\mathbf{r}_R)\right\}, \tag{10.4-27}$$

is a function of *time difference*, that is,

$$R_{p_{n_a}}(\Delta t,\mathbf{r}_R,\mathbf{r}_R) = E\left\{ p_{n_a}(t,\mathbf{r}_R)\, p^*_{n_a}(t',\mathbf{r}_R)\right\}, \tag{10.4-28}$$

where

$$\Delta t = t - t' . \tag{10.4-29}$$

Since $p_{n_a}(t, \mathbf{r}_R)$ is WSS in time, its autocorrelation function $R_{p_{n_a}}(\Delta t, \mathbf{r}_R, \mathbf{r}_R)$ forms a Fourier transform pair with its power spectrum $S_{p_{n_a}}(\eta, \mathbf{r}_R)$ with units of pascals-squared per hertz:

$$R_{p_{n_a}}(\Delta t, \mathbf{r}_R, \mathbf{r}_R) \leftrightarrow S_{p_{n_a}}(\eta, \mathbf{r}_R) , \tag{10.4-30}$$

where

$$S_{p_{n_a}}(\eta, \mathbf{r}_R) = F_{\Delta t}\left\{R_{p_{n_a}}(\Delta t, \mathbf{r}_R, \mathbf{r}_R)\right\} = \int_{-\infty}^{\infty} R_{p_{n_a}}(\Delta t, \mathbf{r}_R, \mathbf{r}_R) \exp(-j2\pi\eta\Delta t)\, d\Delta t \tag{10.4-31}$$

and

$$R_{p_{n_a}}(\Delta t, \mathbf{r}_R, \mathbf{r}_R) = F_{\eta}^{-1}\left\{S_{p_{n_a}}(\eta, \mathbf{r}_R)\right\} = \int_{-\infty}^{\infty} S_{p_{n_a}}(\eta, \mathbf{r}_R) \exp(+j2\pi\eta\Delta t)\, d\eta , \tag{10.4-32}$$

where η is frequency in hertz.

If we let $t' = t$, then $\Delta t = 0$, and as a result, the average power of $p_{n_a}(t, \mathbf{r}_R)$ is given by

$$P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R) = R_{p_{n_a}}(0, \mathbf{r}_R, \mathbf{r}_R) = E\left\{\left| p_{n_a}(t, \mathbf{r}_R)\right|^2\right\}, \tag{10.4-33}$$

or

$$P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R) = \int_{-\infty}^{\infty} S_{p_{n_a}}(\eta, \mathbf{r}_R)\, d\eta , \tag{10.4-34}$$

where $P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R)$ has units of pascals-squared. If we also treat $p_{n_a}(t, \mathbf{r}_R)$ as a *real* function of time and space, its power spectrum $S_{p_{n_a}}(\eta, \mathbf{r}_R)$ is an *even* function of frequency η and (10.4-34) reduces to

$$P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R) = 2\int_{0}^{\infty} S_{p_{n_a}}(\eta, \mathbf{r}_R)\, d\eta . \tag{10.4-35}$$

Therefore, if the acoustic pressure due to ambient noise $p_{n_a}(t, \mathbf{r}_R)$ is WSS in time, and is treated as a real function of time and space, the SNR_a given by (10.4-21) can be rewritten as

$$\boxed{\text{SNR}_\text{a} = \frac{p_{\text{rms},\, s}^2(\mathbf{r}_R)}{P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R)}} \tag{10.4-36}$$

where $p_{\text{rms},\, s}(\mathbf{r}_R)$ is given by (10.4-7) and $P_{\text{avg},\, p_{n_a}}(\mathbf{r}_R)$ is given by (10.4-35).

Since the scattered acoustic pressure $p_s(t, \mathbf{r}_R)$ is time-harmonic in this analysis, the *acoustic signal-to-noise ratio at frequency* f *hertz* can be defined as

$$\boxed{\mathrm{SNR}_{\mathrm{a},\,f} \triangleq \frac{p^2_{\mathrm{rms},\,s}(\mathbf{r}_R)}{P_{\mathrm{avg},\,p_{n_a},\,f}(\mathbf{r}_R)}} \tag{10.4-37}$$

where from (10.4-35), the average power of the ambient noise $p_{n_a}(t, \mathbf{r}_R)$ at frequency f hertz is

$$P_{\mathrm{avg},\,p_{n_a},\,f}(\mathbf{r}_R) = 2\int_{f-(\Delta f/2)}^{f+(\Delta f/2)} S_{p_{n_a}}(\eta, \mathbf{r}_R)\,d\eta\,, \tag{10.4-38}$$

or

$$P_{\mathrm{avg},\,p_{n_a},\,f}(\mathbf{r}_R) = 2\,S_{p_{n_a}}(f, \mathbf{r}_R)\Delta f\,, \tag{10.4-39}$$

where $\Delta f = 1\,\mathrm{Hz}$. Therefore, substituting (10.4-39) into (10.4-37) yields

$$\boxed{\mathrm{SNR}_{\mathrm{a},\,f} = \frac{p^2_{\mathrm{rms},\,s}(\mathbf{r}_R)}{2\,S_{p_{n_a}}(f, \mathbf{r}_R)\Delta f}} \tag{10.4-40}$$

If (10.4-40) is rewritten as

$$\mathrm{SNR}_{\mathrm{a},\,f} = \frac{\left[p_{\mathrm{rms},\,s}(\mathbf{r}_R)/P_{\mathrm{ref}}\right]^2}{2\,S_{p_{n_a}}(f, \mathbf{r}_R)\Delta f\big/P^2_{\mathrm{ref}}}\,, \tag{10.4-41}$$

then the acoustic signal-to-noise ratio (in decibels) at frequency f hertz is given by

$$\boxed{\mathrm{SNR}_{\mathrm{a},\,f}\,(\mathrm{dB}) = \mathrm{SPL} - \mathrm{NL}} \tag{10.4-42}$$

where

$$\mathrm{SNR}_{\mathrm{a},\,f}\,(\mathrm{dB}) = 10\log_{10}\mathrm{SNR}_{\mathrm{a},\,f}\,, \tag{10.4-43}$$

$$\mathrm{SPL} = 20\log_{10}\left[\frac{p_{\mathrm{rms},\,s}(\mathbf{r}_R)}{P_{\mathrm{ref}}}\right]\mathrm{dB\ re}\ P_{\mathrm{ref}} \tag{10.4-44}$$

is the sound-pressure level given by (10.4-16) or (10.4-19),

$$\boxed{\mathrm{NL} = 10\log_{10}\left[\frac{2S_{p_{n_a}}(f,\mathbf{r}_R)\Delta f}{P_{\mathrm{ref}}^2}\right] \text{dB re } P_{\mathrm{ref}}^2} \tag{10.4-45}$$

is the *noise level*[1] where $\Delta f = 1\,\mathrm{Hz}$, and $P_{\mathrm{ref}} = 1\,\mu\mathrm{Pa}\,(\mathrm{rms})$ is the rms reference pressure used in underwater acoustics. The NL can be expressed as a *noise spectrum level*[2] (NSL) as follows:

$$\mathrm{NSL} = 10\log_{10}\left[\frac{2S_{p_{n_a}}(f,\mathbf{r}_R)}{P_{\mathrm{ref}}^2/\Delta f}\right], \tag{10.4-46}$$

or

$$\boxed{\mathrm{NSL} = 10\log_{10}\left[\frac{2S_{p_{n_a}}(f,\mathbf{r}_R)}{S_{\mathrm{ref}}}\right] \text{dB re } S_{\mathrm{ref}}} \tag{10.4-47}$$

where $S_{\mathrm{ref}} = P_{\mathrm{ref}}^2/\mathrm{Hz}$ since $\Delta f = 1\,\mathrm{Hz}$. For a bistatic scattering problem, SPL is given by (10.4-16), and for a monostatic (backscatter) geometry, SPL is given by (10.4-19).

If the scattering function $S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})$ of the object (scatterer) is treated as a *random variable* (see Section 10.6), then the scattered acoustic pressure $p_s(t,\mathbf{r}_R)$ is a *random process*. Therefore, replacing the time-average $\left\langle |p_s(t,\mathbf{r}_R)|^2 \right\rangle$ in the numerator of (10.4-21) with the ensemble-average $E\left\{|p_s(t,\mathbf{r}_R)|^2\right\}$ yields the following general expression for the *acoustic signal-to-noise ratio*:

$$\boxed{\mathrm{SNR}_{\mathrm{a}} \triangleq \frac{E\left\{|p_s(t,\mathbf{r}_R)|^2\right\}}{E\left\{|p_{n_a}(t,\mathbf{r}_R)|^2\right\}}} \tag{10.4-48}$$

With the use of (10.4-24),

$$E\left\{|p_s(t,\mathbf{r}_R)|^2\right\} = \frac{1}{2}E\left\{|p_{f,s}(\mathbf{r}_R)|^2\right\} + \frac{1}{2}E\left\{|p_{f,s}(\mathbf{r}_R)|^2 \cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\}. \tag{10.4-49}$$

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[1] C. S. Clay and H. Medwin, *Acoustical Oceanography*, Wiley, New York, 1977, pp. 150-151.
[2] *Ibid*., pp. 119.

If the *random variables* $\left|p_{f,s}(\mathbf{r}_R)\right|$ and $\angle p_{f,s}(\mathbf{r}_R)$ are *uncorrelated*, then

$$E\left\{\left|p_{f,s}(\mathbf{r}_R)\right|^2 \cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\} = E\left\{\left|p_{f,s}(\mathbf{r}_R)\right|^2\right\} \times E\left\{\cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\}, \tag{10.4-50}$$

and if the random variable $\angle p_{f,s}(\mathbf{r}_R)$ is *uniformly distributed in the interval* $[0, 2\pi]$, then

$$E\left\{\cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\} = 0\,, \tag{10.4-51}$$

and as a result,

$$E\left\{\left|p_{f,s}(\mathbf{r}_R)\right|^2 \cos\left[4\pi f t + 2\angle p_{f,s}(\mathbf{r}_R)\right]\right\} = 0\,. \tag{10.4-52}$$

Substituting (10.4-52) into (10.4-49) yields

$$E\left\{\left|p_s(t, \mathbf{r}_R)\right|^2\right\} = \frac{1}{2} E\left\{\left|p_{f,s}(\mathbf{r}_R)\right|^2\right\}. \tag{10.4-53}$$

Because $p_s(t, \mathbf{r}_R)$ is time-harmonic in this analysis, (10.4-53) can be rewritten as

$$E\left\{\left|p_s(t, \mathbf{r}_R)\right|^2\right\} = E\left\{p^2_{\mathrm{rms},\,s}(\mathbf{r}_R)\right\}, \tag{10.4-54}$$

where $p_{\mathrm{rms},\,s}(\mathbf{r}_R)$ is given by (10.4-7).

Since the acoustic pressure due to ambient noise $p_{n_a}(t, \mathbf{r}_R)$ is assumed to be WSS in time, and is being treated as a real function of time and space, then from (10.4-33)

$$E\left\{\left|p_{n_a}(t, \mathbf{r}_R)\right|^2\right\} = P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)\,, \tag{10.4-55}$$

where $P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)$ is the average power of the ambient noise given by (10.4-35). Substituting (10.4-54) and (10.4-55) into (10.4-48) yields

$$\boxed{\mathrm{SNR}_{\mathrm{a}} = \frac{E\left\{p^2_{\mathrm{rms},\,s}(\mathbf{r}_R)\right\}}{P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)}} \tag{10.4-56}$$

And since the scattered acoustic pressure $p_s(t,\mathbf{r}_R)$ is time-harmonic in this analysis, the *acoustic signal-to-noise ratio at frequency* f *hertz* can be defined as

$$\boxed{\mathrm{SNR}_{\mathrm{a},f} \triangleq \frac{E\left\{p^2_{\mathrm{rms},s}(\mathbf{r}_R)\right\}}{P_{\mathrm{avg},p_{n_a},f}(\mathbf{r}_R)}} \tag{10.4-57}$$

where $P_{\mathrm{avg},p_{n_a},f}(\mathbf{r}_R)$ is the average power of the ambient noise $p_{n_a}(t,\mathbf{r}_R)$ at frequency f hertz given by (10.4-39). Therefore, substituting (10.4-39) into (10.4-57) yields

$$\boxed{\mathrm{SNR}_{\mathrm{a},f} = \frac{E\left\{p^2_{\mathrm{rms},s}(\mathbf{r}_R)\right\}}{2\,S_{p_{n_a}}(f,\mathbf{r}_R)\Delta f}} \tag{10.4-58}$$

where $S_{p_{n_a}}(f,\mathbf{r}_R)$ is the power spectrum (in pascals-squared per hertz) of $p_{n_a}(t,\mathbf{r}_R)$, and $\Delta f = 1\,\mathrm{Hz}$.

If (10.4-58) is rewritten as

$$\mathrm{SNR}_{\mathrm{a},f} = \frac{E\left\{p^2_{\mathrm{rms},s}(\mathbf{r}_R)\right\}\big/P^2_{\mathrm{ref}}}{2\,S_{p_{n_a}}(f,\mathbf{r}_R)\Delta f\big/P^2_{\mathrm{ref}}}\,, \tag{10.4-59}$$

then the acoustic signal-to-noise ratio (in decibels) at frequency f hertz is given by

$$\boxed{\mathrm{SNR}_{\mathrm{a},f}\,(\mathrm{dB}) = \overline{\mathrm{SPL}} - \mathrm{NL}} \tag{10.4-60}$$

where

$$\mathrm{SNR}_{\mathrm{a},f}\,(\mathrm{dB}) = 10\log_{10}\mathrm{SNR}_{\mathrm{a},f}\,, \tag{10.4-61}$$

$$\overline{\mathrm{SPL}} = 10\log_{10}\left[\frac{E\left\{p^2_{\mathrm{rms},s}(\mathbf{r}_R)\right\}}{P^2_{\mathrm{ref}}}\right]\mathrm{dB\ re}\ P^2_{\mathrm{ref}} \tag{10.4-62}$$

is the *average* sound-pressure level given by

$$\boxed{\overline{\mathrm{SPL}} = \mathrm{SL} - 20\log_{10}(r_{TS}r_{SR}) - \alpha'(f)(r_{TS}+r_{SR}-1) + \overline{\mathrm{TS}}} \tag{10.4-63}$$

where $P_{\text{ref}} = 1\,\mu\text{Pa (rms)}$, SL is the source level given by (10.4-12), $\alpha'(f)$ is the attenuation coefficient in decibels per meter given by (10.4-15),

$$\overline{\text{TS}} = 10\log_{10}\left[\frac{\overline{\sigma_{\text{d}}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})}{A_{\text{ref}}}\right]\text{dB re } A_{\text{ref}} \tag{10.4-64}$$

is the *average* target strength where $A_{\text{ref}} = 1\,\text{m}^2$ is the reference cross-sectional area,

$$\overline{\sigma_{\text{d}}}(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) = \frac{1}{(4\pi)^2}E\left\{\left|S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\right|^2\right\} \tag{10.4-65}$$

is the *average* differential scattering cross-section in squared meters, and NL is the noise level given by (10.4-45).

Now that we have an equation for the average sound-pressure level $\overline{\text{SPL}}$, we can compute the *average* transmission loss ($\overline{\text{TL}}$) given by

$$\overline{\text{TL}} = \text{SL} - \overline{\text{SPL}}\,. \tag{10.4-66}$$

Substituting (10.4-63) into (10.4-66) yields

$$\boxed{\overline{\text{TL}} = 20\log_{10}(r_{TS}r_{SR}) + \alpha'(f)(r_{TS} + r_{SR} - 1) - \overline{\text{TS}}} \tag{10.4-67}$$

For the case of a monostatic (backscatter) geometry, where the transmitter (sound-source) and receiver are at the same location, $r_{SR} = r_{TS}$ [see (10.1-77)]. Therefore, (10.4-63) and (10.4-67) reduce as follows:

$$\boxed{\overline{\text{SPL}} = \text{SL} - 40\log_{10}(r_{TS}) - \alpha'(f)(2r_{TS} - 1) + \overline{\text{TS}}} \tag{10.4-68}$$

and

$$\boxed{\overline{\text{TL}} = 40\log_{10}(r_{TS}) + \alpha'(f)(2r_{TS} - 1) - \overline{\text{TS}}} \tag{10.4-69}$$

10.4.2 Direct Path

In this subsection we shall derive equations for the sound-pressure level (SPL), transmission loss (TL), and acoustic signal-to-noise ratio at the location of the receiver for the direct path shown in Fig. 10.3-1. The receiver is an omnidirectional point-element at $\mathbf{r}_R$. In order to derive equations for the SPL and TL, we first have to find the root-mean-square (rms) acoustic pressure for the direct path.

The time-harmonic acoustic pressure (in pascals) incident upon the receiver at $\mathbf{r}_R$ for the direct path is given by

$$p_d(t,\mathbf{r}_R) = p_{f,d}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.4-70}$$

where, by substituting (10.3-3) into (10.3-2),

$$p_{f,d}(\mathbf{r}_R) = -jk\rho_0 c\, S_0\, g_f\left(\mathbf{r}_R \,\middle|\, \mathbf{r}_T\right). \tag{10.4-71}$$

Further substituting (10.3-5) into (10.4-71) yields

$$p_{f,d}(\mathbf{r}_R) = jk\rho_0 c\frac{S_0}{4\pi r_{TR}}\exp[-\alpha(f)r_{TR}]\exp(-jkr_{TR}). \tag{10.4-72}$$

Since the acoustic pressure $p_d(t,\mathbf{r}_R)$ is a time-harmonic acoustic field, its rms value is given by

$$p_{\mathrm{rms},d}(\mathbf{r}_R) = \frac{\sqrt{2}}{2}\left|p_{f,d}(\mathbf{r}_R)\right|. \tag{10.4-73}$$

Computing the magnitude of (10.4-72) yields

$$\left|p_{f,d}(\mathbf{r}_R)\right| = k\rho_0 c\frac{|S_0|}{4\pi r_{TR}}\exp[-\alpha(f)r_{TR}], \tag{10.4-74}$$

and by substituting (10.1-56) into (10.4-74), we obtain

$$\left|p_{f,d}(\mathbf{r}_R)\right| = P_0\frac{R}{r_{TR}}\exp\left[-\alpha(f)(r_{TR}-R)\right]\Big|_{R=1\text{ m}}. \tag{10.4-75}$$

Therefore, substituting (10.4-75) into (10.4-73) yields the rms acoustic pressure for the direct path

$$p_{\mathrm{rms},d}(\mathbf{r}_R) = \frac{\sqrt{2}}{2}P_0\frac{R}{r_{TR}}\exp\left[-\alpha(f)(r_{TR}-R)\right]\Big|_{R=1\text{ m}}. \tag{10.4-76}$$

Now that we have an equation for the rms acoustic pressure, we can begin the SPL calculation by dividing both sides of (10.4-76) by the rms reference pressure P_{ref}. Doing so yields

$$\frac{p_{\text{rms},d}(\mathbf{r}_R)}{P_{\text{ref}}} = \frac{\sqrt{2}P_0/2}{P_{\text{ref}}}\frac{R}{r_{TR}}\exp\left[-\alpha(f)(r_{TR}-R)\right]\bigg|_{R=1\text{ m}} . \tag{10.4-77}$$

Note that both sides of (10.4-77) are *dimensionless*. Taking $20\log_{10}$ of (10.4-77) and setting $R = 1\text{ m}$ yields

$$\text{SPL} = \text{SL} - 20\log_{10}(r_{TR}) + 20\log_{10}\left\{\exp\left[-\alpha(f)(r_{TR}-1)\right]\right\}, \tag{10.4-78}$$

where

$$\text{SPL} = 20\log_{10}\left[\frac{p_{\text{rms},d}(\mathbf{r}_R)}{P_{\text{ref}}}\right]\text{dB re }P_{\text{ref}} \tag{10.4-79}$$

is the sound-pressure level at the receiver, SL is the source level given by (10.4-12), $\alpha(f)$ is the real, nonnegative, frequency-dependent, attenuation coefficient of seawater in nepers per meter, and $P_{\text{ref}} = 1\,\mu\text{Pa (rms)}$ is the rms reference pressure used in underwater acoustics. By referring to (10.4-14) and (10.4-15),

$$20\log_{10}\left\{\exp\left[-\alpha(f)(r_{TR}-1)\right]\right\} = -\alpha'(f)(r_{TR}-1), \tag{10.4-80}$$

where $\alpha'(f)$ is the attenuation coefficient in decibels per meter. Substituting (10.4-80) into (10.4-78) yields the following equation for the sound-pressure level at the location of the receiver for the direct path shown in Fig. 10.3-1:

$$\boxed{\text{SPL} = \text{SL} - 20\log_{10}(r_{TR}) - \alpha'(f)(r_{TR}-1)} \tag{10.4-81}$$

Equation (10.4-81) is another example of an active sonar equation because of the intentional transmission of sound. However, if the sound-source is a submarine, for example, that is *unintentionally* radiating noise at a certain source level, then (10.4-81) would be referred to as a *passive sonar equation*. Substituting (10.4-81) into (10.4-17) yields the transmission loss

$$\boxed{\text{TL} = 20\log_{10}(r_{TR}) + \alpha'(f)(r_{TR}-1)} \tag{10.4-82}$$

Acoustic Signal-to-Noise Ratio

In the development of the sonar equations in this subsection, ambient noise in the ocean was not taken into account. In order to develop sonar equations that include ambient noise, we shall use the following definition for the *acoustic signal-to-noise ratio* (SNR_a) as a starting point:

$$\boxed{\mathrm{SNR}_\mathrm{a} \triangleq \frac{\left\langle \left| p_d(t, \mathbf{r}_R) \right|^2 \right\rangle}{E\left\{ \left| p_{n_a}(t, \mathbf{r}_R) \right|^2 \right\}}} \tag{10.4-83}$$

where $\left\langle \left| p_d(t, \mathbf{r}_R) \right|^2 \right\rangle$ is the time-average second moment (time-average power) of the *deterministic* (nonrandom), direct-path acoustic pressure $p_d(t, \mathbf{r}_R)$ incident upon the receiver at $\mathbf{r}_R$, and $E\left\{ \left| p_{n_a}(t, \mathbf{r}_R) \right|^2 \right\}$ is the statistical second moment (statistical average power) of the acoustic pressure due to ambient noise $p_{n_a}(t, \mathbf{r}_R)$ incident upon the receiver at $\mathbf{r}_R$, where $E\{\bullet\}$ is the ensemble-average (expectation) operator. We know from our discussion of the acoustic signal-to-noise ratio in Subsection 10.4.1 that the time-average of the magnitude-squared of a function of time is equal to its root-mean-square (rms) value squared. Therefore,

$$\left\langle \left| p_d(t, \mathbf{r}_R) \right|^2 \right\rangle = p^2_{\mathrm{rms},\,d}(\mathbf{r}_R)\,, \tag{10.4-84}$$

where $p_{\mathrm{rms},\,d}(\mathbf{r}_R)$ is the rms value of the direct-path acoustic pressure given by (10.4-76).

Since the acoustic pressure due to ambient noise $p_{n_a}(t, \mathbf{r}_R)$ is assumed to be WSS in time, and is being treated as a real function of time and space, then from (10.4-55)

$$E\left\{ \left| p_{n_a}(t, \mathbf{r}_R) \right|^2 \right\} = P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)\,, \tag{10.4-85}$$

where $P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)$ is the average power of the ambient noise given by (10.4-35). Substituting (10.4-84) and (10.4-85) into (10.4-83) yields

$$\boxed{\mathrm{SNR}_\mathrm{a} = \frac{p^2_{\mathrm{rms},\,d}(\mathbf{r}_R)}{P_{\mathrm{avg},\,p_{n_a}}(\mathbf{r}_R)}} \tag{10.4-86}$$

And since the direct-path acoustic pressure $p_d(t, \mathbf{r}_R)$ is time-harmonic in this analysis, the *acoustic signal-to-noise ratio at frequency* f *hertz* can be defined as

$$\boxed{\mathrm{SNR}_{\mathrm{a},f} \triangleq \frac{p^2_{\mathrm{rms},d}(\mathbf{r}_R)}{P_{\mathrm{avg},p_{n_a},f}(\mathbf{r}_R)}} \tag{10.4-87}$$

where $P_{\mathrm{avg},p_{n_a},f}(\mathbf{r}_R)$ is the average power of the ambient noise $p_{n_a}(t,\mathbf{r}_R)$ at frequency f hertz given by (10.4-39). Therefore, substituting (10.4-39) into (10.4-87) yields

$$\boxed{\mathrm{SNR}_{\mathrm{a},f} = \frac{p^2_{\mathrm{rms},d}(\mathbf{r}_R)}{2\,S_{p_{n_a}}(f,\mathbf{r}_R)\Delta f}} \tag{10.4-88}$$

where $S_{p_{n_a}}(f,\mathbf{r}_R)$ is the power spectrum (in pascals-squared per hertz) of $p_{n_a}(t,\mathbf{r}_R)$, and $\Delta f = 1\,\mathrm{Hz}$.

If (10.4-88) is rewritten as

$$\mathrm{SNR}_{\mathrm{a},f} = \frac{\left[p_{\mathrm{rms},d}(\mathbf{r}_R)/P_{\mathrm{ref}}\right]^2}{2\,S_{p_{n_a}}(f,\mathbf{r}_R)\Delta f / P^2_{\mathrm{ref}}}\,, \tag{10.4-89}$$

then the acoustic signal-to-noise ratio (in decibels) at frequency f hertz is given by

$$\boxed{\mathrm{SNR}_{\mathrm{a},f}\ (\mathrm{dB}) = \mathrm{SPL} - \mathrm{NL}} \tag{10.4-90}$$

where

$$\mathrm{SNR}_{\mathrm{a},f}\ (\mathrm{dB}) = 10\log_{10}\mathrm{SNR}_{\mathrm{a},f}\,, \tag{10.4-91}$$

$$\mathrm{SPL} = 20\log_{10}\left[\frac{p_{\mathrm{rms},d}(\mathbf{r}_R)}{P_{\mathrm{ref}}}\right]\mathrm{dB\ re\ }P_{\mathrm{ref}} \tag{10.4-92}$$

is the sound-pressure level at the location of the receiver for the direct path and is given by (10.4-81), $P_{\mathrm{ref}} = 1\,\mu\mathrm{Pa\ (rms)}$ is the rms reference pressure used in underwater acoustics, and NL is the noise level given by (10.4-45).

Optimum Frequency

The solution for the direct path is not only part of the bistatic scattering problem, it can also be used to model underwater acoustic communication between a transmitter and a receiver. For a constant, fixed range between a transmitter and a receiver, the optimum value to use for the carrier frequency of a transmitted signal can be found as follows. Using (10.4-17) to solve for the SPL yields

$$\text{SPL} = \text{SL} - \text{TL}\,, \tag{10.4-93}$$

and by substituting (10.4-93) into (10.4-90), we obtain

$$\text{SNR}_{\text{a},f}\,(\text{dB}) = \text{SL} - (\text{TL} + \text{NL})\,. \tag{10.4-94}$$

Next, plot $\text{TL} + \text{NL}$ in decibels versus frequency in hertz for a constant, fixed range in meters. The *optimum carrier frequency* is that frequency for which the $\text{TL} + \text{NL}$ curve is a *minimum*. Once the optimum carrier frequency has been determined, one can then solve for the SL required for a desired $\text{SNR}_{\text{a},f}\,(\text{dB})$:

$$\text{SL} = \text{SNR}_{\text{a},f}\,(\text{dB}) + \text{TL} + \text{NL}\,. \tag{10.4-95}$$

10.5 Broadband Solutions

Up to now in this chapter, we have dealt with time-harmonic acoustic fields whose values depend on a single frequency. The time-harmonic acoustic fields were produced by applying a time-harmonic, input electrical signal to an omnidirectional point-source. Since a time-harmonic electrical signal is assumed to exist for all time, its complex frequency spectrum contains a single frequency component [see (6C-16) and (6C-17)]. However, all real-world transmitted electrical signals have a finite duration or pulse length and are sometimes referred to as pulses. As a result, they have complex, bandpass frequency spectra that contain many significant frequency components. A bandpass frequency spectrum can be either narrowband or broadband (see Section 11.1). In this section we shall derive pulse solutions (commonly referred to as broadband solutions) for the velocity potentials and acoustic pressures of the scattered and direct-path acoustic fields by generalizing their time-harmonic solutions. The receiver is an omnidirectional point-element at $\mathbf{r}_R$. We begin with the scattered path.

10.5.1 Scattered Path

The time-harmonic, scalar velocity potential (in squared meters per second) of the scattered acoustic field incident upon the receiver at $\mathbf{r}_R$ is given by

$$\varphi_s(t,\mathbf{r}_R) = \varphi_{f,s}(\mathbf{r}_R)\exp(+j2\pi f t)\,, \tag{10.5-1}$$

where, by substituting (10.1-20) into (10.1-23),

$$\varphi_{f,s}(\mathbf{r}_R) = S_0\, g_f\left(\mathbf{r}_S \middle| \mathbf{r}_T\right) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})\, g_f\left(\mathbf{r}_R \middle| \mathbf{r}_S\right). \tag{10.5-2}$$

Further substituting (10.1-6), (10.1-7), (10.1-24), and $k = 2\pi f/c$ into (10.5-2) yields

$$\varphi_{f,s}(\mathbf{r}_R) = \frac{A_x}{(4\pi)^2 r_{TS} r_{SR}} \mathcal{S}_T(f) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \exp\left[-\alpha(f)(r_{TS} + r_{SR})\right] \times \exp\left(-j2\pi f \frac{r_{TS} + r_{SR}}{c}\right). \tag{10.5-3}$$

If we substitute (10.5-3) into (10.5-1), replace the complex amplitude A_x (in volts) of the time-harmonic, input electrical signal applied to the omnidirectional point-source with the complex frequency spectrum $X(f)$ (in volts per hertz) of a broadband input electrical signal, and integrate the right-hand side of the resulting equation over frequency f, then

$$\boxed{\varphi_s(t, \mathbf{r}_R) = \frac{1}{(4\pi)^2 r_{TS} r_{SR}} \int_{-\infty}^{\infty} X(f) \mathcal{S}_T(f) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \times \exp\left[-\alpha(f)(r_{TS} + r_{SR})\right] \exp\left[+j2\pi f(t - \tau)\right] df} \tag{10.5-4}$$

is the broadband solution for the velocity potential of the scattered acoustic field for $t \geq \tau$, where

$$\tau = (r_{TS} + r_{SR})/c \tag{10.5-5}$$

is the *bistatic* time delay in seconds. Equation (10.5-4) indicates that the broadband solution $\varphi_s(t, \mathbf{r}_R)$ is equal to the “sum” of the contributions from each frequency component contained in the complex frequency spectrum $X(f)$. Note that (10.5-4) can be expressed as an inverse Fourier transform, that is,

$$\varphi_s(t, \mathbf{r}_R) = \frac{1}{(4\pi)^2 r_{TS} r_{SR}} F_f^{-1}\left\{X(f) \mathcal{S}_T(f) S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \times \exp\left[-\alpha(f)(r_{TS} + r_{SR})\right] \exp(-j2\pi f \tau)\right\}, \tag{10.5-6}$$

where τ is given by (10.5-5). Therefore, the complex frequency spectrum of the velocity potential of the scattered acoustic field at $\mathbf{r}_R$ is given by

$$\boxed{\begin{aligned}\Phi_s(f,\mathbf{r}_R) &= F_t\{\varphi_s(t,\mathbf{r}_R)\} \\ &= \frac{1}{(4\pi)^2 r_{TS} r_{SR}} X(f)\mathcal{S}_T(f)S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\times \\ &\quad \exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\exp(-j2\pi f\tau)\end{aligned}} \tag{10.5-7}$$

where $\Phi_s(f,\mathbf{r}_R)$ has units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$.

The complex frequency spectrum of the output electrical signal from an omnidirectional point-element at $\mathbf{r}_R$ is given by [see (7.3-27)]

$$Y_s(f,\mathbf{r}_R) = \Phi_s(f,\mathbf{r}_R)\mathcal{S}_R(f), \tag{10.5-8}$$

where $Y_s(f,\mathbf{r}_R)$ has units of $\mathrm{V/Hz}$, and $\mathcal{S}_R(f)$ is the complex, receiver sensitivity function with units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$ (see Table 6.1-2 and Appendix 6B). The time-domain output electrical signal in volts is therefore

$$y_s(t,\mathbf{r}_R) = F_f^{-1}\{Y_s(f,\mathbf{r}_R)\} = F_f^{-1}\{\Phi_s(f,\mathbf{r}_R)\mathcal{S}_R(f)\}. \tag{10.5-9}$$

The time-harmonic, scattered acoustic pressure (in pascals) incident upon the receiver at $\mathbf{r}_R$ is given by

$$p_s(t,\mathbf{r}_R) = p_{f,s}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.5-10}$$

where, by substituting (10.1-6) and $k = 2\pi f/c$ into (10.4-3),

$$\begin{aligned}p_{f,s}(\mathbf{r}_R) = -j\frac{f\rho_0 A_x}{8\pi r_{TS} r_{SR}}\mathcal{S}_T(f)S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})\exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\times \\ \exp\left(-j2\pi f\frac{r_{TS}+r_{SR}}{c}\right).\end{aligned} \tag{10.5-11}$$

If we substitute (10.5-11) into (10.5-10), replace the complex amplitude A_x (in volts) of the time-harmonic, input electrical signal applied to the omnidirectional point-source with the complex frequency spectrum $X(f)$ (in volts per hertz) of a broadband input electrical signal, and integrate the right-hand side of the resulting equation over frequency f, then

$$\boxed{\begin{aligned} p_s(t,\mathbf{r}_R) = -j\frac{\rho_0}{8\pi r_{TS} r_{SR}} \int_{-\infty}^{\infty} f X(f)\, \mathcal{S}_T(f)\, S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) \times \\ \exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\exp\left[+j2\pi f(t-\tau)\right] df \end{aligned}}$$

(10.5-12)

is the broadband solution for the scattered acoustic pressure for $t \geq \tau$, or

$$\begin{aligned} p_s(t,\mathbf{r}_R) = -j\frac{\rho_0}{8\pi r_{TS} r_{SR}} F_f^{-1}\Big\{ f X(f)\, \mathcal{S}_T(f)\, S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) \times \\ \exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\exp(-j2\pi f\tau)\Big\}, \end{aligned}$$

(10.5-13)

where τ is given by (10.5-5). Therefore, the complex frequency spectrum of the scattered acoustic pressure at $\mathbf{r}_R$ is given by

$$\boxed{\begin{aligned} P_s(f,\mathbf{r}_R) &= F_t\{p_s(t,\mathbf{r}_R)\} \\ &= -j\frac{\rho_0}{8\pi r_{TS} r_{SR}} f X(f)\, \mathcal{S}_T(f)\, S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR}) \times \\ &\quad \exp\left[-\alpha(f)(r_{TS}+r_{SR})\right]\exp(-j2\pi f\tau) \end{aligned}}$$

(10.5-14)

where $P_s(f,\mathbf{r}_R)$ has units of $\mathrm{Pa/Hz}$.

Since

$$P_s(f,\mathbf{r}_R) = -j2\pi f\rho_0 \Phi_s(f,\mathbf{r}_R), \tag{10.5-15}$$

the complex frequency spectrum of the output electrical signal from an omnidirectional point-element at $\mathbf{r}_R$ is given by

$$Y_s(f,\mathbf{r}_R) = \Phi_s(f,\mathbf{r}_R)\,\mathcal{S}_R(f) = j\frac{P_s(f,\mathbf{r}_R)}{2\pi f\rho_0}\mathcal{S}_R(f), \tag{10.5-16}$$

where $Y_s(f,\mathbf{r}_R)$ has units of $\mathrm{V/Hz}$, the ratio $P_s(f,\mathbf{r}_R)/(2\pi f\rho_0)$ has units of $\left(\mathrm{m^2/sec}\right)/\mathrm{Hz}$, and $\mathcal{S}_R(f)$ has units of $\mathrm{V}/\left(\mathrm{m^2/sec}\right)$. The time-domain output electrical signal in volts is therefore

$$y_s(t,\mathbf{r}_R) = F_f^{-1}\{Y_s(f,\mathbf{r}_R)\} = F_f^{-1}\left\{ j\frac{P_s(f,\mathbf{r}_R)}{2\pi f\rho_0} \mathcal{S}_R(f) \right\}. \tag{10.5-17}$$

10.5.2 Direct Path

The time-harmonic, scalar velocity potential (in squared meters per second) of the direct-path acoustic field incident upon the receiver at $\mathbf{r}_R$ is given by

$$\varphi_d(t,\mathbf{r}_R) = \varphi_{f,d}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.5-18}$$

where, by substituting (10.3-4), (10.3-5), and $k = 2\pi f/c$ into (10.3-3),

$$\varphi_{f,d}(\mathbf{r}_R) = -\frac{A_x}{4\pi r_{TR}} \mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp\left(-j2\pi f\frac{r_{TR}}{c}\right). \tag{10.5-19}$$

If we substitute (10.5-19) into (10.5-18), replace the complex amplitude A_x (in volts) of the time-harmonic, input electrical signal applied to the omnidirectional point-source with the complex frequency spectrum $X(f)$ (in volts per hertz) of a broadband input electrical signal, and integrate the right-hand side of the resulting equation over frequency f, then

$$\boxed{\varphi_d(t,\mathbf{r}_R) = -\frac{1}{4\pi r_{TR}}\int_{-\infty}^{\infty} X(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp\left[+j2\pi f(t-\tau)\right]df} \tag{10.5-20}$$

is the broadband solution for the velocity potential of the direct-path acoustic field for $t \geq \tau$, or

$$\varphi_d(t,\mathbf{r}_R) = -\frac{1}{4\pi r_{TR}} F_f^{-1}\left\{X(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp(-j2\pi f\tau)\right\}, \tag{10.5-21}$$

where

$$\tau = r_{TR}/c \tag{10.5-22}$$

is the direct-path time delay in seconds. Therefore, the complex frequency spectrum of the velocity potential of the direct-path acoustic field at $\mathbf{r}_R$ is given by

$$\boxed{\Phi_d(f,\mathbf{r}_R) = F_t\{\varphi_d(t,\mathbf{r}_R)\} = -\frac{1}{4\pi r_{TR}} X(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp(-j2\pi f\tau)}$$

(10.5-23)

where $\Phi_d(f,\mathbf{r}_R)$ has units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$.

The complex frequency spectrum of the output electrical signal from an omnidirectional point-element at $\mathbf{r}_R$ is given by [see (7.3-27)]

$$Y_d(f,\mathbf{r}_R) = \Phi_d(f,\mathbf{r}_R)\mathcal{S}_R(f), \tag{10.5-24}$$

where $Y_d(f,\mathbf{r}_R)$ has units of V/Hz, and $\mathcal{S}_R(f)$ is the complex, receiver sensitivity function with units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$ (see Table 6.1-2 and Appendix 6B). The time-domain output electrical signal in volts is therefore

$$y_d(t,\mathbf{r}_R) = F_f^{-1}\{Y_d(f,\mathbf{r}_R)\} = F_f^{-1}\{\Phi_d(f,\mathbf{r}_R)\mathcal{S}_R(f)\}. \tag{10.5-25}$$

The time-harmonic, direct-path acoustic pressure (in pascals) incident upon the receiver at $\mathbf{r}_R$ is given by

$$p_d(t,\mathbf{r}_R) = p_{f,d}(\mathbf{r}_R)\exp(+j2\pi ft), \tag{10.5-26}$$

where, by substituting (10.3-4) and $k = 2\pi f/c$ into (10.4-72),

$$p_{f,d}(\mathbf{r}_R) = j\frac{f\rho_0 A_x}{2r_{TR}}\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp\left(-j2\pi f\frac{r_{TR}}{c}\right). \tag{10.5-27}$$

If we substitute (10.5-27) into (10.5-26), replace the complex amplitude A_x (in volts) of the time-harmonic, input electrical signal applied to the omnidirectional point-source with the complex frequency spectrum $X(f)$ (in volts per hertz) of a broadband input electrical signal, and integrate the right-hand side of the resulting equation over frequency f, then

$$\boxed{p_d(t,\mathbf{r}_R) = j\frac{\rho_0}{2r_{TR}}\int_{-\infty}^{\infty} fX(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp[+j2\pi f(t-\tau)]\,df}$$

(10.5-28)

is the broadband solution for the direct-path acoustic pressure for $t \geq \tau$, or

$$p_d(t,\mathbf{r}_R) = j\frac{\rho_0}{2r_{TR}} F_f^{-1}\left\{ f X(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp(-j2\pi f\tau)\right\}, \tag{10.5-29}$$

where τ is given by (10.5-22). Therefore, the complex frequency spectrum of the direct-path acoustic pressure at $\mathbf{r}_R$ is given by

$$\boxed{P_d(f,\mathbf{r}_R) = F_t\{p_d(t,\mathbf{r}_R)\} = j\frac{\rho_0}{2r_{TR}} f X(f)\mathcal{S}_T(f)\exp[-\alpha(f)r_{TR}]\exp(-j2\pi f\tau)} \tag{10.5-30}$$

where $P_d(f,\mathbf{r}_R)$ has units of Pa/Hz.

Since

$$P_d(f,\mathbf{r}_R) = -j2\pi f\rho_0\Phi_d(f,\mathbf{r}_R), \tag{10.5-31}$$

the complex frequency spectrum of the output electrical signal from an omnidirectional point-element at $\mathbf{r}_R$ is given by

$$Y_d(f,\mathbf{r}_R) = \Phi_d(f,\mathbf{r}_R)\mathcal{S}_R(f) = j\frac{P_d(f,\mathbf{r}_R)}{2\pi f\rho_0}\mathcal{S}_R(f), \tag{10.5-32}$$

where $Y_d(f,\mathbf{r}_R)$ has units of V/Hz, the ratio $P_d(f,\mathbf{r}_R)/(2\pi f\rho_0)$ has units of $\left(\text{m}^2/\text{sec}\right)/\text{Hz}$, and $\mathcal{S}_R(f)$ has units of $\text{V}/\left(\text{m}^2/\text{sec}\right)$. The time-domain output electrical signal in volts is therefore

$$y_d(t,\mathbf{r}_R) = F_f^{-1}\{Y_d(f,\mathbf{r}_R)\} = F_f^{-1}\left\{ j\frac{P_d(f,\mathbf{r}_R)}{2\pi f\rho_0}\mathcal{S}_R(f)\right\}. \tag{10.5-33}$$

10.6 A Statistical Model of the Scattering Function

In this section we shall develop a statistical model for the scattering function $S_S(f,\theta_{TS},\psi_{TS},\theta_{SR},\psi_{SR})$ of an object by treating it as a *random function*. Therefore, for a given frequency and geometry, the value of the scattering function, which we shall call Z, will be a *complex*, *random variable* rather than a complex, nonrandom constant. As part of the model, we shall derive the probability density functions of the magnitude, magnitude-squared, and phase of Z.

We begin by equating the value of the scattering function Z to a sum of deterministic (nonrandom) and random components, that is,

$$Z = Z_D + Z_R , \tag{10.6-1}$$

where

$$Z = S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) \tag{10.6-2}$$

is a complex, random variable (RV),

$$Z_D = X_D + jY_D \tag{10.6-3}$$

is a complex, deterministic (nonrandom) constant, where X_D and Y_D are real, nonrandom constants, and

$$Z_R = X_R + jY_R \tag{10.6-4}$$

is a complex RV, where X_R and Y_R are real RVs. The complex, nonrandom constant Z_D can be thought of as being the "signal" and the complex RV Z_R can be thought of as being "noise." Substituting (10.6-3) and (10.6-4) into (10.6-1) yields

$$Z = X + jY , \tag{10.6-5}$$

where

$$X = X_D + X_R \tag{10.6-6}$$

and

$$Y = Y_D + Y_R \tag{10.6-7}$$

are real RVs.

The next step is to decide how to model the complex RV Z_R. We shall model Z_R as a *complex*, *zero-mean*, *Gaussian RV*, where X_R and Y_R are *real*, *zero-mean*, *uncorrelated*, *Gaussian RVs*. Since [see (10.6-4)]

$$E\{Z_R\} = E\{X_R\} + jE\{Y_R\} , \tag{10.6-8}$$

if

$$E\{Y_R\} = E\{X_R\} = 0 , \tag{10.6-9}$$

then

$$E\{Z_R\} = 0 . \tag{10.6-10}$$

Also, since the variance of Z_R is given by

$$\begin{aligned}\sigma_{Z_R}^2 &= E\left\{\left|Z_R\right|^2\right\} - \left|E\left\{Z_R\right\}\right|^2 \\ &= E\left\{\left|Z_R\right|^2\right\} \\ &= E\left\{X_R^2\right\} + E\left\{Y_R^2\right\},\end{aligned} \tag{10.6-11}$$

if

$$E\left\{Y_R^2\right\} = E\left\{X_R^2\right\} = \sigma^2 , \tag{10.6-12}$$

then

$$\sigma_{Z_R}^2 = 2\sigma^2 . \tag{10.6-13}$$

Furthermore, since X_R and Y_R are zero-mean [see (10.6-9)], their variances are equal to their second moments given by (10.6-12), and because they are uncorrelated Gaussian RVs, they are also *statistically independent*.

If Z_R is a complex, zero-mean, Gaussian random variable (RV), then Z will be a *complex*, *non-zero mean*, *Gaussian RV* with mean value [see (10.6-1)]

$$m_Z = E\left\{Z\right\} = Z_D \tag{10.6-14}$$

and variance [see (10.6-1) and (10.6-13)]

$$\sigma_Z^2 = \sigma_{Z_R}^2 = 2\sigma^2 , \tag{10.6-15}$$

where σ^2 is the variance of both X_R and Y_R [see (10.6-12)]. With the use of (10.6-2) and (10.6-5), the magnitude of Z can be expressed as

$$\left|Z\right| = \left|S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})\right| = \sqrt{X^2 + Y^2} , \tag{10.6-16}$$

where X and Y are Gaussian RVs because X_R and Y_R are Gaussian RVs [see (10.6-6) and (10.6-7)]. The statistics of X and Y are as follows:

$$m_X = E\left\{X\right\} = X_D , \tag{10.6-17}$$

$$\sigma_X^2 = \sigma_{X_R}^2 = \sigma^2 , \tag{10.6-18}$$

$$m_Y = E\left\{Y\right\} = Y_D , \tag{10.6-19}$$

$$\sigma_Y^2 = \sigma_{Y_R}^2 = \sigma^2 , \tag{10.6-20}$$

and

$$\begin{aligned}\operatorname{Cov}(X,Y) &= E\{XY\} - m_X m_Y \\ &= E\{X_R Y_R\} \\ &= E\{X_R\}E\{Y_R\} \\ &= 0\end{aligned} \tag{10.6-21}$$

because X_R and Y_R are uncorrelated and zero-mean. Therefore, since X is $N(X_D, \sigma^2)$ and Y is $N(Y_D, \sigma^2)$, where X and Y have the *same* variance σ^2, and X and Y are *statistically independent* because they are uncorrelated Gaussian RVs [see (10.6-21)], $|Z|$ is *Rice* distributed with probability density function[1]

$$\boxed{p_\zeta(\zeta) = \frac{\zeta}{\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left(\zeta^2 + \zeta_D^2\right)\right] I_0\left(\frac{\zeta_D}{\sigma^2}\zeta\right), \qquad \zeta \geq 0} \tag{10.6-22}$$

where the RV

$$\zeta = |Z| = \sqrt{X^2 + Y^2}\,, \tag{10.6-23}$$

the nonrandom constant

$$\zeta_D = |Z_D| = \sqrt{X_D^2 + Y_D^2}\,, \tag{10.6-24}$$

and

$$I_0(u) = \sum_{n=0}^{\infty} \frac{u^{2n}}{2^{2n}(n!)^2} \tag{10.6-25}$$

is the zeroth-order modified Bessel function of the first kind. Figure 10.6-1 is a plot of the scaled Rician probability density function (PDF) $\sigma p_\zeta(\zeta)$ versus ζ/σ for several different values of ζ_D/σ. The ratio $\zeta_D^2/(2\sigma^2) = |Z_D|^2 / \sigma_{Z_R}^2$ can be thought of as being a signal-to-noise ratio (SNR).

If the nonrandom component of the scattering function $Z_D = 0$, then $Z = Z_R$, that is, the value of the scattering function is equal to a complex, zero-mean, Gaussian RV. Also, if $Z_D = 0$, then $\zeta_D = 0$, and since $I_0(0) = 1$, the Rician PDF given by (10.6-22) reduces to the *Rayleigh* PDF

$$\boxed{p_\zeta(\zeta) = \frac{\zeta}{\sigma^2}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right), \qquad \zeta \geq 0} \tag{10.6-26}$$

[1] T. A. Schonhoff and A. A. Giordano, *Detection and Estimation Theory*, Pearson Prentice Hall, Upper Saddle River, NJ, 2006, pg. 21.

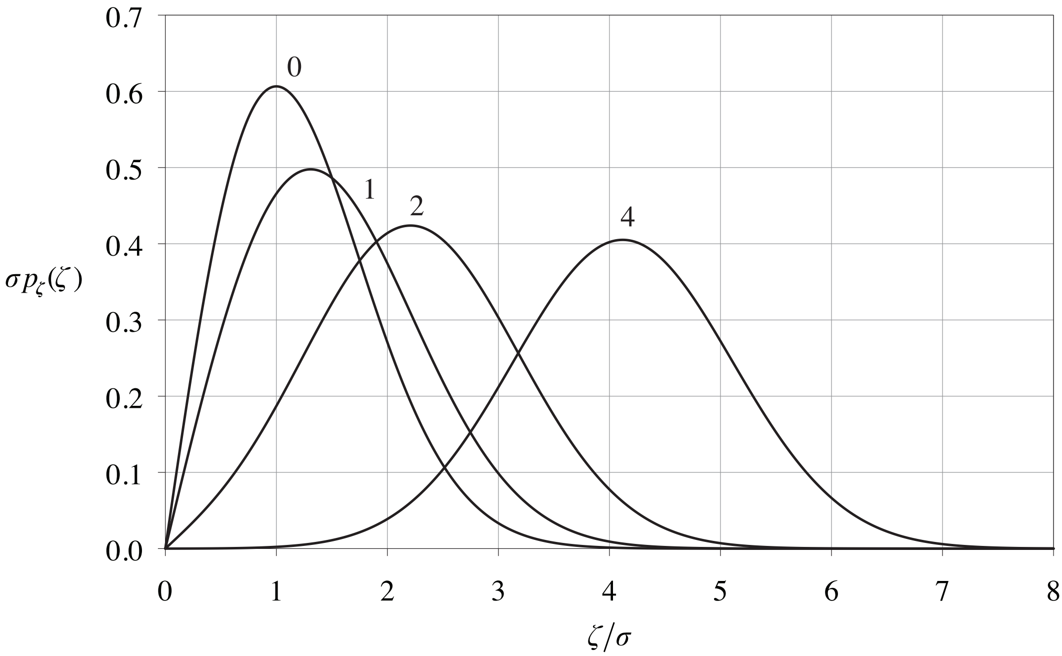

Figure 10.6-1 Plot of the scaled Rician probability density function $\sigma p_\zeta(\zeta)$ versus ζ/σ for $\zeta_D/\sigma = 0$, 1 , 2 , and 4 .

Therefore, in the limit as $\zeta_D/\sigma \to 0$, or equivalently, as the SNR $\zeta_D^2/(2\sigma^2) \to 0$, $\zeta = |Z|$ approaches being Rayleigh distributed.

As ζ_D/σ becomes large, the Rician PDF given by (10.6-22) reduces to an approximate *Gaussian* PDF [see Fig. 10.6-1], as we shall show next. Since

$$I_0(u) \approx \frac{1}{\sqrt{2\pi u}}\exp(u), \qquad u >> 0.25\,, \tag{10.6-27}$$

or equivalently,

$$I_0(u) \approx \frac{1}{\sqrt{2\pi u}}\exp(u), \qquad u \geq 2.5\,, \tag{10.6-28}$$

$$I_0\left(\frac{\zeta_D}{\sigma^2}\zeta\right) \approx \frac{\sigma}{\sqrt{2\pi}\sqrt{\zeta_D\zeta}}\exp\left(\frac{\zeta_D}{\sigma^2}\zeta\right), \qquad \zeta \geq 2.5\frac{\sigma^2}{\zeta_D}\,. \tag{10.6-29}$$

Substituting (10.6-29) into (10.6-22) yields

$$\boxed{p_\zeta(\zeta) \approx \sqrt{\frac{\zeta}{\zeta_D}}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(\zeta-\zeta_D)^2\right], \qquad \zeta \geq 2.5\frac{\sigma^2}{\zeta_D}} \tag{10.6-30}$$

Therefore, as ζ_D/σ becomes large, or equivalently, as the SNR $\zeta_D^2/(2\sigma^2)$ becomes large, $\zeta = |Z|$ is approximately Gaussian distributed with mean value ζ_D and variance σ^2.

With the use of (10.6-2) and (10.6-5), the phase of the scattering function can be expressed as

$$\angle Z = \angle S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR}) = \tan^{-1}(Y/X), \tag{10.6-31}$$

with probability density function[2]

$$\boxed{\begin{aligned} p_\phi(\phi) = \frac{1}{2\pi}\exp\left(-\frac{\zeta_D^2}{2\sigma^2}\right) + \Bigg\{ \frac{\zeta_D}{\sqrt{2\pi}\,\sigma}\cos(\phi-\phi_D)\exp\left[-\frac{\zeta_D^2}{2\sigma^2}\sin^2(\phi-\phi_D)\right]\times \\ \Phi\left[\frac{\zeta_D}{\sigma}\cos(\phi-\phi_D)\right]\Bigg\}, \qquad 0 \le \phi \le 2\pi \end{aligned}} \tag{10.6-32}$$

where the RV

$$\phi = \angle Z = \tan^{-1}(Y/X), \tag{10.6-33}$$

the nonrandom constant ζ_D is given by (10.6-24), the nonrandom constant

$$\phi_D = \angle Z_D = \tan^{-1}(Y_D/X_D), \tag{10.6-34}$$

and

$$\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp\left(-\xi^2/2\right)d\xi \tag{10.6-35}$$

is the cumulative distribution function (CDF) for a zero-mean, unit-variance, Gaussian RV. Figure 10.6-2 is a plot of (10.6-32) for $\phi_D = 210°$ and several different values of ζ_D/σ. If $Z_D = 0$, then $Z = Z_R$ and $\zeta_D = 0$. With $\zeta_D = 0$, the probability density function given by (10.6-32) reduces to the *uniform* probability density function

$$\boxed{p_\phi(\phi) = \frac{1}{2\pi}, \qquad 0 \le \phi \le 2\pi} \tag{10.6-36}$$

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[2] P. Z. Peebles, Jr., *Probability, Random Variables, and Random Signal Principles*, 4th ed., McGraw-Hill, New York, 2001, pp. 400-401.

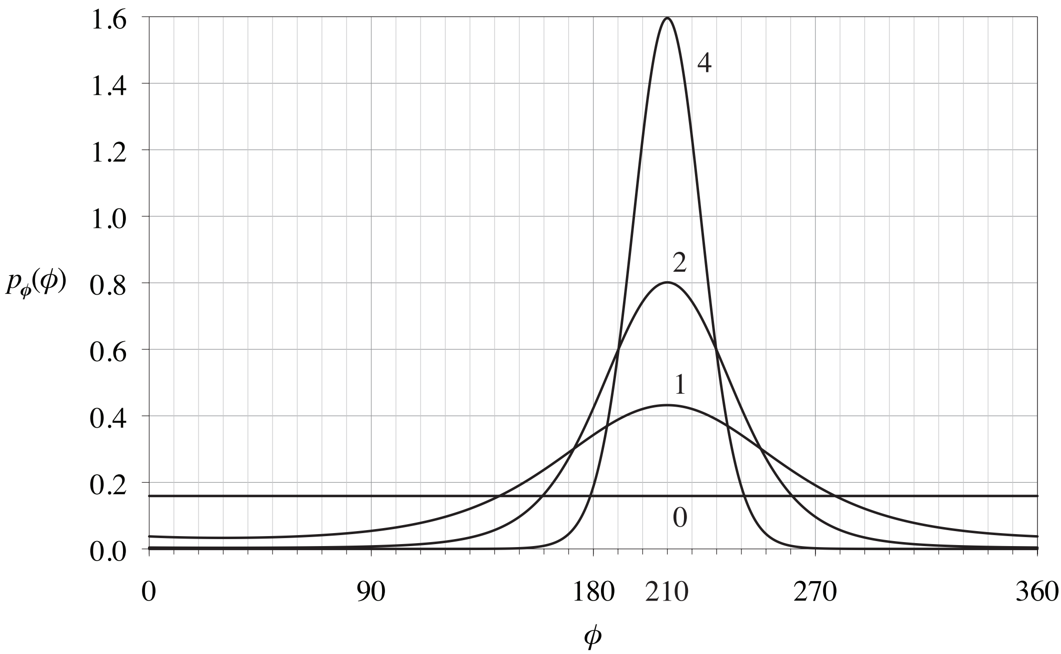

Figure 10.6-2 Plot of the probability density function $p_\phi(\phi)$ versus ϕ (in degrees) for $\phi_D = 210°$ and $\zeta_D/\sigma = 0$, 1 , 2 , and 4 .

Therefore, in the limit as $\zeta_D/\sigma \to 0$, or equivalently, as the SNR $\zeta_D^2/(2\sigma^2) \to 0$, $\phi = \angle Z$ approaches being uniformly distributed in the interval $[0, 2\pi]$. However, as can be seen from Fig. 10.6-2, as ζ_D/σ increases in value, $p_\phi(\phi)$ becomes more concentrated about the deterministic phase angle $\phi_D = 210°$.

And finally, since X is $N(X_D, \sigma^2)$ and Y is $N(Y_D, \sigma^2)$, where X and Y have the *same* variance σ^2 , and X and Y are *statistically independent* because they are uncorrelated Gaussian RVs, the magnitude-squared of the scattering function, given by

$$|Z|^2 = \left|S_S(f, \theta_{TS}, \psi_{TS}, \theta_{SR}, \psi_{SR})\right|^2 = X^2 + Y^2 , \tag{10.6-37}$$

is a *noncentral*, *chi-squared RV* with 2 degrees of freedom and probability density function[3]

$$\boxed{p_\gamma(\gamma) = \frac{1}{2\sigma^2}\exp\left(-\frac{\gamma + \gamma_D}{2\sigma^2}\right) I_0\left(\frac{\sqrt{\gamma_D}}{\sigma^2}\sqrt{\gamma}\right), \qquad \gamma \ge 0} \tag{10.6-38}$$

[3] T. A. Schonhoff and A. A. Giordano, *Detection and Estimation Theory*, Pearson Prentice Hall, Upper Saddle River, NJ, 2006, pp. 602-604.

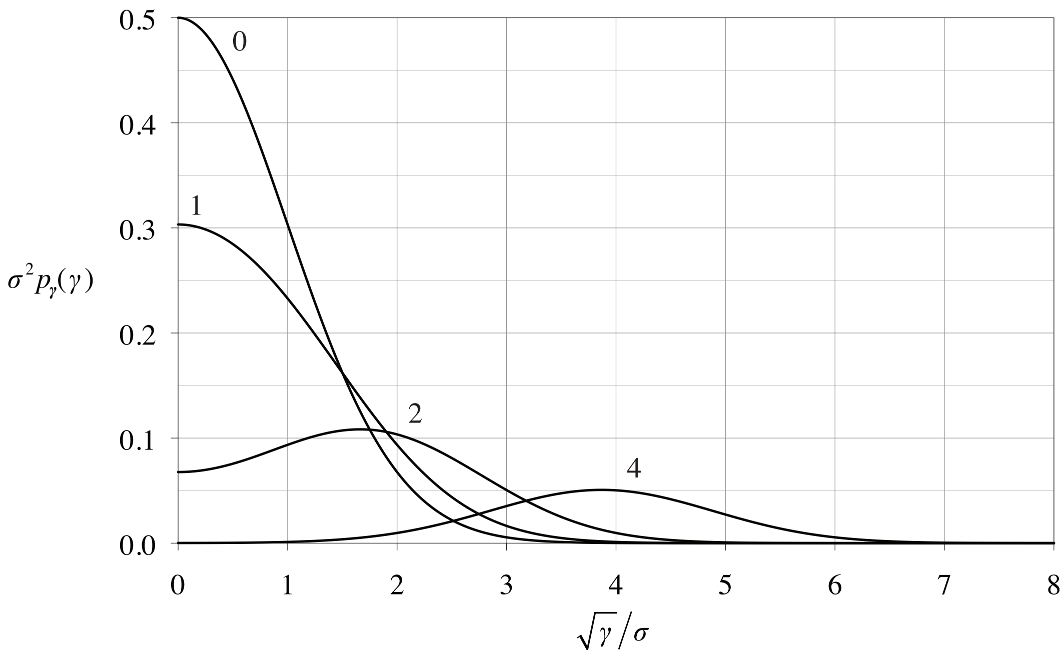

Figure 10.6-3 Plot of the scaled, noncentral, chi-squared probability density function $\sigma^2 p_\gamma(\gamma)$ versus $\sqrt{\gamma}/\sigma$ for $\sqrt{\gamma_D}/\sigma = 0$, 1, 2, and 4.

where the RV

$$\gamma = |Z|^2 = X^2 + Y^2, \tag{10.6-39}$$

and the nonrandom constant

$$\gamma_D = |Z_D|^2 = X_D^2 + Y_D^2 \tag{10.6-40}$$

is the *noncentrality parameter*. Figure 10.6-3 is a plot of the scaled, noncentral, chi-squared probability density function $\sigma^2 p_\gamma(\gamma)$ versus $\sqrt{\gamma}/\sigma$ for several different values of $\sqrt{\gamma_D}/\sigma$. The ratio $\gamma_D/(2\sigma^2) = |Z_D|^2/\sigma_{Z_R}^2$ can be thought of as being a SNR. If $Z_D = 0$, then $Z = Z_R$ and $\gamma_D = 0$. With $\gamma_D = 0$, and since $I_0(0) = 1$, the noncentral, chi-squared probability density function given by (10.6-38) reduces to the *exponential* probability density function

$$\boxed{p_\gamma(\gamma) = \frac{1}{2\sigma^2}\exp\left(-\frac{\gamma}{2\sigma^2}\right), \qquad \gamma \geq 0} \tag{10.6-41}$$

Therefore, in the limit as $\sqrt{\gamma_D}/\sigma \to 0$, or equivalently, as the SNR

$\gamma_D/(2\sigma^2) \to 0$, $\gamma = |Z|^2$ approaches being exponentially distributed.

In summary, if the nonrandom component of the scattering function $Z_D = 0$, then $Z = Z_R$, that is, the value of the scattering function is equal to a complex, zero-mean, Gaussian RV, where the magnitude $|Z| = |Z_R|$ is Rayleigh distributed, the phase $\angle Z = \angle Z_R$ is uniformly distributed in the interval $[0, 2\pi]$, and the magnitude-squared $|Z|^2 = |Z_R|^2$ is exponentially distributed.

10.7 Moving Platforms

In this section we shall derive *exact* equations for the time delay, time-compression/time-expansion factor, and Doppler shift at the receiver for the bistatic scattering problem shown in Fig. 10.1-1 when the transmitter, scatterer (target), and receiver are in *three-dimensional*, *translational motion*. These equations shall be derived for both the scattered and direct paths. We shall also derive *exact* equations for the time-varying angles of incidence at the scatterer, and the time-varying angles of scatter at the receiver for the scattered path; and *exact* equations for the time-varying angles of incidence at the receiver for the direct path. For our purposes, the transmitter is an omnidirectional point-source, and the receiver is an omnidirectional point-element.

10.7.1 Scattered Path

The velocity vectors of the transmitter, $\mathbf{V}_T$, the scatterer, $\mathbf{V}_S$, and the receiver, $\mathbf{V}_R$, are given by

$$\mathbf{V}_T = V_T \hat{n}_{\mathbf{V}_T}, \tag{10.7-1}$$

$$\mathbf{V}_S = V_S \hat{n}_{\mathbf{V}_S}, \tag{10.7-2}$$

and

$$\mathbf{V}_R = V_R \hat{n}_{\mathbf{V}_R}, \tag{10.7-3}$$

where V_T, V_S, and V_R are the speeds in meters per second of the transmitter, scatterer, and receiver, respectively, and $\hat{n}_{\mathbf{V}_T}$, $\hat{n}_{\mathbf{V}_S}$, and $\hat{n}_{\mathbf{V}_R}$ are the dimensionless unit vectors in the directions of $\mathbf{V}_T$, $\mathbf{V}_S$, and $\mathbf{V}_R$, respectively (see Fig. 10.7-1). The velocity vectors given by (10.7-1) through (10.7-3) are *constants*, that is, the speeds and directions are *constants* – acceleration is *not* being modeled. Transmission begins at time $t = t_0$ seconds. The origin of the coordinate system is *not* in motion.

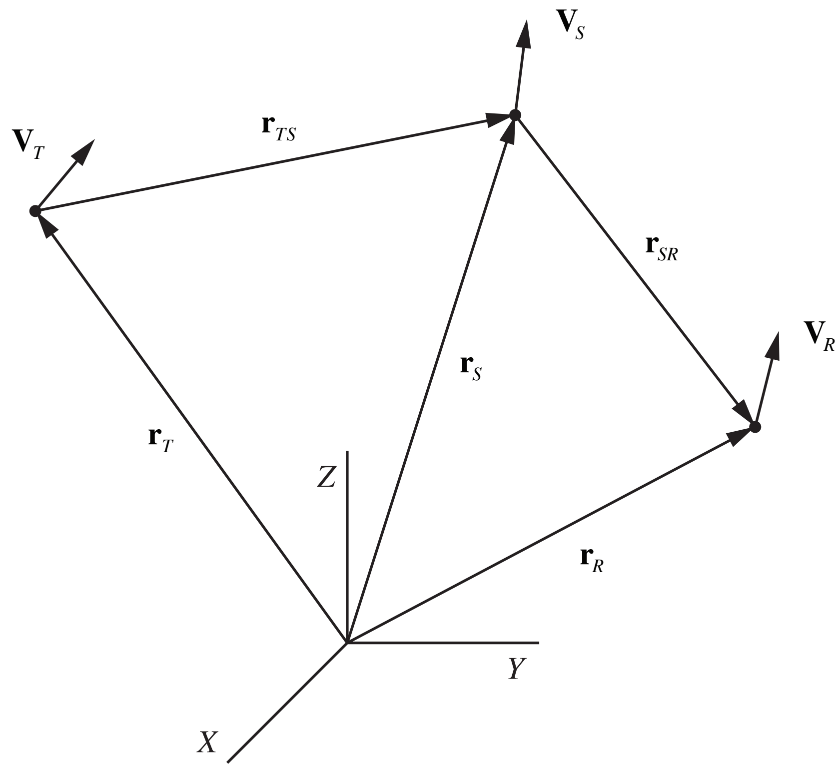

Figure 10.7-1 Bistatic scattering geometry when transmission begins at time $t = t_0$ seconds, where $\mathbf{V}_T$, $\mathbf{V}_S$, and $\mathbf{V}_R$ are the velocity vectors of the transmitter, scatterer, and receiver, respectively.

Since all three platforms are in motion, the position vectors from the origin to the transmitter, scatterer, and receiver – denoted by $\mathcal{R}_T(t)$, $\mathcal{R}_S(t)$, and $\mathcal{R}_R(t)$, respectively – are functions of time given by

$$\mathcal{R}_T(t) = \mathbf{r}_T + \mathbf{V}_T \Delta t, \qquad t \geq t_0\,, \tag{10.7-4}$$

$$\mathcal{R}_S(t) = \mathbf{r}_S + \mathbf{V}_S \Delta t, \qquad t \geq t_0\,, \tag{10.7-5}$$

and

$$\mathcal{R}_R(t) = \mathbf{r}_R + \mathbf{V}_R \Delta t, \qquad t \geq t_0\,, \tag{10.7-6}$$

where $\mathbf{r}_T = (x_T, y_T, z_T)$, $\mathbf{r}_S = (x_S, y_S, z_S)$, and $\mathbf{r}_R = (x_R, y_R, z_R)$ are the position vectors from the origin to the transmitter, scatterer, and receiver, respectively, when transmission begins (see Fig. 10.7-1), and

$$\Delta t = t - t_0, \qquad t \geq t_0\,. \tag{10.7-7}$$

Note that $\mathcal{R}_T(t_0) = \mathbf{r}_T$, $\mathcal{R}_S(t_0) = \mathbf{r}_S$, and $\mathcal{R}_R(t_0) = \mathbf{r}_R$.

The time-varying position vector from the transmitter to the scatterer is given by

$$\boldsymbol{\mathcal{R}}_{TS}(t) = \boldsymbol{\mathcal{R}}_S(t) - \boldsymbol{\mathcal{R}}_T(t), \tag{10.7-8}$$

and by substituting (10.7-4) and (10.7-5) into (10.7-8), we obtain

$$\boldsymbol{\mathcal{R}}_{TS}(t) = \mathbf{r}_{TS} + \mathbf{V}_{ST}\,\Delta t, \qquad t \geq t_0, \tag{10.7-9}$$

where

$$\mathbf{r}_{TS} = \mathbf{r}_S - \mathbf{r}_T \tag{10.7-10}$$

is the position vector from the transmitter to the scatterer when transmission begins (see Fig. 10.7-1), and

$$\mathbf{V}_{ST} = \mathbf{V}_S - \mathbf{V}_T = V_{ST}\,\hat{n}_{\mathbf{V}_{ST}} \tag{10.7-11}$$

is the velocity vector of the scatterer *relative to* the velocity vector of the transmitter, where $V_{ST} = |\mathbf{V}_{ST}|$ and $\hat{n}_{\mathbf{V}_{ST}}$ is the dimensionless unit vector in the direction of $\mathbf{V}_{ST}$. Note that $\boldsymbol{\mathcal{R}}_{TS}(t_0) = \mathbf{r}_{TS}$.

Similarly, the time-varying position vector from the scatterer to the receiver is given by

$$\boldsymbol{\mathcal{R}}_{SR}(t) = \boldsymbol{\mathcal{R}}_R(t) - \boldsymbol{\mathcal{R}}_S(t), \tag{10.7-12}$$

and by substituting (10.7-5) and (10.7-6) into (10.7-12), we obtain

$$\boldsymbol{\mathcal{R}}_{SR}(t) = \mathbf{r}_{SR} + \mathbf{V}_{RS}\,\Delta t, \qquad t \geq t_0, \tag{10.7-13}$$

where

$$\mathbf{r}_{SR} = \mathbf{r}_R - \mathbf{r}_S \tag{10.7-14}$$

is the position vector from the scatterer to the receiver when transmission begins (see Fig. 10.7-1), and

$$\mathbf{V}_{RS} = \mathbf{V}_R - \mathbf{V}_S = V_{RS}\,\hat{n}_{\mathbf{V}_{RS}} \tag{10.7-15}$$

is the velocity vector of the receiver *relative to* the velocity vector of the scatterer, where $V_{RS} = |\mathbf{V}_{RS}|$ and $\hat{n}_{\mathbf{V}_{RS}}$ is the dimensionless unit vector in the direction of $\mathbf{V}_{RS}$. Note that $\boldsymbol{\mathcal{R}}_{SR}(t_0) = \mathbf{r}_{SR}$.

Time Delay

When the transmitted acoustic field is *first* incident upon the scatterer at some time t' seconds, where $t' > t_0$, the position vector from the origin to the scatterer is given by [see (10.7-5)]

$$\mathcal{R}_S(t') = \mathbf{r}_S + \mathbf{V}_S \Delta t', \qquad t' > t_0 , \tag{10.7-16}$$

where

$$\Delta t' = t' - t_0 , \qquad t' > t_0 . \tag{10.7-17}$$

The position vector from the transmitter to the scatterer at time t' is given by [see (10.7-9)]

$$\boxed{\mathcal{R}_{TS}(t') = \mathbf{r}_{TS} + \mathbf{V}_{ST} \Delta t', \qquad t' > t_0} \tag{10.7-18}$$

where the position vector $\mathbf{r}_{TS}$ is given by (10.7-10), and the relative velocity vector $\mathbf{V}_{ST}$ is given by (10.7-11).

When the scattered acoustic field is *first* incident upon the receiver at some time t'' seconds, where $t'' > t' > t_0$, the position vector from the origin to the receiver is given by [see (10.7-6)]

$$\mathcal{R}_R(t'') = \mathbf{r}_R + \mathbf{V}_R \Delta t'', \qquad t'' > t' > t_0 , \tag{10.7-19}$$

where

$$\Delta t'' = t'' - t_0 , \qquad t'' > t' > t_0 . \tag{10.7-20}$$

The position vector from the scatterer to the receiver at time t'' is given by [see (10.7-13)]

$$\boxed{\mathcal{R}_{SR}(t'') = \mathbf{r}_{SR} + \mathbf{V}_{RS} \Delta t'', \qquad t'' > t' > t_0} \tag{10.7-21}$$

where the position vector $\mathbf{r}_{SR}$ is given by (10.7-14), and the relative velocity vector $\mathbf{V}_{RS}$ is given by (10.7-15).

If (10.7-20) is rewritten as

$$\begin{aligned} \Delta t'' &= t'' - t_0 \\ &= t' - t_0 + t'' - t' \\ &= \Delta t' + (t'' - t') \\ &= \tau , \end{aligned} \tag{10.7-22}$$

then substituting

$$\Delta t' = t' - t_0 = \frac{\left|\mathcal{R}_{TS}(t')\right|}{c}, \qquad t' > t_0 \tag{10.7-23}$$

and

$$t'' - t' = \frac{|\boldsymbol{\mathcal{R}}_{SR}(t'')|}{c}, \qquad t'' > t' \tag{10.7-24}$$

into (10.7-22) yields

$$\boxed{\tau = \frac{|\boldsymbol{\mathcal{R}}_{TS}(t')| + |\boldsymbol{\mathcal{R}}_{SR}(t'')|}{c}} \tag{10.7-25}$$

where c is the constant speed of sound in meters per second. Equation (10.7-25) is the *time delay* (or travel time) in seconds associated with the scattered path when the scattered acoustic field is *first* incident upon the receiver. In order to evaluate (10.7-25), we need to find solutions for the ranges $|\boldsymbol{\mathcal{R}}_{TS}(t')|$ and $|\boldsymbol{\mathcal{R}}_{SR}(t'')|$, which we shall do next.

Exact Solution for $|\boldsymbol{\mathcal{R}}_{TS}(t')|$

Since

$$|\boldsymbol{\mathcal{R}}_{TS}(t')|^2 = \boldsymbol{\mathcal{R}}_{TS}(t') \bullet \boldsymbol{\mathcal{R}}_{TS}(t'), \tag{10.7-26}$$

substituting (10.7-18) into (10.7-26) yields

$$|\boldsymbol{\mathcal{R}}_{TS}(t')|^2 = r_{TS}^2 + 2 r_{TS} \left(\hat{r}_{TS} \bullet \mathbf{V}_{ST} \right) \Delta t' + V_{ST}^2 \, (\Delta t')^2, \tag{10.7-27}$$

where $r_{TS} = |\mathbf{r}_{TS}|$ is given by (10.1-9), $\hat{r}_{TS}$ is the unit vector in the direction of $\mathbf{r}_{TS}$, and $V_{ST} = |\mathbf{V}_{ST}|$, where $\mathbf{V}_{ST}$ is given by (10.7-11). Substituting (10.7-23) into (10.7-27) yields the second-order polynomial

$$A|\boldsymbol{\mathcal{R}}_{TS}(t')|^2 - B|\boldsymbol{\mathcal{R}}_{TS}(t')| - C = 0 \tag{10.7-28}$$

with *exact* solution

$$\boxed{|\boldsymbol{\mathcal{R}}_{TS}(t')| = \frac{B + \sqrt{B^2 + 4AC}}{2A}} \tag{10.7-29}$$

where

$$A = 1 - \left(V_{ST} / c \right)^2, \tag{10.7-30}$$

$$B = 2r_{TS} \frac{\hat{r}_{TS} \bullet \mathbf{V}_{ST}}{c} , \tag{10.7-31}$$

and

$$C = r_{TS}^2 . \tag{10.7-32}$$

Equation (10.7-29) is the *exact* range between the transmitter and scatterer when the transmitted acoustic field is *first* incident upon the scatterer at time t', where $t' > t_0$. For the problems that we are interested in,

$$V_{ST} / c << 1 , \tag{10.7-33}$$

and as a result, $A > 0$. The decision to use the plus sign in front of the square-root term in (10.7-29) instead of the minus sign was dictated by the fact that since $4AC > 0$, the square-root term is greater than B, and that the range $\left|\mathcal{R}_{TS}(t')\right|$ must be positive. Note that if the transmitter and scatterer are *not* in motion, then $\mathbf{V}_T = \mathbf{0}$, $\mathbf{V}_S = \mathbf{0}$, $\mathbf{V}_{ST} = \mathbf{0}$, $V_{ST} = 0$, and as expected, (10.7-29) reduces to $\left|\mathcal{R}_{TS}(t')\right| = r_{TS}$.

Exact Solution for $\left|\mathcal{R}_{SR}(t'')\right|$

Since

$$\left|\mathcal{R}_{SR}(t'')\right|^2 = \mathcal{R}_{SR}(t'') \bullet \mathcal{R}_{SR}(t'') , \tag{10.7-34}$$

substituting (10.7-21) into (10.7-34) yields

$$\left|\mathcal{R}_{SR}(t'')\right|^2 = r_{SR}^2 + 2r_{SR}\left(\hat{r}_{SR} \bullet \mathbf{V}_{RS}\right)\Delta t'' + V_{RS}^2 (\Delta t'')^2 , \tag{10.7-35}$$

where $r_{SR} = \left|\mathbf{r}_{SR}\right|$ is given by (10.1-26), $\hat{r}_{SR}$ is the unit vector in the direction of $\mathbf{r}_{SR}$, and $V_{RS} = \left|\mathbf{V}_{RS}\right|$, where $\mathbf{V}_{RS}$ is given by (10.7-15). Since $\Delta t'' = \tau$ [see (10.7-22)], substituting (10.7-25) into (10.7-35) yields the second-order polynomial

$$\mathcal{A}\left|\mathcal{R}_{SR}(t'')\right|^2 - \mathcal{B}\left|\mathcal{R}_{SR}(t'')\right| - \mathcal{C} = 0 \tag{10.7-36}$$

with *exact* solution

$$\boxed{\left|\mathcal{R}_{SR}(t'')\right| = \frac{\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}}} \tag{10.7-37}$$

where

$$\mathcal{A} = 1 - \left(V_{RS}/c\right)^2 , \tag{10.7-38}$$

$$\mathcal{B} = 2\left[r_{SR} \frac{\hat{r}_{SR} \bullet \mathbf{V}_{RS}}{c} + \left(\frac{V_{RS}}{c}\right)^2 \left|\mathcal{R}_{TS}(t')\right| \right], \tag{10.7-39}$$

$$\mathcal{C} = r_{SR}^2 + 2 r_{SR} \frac{\hat{r}_{SR} \bullet \mathbf{V}_{RS}}{c} \left|\mathcal{R}_{TS}(t')\right| + \left(\frac{V_{RS}}{c}\right)^2 \left|\mathcal{R}_{TS}(t')\right|^2 , \tag{10.7-40}$$

and $\left|\mathcal{R}_{TS}(t')\right|$ is given by (10.7-29). Equation (10.7-37) is the *exact* range between the scatterer and receiver when the scattered acoustic field is *first* incident upon the receiver at time t'', where $t'' > t' > t_0$. For the problems that we are interested in,

$$V_{RS}/c << 1 , \tag{10.7-41}$$

and as a result, $\mathcal{A} > 0$. The decision to use the plus sign in front of the square-root term in (10.7-37) instead of the minus sign was dictated by the fact that since $4\mathcal{A}\mathcal{C} > 0$ (see Appendix 10C), the square-root term is greater than $\mathcal{B}$, and that the range $\left|\mathcal{R}_{SR}(t'')\right|$ must be positive. Note that if the scatterer and receiver are *not* in motion, then $\mathbf{V}_S = \mathbf{0}$, $\mathbf{V}_R = \mathbf{0}$, $\mathbf{V}_{RS} = \mathbf{0}$, $V_{RS} = 0$, and as expected, (10.7-37) reduces to $\left|\mathcal{R}_{SR}(t'')\right| = r_{SR}$. With the use of (10.7-29) and (10.7-37), the time delay τ given by (10.7-25) can now be evaluated.

Time-Compression/Time-Expansion Factor and Doppler Shift

The dimensionless *time-compression/time-expansion factor* s is defined as follows:

$$\boxed{s \triangleq \tau_0/\tau} \tag{10.7-42}$$

where for the scattered path

$$\tau_0 = (r_{TS} + r_{SR})/c \tag{10.7-43}$$

is the *bistatic* time delay in seconds when there is *no* motion, and τ is given by (10.7-25), where the ranges $\left|\mathcal{R}_{TS}(t')\right|$ and $\left|\mathcal{R}_{SR}(t'')\right|$ are given by (10.7-29) and (10.7-37), respectively. Note that $s > 0$. The *Doppler shift* η_D in hertz is related to the dimensionless time-compression/time-expansion factor s and is defined as follows:

$$\boxed{\eta_D \triangleq (s-1)\,f_c} \tag{10.7-44}$$

where f_c is referred to as either the *center frequency* or *carrier frequency* in hertz of the transmitted electrical signal (see Chapter 11).

There are three ranges of values for the time-compression/time-expansion factor s; namely, $s=1$, $s>1$, and $0<s<1$. When there is *no* motion, $\left|\boldsymbol{\mathcal{R}}_{TS}(t')\right| = r_{TS}$ and $\left|\boldsymbol{\mathcal{R}}_{SR}(t'')\right| = r_{SR}$ as we discussed, and as a result, $\tau = \tau_0$ [see (10.7-25) and (10.7-43)]. Substituting $\tau = \tau_0$ into (10.7-42) yields $s=1$. Substituting $s=1$ into (10.7-44) yields $\eta_D = 0$, that is, there is *zero* Doppler shift when there is no motion (and no relative motion). When $s=1$, there is *neither* time compression *nor* time expansion, that is, the duration of the received signal is *equal to* the duration of the transmitted signal resulting in *no* change in signal bandwidth at the receiver. If $\tau < \tau_0$, then $s>1$ (*time compression*) and $\eta_D > 0$ (*positive* Doppler shift). Time compression means that the duration of the received signal is *less than* the duration of the transmitted signal resulting in an *increase* in signal bandwidth. A positive Doppler shift means that the scatterer (target) and receiver are moving *towards* one another. If $\tau > \tau_0$, then $0<s<1$ (*time expansion*) and $\eta_D < 0$ (*negative* Doppler shift). Time expansion means that the duration of the received signal is *greater than* the duration of the transmitted signal resulting in a *decrease* in signal bandwidth. A negative Doppler shift means that the scatterer (target) and receiver are moving *away* from one another.

For a *monostatic* (*backscatter*) geometry, where both the transmitter and receiver are physically collocated,

$$\mathbf{V}_R = \mathbf{V}_T\,, \tag{10.7-45}$$

$$\mathbf{V}_{RS} = -\mathbf{V}_{ST}\,, \tag{10.7-46}$$

$$V_{RS} = V_{ST}\,, \tag{10.7-47}$$

$$\mathbf{r}_{SR} = -\mathbf{r}_{TS}\,, \tag{10.7-48}$$

$$r_{SR} = r_{TS}\,, \tag{10.7-49}$$

$$\hat{r}_{SR} = -\hat{r}_{TS}\,, \tag{10.7-50}$$

and

$$\tau_0 = 2r_{TS}/c \tag{10.7-51}$$

is the *round-trip* time delay in seconds when there is *no* motion.

Example 10.7-1

In this example we shall demonstrate how to model a received signal using the time-compression/time-expansion factor s. For example purposes, let the transmitted signal $x(t)$ be a rectangular-envelope, CW (continuous-wave) pulse given by

$$x(t) = A\cos(2\pi f_c t + \theta_0)\text{rect}\left[(t - 0.5T)/T\right], \tag{10.7-52}$$

where A is an amplitude factor in volts, $\cos(2\pi f_c t)$ is the carrier waveform, f_c is the carrier frequency in hertz, θ_0 is a phase term in radians,

$$\text{rect}\left(\frac{t - 0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{10.7-53}$$

is the time-shifted rectangle function, and T is the pulse length in seconds.

A simple model for the received signal is given by

$$y(t) = Kx(s[t - \tau]), \tag{10.7-54}$$

or

$$y(t) = KA\cos\left(2\pi f_c s[t - \tau] + \theta_0\right)\text{rect}\left[\left(s[t - \tau] - 0.5T\right)/T\right], \tag{10.7-55}$$

where $0 < K < 1$ is a dimensionless constant that is meant to model amplitude attenuation. Solving for s using (10.7-44) yields

$$s = 1 + \frac{\eta_D}{f_c} = \frac{f_c + \eta_D}{f_c}, \tag{10.7-56}$$

and by substituting (10.7-56) into the first term in the argument of the cosine function in (10.7-55), we obtain

$$y(t) = KA\cos\left(2\pi[f_c + \eta_D][t - \tau] + \theta_0\right)\text{rect}\left[\left(s[t - \tau] - 0.5T\right)/T\right]. \tag{10.7-57}$$

Using the definition of the time-shifted rectangle function given by (10.7-53),

$$\text{rect}\left(\frac{s[t - \tau] - 0.5T}{T}\right) = \begin{cases} 1, & 0 \le s[t - \tau] \le T \\ 0, & \text{otherwise,} \end{cases} \tag{10.7-58}$$

or

$$\mathrm{rect}\left(\frac{s[t-\tau]-0.5T}{T}\right)=\begin{cases}1, & \tau \le t \le \tau+(T/s)\\ 0, & \text{otherwise.}\end{cases} \tag{10.7-59}$$

Therefore, (10.7-57) can be rewritten as

$$y(t)=KA\cos\left(2\pi[f_c+\eta_D][t-\tau]+\theta_0\right), \qquad \tau \le t \le \tau+T_y, \tag{10.7-60}$$

where the Doppler shift η_D is given by (10.7-44), the time delay τ is given by (10.7-25), and the pulse length of the received signal T_y is given by

$$\boxed{T_y=T/s} \tag{10.7-61}$$

where the time-compression/time-expansion factor s is given by (10.7-42). Equation (10.7-61) is a general result for any transmitted signal $x(t)$ with pulse length T seconds when the received signal $y(t)$ is modeled by (10.7-54). From (10.7-61), and as was stated before, if $s=1$ (neither time compression nor time expansion), $s>1$ (time compression), or $0<s<1$ (time expansion), then the pulse length of the received signal T_y is equal to, less than, or greater than the pulse length T of the transmitted signal, respectively. We shall compute the bandwidth of the received signal $y(t)$ next.

In order to compute the bandwidth of $y(t)$, we need the magnitude spectrum of $y(t)$ along the positive frequency axis. The Fourier transform of (10.7-54) is equal to

$$Y(\eta)=\frac{K}{s}X(\eta/s)\exp(-j2\pi\eta\tau), \tag{10.7-62}$$

with magnitude spectrum

$$|Y(\eta)|=\frac{K}{s}|X(\eta/s)|, \tag{10.7-63}$$

where η represents output (received) frequencies in hertz, $K>0$, and $s>0$. The magnitude spectrum along the positive frequency axis of the transmitted signal $x(t)$ given by (10.7-52) is equal to

$$|X(f)|=\frac{A}{2}T\left|\mathrm{sinc}[(f-f_c)T]\right|, \qquad f>0, \tag{10.7-64}$$

where f represents input (transmitted) frequencies in hertz. Substituting (10.7-64) into (10.7-63) yields

$$\left|Y(\eta)\right| = \frac{K}{s}\frac{A}{2}T\left|\operatorname{sinc}\left[\left(\frac{\eta - sf_c}{s}\right)T\right]\right|, \qquad \eta > 0\,, \tag{10.7-65}$$

or, with the use of (10.7-56),

$$\left|Y(\eta)\right| = \frac{K}{s}\frac{A}{2}T\left|\operatorname{sinc}\left[\left(\frac{\eta - [f_c + \eta_D]}{s}\right)T\right]\right|, \qquad \eta > 0\,. \tag{10.7-66}$$

Equation (10.7-65), or equivalently (10.7-66), is the magnitude spectrum along the positive frequency axis of the received signal $y(t)$ given by (10.7-60).

The zero-crossings of the sinc function in (10.7-65) occur at

$$\frac{\eta}{s} - f_c = \pm\frac{i}{T}\ \text{Hz}, \qquad i = 1, 2, \ldots\,, \tag{10.7-67}$$

or

$$\eta = s\left(f_c \pm \frac{i}{T}\right)\text{Hz}, \qquad i = 1, 2, \ldots\,. \tag{10.7-68}$$

Since $\left|\operatorname{sinc}(\eta T)\right| < 0.1$ for $\eta > 3/T$, estimates of the maximum and minimum frequency components are given by

$$\eta_{\max} = s\left(f_c + \frac{\text{NZC}}{T}\right) \tag{10.7-69}$$

and

$$\eta_{\min} = s\left(f_c - \frac{\text{NZC}}{T}\right), \tag{10.7-70}$$

respectively, where NZC is the integer number of zero-crossings of the sinc function that is used to estimate both the maximum and minimum frequency components. The parameter NZC should be at least 3, with 5 being a conservative choice. Therefore, the bandwidth BW_y (in hertz) of the received signal $y(t)$ given by (10.7-60) is

$$\text{BW}_y = \eta_{\max} - \eta_{\min} = s\,2\,\text{NZC}/T\,, \tag{10.7-71}$$

or

$$\boxed{\text{BW}_y = s\,\text{BW}_x} \tag{10.7-72}$$

where

$$\text{BW}_x = f_{\max} - f_{\min} = 2\,\text{NZC}/T \tag{10.7-73}$$

is the bandwidth (in hertz) of the transmitted signal $x(t)$ given by (10.7-52). Equation (10.7-72) is a general result for any transmitted signal $x(t)$ with bandwidth BW_x hertz when the received signal $y(t)$ is modeled by (10.7-54). From (10.7-72), and as was stated before, if $s = 1$ (neither time compression nor time expansion), $s > 1$ (time compression), or $0 < s < 1$ (time expansion), then the bandwidth of the received signal BW_y is equal to, greater than, or less than the bandwidth BW_x of the transmitted signal, respectively.

Let us conclude this example by proving (10.7-61) and (10.7-72) for any transmitted signal $x(t)$ when the received signal $y(t)$ is modeled by (10.7-54). If $x(t)$ is defined in the time interval $0 \le t \le T$, then [see (10.7-54)]

$$y(\tau) = Kx(0) \tag{10.7-74}$$

and

$$y\left(\tau + [T/s]\right) = Kx(T). \tag{10.7-75}$$

Therefore, the pulse length of the received signal T_y is given by (10.7-61). If the maximum and minimum frequency components of $x(t)$ are $f_{\max}$ and $f_{\min}$, respectively, then [see (10.7-63)]

$$|Y(sf_{\max})| = \frac{K}{s}|X(f_{\max})| \tag{10.7-76}$$

and

$$|Y(sf_{\min})| = \frac{K}{s}|X(f_{\min})|. \tag{10.7-77}$$

Therefore, the bandwidth of the received signal BW_y is given by (10.7-72). ■

Time-Varying Ranges and Angles of Incidence and Scatter

The leading edge of the scattered acoustic field that is *first* incident upon the receiver at time t'' is the leading edge of the transmitted acoustic field that *was first* incident upon the scatterer at time t'. The time-varying position vector from transmitter to scatterer that describes this situation is defined as follows:

$$\boldsymbol{\mathcal{R}}'_{TS}(t) \triangleq \boldsymbol{\mathcal{R}}_{TS}(t') + \mathbf{V}_{ST}(t - t''), \qquad t \ge t'', \tag{10.7-78}$$

where $\boldsymbol{\mathcal{R}}_{TS}(t')$ is given by (10.7-18). Note that $\boldsymbol{\mathcal{R}}'_{TS}(t'') = \boldsymbol{\mathcal{R}}_{TS}(t')$, or $\left|\boldsymbol{\mathcal{R}}'_{TS}(t'')\right| = \left|\boldsymbol{\mathcal{R}}_{TS}(t')\right|$. Therefore, when the scattered acoustic field is *first* incident upon the receiver at time t'', the range between the transmitter and scatterer *was* $\left|\mathcal{R}_{TS}(t')\right|$. Since [see (10.7-22)]

$$t'' = t_0 + \tau , \tag{10.7-79}$$

substituting (10.7-18), (10.7-23), and (10.7-79) into (10.7-78) yields

$$\boxed{\mathcal{R}'_{TS}(t) = \mathbf{r}_{TS} + \mathbf{V}_{ST}\frac{\left|\mathcal{R}_{TS}(t')\right|}{c} + \mathbf{V}_{ST}(t - [t_0 + \tau]), \qquad t \ge t_0 + \tau} \tag{10.7-80}$$

where the position vector $\mathbf{r}_{TS}$ is given by (10.7-10), the relative velocity vector $\mathbf{V}_{ST}$ is given by (10.7-11), and $\left|\mathcal{R}_{TS}(t')\right|$ is given by (10.7-29). The *time-varying range* is equal to $\left|\mathcal{R}'_{TS}(t)\right|$, where $\mathcal{R}'_{TS}(t)$ is given by (10.7-80). We shall compute the time-varying angles of incidence at the scatterer next.

The time-varying, dimensionless unit vector $\hat{r}'_{TS}(t)$ in the direction of $\mathcal{R}'_{TS}(t)$ is given by

$$\boxed{\hat{r}'_{TS}(t) = \frac{\mathcal{R}'_{TS}(t)}{\left|\mathcal{R}'_{TS}(t)\right|} = u'_{TS}(t)\hat{x} + v'_{TS}(t)\hat{y} + w'_{TS}(t)\hat{z}, \qquad t \ge t_0 + \tau} \tag{10.7-81}$$

where $\mathcal{R}'_{TS}(t)$ is given by (10.7-80), and

$$u'_{TS}(t) = \sin\theta'_{TS}(t)\cos\psi'_{TS}(t) , \tag{10.7-82}$$

$$v'_{TS}(t) = \sin\theta'_{TS}(t)\sin\psi'_{TS}(t) , \tag{10.7-83}$$

and

$$w'_{TS}(t) = \cos\theta'_{TS}(t) \tag{10.7-84}$$

are time-varying, dimensionless direction cosines with respect to the X, Y, and Z axes, respectively. Therefore, the *time-varying angles of incidence* at the scatterer for $t \ge t_0 + \tau$ are given by

$$\boxed{\theta'_{TS}(t) = \cos^{-1} w'_{TS}(t), \qquad t \ge t_0 + \tau} \tag{10.7-85}$$

and

$$\boxed{\psi'_{TS}(t) = \tan^{-1}\left[v'_{TS}(t) / u'_{TS}(t) \right], \qquad t \ge t_0 + \tau} \tag{10.7-86}$$

Note that $\theta'_{TS}(t_0+\tau)=\theta'_{TS}$ and $\psi'_{TS}(t_0+\tau)=\psi'_{TS}$ are the angles of incidence at the scatterer when the transmitted acoustic field is *first* incident upon the scatterer at time t' because $\boldsymbol{\mathcal{R}}'_{TS}(t'')=\boldsymbol{\mathcal{R}}_{TS}(t')$ [see (10.7-78)], where $t''=t_0+\tau$ [see (10.7-79)]. If there is *no* motion, then the angles of incidence at the scatterer, θ_{TS} and ψ_{TS}, are given by (10.1-13) and (10.1-14), respectively.

The time-varying position vector from the scatterer to the receiver for $t \ge t''$ is defined as follows:

$$\boldsymbol{\mathcal{R}}''_{SR}(t) \triangleq \boldsymbol{\mathcal{R}}_{SR}(t'') + \mathbf{V}_{RS}(t-t''), \qquad t \ge t'', \tag{10.7-87}$$

where $\boldsymbol{\mathcal{R}}_{SR}(t'')$ is given by (10.7-21). Substituting (10.7-21), (10.7-20), and (10.7-79) into (10.7-87) yields

$$\boxed{\boldsymbol{\mathcal{R}}''_{SR}(t) = \mathbf{r}_{SR} + \mathbf{V}_{RS}(t-t_0), \qquad t \ge t_0+\tau} \tag{10.7-88}$$

where the position vector $\mathbf{r}_{SR}$ is given by (10.7-14), and the relative velocity vector $\mathbf{V}_{RS}$ is given by (10.7-15). Note that

$$\boldsymbol{\mathcal{R}}''_{SR}(t_0+\tau) = \mathbf{r}_{SR} + \mathbf{V}_{RS}\tau = \boldsymbol{\mathcal{R}}_{SR}(t'') \tag{10.7-89}$$

since $\Delta t''=\tau$ [see (10.7-22)]. The *time-varying range* from the scatterer to the receiver for $t \ge t_0+\tau$ is equal to $\left|\boldsymbol{\mathcal{R}}''_{SR}(t)\right|$, where $\boldsymbol{\mathcal{R}}''_{SR}(t)$ is given by (10.7-88). Therefore, the *total time-varying distance* from transmitter to scatterer to receiver for $t \ge t_0+\tau$ is equal to

$$\boxed{d_{TSR}(t) = \left|\boldsymbol{\mathcal{R}}'_{TS}(t)\right| + \left|\boldsymbol{\mathcal{R}}''_{SR}(t)\right|, \qquad t \ge t_0+\tau} \tag{10.7-90}$$

where $\boldsymbol{\mathcal{R}}'_{TS}(t)$ is given by (10.7-80), and $\boldsymbol{\mathcal{R}}''_{SR}(t)$ is given by (10.7-88). Note that $\tau = d_{TSR}(t_0+\tau)/c$. We shall compute the time-varying angles of scatter at the receiver next.

The time-varying, dimensionless unit vector $\hat{r}''_{SR}(t)$ in the direction of $\boldsymbol{\mathcal{R}}''_{SR}(t)$ is given by

$$\boxed{\hat{r}''_{SR}(t) = \frac{\boldsymbol{\mathcal{R}}''_{SR}(t)}{\left|\boldsymbol{\mathcal{R}}''_{SR}(t)\right|} = u''_{SR}(t)\hat{x} + v''_{SR}(t)\hat{y} + w''_{SR}(t)\hat{z}, \qquad t \ge t_0+\tau} \tag{10.7-91}$$

where $\boldsymbol{\mathcal{R}}''_{SR}(t)$ is given by (10.7-88), and

$$u''_{SR}(t) = \sin\theta''_{SR}(t)\cos\psi''_{SR}(t) \,, \tag{10.7-92}$$

$$v''_{SR}(t) = \sin\theta''_{SR}(t)\sin\psi''_{SR}(t) \,, \tag{10.7-93}$$

and

$$w''_{SR}(t) = \cos\theta''_{SR}(t) \tag{10.7-94}$$

are time-varying, dimensionless direction cosines with respect to the X, Y, and Z axes, respectively. Therefore, the *time-varying angles of scatter* at the receiver for $t \geq t_0 + \tau$ are given by

$$\boxed{\theta''_{SR}(t) = \cos^{-1} w''_{SR}(t), \qquad t \geq t_0 + \tau} \tag{10.7-95}$$

and

$$\boxed{\psi''_{SR}(t) = \tan^{-1}\left[v''_{SR}(t)/u''_{SR}(t)\right], \qquad t \geq t_0 + \tau} \tag{10.7-96}$$

Note that $\theta''_{SR}(t_0 + \tau) = \theta''_{SR}$ and $\psi''_{SR}(t_0 + \tau) = \psi''_{SR}$ are the angles of scatter at the receiver when the scattered acoustic field is *first* incident upon the receiver at time $t'' = t_0 + \tau$ because $\mathcal{R}''_{SR}(t_0 + \tau) = \mathcal{R}_{SR}(t'')$ [see (10.7-89)]. If there is *no* motion, then the angles of scatter at the receiver, θ_{SR} and ψ_{SR}, are given by (10.1-31) and (10.1-32), respectively.

10.7.2 Direct Path

The time-varying position vector from the transmitter to the receiver is given by

$$\mathcal{R}_{TR}(t) = \mathcal{R}_R(t) - \mathcal{R}_T(t) \,, \tag{10.7-97}$$

and by substituting (10.7-4) and (10.7-6) into (10.7-97), we obtain

$$\mathcal{R}_{TR}(t) = \mathbf{r}_{TR} + \mathbf{V}_{RT}\,\Delta t, \qquad t \geq t_0 \,, \tag{10.7-98}$$

where

$$\mathbf{r}_{TR} = \mathbf{r}_R - \mathbf{r}_T \tag{10.7-99}$$

is the position vector from the transmitter to the receiver when transmission begins at time $t = t_0$ seconds (see Fig. 10.7-2), and

$$\mathbf{V}_{RT} = \mathbf{V}_R - \mathbf{V}_T = V_{RT}\,\hat{n}_{\mathbf{V}_{RT}} \tag{10.7-100}$$

is the velocity vector of the receiver *relative to* the velocity vector of the

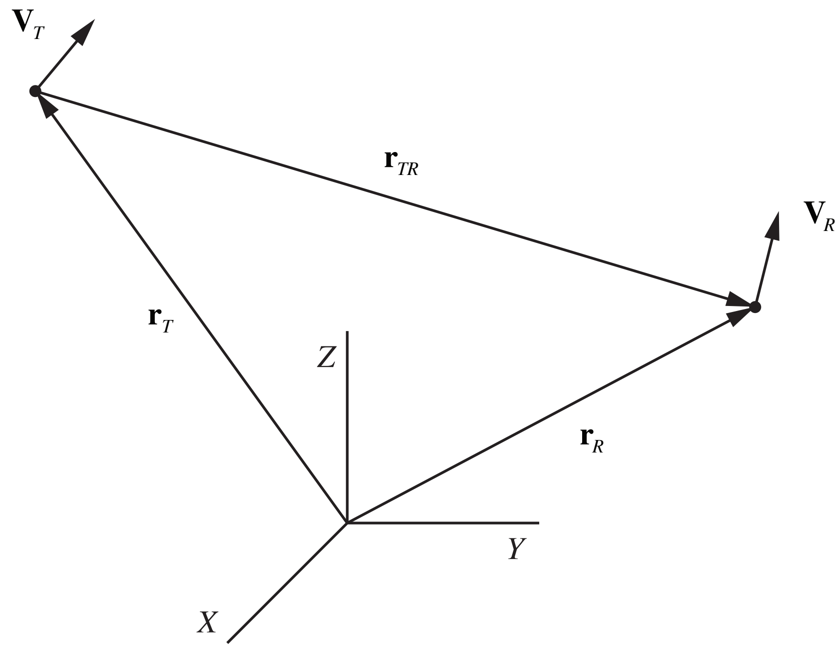

Figure 10.7-2 Direct path geometry when transmission begins at time $t = t_0$ seconds, where $\mathbf{V}_T$ and $\mathbf{V}_R$ are the velocity vectors of the transmitter and receiver, respectively.

transmitter, where $V_{RT} = |\mathbf{V}_{RT}|$ and $\hat{n}_{\mathbf{V}_{RT}}$ is the dimensionless unit vector in the direction of $\mathbf{V}_{RT}$. Note that $\mathcal{R}_{TR}(t_0) = \mathbf{r}_{TR}$.

Time Delay

When the transmitted acoustic field is *first* incident upon the receiver at some time t''' seconds, where $t''' > t_0$, the position vector from the origin to the receiver is given by [see (10.7-6)]

$$\mathcal{R}_R(t''') = \mathbf{r}_R + \mathbf{V}_R \Delta t''', \qquad t''' > t_0 , \tag{10.7-101}$$

where

$$\Delta t''' = t''' - t_0 , \qquad t''' > t_0 . \tag{10.7-102}$$

The position vector from the transmitter to the receiver at time t''' is given by [see (10.7-98)]

$$\boxed{\mathcal{R}_{TR}(t''') = \mathbf{r}_{TR} + \mathbf{V}_{RT} \Delta t''', \qquad t''' > t_0} \tag{10.7-103}$$

where the position vector $\mathbf{r}_{TR}$ is given by (10.7-99), and the relative velocity vector $\mathbf{V}_{RT}$ is given by (10.7-100).

If (10.7-102) is rewritten as

$$\Delta t''' = t''' - t_0 = \tau \,, \tag{10.7-104}$$

then

$$t''' = t_0 + \tau \,, \tag{10.7-105}$$

where

$$\boxed{\tau = \frac{\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|}{c}} \tag{10.7-106}$$

is the *time delay* (or travel time) in seconds associated with the direct path when the transmitted acoustic field is *first* incident upon the receiver, where c is the constant speed of sound in meters per second. In order to evaluate (10.7-106), we need to find the solution for the range $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|$, which we shall do next.

Exact Solution for $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|$

Since

$$\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|^2 = \boldsymbol{\mathcal{R}}_{TR}(t''')\bullet\boldsymbol{\mathcal{R}}_{TR}(t''') \,, \tag{10.7-107}$$

substituting (10.7-103) into (10.7-107) yields

$$\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|^2 = r_{TR}^2 + 2r_{TR}\left(\hat{r}_{TR}\bullet\mathbf{V}_{RT}\right)\Delta t''' + V_{RT}^2\,(\Delta t''')^2 \,, \tag{10.7-108}$$

where $r_{TR} = \left|\mathbf{r}_{TR}\right|$ is given by (10.3-7), $\hat{r}_{TR}$ is the unit vector in the direction of $\mathbf{r}_{TR}$, and $V_{RT} = \left|\mathbf{V}_{RT}\right|$, where $\mathbf{V}_{RT}$ is given by (10.7-100). Substituting (10.7-104) and (10.7-106) into (10.7-108) yields the second-order polynomial

$$A\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|^2 - B\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right| - C = 0 \tag{10.7-109}$$

with *exact* solution

$$\boxed{\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right| = \frac{B + \sqrt{B^2 + 4AC}}{2A}} \tag{10.7-110}$$

where

$$A = 1 - \left(V_{RT}/c\right)^2 , \tag{10.7-111}$$

$$B = 2r_{TR}\frac{\hat{r}_{TR} \bullet \mathbf{V}_{RT}}{c} , \tag{10.7-112}$$

and

$$C = r_{TR}^2 . \tag{10.7-113}$$

Equation (10.7-110) is the *exact* range between the transmitter and receiver when the transmitted acoustic field is *first* incident upon the receiver at time t''', where $t''' > t_0$. For the problems that we are interested in,

$$V_{RT}/c \ll 1 , \tag{10.7-114}$$

and as a result, $A > 0$. The decision to use the plus sign in front of the square-root term in (10.7-110) instead of the minus sign was dictated by the fact that since $4AC > 0$, the square-root term is greater than B, and that the range $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|$ must be positive. Note that if the transmitter and receiver are *not* in motion, then $\mathbf{V}_T = \mathbf{0}$, $\mathbf{V}_R = \mathbf{0}$, $\mathbf{V}_{RT} = \mathbf{0}$, $V_{RT} = 0$, and as expected, (10.7-110) reduces to $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right| = r_{TR}$.

Time-Compression/Time-Expansion Factor and Doppler Shift

The dimensionless *time-compression/time-expansion factor* s is defined as [see (10.7-42)]

$$\boxed{s \triangleq \tau_0/\tau} \tag{10.7-115}$$

where for the direct path

$$\tau_0 = r_{TR}/c \tag{10.7-116}$$

is the *one-way* time delay in seconds when there is *no* motion, and τ is given by (10.7-106), where the range $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right|$ is given by (10.7-110). Note that $s > 0$. The *Doppler shift* η_D in hertz is related to the dimensionless time-compression/time-expansion factor s and is defined as [see (10.7-44)]

$$\boxed{\eta_D \triangleq (s-1)\,f_c} \tag{10.7-117}$$

where f_c is referred to as either the *center frequency* or *carrier frequency* in hertz of the transmitted electrical signal (see Chapter 11).

Recall from the discussion of the scattered path that there are three ranges of values for s; namely, $s=1$, $s>1$, and $0<s<1$. When there is *no* motion, $\left|\boldsymbol{\mathcal{R}}_{TR}(t''')\right| = r_{TR}$, $\tau=\tau_0$, $s=1$, and $\eta_D=0$, that is, there is *zero* Doppler shift when there is no motion (and no relative motion). When $s=1$, there is *neither* time compression *nor* time expansion, that is, the duration of the received signal is *equal to* the duration of the transmitted signal resulting in *no* change in signal bandwidth at the receiver. If $\tau<\tau_0$, then $s>1$ (*time compression*) and $\eta_D>0$ (*positive* Doppler shift). Time compression means that the duration of the received signal is *less than* the duration of the transmitted signal resulting in an *increase* in signal bandwidth. A positive Doppler shift means that the transmitter and receiver are moving *towards* one another. If $\tau>\tau_0$, then $0<s<1$ (*time expansion*) and $\eta_D<0$ (*negative* Doppler shift). Time expansion means that the duration of the received signal is *greater than* the duration of the transmitted signal resulting in a *decrease* in signal bandwidth. A negative Doppler shift means that the transmitter and receiver are moving *away* from one another.

Time-Varying Range and Angles of Incidence

The time-varying position vector from the transmitter to the receiver for $t \geq t'''$ is defined as follows:

$$\boldsymbol{\mathcal{R}}'''_{TR}(t) \triangleq \boldsymbol{\mathcal{R}}_{TR}(t''') + \mathbf{V}_{RT}(t-t'''), \qquad t \geq t''', \tag{10.7-118}$$

where $\boldsymbol{\mathcal{R}}_{TR}(t''')$ is given by (10.7-103). Substituting (10.7-103), (10.7-102), and (10.7-105) into (10.7-118) yields

$$\boxed{\boldsymbol{\mathcal{R}}'''_{TR}(t) = \mathbf{r}_{TR} + \mathbf{V}_{RT}(t-t_0), \qquad t \geq t_0+\tau} \tag{10.7-119}$$

where the position vector $\mathbf{r}_{TR}$ is given by (10.7-99), and the relative velocity vector $\mathbf{V}_{RT}$ is given by (10.7-100). Note that

$$\boldsymbol{\mathcal{R}}'''_{TR}(t_0+\tau) = \mathbf{r}_{TR} + \mathbf{V}_{RT}\,\tau = \boldsymbol{\mathcal{R}}_{TR}(t''') \tag{10.7-120}$$

since $\Delta t''' = \tau$ [see (10.7-104)]. The *time-varying range* from the transmitter to the receiver for $t \geq t_0+\tau$ is equal to $\left|\boldsymbol{\mathcal{R}}'''_{TR}(t)\right|$, where $\boldsymbol{\mathcal{R}}'''_{TR}(t)$ is given by (10.7-119). We shall compute the time-varying angles of incidence at the receiver next.

The time-varying, dimensionless unit vector $\hat{r}'''_{TR}(t)$ in the direction of $\boldsymbol{\mathcal{R}}'''_{TR}(t)$ is given by

$$\boxed{\hat{r}'''_{TR}(t) = \frac{\boldsymbol{\mathcal{R}}'''_{TR}(t)}{\left|\boldsymbol{\mathcal{R}}'''_{TR}(t)\right|} = u'''_{TR}(t)\hat{x} + v'''_{TR}(t)\hat{y} + w'''_{TR}(t)\hat{z}\,, \qquad t \geq t_0 + \tau}$$

(10.7-121)

where $\boldsymbol{\mathcal{R}}'''_{TR}(t)$ is given by (10.7-119), and

$$u'''_{TR}(t) = \sin\theta'''_{TR}(t)\cos\psi'''_{TR}(t)\,, \tag{10.7-122}$$

$$v'''_{TR}(t) = \sin\theta'''_{TR}(t)\sin\psi'''_{TR}(t)\,, \tag{10.7-123}$$

and

$$w'''_{TR}(t) = \cos\theta'''_{TR}(t) \tag{10.7-124}$$

are time-varying, dimensionless direction cosines with respect to the X, Y, and Z axes, respectively. Therefore, the *time-varying angles of incidence* at the receiver for $t \geq t_0 + \tau$ are given by

$$\boxed{\theta'''_{TR}(t) = \cos^{-1} w'''_{TR}(t)\,, \qquad t \geq t_0 + \tau} \tag{10.7-125}$$

and

$$\boxed{\psi'''_{TR}(t) = \tan^{-1}\left[v'''_{TR}(t) / u'''_{TR}(t) \right], \qquad t \geq t_0 + \tau} \tag{10.7-126}$$

Note that $\theta'''_{TR}(t_0 + \tau) = \theta'''_{TR}$ and $\psi'''_{TR}(t_0 + \tau) = \psi'''_{TR}$ are the angles of incidence at the receiver when the transmitted acoustic field is *first* incident upon the receiver at time $t''' = t_0 + \tau$ because $\boldsymbol{\mathcal{R}}'''_{TR}(t_0 + \tau) = \boldsymbol{\mathcal{R}}_{TR}(t''')$ [see (10.7-120)]. If there is *no* motion, then the angles of incidence at the receiver, θ_{TR} and ψ_{TR}, are given by (10.3-11) and (10.3-12), respectively.

Problems

Section 10.4

10-1 A side-looking sonar (SLS) is being used to detect a mine on the ocean bottom. The frequency of operation and source level of the SLS are 30 kHz and $215\text{ dB re }1\,\mu\text{Pa (rms)}$, respectively, and the range from the SLS to the mine is 218 m. If the sound-pressure level at the SLS after scatter from the mine is $98.79\text{ dB re }1\,\mu\text{Pa (rms)}$, the attenuation coefficient is 6.14 dB/km, and we treat this problem as a backscatter problem, then

(a) find the target strength of the mine.

(b) What is the differential scattering cross-section of the mine?

(c) What is the *magnitude* of the scattering function of the mine?

(d) What is the backscattering cross-section of the mine?

(e) What is the effective radius of the mine based on the backscattering cross-section?

(f) What is the transmission loss?

10-2 The range between a sound-source in the ocean and a receiver is 10 km. Find the transmission loss at the receiver if

(a) the frequency of operation of the sound-source is 200 Hz and the attenuation coefficient is 3.41×10^{-3} dB/km.

(b) the frequency of operation of the sound-source is 1 kHz and the attenuation coefficient is 5.7×10^{-2} dB/km.

(c) the frequency of operation of the sound-source is 10 kHz and the attenuation coefficient is 0.849 dB/km.

Section 10.7

10-3 Compute the Fourier transform of the received signal given by (10.7-54) using η as the output frequency in hertz. Compare your answer with (10.7-62).

Appendix 10A Radiation from a Time-Harmonic, Omni-directional Point-Source

In order to derive an equation for the target strength of an object for the bistatic scattering problem shown in Fig. 10.1-1, we need some basic equations that describe the acoustic field radiated by a time-harmonic, omnidirectional point-source. An exact solution of the linear wave equation given by (10.1-2) for the source distribution given by (10.1-1) for free-space propagation in an ideal (nonviscous), homogeneous, fluid medium was derived in Appendix 6C and is summarized below:

$$\varphi(t,\mathbf{r}) = \varphi_f(\mathbf{r})\exp(+j2\pi f t), \qquad (10\text{A-}1)$$

where $\varphi(t,\mathbf{r})$ is the time-harmonic, scalar velocity potential in squared meters per second of the radiated acoustic field,

$$\varphi_f(\mathbf{r}) = S_0\, g_f\left(\mathbf{r} \,|\, \mathbf{r}_0\right), \tag{10A-2}$$

$$S_0 = A_x \mathcal{S}_T(f) \tag{10A-3}$$

is the source strength of the omnidirectional point-source in cubic meters per second at frequency f hertz, A_x is the complex amplitude in volts of the time-harmonic, input electrical signal applied to the omnidirectional point-source, $\mathcal{S}_T(f)$ is the complex, transmitter sensitivity function of the omnidirectional point-source in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$ (see Table 6.1-2 and Appendix 6B),

$$g_f\left(\mathbf{r} \,|\, \mathbf{r}_0\right) = -\frac{\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}-\mathbf{r}_0\right|} = -\frac{\exp(-jkR)}{4\pi R} \tag{10A-4}$$

is the time-independent, free-space, Green's function of an unbounded, ideal (nonviscous), homogeneous, fluid medium with units of inverse meters,

$$k = 2\pi f/c = 2\pi/\lambda \tag{10A-5}$$

is the wavenumber in radians per meter, $c = f\lambda$ is the constant speed of sound in the fluid medium in meters per second, λ is the wavelength in meters,

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{10A-6}$$

is the position vector to a field point,

$$\mathbf{r}_0 = x_0\hat{x} + y_0\hat{y} + z_0\hat{z} \tag{10A-7}$$

is the position vector to the point-source, and

$$R = \left|\mathbf{r}-\mathbf{r}_0\right| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \tag{10A-8}$$

is the range in meters between the point-source and a field point. Frequency-dependent attenuation due to viscosity can be taken into account by replacing the real wavenumber k in (10A-4) with the complex wavenumber

$$K = k - j\alpha(f), \tag{10A-9}$$

where K has units of inverse meters and $\alpha(f)$ is the real, nonnegative, frequency-dependent, attenuation coefficient of seawater in nepers per meter (see Section 7.2 and Appendix 7A). Doing so yields

$$\boxed{g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) = -\frac{\exp\left(-jK\left|\mathbf{r}-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}-\mathbf{r}_0\right|} = -\frac{\exp(-jKR)}{4\pi R}} \tag{10A-10}$$

or

$$\boxed{g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) = -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}-\mathbf{r}_0\right|\right]}{4\pi\left|\mathbf{r}-\mathbf{r}_0\right|}\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right) = -\frac{\exp[-\alpha(f)R]}{4\pi R}\exp(-jkR)} \tag{10A-11}$$

Note that if there is no viscosity, that is, if $\alpha(f)=0$, then (10A-9) reduces to (10A-5) and both (10A-10) and (10A-11) reduce to (10A-4).

The corresponding solution for the time-harmonic, radiated acoustic pressure $p(t,\mathbf{r})$ in pascals can be obtained from the scalar velocity potential $\varphi(t,\mathbf{r})$ in squared meters per second as follows:

$$p(t,\mathbf{r}) = -\rho_0(\mathbf{r})\frac{\partial}{\partial t}\varphi(t,\mathbf{r}) = -\rho_0\frac{\partial}{\partial t}\varphi(t,\mathbf{r})\,, \tag{10A-12}$$

and by substituting (10A-1) into (10A-12), we obtain

$$p(t,\mathbf{r}) = p_f(\mathbf{r})\exp(+j2\pi ft)\,, \tag{10A-13}$$

where

$$p_f(\mathbf{r}) = -j2\pi f\rho_0\varphi_f(\mathbf{r}) = -jk\rho_0 c\,\varphi_f(\mathbf{r})\,, \tag{10A-14}$$

k is the real wavenumber given by (10A-5), ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter, the factor $\rho_0 c$ is the characteristic impedance of the fluid medium in rayls ($1\text{ rayl} = 1\text{ Pa-sec/m}$), $\varphi_f(\mathbf{r})$ is given by (10A-2), and $g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)$ is given by either (10A-10) or (10A-11). We also need an equation for the time-average intensity vector of the radiated acoustic field, which requires equations for the acoustic pressure – which we have [(10A-13) and (10A-14)] – and the acoustic fluid-velocity-vector (a.k.a. the acoustic particle-velocity-vector). Therefore, we shall compute the acoustic fluid-velocity-vector next.

The acoustic fluid-velocity-vector $\mathbf{u}(t,\mathbf{r})$ in meters per second can be obtained from the scalar velocity potential $\varphi(t,\mathbf{r})$ in squared meters per second as follows:

$$\mathbf{u}(t,\mathbf{r}) = \nabla\varphi(t,\mathbf{r}), \tag{10A-15}$$

where

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{10A-16}$$

is the gradient expressed in the rectangular coordinates (x, y, z). Substituting (10A-1) and (10A-2) into (10A-15) yields

$$\mathbf{u}(t,\mathbf{r}) = S_0\nabla\left[g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)\right]\exp(+j2\pi ft). \tag{10A-17}$$

In Appendix 10B it is shown that the gradient of the Green's function given by (10A-10) is

$$\nabla g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) = -\left(\frac{1}{R} + jK\right)g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)\hat{R}, \tag{10A-18}$$

where

$$\hat{R} = \mathbf{R}/R \tag{10A-19}$$

is the dimensionless unit vector in the direction of the vector

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0 = (x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z}, \tag{10A-20}$$

and $R = |\mathbf{R}|$ is given by (10A-8). Substituting (10A-18) into (10A-17) yields the time-harmonic, radiated, acoustic fluid-velocity-vector in meters per second

$$\mathbf{u}(t,\mathbf{r}) = \mathbf{u}_f(\mathbf{r})\exp(+j2\pi ft), \tag{10A-21}$$

where

$$\mathbf{u}_f(\mathbf{r}) = -\left(\frac{1}{R} + jK\right)\varphi_f(\mathbf{r})\hat{R} \tag{10A-22}$$

and $\varphi_f(\mathbf{r})$ is given by (10A-2). We are now in a position to compute the time-average intensity vector.

Recall from Section 4.1 that the time-average intensity vector in watts per squared meter for time-harmonic acoustic fields is given by

$$\mathbf{I}_{\text{avg}}(\mathbf{r}) = \frac{1}{2}\text{Re}\left\{p_f(\mathbf{r})\mathbf{u}_f^*(\mathbf{r})\right\}, \tag{10A-23}$$

where $p_f(\mathbf{r})$ and $\mathbf{u}_f(\mathbf{r})$ are the spatial-dependent parts of the time-harmonic acoustic pressure and acoustic fluid-velocity-vector, respectively, and the asterisk denotes complex conjugate. Substituting (10A-14) and (10A-22) into (10A-23) yields

$$\mathbf{I}_{\text{avg}}(\mathbf{r}) = I_{\text{avg}}(\mathbf{r})\hat{R}\,, \tag{10A-24}$$

where

$$I_{\text{avg}}(\mathbf{r}) = \frac{1}{2}k^2\rho_0 c\left|\varphi_f(\mathbf{r})\right|^2 \tag{10A-25}$$

is the magnitude of the time-average intensity vector of the radiated acoustic field, and $\varphi_f(\mathbf{r})$ is given by (10A-2). Since

$$\left|p_f(\mathbf{r})\right|^2 = k^2(\rho_0 c)^2\left|\varphi_f(\mathbf{r})\right|^2, \tag{10A-26}$$

where $p_f(\mathbf{r})$ is given by (10A-14), (10A-25) can be rewritten as

$$I_{\text{avg}}(\mathbf{r}) = \frac{\left|p_f(\mathbf{r})\right|^2}{2\rho_0 c}\,. \tag{10A-27}$$

And since

$$\left|\varphi_f(\mathbf{r})\right|^2 = \left(\frac{|S_0|}{4\pi R}\right)^2 \exp[-2\alpha(f)R]\,, \tag{10A-28}$$

where $\varphi_f(\mathbf{r})$ is given by (10A-2) and $g_f\!\left(\mathbf{r}\,|\,\mathbf{r}_0\right)$ given by (10A-11) was used, substituting (10A-28) into (10A-25) and (10A-26) yields

$$I_{\text{avg}}(\mathbf{r}) = \frac{1}{2}k^2\rho_0 c\left(\frac{|S_0|}{4\pi R}\right)^2 \exp[-2\alpha(f)R] \tag{10A-29}$$

and

$$\left|p_f(\mathbf{r})\right|^2 = k^2(\rho_0 c)^2\left(\frac{|S_0|}{4\pi R}\right)^2 \exp[-2\alpha(f)R]\,, \tag{10A-30}$$

respectively.

The next set of equations that we need are equations for the spherical angles θ and ψ that describe the direction of the unit vector $\hat{R}$ as measured from the location of the omnidirectional point-source to a field point. Substituting (10A-20) into (10A-19) yields

$$\hat{R} = \frac{x - x_0}{R}\hat{x} + \frac{y - y_0}{R}\hat{y} + \frac{z - z_0}{R}\hat{z}. \tag{10A-31}$$

The unit vector $\hat{R}$ can also be expressed as

$$\hat{R} = u\hat{x} + v\hat{y} + w\hat{z}, \tag{10A-32}$$

where

$$u = \sin\theta\cos\psi, \tag{10A-33}$$

$$v = \sin\theta\sin\psi, \tag{10A-34}$$

and

$$w = \cos\theta \tag{10A-35}$$

are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively. Equating the right-hand sides of (10A-31) and (10A-32) yields

$$u = \frac{x - x_0}{R}, \tag{10A-36}$$

$$v = \frac{y - y_0}{R}, \tag{10A-37}$$

and

$$w = \frac{z - z_0}{R}. \tag{10A-38}$$

Therefore, if the origin of a Cartesian coordinate system (with the same orientation as the one used to measure $\mathbf{r}_0$ and $\mathbf{r}$) is placed at $\mathbf{r}_0$, then the spherical angles θ and ψ, which are measured as shown in Fig. 1.2-2, can be computed as follows:

$$\theta = \cos^{-1} w = \cos^{-1}\left(\frac{z - z_0}{R}\right) \tag{10A-39}$$

and

$$\psi = \tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}\left(\frac{y - y_0}{x - x_0}\right). \tag{10A-40}$$

The last equation that we need is the equation for the time-average power radiated by a time-harmonic, omnidirectional point-source. If we enclose the time-harmonic, omnidirectional point-source at $\mathbf{r}_0$ with a sphere of radius R meters, where R is given by (10A-8), then the time-average radiated power in watts is given by

$$P_{\text{avg}} = \oint_S \mathbf{I}_{\text{avg}}(\mathbf{r}) \bullet \mathbf{dS} \,, \tag{10A-41}$$

where

$$\mathbf{dS} = R^2 \sin\theta \, d\theta \, d\psi \hat{R} \,. \tag{10A-42}$$

Substituting (10A-24), (10A-29), and (10A-42) into (10A-41) yields

$$P_{\text{avg}} = \frac{1}{2} k^2 \rho_0 c \left(\frac{|S_0|}{4\pi} \right)^2 \exp[-2\alpha(f)R] \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\psi \,, \tag{10A-43}$$

and since

$$\int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\psi = 4\pi \,, \tag{10A-44}$$

(10A-43) reduces to

$$P_{\text{avg}} = \frac{1}{2} k^2 \rho_0 c \frac{|S_0|^2}{4\pi} \exp[-2\alpha(f)R] = 4\pi R^2 I_{\text{avg}}(\mathbf{r}) \,, \tag{10A-45}$$

or

$$I_{\text{avg}}(\mathbf{r}) = \frac{P_{\text{avg}}}{4\pi R^2} \,, \tag{10A-46}$$

where $4\pi R^2$ is the surface area (in squared meters) of a sphere with radius R meters. Furthermore, by solving for the magnitude-squared of the source strength using (10A-30), we obtain

$$|S_0|^2 = \frac{(4\pi)^2 R^2}{k^2 (\rho_0 c)^2} \exp[+2\alpha(f)R] \left| p_f(\mathbf{r}) \right|^2 \,, \tag{10A-47}$$

and if we let

$$P_0 = \left| p(t, \mathbf{r}) \right|_{R = R_0 = 1\,\text{m}} = \left| p_f(\mathbf{r}) \right|_{R = R_0 = 1\,\text{m}} \,, \tag{10A-48}$$

then

$$|S_0|^2 = \left. \frac{(4\pi)^2 R_0^2}{k^2 (\rho_0 c)^2} \exp[+2\alpha(f)R_0] P_0^2 \right|_{R_0 = 1\,\text{m}} \,. \tag{10A-49}$$

Substituting (10A-49) into (10A-45) yields

$$P_{\text{avg}} = \left. 4\pi R_0^2 \frac{P_0^2}{2\rho_0 c} \exp\left[-2\alpha(f)(R - R_0) \right] \right|_{R_0 = 1\,\text{m}} \,. \tag{10A-50}$$

Appendix 10B Gradient of the Time-Independent, Free-Space, Green's Function

In this Appendix we shall compute $\nabla g_f(\mathbf{r}\,|\,\mathbf{r}_0)$, where

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{10B-1}$$

is the gradient expressed in the rectangular coordinates (x, y, z),

$$g_f(\mathbf{r}\,|\,\mathbf{r}_0) = -\frac{\exp(-jK|\mathbf{r}-\mathbf{r}_0|)}{4\pi|\mathbf{r}-\mathbf{r}_0|} = -\frac{\exp(-jKR)}{4\pi R} \tag{10B-2}$$

is the time-independent, free-space, Green's function with units of inverse meters,

$$K = k - j\alpha(f) \tag{10B-3}$$

is the complex wavenumber in inverse meters, and

$$R = |\mathbf{r}-\mathbf{r}_0| = \left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{1/2} \tag{10B-4}$$

is the range in meters between source and field points. Expanding $\nabla g_f(\mathbf{r}\,|\,\mathbf{r}_0)$ using (10B-1) and (10B-2) yields

$$\begin{aligned}\nabla g_f(\mathbf{r}\,|\,\mathbf{r}_0) &= \nabla\left[-\frac{1}{4\pi R}\exp(-jKR)\right] \\ &= -\frac{1}{4\pi R}\nabla\left[\exp(-jKR)\right] + \exp(-jKR)\nabla\left(-\frac{1}{4\pi R}\right),\end{aligned} \tag{10B-5}$$

where

$$\nabla\exp(-jKR) = \frac{\partial}{\partial x}\exp(-jKR)\,\hat{x} + \frac{\partial}{\partial y}\exp(-jKR)\,\hat{y} + \frac{\partial}{\partial z}\exp(-jKR)\,\hat{z}\,, \tag{10B-6}$$

$$\nabla\left(-\frac{1}{4\pi R}\right) = -\frac{1}{4\pi}\nabla\frac{1}{R}\,, \tag{10B-7}$$

and

$$\nabla\frac{1}{R}=\frac{\partial}{\partial x}\frac{1}{R}\hat{x}+\frac{\partial}{\partial y}\frac{1}{R}\hat{y}+\frac{\partial}{\partial z}\frac{1}{R}\hat{z}\,. \tag{10B-8}$$

Next we shall compute $\nabla\exp(-jKR)$ as given by (10B-6).

Since

$$\frac{\partial}{\partial x}\exp(-jKR)=-jK\exp(-jKR)\frac{\partial R}{\partial x}\,, \tag{10B-9}$$

$$\frac{\partial}{\partial y}\exp(-jKR)=-jK\exp(-jKR)\frac{\partial R}{\partial y}\,, \tag{10B-10}$$

and

$$\frac{\partial}{\partial z}\exp(-jKR)=-jK\exp(-jKR)\frac{\partial R}{\partial z}\,, \tag{10B-11}$$

substituting (10B-9) through (10B-11) into (10B-6) yields

$$\nabla\exp(-jKR)=-jK\exp(-jKR)\nabla R\,, \tag{10B-12}$$

where

$$\nabla R=\frac{\partial R}{\partial x}\hat{x}+\frac{\partial R}{\partial y}\hat{y}+\frac{\partial R}{\partial z}\hat{z}\,. \tag{10B-13}$$

From (10B-4) it can be seen that

$$\frac{\partial R}{\partial x}=\frac{x-x_0}{R}\,, \tag{10B-14}$$

$$\frac{\partial R}{\partial y}=\frac{y-y_0}{R}\,, \tag{10B-15}$$

and

$$\frac{\partial R}{\partial z}=\frac{z-z_0}{R}\,. \tag{10B-16}$$

Substituting (10B-14) through (10B-16) into (10B-13) yields

$$\boxed{\nabla R=\hat{R}=\mathbf{R}/R} \tag{10B-17}$$

where $\hat{R}$ is the dimensionless unit vector in the direction of the vector

$$\mathbf{R}=\mathbf{r}-\mathbf{r}_0=(x-x_0)\hat{x}+(y-y_0)\hat{y}+(z-z_0)\hat{z}\,, \tag{10B-18}$$

and $R = |\mathbf{R}|$ is given by (10B-4). And by substituting (10B-17) into (10B-12), we obtain

$$\boxed{\nabla \exp(-jKR) = -jK \exp(-jKR)\,\hat{R}} \tag{10B-19}$$

Next we shall compute $\nabla(1/R)$ as given by (10B-8).

Since

$$\frac{\partial}{\partial x}\frac{1}{R} = -\frac{1}{R^2}\frac{\partial R}{\partial x}, \tag{10B-20}$$

$$\frac{\partial}{\partial y}\frac{1}{R} = -\frac{1}{R^2}\frac{\partial R}{\partial y}, \tag{10B-21}$$

and

$$\frac{\partial}{\partial z}\frac{1}{R} = -\frac{1}{R^2}\frac{\partial R}{\partial z}, \tag{10B-22}$$

substituting (10B-20) through (10B-22), and (10B-17) into (10B-8) yields

$$\boxed{\nabla \frac{1}{R} = -\frac{1}{R^2}\nabla R = -\frac{1}{R^2}\hat{R}} \tag{10B-23}$$

and by substituting (10B-7), (10B-19), and (10B-23) into (10B-5), we finally obtain

$$\boxed{\nabla g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) = -\left(\frac{1}{R} + jK\right) g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)\hat{R}} \tag{10B-24}$$

Appendix 10C

Since we have already shown that $\mathcal{A} > 0$ for the problems that we are interested in, in order to prove that $4\mathcal{A}\mathcal{C} > 0$, where

$$\mathcal{C} = r_{SR}^2 + 2r_{SR}\frac{\hat{r}_{SR} \bullet \mathbf{V}_{RS}}{c}\left|\boldsymbol{\mathcal{R}}_{TS}(t')\right| + \left(\frac{V_{RS}}{c}\right)^2 \left|\boldsymbol{\mathcal{R}}_{TS}(t')\right|^2, \tag{10.7-40}$$

we need to prove that $\mathcal{C} > 0$.

We begin by noting that