Appendix 2E Summary of One-Dimensional Spatial Fourier Transforms

Table 2E-1 Summary of One-Dimensional Spatial Fourier Transforms

$A(f,x_A)=F_{f_X}^{-1}\{D(f,f_X)\}$ $=\int_{-\infty}^{\infty} D(f,f_X)\exp(-j2\pi f_X x_A)\,df_X$ $f_X=\frac{u}{\lambda}=\frac{\sin\theta\cos\psi}{\lambda}$	$D(f,f_X)=F_{x_A}\{A(f,x_A)\}$ $=\int_{-\infty}^{\infty} A(f,x_A)\exp(+j2\pi f_X x_A)\,dx_A$ $f_X=\frac{u}{\lambda}=\frac{\sin\theta\cos\psi}{\lambda}$

$A(f,x_A)$	$D(f,f_X)$
$A_A(f)$	$A_A(f)\delta(f_X)$
$A_A(f)\delta(x_A \pm d)$	$A_A(f)\exp(\mp j2\pi f_X d)$
$A_A(f)\exp(\pm j2\pi f'_X x_A)$	$A_A(f)\delta(f_X \pm f'_X)$
$A_A(f)\mathrm{rect}(x_A/L)$	$A_A(f)L\,\mathrm{sinc}(f_X L)$
$A_A(f)\mathrm{tri}(x_A/L)$	$[A_A(f)L/2]\mathrm{sinc}^2(f_X L/2)$
$A_A(f)\cos(2\pi x_A/d)$	$\frac{A_A(f)}{2}\left[\delta\left(f_X+\frac{1}{d}\right)+\delta\left(f_X-\frac{1}{d}\right)\right]$
$A_A(f)\cos(2\pi x_A/d)\mathrm{rect}(x_A/L)$	$\frac{A_A(f)L}{2}\left\{\mathrm{sinc}\left[\left(f_X+\frac{1}{d}\right)L\right]+\mathrm{sinc}\left[\left(f_X-\frac{1}{d}\right)L\right]\right\}$
$A_A(f)\cos(\pi x_A/L)\mathrm{rect}(x_A/L)$	$\frac{2A_A(f)L}{\pi}\frac{\cos(\pi f_X L)}{1-(2f_X L)^2}$

Comments

1. The function $A_A(f)$ can also be equal to a constant A_A.

Chapter 3

Complex Aperture Theory – Planar Apertures

3.1 The Far-Field Beam Pattern of a Planar Aperture

In this section we shall demonstrate that the far-field beam pattern of a planar aperture is just a special case of the far-field beam pattern of a volume aperture. In Section 1.3 it was shown that the far-field beam pattern (directivity function) of a closed-surface, volume aperture – based on the Fraunhofer approximation of a time-independent, free-space, Green's function – is given by the following three-dimensional spatial Fourier transform:

$$D(f,\boldsymbol{\alpha}) = F_{\mathbf{r}_A}\{A(f,\mathbf{r}_A)\} = \int_{-\infty}^{\infty} A(f,\mathbf{r}_A)\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_A)\,d\mathbf{r}_A\,, \tag{3.1-1}$$

where

$$\mathbf{r}_A = (x_A, y_A, z_A)\,, \tag{3.1-2}$$

$$\boldsymbol{\alpha} = (f_X, f_Y, f_Z)\,, \tag{3.1-3}$$

$$\boldsymbol{\alpha}\bullet\mathbf{r}_A = f_X x_A + f_Y y_A + f_Z z_A\,, \tag{3.1-4}$$

$$d\mathbf{r}_A = dx_A\,dy_A\,dz_A\,, \tag{3.1-5}$$

and $F_{\mathbf{r}_A}\{\bullet\}$ is shorthand notation for $F_{x_A}F_{y_A}F_{z_A}\{\bullet\}$. The complex aperture function $A(f,\mathbf{r}_A)$ can be expressed as

$$A(f,\mathbf{r}_A) = a(f,\mathbf{r}_A)\exp[+j\theta(f,\mathbf{r}_A)]\,, \tag{3.1-6}$$

where $a(f,\mathbf{r}_A)$ is the *amplitude response* and $\theta(f,\mathbf{r}_A)$ is the *phase response* of the aperture at $\mathbf{r}_A$. Both $a(f,\mathbf{r}_A)$ and $\theta(f,\mathbf{r}_A)$ are *real* functions. The function $a(f,\mathbf{r}_A)$ is also known as the *amplitude window*. The units of $a(f,\mathbf{r}_A)$ and, hence, $A(f,\mathbf{r}_A)$ depend on whether the aperture is a transmit aperture (see Table 1B-1 in Appendix 1B) or a receive aperture (see Table 1B-2 in Appendix 1B). The units of $\theta(f,\mathbf{r}_A)$ are radians.

Now consider the case of a *planar aperture* of arbitrary shape lying in the XY plane as shown in Fig. 3.1-1. A planar aperture can represent either a single electroacoustic transducer (e.g., a rectangular or circular piston) or a planar array of many individual electroacoustic transducers. Also shown in Fig. 3.1-1 is a field point with spherical coordinates (r,θ,ψ). The field point is in the far-field region of the aperture, where the range r satisfies the Fraunhofer (far-field) range criterion given by $r > \pi R_A^2/\lambda > 2.414 R_A$ [see either (1.2-39) or (1.2-93)], and R_A is the maximum radial extent of the planar aperture.

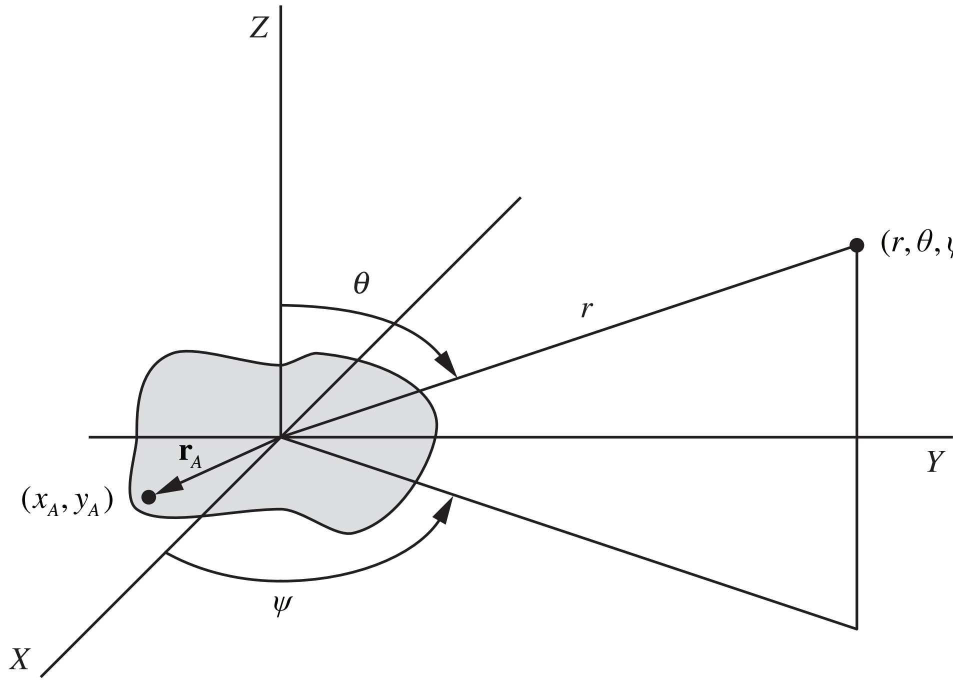

Figure 3.1-1 Planar aperture of arbitrary shape lying in the XY plane. Also shown is a field point with spherical coordinates (r,θ,ψ).

Since the planar aperture in Fig. 3.1-1 is lying in the XY plane, the position vector to a point on the surface of the aperture is given by

$$\mathbf{r}_A = (x_A, y_A, 0)\,, \tag{3.1-7}$$

and as a result,

$$A(f,\mathbf{r}_A) = A(f,x_A,y_A,z_A) \rightarrow A(f,x_A,y_A)\delta(z_A)\,. \tag{3.1-8}$$

Therefore, by substituting (3.1-8) into (3.1-1) and using the sifting property of impulse functions, we obtain the following two-dimensional spatial Fourier transform for the far-field beam pattern of a planar aperture lying in the XY plane:

$$\boxed{\begin{aligned} D(f,f_X,f_Y) &= F_{x_A}F_{y_A}\{A(f,x_A,y_A)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f,x_A,y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\,dy_A \end{aligned}}$$

(3.1-9)

where

$$A(f,x_A,y_A) = a(f,x_A,y_A)\exp[+j\theta(f,x_A,y_A)] \tag{3.1-10}$$

is the complex frequency response (complex aperture function) of the planar aperture, and

$$f_X = u/\lambda = \sin\theta\cos\psi/\lambda \tag{3.1-11}$$

and

$$f_Y = v/\lambda = \sin\theta\sin\psi/\lambda \tag{3.1-12}$$

are spatial frequencies in the X and Y directions, respectively, with units of cycles per meter [see (1.2-43) and (1.2-44)]. Since f_X and f_Y can be expressed in terms of the spherical angles θ and ψ, the far-field beam pattern can ultimately be expressed as a function of frequency f and the spherical angles θ and ψ, that is, $D(f,f_X,f_Y) \rightarrow D(f,\theta,\psi)$. If the planar aperture is a transmit aperture, then the aperture function $A(f,x_A,y_A)$ has units of $\left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}\right)/\mathrm{m}^2$ and the far-field beam pattern $D(f,f_X,f_Y)$ has units of $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$. However, if the planar aperture is a receive aperture, then $A(f,x_A,y_A)$ has units of $\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^2$ and $D(f,f_X,f_Y)$ has units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$. These units are summarized in Table 3.1-1. In addition to describing the performance of an electroacoustic transducer by its far-field beam pattern, values of the frequency-dependent, *transmitter* and *receiver sensitivity functions* are also used. The relationships between the complex frequency response $A(f,x_A,y_A)$ of a single electroacoustic transducer and complex, transmitter and receiver sensitivity functions are discussed in Appendix 3A.

Table 3.1-1 Units of the Complex Aperture Function $A(f,x_A,y_A)$ and Corresponding Far-Field Beam Pattern $D(f,f_X,f_Y)$ for a Planar Aperture

Planar Aperture	$A(f,x_A,y_A)$	$D(f,f_X,f_Y)$
transmit	$\left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}\right)/\mathrm{m}^2$	$\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$
receive	$\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^2$	$\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$

Finally, if a planar aperture lies in the XZ plane instead of the XY plane, then simply replace y_A and f_Y with z_A and f_Z, respectively, in (3.1-9), where spatial frequency f_Z is given by (1.2-45). And if a planar aperture lies in the YZ plane instead of the XY plane, then simply replace x_A and f_X with z_A and f_Z, respectively, in (3.1-9). With the use of (3.1-9), we will derive the normalized, far-field beam patterns of rectangular and circular pistons in Sections 3.2 and 3.3, respectively. Although we shall be concentrating on single, planar, electroacoustic transducers in this chapter, as we shall discover in Chapter 8, some of the results obtained in this chapter are directly applicable to planar arrays. In other words, we are laying the foundation for planar array theory in this chapter as well.

3.2 The Far-Field Beam Pattern of a Rectangular Piston

One of the most common examples of a planar aperture lying in the XY plane is a single electroacoustic transducer, rectangular in shape, with sides equal to L_X and L_Y meters in length, as shown in Fig. 3.2-1. The simplest mathematical model for the complex frequency response (complex aperture function) for this rectangular-shaped transducer is given by

$$A(f, x_A, y_A) = A_A \operatorname{rect}(x_A / L_X) \operatorname{rect}(y_A / L_Y)\,, \tag{3.2-1}$$

where the *real*, *positive* constant A_A carries the correct units of $A(f, x_A, y_A)$ (see Table 3.1-1) and the rectangle function is defined as follows:

$$\operatorname{rect}\left(\frac{x}{L}\right) \triangleq \begin{cases} 1, & |x| \le L/2 \\ 0, & |x| > L/2\,. \end{cases} \tag{3.2-2}$$

According to (3.2-1), the complex frequency response of the transducer is constant across the entire surface of the transducer, regardless of the value of frequency f. It is important to note that although A_A is shown here as a constant, it is, in general, a *real*, *nonnegative* function of frequency, that is, $A_A \to A_A(f)$. A single electroacoustic transducer, rectangular in shape, with complex frequency response given by (3.2-1), is referred to as a *rectangular piston*. The maximum radial extent of a rectangular-shaped, planar aperture lying in the XY plane is given by

$$R_A = \sqrt{\left(\frac{L_X}{2}\right)^2 + \left(\frac{L_Y}{2}\right)^2}\,. \tag{3.2-3}$$

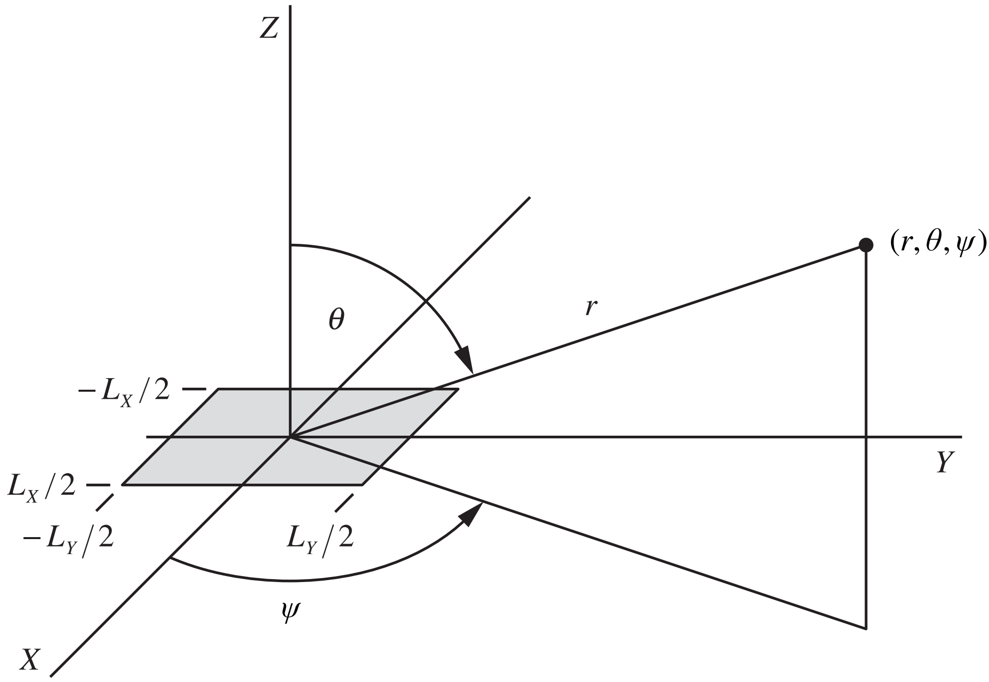

Figure 3.2-1 A single electroacoustic transducer, rectangular in shape, with sides equal to L_X and L_Y meters in length, lying in the XY plane. Also shown is a field point with spherical coordinates (r,θ,ψ).

The far-field beam pattern of the rectangular piston can be obtained by substituting (3.2-1) into (3.1-9). Doing so yields

$$\begin{aligned} D(f,f_X,f_Y) &= F_{x_A}F_{y_A}\{A(f,x_A,y_A)\} \\ &= A_A F_{x_A}F_{y_A}\{\text{rect}(x_A/L_X)\text{rect}(y_A/L_Y)\} \\ &= A_A F_{x_A}\{\text{rect}(x_A/L_X)\}F_{y_A}\{\text{rect}(y_A/L_Y)\}, \end{aligned} \tag{3.2-4}$$

or

$$\boxed{D(f,f_X,f_Y) = A_A L_X L_Y \,\text{sinc}(f_X L_X)\text{sinc}(f_Y L_Y)} \tag{3.2-5}$$

where use was made of (2.2-5) (Table 2E-1 can also be used). Equation (3.2-5) is the *unnormalized*, far-field beam pattern of a rectangular piston lying in the XY plane, where $L_X L_Y$ is the area of the rectangular piston in squared meters. Since $\text{sinc}(0)=1$ is the maximum value of a sinc function, the normalization factor $D_{\max} = \max|D(f,f_X,f_Y)|$ – which is the *maximum* value of the *magnitude* of the unnormalized, far-field beam pattern – is given by

$$D_{\max} = |D(f,0,0)| = A_A L_X L_Y \,. \tag{3.2-6}$$

Whether or not the normalization factor is equal to the magnitude of the unnormalized, far-field beam pattern evaluated at broadside, $D_{\max}$ can also be found by using the standard calculus approach for finding the maximum value of a function, or by using the formulas in Appendix 3C. Dividing (3.2-5) by (3.2-6) yields the following expression for the *normalized*, far-field beam pattern of a rectangular piston lying in the XY plane:

$$\boxed{D_N(f, f_X, f_Y) = \text{sinc}(f_X L_X)\,\text{sinc}(f_Y L_Y)} \tag{3.2-7}$$

Note that $|D_N(f,0,0)| = 1$. Since $f_X = u/\lambda$ and $f_Y = v/\lambda$, (3.2-7) can also be expressed as

$$\boxed{D_N(f,u,v) = \text{sinc}\left(\frac{L_X}{\lambda}u\right)\text{sinc}\left(\frac{L_Y}{\lambda}v\right)} \tag{3.2-8}$$

The magnitude of the normalized, far-field beam pattern given by (3.2-8) is plotted as a function of direction cosines u and v in Fig. 3.2-2 for $L_Y = L_X = L$ and $L/\lambda = 2.5$. As can be seen from the equations developed in this section, the rectangular piston is just a two-dimensional generalization of a linear aperture lying along the X axis with a rectangular amplitude window for a complex frequency response (see Subsection 2.2.1).

When plotting the far-field beam pattern of a planar aperture in direction-cosine space, care must be taken when trying to solve for the spherical angles θ and ψ from (u,v) coordinates. Since $u = \sin\theta\cos\psi$ and $v = \sin\theta\sin\psi$, it can be shown that

$$\theta = \sin^{-1}\sqrt{u^2+v^2}\,, \qquad \sqrt{u^2+v^2} \le 1\,, \tag{3.2-9}$$

and

$$\psi = \tan^{-1}(v/u)\,. \tag{3.2-10}$$

The inequality $\sqrt{u^2+v^2} \le 1$ appearing in (3.2-9) is illustrated in Fig. 3.2-3. Only those (u,v) coordinates inside or on the unit circle shown in Fig. 3.2-3 can be used for computing θ and ψ. Also, since the azimuthal (bearing) angle ψ is measured in a counter-clockwise direction from the positive u (X) axis, it takes on values in the range $0° \le \psi \le 360°$. However, the arctangent function in (3.2-10) may only give values for ψ in the range $-90° \le \psi \le 90°$ or

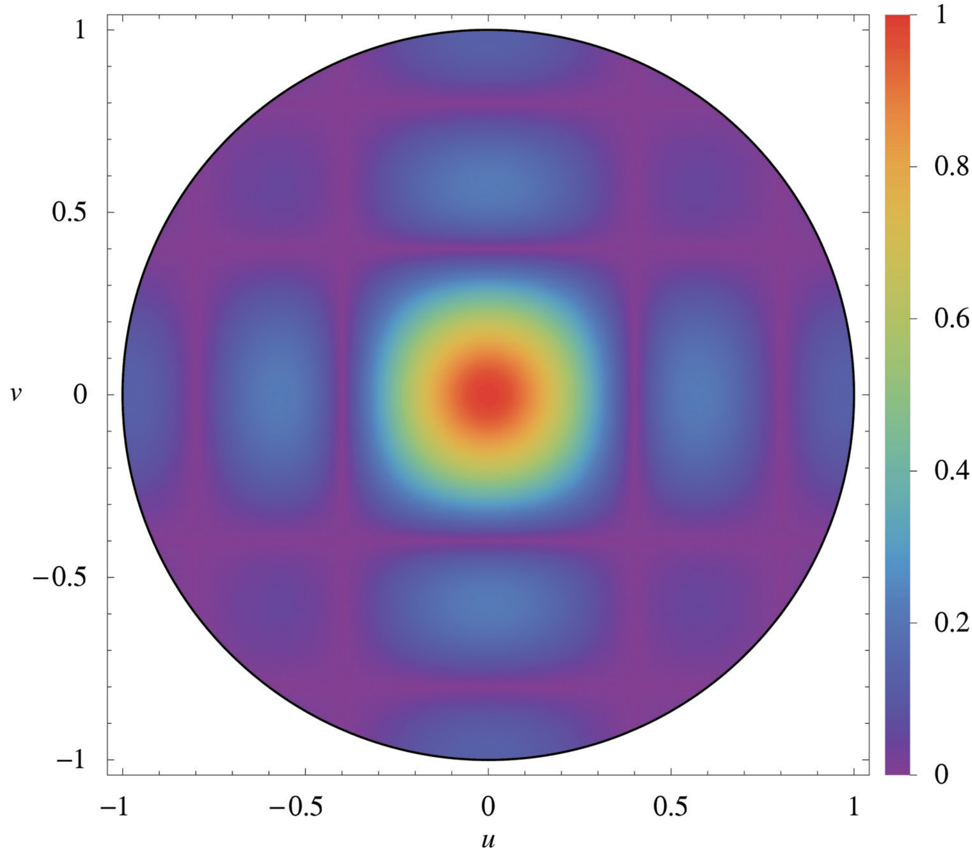

Figure 3.2-2 Magnitude of the normalized, far-field beam pattern of a rectangular piston lying in the XY plane, plotted as a function of direction cosines u and v for $L_Y = L_X = L$ and $L/\lambda = 2.5$.

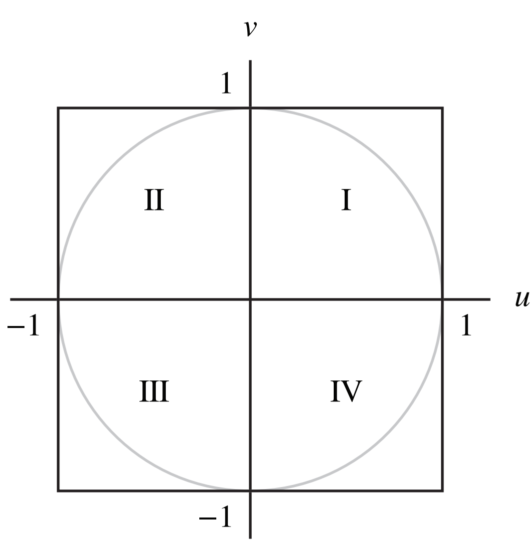

Figure 3.2-3 Unit circle with radius $\sqrt{u^2+v^2}=1$ in the uv plane. Points inside or on the unit circle satisfy the inequality $\sqrt{u^2+v^2}\le 1$.

$-180° \le \psi \le 180°$. Therefore, by taking into account the signs (quadrant) of the (u,v) coordinates, one may have to add either $180°$ or $360°$ to the value obtained from (3.2-10) so that ψ has the correct value (see Table 3.2-1).

Table 3.2-1 Range of Values for Azimuthal (Bearing) Angle ψ Versus Quadrant Number

Quadrant	u	v	ψ
I	+	+	$0° \le \psi \le 90°$
II	–	+	$90° \le \psi \le 180°$
III	–	–	$180° \le \psi \le 270°$
IV	+	–	$270° \le \psi \le 360°$

Example 3.2-1 3-dB Beamwidths of the Vertical Far-Field Beam Patterns of a Rectangular Piston

In this example we shall derive equations for the 3-dB beamwidths of the vertical far-field beam patterns in the XZ and YZ planes for a rectangular piston lying in the XY plane. Since direction cosine $v=0$ in the XZ plane, substituting $v=0$ into (3.2-8) yields the following expression for the normalized, *vertical*, far-field beam pattern in the XZ plane of a rectangular piston lying in the XY plane:

$$D_N(f,u,0) = \operatorname{sinc}\left(\frac{L_X}{\lambda}u\right). \tag{3.2-11}$$

Equation (3.2-11) is also the normalized, far-field beam pattern of a linear aperture lying along the X axis with a rectangular amplitude window for a complex frequency response (see Subsection 2.2.1). Therefore, by combining the results from Examples 2.3-1 and 2.3-3, we obtain

$$\Delta\theta = 2\sin^{-1}\left(0.443\lambda/L_X\right), \tag{3.2-12}$$

where $\Delta\theta$ is the 3-dB beamwidth in degrees of the vertical, far-field beam pattern in the XZ plane.

Since direction cosine $u=0$ in the YZ plane, substituting $u=0$ into (3.2-8) yields the following expression for the normalized, *vertical*, far-field beam pattern in the YZ plane of a rectangular piston lying in the XY plane:

$$D_N(f,0,v) = \operatorname{sinc}\left(\frac{L_Y}{\lambda}v\right). \tag{3.2-13}$$

Equation (3.2-13) is also the normalized, far-field beam pattern of a linear aperture lying along the Y axis with a rectangular amplitude window for a complex frequency response. Therefore,

$$\Delta\theta = 2\sin^{-1}\left(0.443\lambda/L_Y\right), \tag{3.2-14}$$

where $\Delta\theta$ is the 3-dB beamwidth in degrees of the vertical, far-field beam pattern in the YZ plane. By comparing (3.2-12) and (3.2-14), it can be seen that the 3-dB beamwidths in the XZ and YZ planes are not equal as long as $L_Y \neq L_X$. This implies that the 3-dB beamwidth in the XZ plane can be made smaller than the 3-dB beamwidth in the YZ plane ($L_X > L_Y$), or vice-versa ($L_Y > L_X$). ■

3.3 The Far-Field Beam Pattern of a Circular Piston

We know from our discussion in Section 3.1 that the far-field beam pattern of an arbitrarily shaped planar aperture lying in the XY plane is given by the following two-dimensional spatial Fourier transform [see (3.1-9)]:

$$D(f, f_X, f_Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f, x_A, y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\,dy_A\,, \tag{3.3-1}$$

where

$$f_X = u/\lambda = \sin\theta\cos\psi/\lambda \tag{3.3-2}$$

and

$$f_Y = v/\lambda = \sin\theta\sin\psi/\lambda\,. \tag{3.3-3}$$

Now consider the problem of computing the far-field beam pattern of a planar aperture lying in the XY plane that is circular in shape, with radius a meters and complex frequency response $A(f, r_A, \phi_A)$ expressed in polar coordinates r_A and ϕ_A (see Fig. 3.3-1). A circular-shaped, planar aperture can represent either a single electroacoustic transducer (e.g., a circular piston) or a planar array of concentric circular arrays composed of many individual electroacoustic transducers. The maximum radial extent of a circular-shaped, planar aperture is $R_A = a$. In order to solve this problem, we need to transform (3.3-1) from rectangular coordinates to polar coordinates.

By referring to Fig. 3.3-2, we can write that

$$x_A = r_A\cos\phi_A\,, \tag{3.3-4}$$

$$y_A = r_A\sin\phi_A\,, \tag{3.3-5}$$

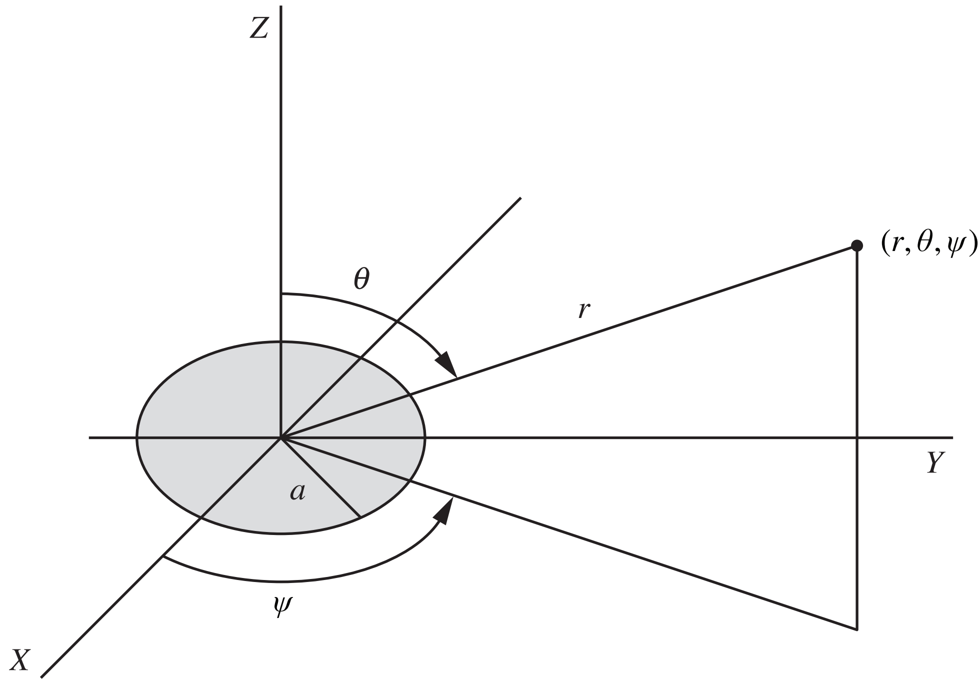

Figure 3.3-1 A single electroacoustic transducer, circular in shape, with radius a meters, lying in the XY plane. Also shown is a field point with spherical coordinates (r,θ,ψ).

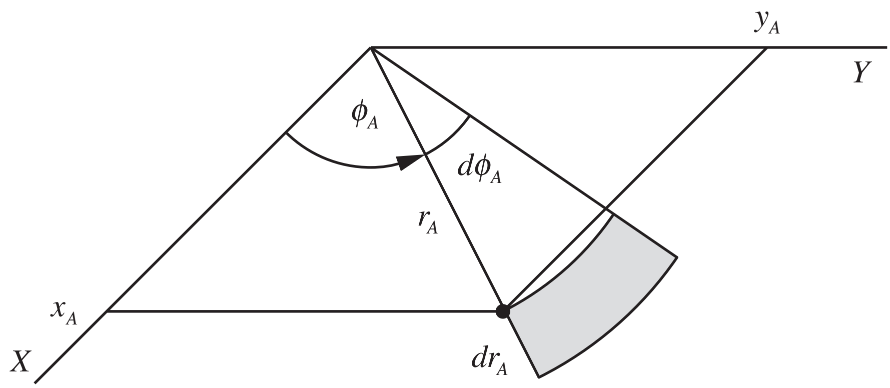

Figure 3.3-2 Relationship between the rectangular coordinates (x_A, y_A) and the polar coordinates (r_A,ϕ_A). Also shown is the infinitesimal area $r_A\,dr_A\,d\phi_A$.

$$dx_A\,dy_A \rightarrow r_A\,dr_A\,d\phi_A\,, \tag{3.3-6}$$

and

$$A(f,x_A,y_A) = A(f,r_A\cos\phi_A, r_A\sin\phi_A) \rightarrow A(f,r_A,\phi_A)\,. \tag{3.3-7}$$

Substituting (3.3-2) through (3.3-7) into (3.3-1), and using the trigonometric identity

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta\,, \tag{3.3-8}$$

yields

$$\boxed{D(f,\theta,\psi)=\int_0^{2\pi}\int_0^{a} A(f,r_A,\phi_A)\exp\left[+j\frac{2\pi r_A}{\lambda}\sin\theta\cos(\psi-\phi_A)\right]r_A\,dr_A\,d\phi_A} \tag{3.3-9}$$

where

$$A(f,r_A,\phi_A)=a(f,r_A,\phi_A)\exp[+j\theta(f,r_A,\phi_A)] \tag{3.3-10}$$

is the complex frequency response (complex aperture function) of the circular aperture. Note that $2\pi r_A$ in (3.3-9) is the circumference of a circle with radius r_A meters. Equation (3.3-9) will be used in Example 8.1-6 to derive the far-field beam pattern of a planar array of concentric circular arrays.

Circular Symmetry

Before we derive the far-field beam pattern of a circular piston, let us first consider the important special case when the complex frequency response $A(f,r_A,\phi_A)$ is *circularly symmetric*, that is,

$$A(f,r_A,\phi_A)=A_r(f,r_A)\,, \tag{3.3-11}$$

where

$$A_r(f,r_A)=a_r(f,r_A)\exp[+j\theta_r(f,r_A)]\,. \tag{3.3-12}$$

The complex frequency response given by (3.3-11) is said to be *circularly symmetric*, or to have *circular symmetry*, because it's value does not depend on the polar angle ϕ_A to an aperture point. Substituting (3.3-11) into (3.3-9) and interchanging the order of integration yields

$$D(f,\theta,\psi)=\int_0^{a} A_r(f,r_A)\int_0^{2\pi}\exp\left[+j\frac{2\pi r_A}{\lambda}\sin\theta\cos(\psi-\phi_A)\right]d\phi_A\,r_A\,dr_A\,. \tag{3.3-13}$$

If we let

$$b=\frac{2\pi r_A}{\lambda}\sin\theta\,, \tag{3.3-14}$$

and since

$$\cos(\psi - \phi_A) = \sin\left(\psi - \phi_A + \frac{\pi}{2}\right), \tag{3.3-15}$$

substituting (3.3-14) and (3.3-15) into (3.3-13) yields

$$D(f,\theta,\psi) = \int_0^a A_r(f,r_A) \int_0^{2\pi} \exp(+jb\sin\alpha)\, d\phi_A\, r_A\, dr_A\,, \tag{3.3-16}$$

where

$$\alpha = \psi - \phi_A + \frac{\pi}{2}\,. \tag{3.3-17}$$

We next take advantage of the following identity:

$$\exp(+jb\sin\alpha) = \sum_{n=-\infty}^{\infty} J_n(b)\exp(+jn\alpha)\,, \tag{3.3-18}$$

where

$$J_n(b) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[\pm j(b\sin\beta - n\beta)]\,d\beta \tag{3.3-19}$$

or

$$J_n(b) = \frac{1}{\pi}\int_0^{\pi} \cos(b\sin\beta - n\beta)\,d\beta \tag{3.3-20}$$

is the *nth-order Bessel function of the first kind*. With the use of (3.3-17) and (3.3-18), the innermost integral in (3.3-16) can be rewritten as

$$\int_0^{2\pi} \exp(+jb\sin\alpha)\,d\phi_A = \sum_{n=-\infty}^{\infty} J_n(b)\exp\{+jn[\psi + (\pi/2)]\} \int_0^{2\pi} \exp(-jn\phi_A)\,d\phi_A\,. \tag{3.3-21}$$

Since

$$\int_0^{2\pi} \exp(-jn\phi_A)\,d\phi_A = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0, \end{cases} \tag{3.3-22}$$

(3.3-21) reduces to

$$\int_0^{2\pi} \exp(+jb\sin\alpha)\,d\phi_A = 2\pi J_0(b)\,, \tag{3.3-23}$$

and by substituting (3.3-23) into (3.3-16) and then making use of (3.3-14), we obtain

$$\boxed{D(f,\theta,\psi) = 2\pi \int_0^a A_r(f,r_A) J_0\left(\frac{2\pi r_A}{\lambda}\sin\theta\right) r_A \, dr_A} \tag{3.3-24}$$

Equation (3.3-24) is the far-field beam pattern of a circular-shaped, planar aperture lying in the XY plane with radius a meters and circularly symmetric complex frequency response given by (3.3-11). The far-field beam pattern given by (3.3-24) is *axisymmetric* because it's value does not depend on the azimuthal (bearing) angle ψ to a field point. Equation (3.3-24) can also be expressed as follows:

$$D(f,f_r) = H_0\{A_r(f,r_A)\} = 2\pi \int_0^a A_r(f,r_A) J_0(2\pi f_r r_A) r_A \, dr_A \, , \tag{3.3-25}$$

where

$$f_r = \sqrt{f_X^2 + f_Y^2} = \sin\theta/\lambda \tag{3.3-26}$$

is the spatial frequency in the horizontal (polar) radial direction with units of cycles per meter, and $H_0\{A_r(f,r_A)\}$ is the *zeroth-order Hankel transform* of $A_r(f,r_A)$, also known as the *Fourier-Bessel transform* of $A_r(f,r_A)$. Note that $k_r = 2\pi f_r$ is the propagation-vector component in the horizontal (polar) radial direction with units of radians per meter. The circular piston is considered next.

Circular Piston

Another very common example of a planar aperture lying in the XY plane (analogous to the rectangular piston discussed in Section 3.2) is a single electroacoustic transducer, circular in shape, with radius a meters (see Fig. 3.3-1). The simplest mathematical model for the complex frequency response (complex aperture function) for this circular-shaped transducer is given by

$$A(f,r_A,\phi_A) = A_r(f,r_A) = A_A \operatorname{circ}(r_A/a) \, , \tag{3.3-27}$$

where the real, positive constant A_A carries the correct units of $A(f,r_A,\phi_A)$ (see Table 3.1-1) and the circle function is defined as follows:

$$\operatorname{circ}\left(\frac{r}{a}\right) \triangleq \begin{cases} 1, & r \le a \\ 0, & r > a. \end{cases} \tag{3.3-28}$$

Therefore, according to (3.3-27), the complex frequency response of the transducer is circularly symmetric and is equal to a constant across the entire surface of the transducer, regardless of the value of frequency f. It is important to note that although A_A is shown here as a constant, it is, in general, a real, nonnegative function of frequency, that is, $A_A \to A_A(f)$. A single electroacoustic transducer, circular in shape, with complex frequency response given by (3.3-27), is referred to as a *circular piston*.

The far-field beam pattern of the circular piston can be obtained by substituting

$$A_r(f, r_A) = A_A \operatorname{circ}(r_A/a) \tag{3.3-29}$$

into (3.3-24). Doing so yields

$$D(f, \theta, \psi) = 2\pi A_A \int_0^a J_0\left(\frac{2\pi r_A}{\lambda} \sin\theta\right) r_A \, dr_A\,, \tag{3.3-30}$$

and by using the identity

$$\int_0^x J_0(\alpha)\alpha \, d\alpha = xJ_1(x)\,, \tag{3.3-31}$$

(3.3-30) reduces to

$$\boxed{D(f, \theta, \psi) = 2A_A \pi a^2 \frac{J_1\left(\dfrac{2\pi a}{\lambda}\sin\theta\right)}{\dfrac{2\pi a}{\lambda}\sin\theta}} \tag{3.3-32}$$

Equation (3.3-32) is the *unnormalized*, far-field beam pattern of a circular piston with radius a meters lying in the XY plane. It is axisymmetric because it's value does not depend on the azimuthal (bearing) angle ψ to a field point. The factor $2\pi a$ is the circumference of the circular piston in meters and πa^2 is the area in squared meters. Equation (3.3-32) can also be expressed as

$$D(f, f_r) = H_0\{A_A \operatorname{circ}(r_A/a)\} = 2A_A \pi a^2 \frac{J_1(2\pi f_r a)}{2\pi f_r a}\,, \tag{3.3-33}$$

where f_r is given by (3.3-26). Since the maximum value of the magnitude of (3.3-32) is at broadside where $\theta = 0°$, the normalization factor is given by

$$D_{\max} = |D(f, 0°, \psi)| = A_A \pi a^2\,. \tag{3.3-34}$$

Whether or not the normalization factor is equal to the magnitude of the unnormalized, far-field beam pattern evaluated at broadside, $D_{\max}$ can also be found by using the standard calculus approach for finding the maximum value of a function, or by using the formulas in Appendix 3C. Dividing (3.3-32) by (3.3-34) yields the following expression for the *normalized*, far-field beam pattern of a circular piston with radius a meters lying in the XY plane:

$$\boxed{D_N(f,\theta,\psi) = 2\frac{J_1\left(\frac{2\pi a}{\lambda}\sin\theta\right)}{\frac{2\pi a}{\lambda}\sin\theta}} \tag{3.3-35}$$

or

$$D_N(f,f_r) = 2\frac{J_1(2\pi f_r a)}{2\pi f_r a} \tag{3.3-36}$$

where f_r is given by (3.3-26). Figure 3.3-3 is a plot of the magnitude of (3.3-35) for $ka = 11$, where the level of the first sidelobe is approximately $-17.6\,\text{dB}$. Figure 3.3-4 is a plot of the magnitude of (3.3-36) for $ka = 11$, where f_r was replaced by $\sqrt{u^2+v^2}\big/\lambda$ [see (3.3-26)].

Example 3.3-1 3-dB Beamwidth of the Vertical Far-Field Beam Pattern of a Circular Piston

In this example we shall compute the 3-dB beamwidth $\Delta\theta$ of the vertical, far-field beam pattern of a circular piston lying in the XY plane using the procedure discussed in Example 2.3-3. We shall then compare this beamwidth with the 3-dB beamwidth of the vertical, far-field beam pattern of a rectangular piston that has the same area as the circular piston. If we evaluate (3.3-35) at the location of a 3-dB-down point, then

$$D_N(f,\theta_+,\psi) = 2\frac{J_1\left(\frac{2\pi a}{\lambda}\sin\theta_+\right)}{\frac{2\pi a}{\lambda}\sin\theta_+} = \frac{\sqrt{2}}{2}, \tag{3.3-37}$$

or

$$J_1(\pi x) - \frac{\sqrt{2}}{4}\pi x = 0, \tag{3.3-38}$$

where

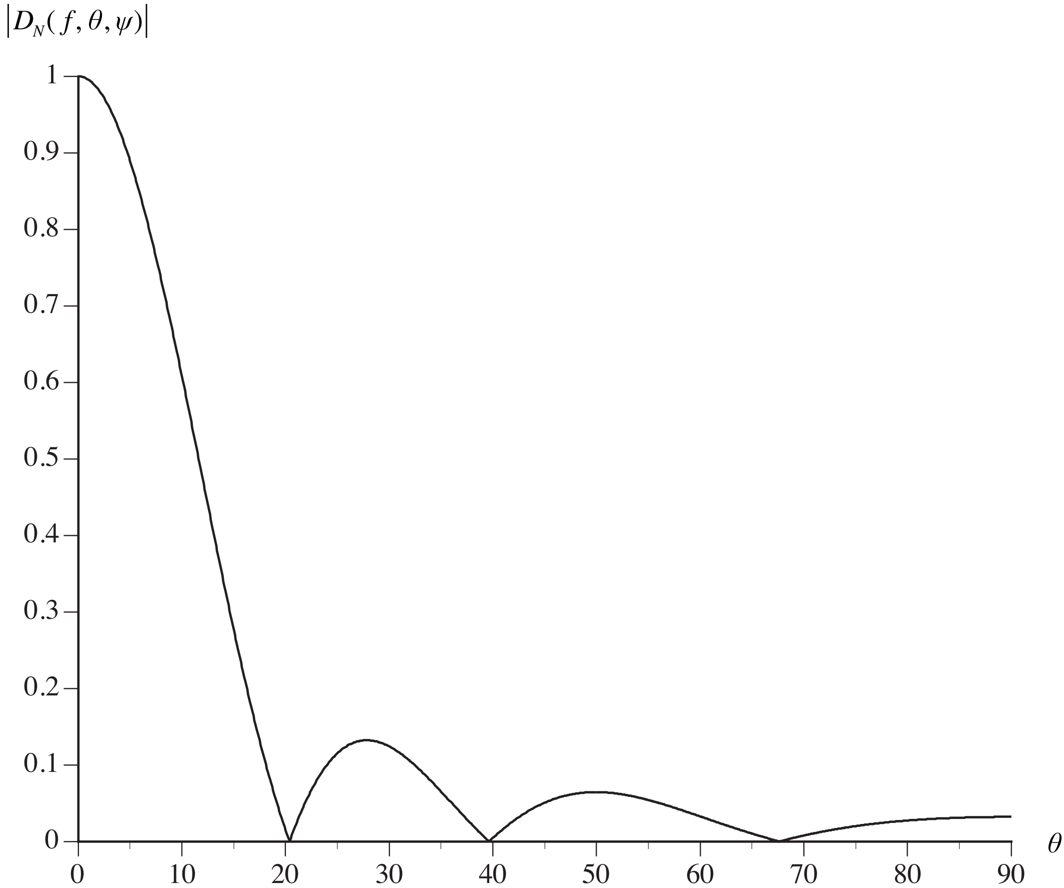

Figure 3.3-3 Magnitude of the normalized, far-field beam pattern of a circular piston lying in the XY plane given by (3.3-35) for $ka = 11$, where θ is the vertical angle in degrees.

$$x = (2a/\lambda)\sin\theta_+ . \tag{3.3-39}$$

Since

$$\theta_+ = \Delta\theta/2 , \tag{3.3-40}$$

substituting (3.3-40) into (3.3-39) yields

$$\Delta\theta = 2\sin^{-1}\left(x\frac{\lambda}{2a}\right). \tag{3.3-41}$$

The solution (root) of (3.3-38) is $x \approx 0.514$, and by substituting this result into (3.3-41), we obtain the following expression for the 3-dB beamwidth $\Delta\theta$ (in degrees) of the vertical, far-field beam pattern of a circular piston lying in the XY plane:

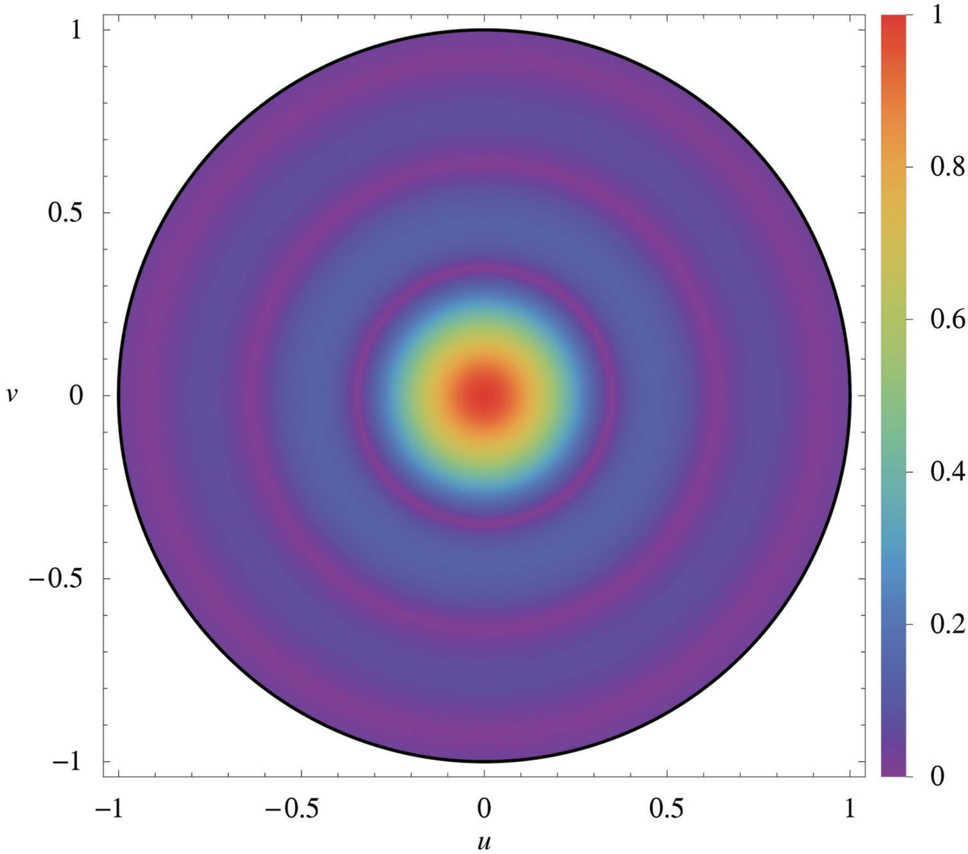

Figure 3.3-4 Magnitude of the normalized, far-field beam pattern of a circular piston lying in the XY plane given by (3.3-36) for $ka = 11$, where f_r was replaced by $\sqrt{u^2 + v^2} / \lambda$.

$$\boxed{\Delta\theta = 2\sin^{-1}(0.257\,\lambda/a)} \tag{3.3-42}$$

where a is the radius of the circular piston in meters.

Let us conclude this example by comparing (3.3-42) for a circular piston with the 3-dB beamwidth $\Delta\theta$ of the vertical, far-field beam pattern in the XZ plane of a rectangular piston lying in the XY plane given by

$$\Delta\theta = 2\sin^{-1}(0.443\,\lambda/L_X), \tag{3.3-43}$$

where L_X is the length (in meters) of the rectangular piston in the X direction (see Example 3.2-1). In order to make a fair comparison, consider rectangular and circular pistons of the same area. Therefore, if we let $L_Y = L_X$, then the area of the rectangular piston is L_X^2. If we then let

$$L_X^2 = \pi a^2 , \tag{3.3-44}$$

where πa^2 is the area of the circular piston, then

$$L_X = \sqrt{\pi}\, a , \tag{3.3-45}$$

and by substituting (3.3-45) into (3.3-43), we finally obtain

$$\Delta\theta = 2\sin^{-1}\left(0.250\,\lambda/a\right) \tag{3.3-46}$$

for the rectangular piston. By comparing (3.3-42) for a circular piston with (3.3-46) for a rectangular piston with the same area as a circular piston, it can be seen that a circular piston will have a slightly larger 3-dB beamwidth. However, recall that the levels of the first sidelobes of the normalized, far-field beam patterns of a circular piston and a rectangular amplitude window are approximately $-17.6\,\text{dB}$ and $-13.3\,\text{dB}$, respectively. The slight increase in beamwidth is offset by an additional, approximate, $4.3\,\text{dB}$ sidelobe suppression provided by a circular piston. ■

3.4 Beam Steering

Following the same procedure used in Section 2.4, but generalizing it for the planar aperture problem, let $D(f, f_X, f_Y)$ be the far-field beam pattern of the amplitude window $a(f, x_A, y_A)$:

$$\begin{aligned} D(f, f_X, f_Y) &= F_{x_A} F_{y_A}\left\{a(f, x_A, y_A)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\, dy_A . \end{aligned} \tag{3.4-1}$$

Let $D'(f, f_X, f_Y)$ be the far-field beam pattern of the aperture function $A(f, x_A, y_A)$ given by (3.1-10):

$$\begin{aligned} D'(f, f_X, f_Y) &= F_{x_A} F_{y_A}\left\{a(f, x_A, y_A)\exp[+j\theta(f, x_A, y_A)]\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\exp[+j\theta(f, x_A, y_A)]\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\, dy_A . \end{aligned} \tag{3.4-2}$$

Now let the phase response across the surface of the aperture be a linear function of both x_A and y_A, that is, let

$$\theta(f, x_A, y_A) = -2\pi f_X' x_A - 2\pi f_Y' y_A \,, \tag{3.4-3}$$

where

$$f_X' = u'/\lambda = \sin\theta' \cos\psi'/\lambda \tag{3.4-4}$$

and

$$f_Y' = v'/\lambda = \sin\theta' \sin\psi'/\lambda \,. \tag{3.4-5}$$

If (3.4-3) is substituted into (3.4-2), then

$$\begin{aligned} D'(f, f_X, f_Y) &= F_{x_A} F_{y_A} \left\{ a(f, x_A, y_A) \exp\left[-j2\pi(f_X' x_A + f_Y' y_A)\right] \right\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(f, x_A, y_A) \exp\left\{+j2\pi\left[(f_X - f_X')x_A + (f_Y - f_Y')y_A\right]\right\} dx_A \, dy_A \,, \end{aligned} \tag{3.4-6}$$

and by comparing (3.4-6) with (3.4-1),

$$D'(f, f_X, f_Y) = D(f, f_X - f_X', f_Y - f_Y') \,, \tag{3.4-7}$$

or

$$F_{x_A} F_{y_A} \left\{ a(f, x_A, y_A) \exp\left[-j2\pi(f_X' x_A + f_Y' y_A)\right] \right\} = D(f, f_X - f_X', f_Y - f_Y') \,, \tag{3.4-8}$$

where $D(f, f_X, f_Y)$ is given by (3.4-1). Equation (3.4-7) can also be expressed as

$$D'(f, u, v) = D(f, u - u', v - v') \,, \tag{3.4-9}$$

since $D(f, f_X, f_Y) = D(f, u/\lambda, v/\lambda) \rightarrow D(f, u, v)$. Therefore, a linear phase response across the surface of the aperture in both the X and Y directions will cause the beam pattern $D(f, u, v)$ to be steered in the direction $u = u'$ and $v = v'$ in direction-cosine space, which is equivalent to steering (tilting) the beam pattern to $\theta = \theta'$ and $\psi = \psi'$. Equation (3.4-9) indicates that the far-field beam pattern $D'(f, u, v)$ is simply a translated or shifted version of $D(f, u, v)$ in the uv plane, implying that the shape and beamwidth of the mainlobe of the beam pattern remain unchanged (see Fig. 3.4-1). However, recall from linear aperture theory, that when a beam pattern is steered, the shape and beamwidth of the mainlobe of the beam pattern *do change* when the magnitude of the beam pattern is plotted as a function of the spherical angles θ and ψ (see Section 2.5).

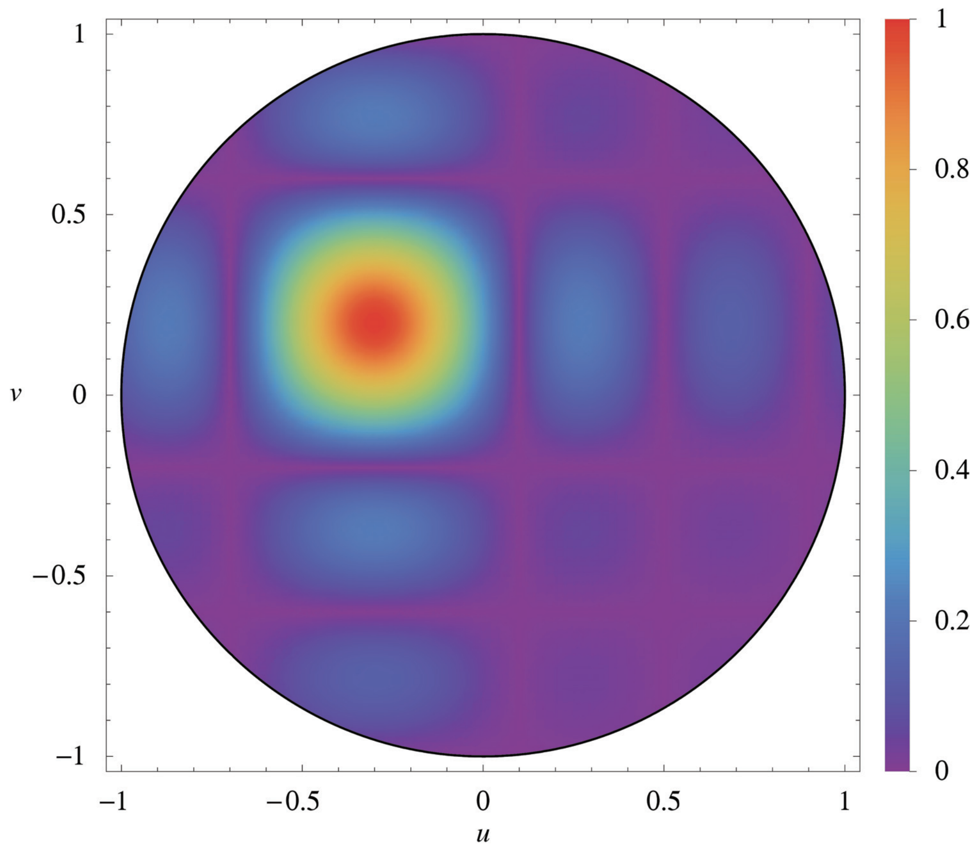

Figure 3.4-1 Magnitude of the normalized, far-field beam pattern of a rectangular piston lying in the XY plane, steered in the direction $u = u' = -0.3$ and $v = v' = 0.2$ in direction-cosine space, for $L_Y = L_X = L$ and $L/\lambda = 2.5$.

3.5 The Near-Field Beam Pattern of a Planar Aperture

In this section we shall demonstrate that the near-field beam pattern of a planar aperture is just a special case of the near-field beam pattern of a volume aperture. In Section 1.2 it was shown that the near-field beam pattern (directivity function) of a closed-surface, volume aperture – based on the Fresnel approximation of a time-independent, free-space, Green's function – is given by the following three-dimensional spatial Fresnel transform:

$$\mathcal{D}(f, r, \boldsymbol{\alpha}) = F_{\mathbf{r}_A}\left\{A(f, \mathbf{r}_A)\exp\left(-j\frac{k}{2r}r_A^2\right)\right\}$$
$$= \int_{-\infty}^{\infty} A(f, \mathbf{r}_A)\exp\left(-j\frac{k}{2r}r_A^2\right)\exp(+j2\pi\boldsymbol{\alpha}\bullet\mathbf{r}_A)\,d\mathbf{r}_A, \tag{3.5-1}$$

where

$$\mathbf{r}_A = (x_A, y_A, z_A)\,, \tag{3.5-2}$$

$$k = 2\pi f / c = 2\pi / \lambda\,, \tag{3.5-3}$$

$$r_A^2 = \left|\mathbf{r}_A\right|^2 = x_A^2 + y_A^2 + z_A^2\,, \tag{3.5-4}$$

$$r = \left|\mathbf{r}\right| = \sqrt{x^2 + y^2 + z^2}\,, \tag{3.5-5}$$

$$\boldsymbol{\alpha} = (f_X, f_Y, f_Z)\,, \tag{3.5-6}$$

$$\boldsymbol{\alpha} \bullet \mathbf{r}_A = f_X x_A + f_Y y_A + f_Z z_A\,, \tag{3.5-7}$$

$$d\mathbf{r}_A = dx_A\, dy_A\, dz_A\,, \tag{3.5-8}$$

and $F_{\mathbf{r}_A}\{\bullet\}$ is shorthand notation for $F_{x_A} F_{y_A} F_{z_A}\{\bullet\}$. The complex aperture function $A(f, \mathbf{r}_A)$ can be expressed as

$$A(f, \mathbf{r}_A) = a(f, \mathbf{r}_A)\exp[+j\theta(f, \mathbf{r}_A)]\,, \tag{3.5-9}$$

where $a(f, \mathbf{r}_A)$ is the *amplitude response* and $\theta(f, \mathbf{r}_A)$ is the *phase response* of the aperture at $\mathbf{r}_A$. Both $a(f, \mathbf{r}_A)$ and $\theta(f, \mathbf{r}_A)$ are *real* functions. The function $a(f, \mathbf{r}_A)$ is also known as the *amplitude window*. The units of $a(f, \mathbf{r}_A)$ and, hence, $A(f, \mathbf{r}_A)$ depend on whether the aperture is a transmit aperture (see Table 1B-1 in Appendix 1B) or a receive aperture (see Table 1B-2 in Appendix 1B). The units of $\theta(f, \mathbf{r}_A)$ are radians.

Now consider the case of a *planar aperture* of arbitrary shape lying in the XY plane as shown in Fig. 3.1-1. As was mentioned earlier, a planar aperture can represent either a single electroacoustic transducer (e.g., a rectangular or circular piston) or a planar array of many individual electroacoustic transducers. Also shown in Fig. 3.1-1 is a field point with spherical coordinates (r, θ, ψ). The field point is in the Fresnel region of the aperture, where the Fresnel angle criterion given by either (1.2-35) or (1.2-91) is satisfied, and the range r satisfies the Fresnel range criterion given by $1.356 R_A < r < \pi R_A^2 / \lambda$ [see either (1.2-40) or (1.2-92)], where R_A is the maximum radial extent of the planar aperture.

Since the planar aperture in Fig. 3.1-1 is lying in the XY plane, the position vector to a point on the surface of the aperture is given by

$$\mathbf{r}_A = (x_A, y_A, 0)\,, \tag{3.5-10}$$

and as a result,

$$A(f,\mathbf{r}_A) = A(f,x_A,y_A,z_A) \to A(f,x_A,y_A)\delta(z_A)\,. \tag{3.5-11}$$

Therefore, by substituting (3.5-11) into (3.5-1) and using the sifting property of impulse functions, we obtain the following two-dimensional spatial Fresnel transform for the near-field beam pattern of a planar aperture lying in the XY plane:

$$\boxed{\begin{aligned}\mathcal{D}(f,r,f_X,f_Y) &= F_{x_A}F_{y_A}\left\{A(f,x_A,y_A)\exp\left[-j\frac{k}{2r}(x_A^2+y_A^2)\right]\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f,x_A,y_A)\exp\left[-j\frac{k}{2r}(x_A^2+y_A^2)\right]\times \\ &\qquad \exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\,dy_A\end{aligned}} \tag{3.5-12}$$

where

$$A(f,x_A,y_A) = a(f,x_A,y_A)\exp[+j\theta(f,x_A,y_A)] \tag{3.5-13}$$

is the complex frequency response (complex aperture function) of the planar aperture,

$$k = 2\pi f/c = 2\pi/\lambda \tag{3.5-14}$$

is the wavenumber in radians per meter, and

$$f_X = u/\lambda = \sin\theta\cos\psi/\lambda \tag{3.5-15}$$

and

$$f_Y = v/\lambda = \sin\theta\sin\psi/\lambda \tag{3.5-16}$$

are spatial frequencies in the X and Y directions, respectively, with units of cycles per meter [see (1.2-43) and (1.2-44)]. Since f_X and f_Y can be expressed in terms of the spherical angles θ and ψ, the near-field beam pattern can ultimately be expressed as a function of frequency f, the range r to a field point, and the spherical angles θ and ψ, that is, $\mathcal{D}(f,r,f_X,f_Y) \to \mathcal{D}(f,r,\theta,\psi)$. If the planar aperture is a transmit aperture, then the aperture function $A(f,x_A,y_A)$ has units of $\left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}\right)/\mathrm{m}^2$ and the near-field beam pattern $\mathcal{D}(f,r,f_X,f_Y)$ has units of $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$. However, if the planar aperture is a receive aperture, then $A(f,x_A,y_A)$ has units of $\left(\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)\right)/\mathrm{m}^2$ and $\mathcal{D}(f,r,f_X,f_Y)$ has units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$. These units are summarized in Table 3.5-1.

Table 3.5-1 Units of the Complex Aperture Function $A(f, x_A, y_A)$ and Corresponding Near-Field Beam Pattern $\mathcal{D}(f, r, f_X, f_Y)$ for a Planar Aperture

Planar Aperture	$A(f, x_A, y_A)$	$\mathcal{D}(f, r, f_X, f_Y)$
transmit	$\left(\left(\text{m}^3/\text{sec}\right)/\text{V}\right)/\text{m}^2$	$\left(\text{m}^3/\text{sec}\right)/\text{V}$
receive	$\left(\text{V}/\left(\text{m}^2/\text{sec}\right)\right)/\text{m}^2$	$\text{V}/\left(\text{m}^2/\text{sec}\right)$

Equations (3.5-1) and (3.5-12) are most accurate in the Fresnel region of an aperture where both the Fresnel angle criterion and the Fresnel range criterion are satisfied. They are less accurate in the near-field region outside the Fresnel region where $r < r_{\text{NF/FF}}$, but the Fresnel angle criterion is *not* satisfied. As was discussed in Subsection 1.2.1, the Fresnel region is only a subset of the near-field.

Finally, if a planar aperture lies in the XZ plane instead of the XY plane, then simply replace y_A and f_Y with z_A and f_Z, respectively, in (3.5-12), where spatial frequency f_Z is given by (1.2-45). And if a planar aperture lies in the YZ plane instead of the XY plane, then simply replace x_A and f_X with z_A and f_Z, respectively, in (3.5-12).

3.5.1 Beam Steering and Aperture Focusing

Following the same procedure used in Subsection 2.6.2, but generalizing it for the planar aperture problem, let the phase response across the surface of the aperture be equal to the sum of linear and quadratic functions of both x_A and y_A, that is, let

$$\theta(f, x_A, y_A) = -2\pi f_X' x_A - 2\pi f_Y' y_A + \frac{k}{2r'}(x_A^2 + y_A^2), \tag{3.5-17}$$

where

$$f_X' = u'/\lambda = \sin\theta' \cos\psi'/\lambda, \tag{3.5-18}$$

$$f_Y' = v'/\lambda = \sin\theta' \sin\psi'/\lambda, \tag{3.5-19}$$

and

$$k = 2\pi f/c = 2\pi/\lambda. \tag{3.5-20}$$

The parameter r' in (3.5-17) is referred to as the *focal range*. It is the near-field range from the aperture where the far-field beam pattern of the aperture will be in focus. Note that the third term on the right-hand side of (3.5-17) is a paraboloid that passes through the origin.

If (3.5-13) and (3.5-17) are substituted into (3.5-12), then

$$\mathcal{D}(f,r,f_X,f_Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f,x_A,y_A)\exp\left[-j\frac{k}{2}\left(\frac{1}{r}-\frac{1}{r'}\right)(x_A^2+y_A^2)\right]\times$$
$$\exp\left\{+j2\pi\left[(f_X-f_X')x_A+(f_Y-f_Y')y_A\right]\right\}dx_A\,dy_A, \tag{3.5-21}$$

and if (3.5-21) is evaluated at the near-field range $r=r'$, then it reduces to

$$\mathcal{D}(f,r',f_X,f_Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f,x_A,y_A)\exp\left\{+j2\pi\left[(f_X-f_X')x_A+(f_Y-f_Y')y_A\right]\right\}dx_A\,dy_A \tag{3.5-22}$$

or

$$\begin{aligned}\mathcal{D}(f,r',f_X,f_Y)&=F_{x_A}F_{y_A}\left\{a(f,x_A,y_A)\exp\left[-j2\pi(f_X'x_A+f_Y'y_A)\right]\right\}\\&=D(f,f_X-f_X',f_Y-f_Y').\end{aligned} \tag{3.5-23}$$

Equation (3.5-23) indicates that the far-field beam pattern $D(f,f_X,f_Y)$ of the amplitude response (amplitude window) $a(f,x_A,y_A)$ is *in focus* at the near-field range $r=r'$ meters from the aperture, and has been *steered* in the direction $u=u'$ and $v=v'$ in direction-cosine space, which is equivalent to steering (tilting) the beam pattern to $\theta=\theta'$ and $\psi=\psi'$ [see (3.5-18) and (3.5-19)]. However, in the near-field region of the aperture outside the Fresnel region, a quadratic phase response can only approximately focus the far-field beam pattern.

Problems

Section 3.1

3-1 Find both the unnormalized and normalized, far-field beam patterns of the following aperture function, where $A_A(f)$ is a real, nonnegative function of frequency:

$$A(f,x_A,y_A)=A_A(f)\cos\left(\frac{\pi x_A}{L_X}\right)\operatorname{rect}\left(\frac{x_A}{L_X}\right)\cos\left(\frac{\pi y_A}{L_Y}\right)\operatorname{rect}\left(\frac{y_A}{L_Y}\right).$$

Section 3.1 Appendix 3A

3-2 If the aperture function given in Problem 3-1 is a *transmit* aperture function, then find the corresponding transmitter sensitivity function.

Section 3.2 Appendix 3A

3-3 Find the transmitter sensitivity function for the following complex frequency response of a rectangular piston, where $A_A(f)$ is a real, nonnegative function of frequency:

$$A_T(f, x_A, y_A) = A_A(f)\operatorname{rect}(x_A/L_X)\operatorname{rect}(y_A/L_Y).$$

Section 3.2

3-4 Find the spherical angles θ and ψ from the following (u,v) coordinates:

(a) $u = -0.3$ and $v = 0.2$

(b) $u = 0.5$ and $v = -0.5$

(c) $u = 0.1$ and $v = 0.4$

(d) $u = -0.2$ and $v = -0.7$

3-5 Consider a rectangular-shaped, planar array of electroacoustic transducers (an example of a rectangular-shaped, planar aperture) lying in the XY plane. If the operating frequency is $31\,\text{kHz}$ and the aperture is modeled as a rectangular piston, then what should the lengths of the aperture be in the X and Y directions if the 3-dB beamwidths of the vertical, far-field beam patterns in the XZ and YZ planes are required to be $1°$ and $25°$, respectively.

Section 3.3

3-6 Verify (3.3-22).

3-7 Verify (3.3-32).

3-8 Verify (3.3-34). Note:

$$J_0(0) = 1,$$

$$J_n(0) = 0, \qquad n \neq 0,$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x),$$

and

$$J_n'(x) = \frac{d}{dx} J_n(x) .$$

Section 3.3 Appendix 3A

3-9 Find the transmitter sensitivity function for the following complex frequency response of a circular piston, where $A_A(f)$ is a real, nonnegative function of frequency:

$$A_T(f, r_A, \phi_A) = A_A(f)\,\text{circ}(r_A/a) .$$

3-10 If the transmitter sensitivity level (transmitting voltage response) of a circular piston at 14 kHz is $150\text{ dB re }1\,\mu\text{Pa/V}$ at 1 m, then what is the value of the magnitude of the transmitter sensitivity function in $\left(\text{m}^3/\text{sec}\right)/\text{V}$? Use $\rho_0 = 1026\text{ kg/m}^3$ for the ambient density of seawater. The radius a of the circular piston is less than 0.316 m or 12.4 in.

3-11 If the receiver sensitivity level (open circuit receiving response) of a circular piston at 14 kHz is $-170\text{ dB re }1\,\text{V}/\mu\text{Pa}$, then what is the value of the magnitude of the receiver sensitivity function in $\text{V}/\left(\text{m}^2/\text{sec}\right)$? Use $\rho_0 = 1026\text{ kg/m}^3$ for the ambient density of seawater.

Section 3.3

3-12 Consider a thin, planar electroacoustic transducer, circular in shape with radius b meters, lying in the XY plane. The complex frequency response (complex aperture function) of the transducer is circularly symmetric and is given by

$$A_r(f, r_A) = \begin{cases} +A_A, & 0 \le r_A < a, \\ -A_A, & a \le r_A \le b, \end{cases}$$

where A_A is a real, positive constant and $b > a$. Find the unnormalized, far-field beam pattern of this transducer.

3-13 Consider a rectangular and circular piston of *equal areas* lying in the XY plane. With $L_Y = L_X$ and for $a/\lambda = 0.5$, 1, 2, and 4, compute the 3-dB beamwidths in degrees of the vertical, far-field beam patterns of both pistons in the XZ plane.

Section 3.4

3-14 Repeat Problem 3-1 using the following modified aperture function, where $A_A(f)$ is a real, nonnegative function of frequency:

$$A(f,x_A,y_A) = A_A(f)\cos\left(\frac{\pi x_A}{L_X}\right)\text{rect}\left(\frac{x_A}{L_X}\right)\exp(-j2\pi f_X' x_A)\times$$
$$\cos\left(\frac{\pi y_A}{L_Y}\right)\text{rect}\left(\frac{y_A}{L_Y}\right)\exp(-j2\pi f_Y' y_A)$$

Section 3.5

3-15 The spherical coordinates of a sound source (target) as measured from the center of a rectangular-shaped, planar array of electroacoustic transducers (an example of a rectangular-shaped, planar aperture) lying in the XY plane are $r_S = 210$ m, $\theta_S = 60°$, and $\psi_S = 105°$. If $L_Y = L_X = 3$ m, $f = 25$ kHz, and $c = 1500$ m/sec,

(a) is the sound source in the aperture's Fresnel region or far-field?

(b) What must the phase response across the surface of the aperture be if the aperture's far-field beam pattern is to be focused (if required) and steered to coordinates (r_S, θ_S, ψ_S)?

Appendix 3A Transmitter and Receiver Sensitivity Functions of a Planar Transducer

Transmitter Sensitivity Function

For the case of a planar electroacoustic transducer lying in the XY plane – such as a rectangular or circular piston – that is being used as a transmitter, the complex, *transmitter sensitivity function* $\mathcal{S}_T(f)$ is related to the complex, transmit frequency response $A_T(f,x_A,y_A)$ of the transducer as follows (see Example 3B-1 in Appendix 3B):

$$\boxed{\mathcal{S}_T(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f,x_A,y_A)\,dx_A\,dy_A} \qquad (3A\text{-}1)$$

where $\mathcal{S}_T(f)$ has units of $\left(\text{m}^3/\text{sec}\right)/\text{V}$ and $A_T(f,x_A,y_A)$ has units of

$\left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}\right)/\mathrm{m}^2$ (see Table 3.1-1). However, manufacturers of electroacoustic transducers usually describe the *transmitting voltage response* of a transducer by plotting the *transmitter sensitivity level* $\mathrm{TSL}(f)$ in $\mathrm{dB\,re\,TS_{ref}}$ at $1\,\mathrm{m}$ versus frequency f in hertz. The notation "$\mathrm{dB\,re\,TS_{ref}}$" means decibels relative to a reference transmitter sensitivity. In underwater acoustics, the reference transmitter sensitivity $\mathrm{TS_{ref}}$ is usually $1\,\mu\mathrm{Pa/V}$. The corresponding *transmitter sensitivity* $\mathrm{TS}(f)$ with units of $\mathrm{Pa/V}$ is given by

$$\boxed{\mathrm{TS}(f) = \mathrm{TS_{ref}}\,10^{[\mathrm{TSL}(f)/20]}} \tag{3A-2}$$

The transmitter sensitivity $\mathrm{TS}(f)$ is a *real*, *nonnegative* function of frequency.

Values for the magnitude of the transmitter sensitivity function, $|\mathcal{S}_T(f)|$ in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$, for a planar electroacoustic transducer lying in the XY plane and centered at the origin of the coordinate system as shown in Figures 3.2-1 and 3.3-1, can be obtained from measured values of the transmitter sensitivity, $\mathrm{TS}(f)$ in $\mathrm{Pa/V}$, by using the following formula (see Example 3B-1 in Appendix 3B):

$$\boxed{|\mathcal{S}_T(f)| = \frac{2z}{f\rho_0}\mathrm{TS}(f)\Big|_{z=1\,\mathrm{m}}, \qquad R_A < 0.316\,\mathrm{m}} \tag{3A-3}$$

where the field point is located at coordinates $(0,0,1\,\mathrm{m})$ and is at *broadside* relative to the transducer, ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter, $\mathrm{TS}(f)$ is given by (3A-2), and R_A is the maximum *radial* extent of the transducer in meters. If the transducer is rectangular in shape, then R_A is given by (3B-28). If the transducer is circular in shape, then R_A is equal to the radius of the transducer. Note that (3A-3) is valid only if R_A is less than $0.316\,\mathrm{m}$ or $12.4\,\mathrm{in}$.

Receiver Sensitivity Function

For the case of a planar electroacoustic transducer lying in the XY plane – such as a rectangular or circular piston – that is being used as a receiver, the complex, *receiver sensitivity function* $\mathcal{S}_R(f)$ is related to the complex, receive frequency response $A_R(f, x_A, y_A)$ of the transducer as follows:

$$\boxed{\mathcal{S}_R(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_R(f, x_A, y_A)\,dx_A\,dy_A} \tag{3A-4}$$

where $\mathcal{S}_R(f)$ has units of $\mathrm{V}/\left(\mathrm{m}^2/\sec\right)$ and $A_R(f, x_A, y_A)$ has units of $\left(\mathrm{V}/\left(\mathrm{m}^2/\sec\right)\right)/\mathrm{m}^2$ (see Table 3.1-1). However, manufacturers of electroacoustic transducers usually describe the *open circuit receiving response* of a transducer by plotting the *receiver sensitivity level* $\mathrm{RSL}(f)$ in $\mathrm{dB\ re\ RS_{ref}}$ versus frequency f in hertz. The notation "$\mathrm{dB\ re\ RS_{ref}}$" means decibels relative to a reference receiver sensitivity. In underwater acoustics, the reference receiver sensitivity $\mathrm{RS_{ref}}$ is usually $1\,\mathrm{V}/\mu\mathrm{Pa}$. The corresponding *receiver sensitivity* $\mathrm{RS}(f)$ with units of V/Pa is given by

$$\boxed{\mathrm{RS}(f) = \mathrm{RS_{ref}}\, 10^{[\mathrm{RSL}(f)/20]}} \tag{3A-5}$$

The receiver sensitivity $\mathrm{RS}(f)$ is a *real*, *nonnegative* function of frequency. What we need to do next is to derive the conversion factor that will allow us to convert $\mathrm{m}^2/\sec$ to Pa for a time-harmonic acoustic field.

The derivation of the desired conversion factor in this case is simple. By computing the magnitude of (3B-24), we obtain

$$\left|p_f(\mathbf{r})\right| = 2\pi f \rho_0 \left|\varphi_f(\mathbf{r})\right|, \tag{3A-6}$$

from which we obtain the conversion factor

$$\boxed{\frac{\left|p_f(\mathbf{r})\right|}{\left|\varphi_f(\mathbf{r})\right|} = 2\pi f \rho_0 \, \frac{\mathrm{Pa}}{\mathrm{m}^2/\sec}} \tag{3A-7}$$

where $p_f(\mathbf{r}) \equiv p_f(x, y, z)$ and $\varphi_f(\mathbf{r}) \equiv \varphi_f(x, y, z)$. The conversion factor on the right-hand-side of (3A-7) is used to convert velocity potential in $\mathrm{m}^2/\sec$ to acoustic pressure in Pa for a time-harmonic acoustic field. Therefore, values for the magnitude of the receiver sensitivity function, $\left|\mathcal{S}_R(f)\right|$ in $\mathrm{V}/\left(\mathrm{m}^2/\sec\right)$, can be obtained from measured values of the receiver sensitivity, $\mathrm{RS}(f)$ in V/Pa, by using the following formula:

$$\boxed{\left|\mathcal{S}_R(f)\right| = 2\pi f \rho_0 \mathrm{RS}(f)} \tag{3A-8}$$

where ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter and $\mathrm{RS}(f)$ is given by (3A-5).

Appendix 3B Radiation from a Planar Aperture

An exact solution of the linear wave equation given by (1.2-1) for free-space propagation in an ideal (nonviscous), homogeneous, fluid medium is given by [see (1.2-3)]

$$\varphi(t,\mathbf{r}) = \int_{-\infty}^{\infty} \int_{V_0} X(f,\mathbf{r}_0) A_T(f,\mathbf{r}_0) g_f\left(\mathbf{r} \,|\, \mathbf{r}_0\right) dV_0 \exp(+j2\pi f t)\, df \;, \tag{3B-1}$$

where $\varphi(t,\mathbf{r})$ is the scalar velocity potential in squared meters per second, $X(f,\mathbf{r}_0)$ is the complex frequency spectrum of the input electrical signal at location $\mathbf{r}_0$ of the transmit aperture with units of volts per hertz, $A_T(f,\mathbf{r}_0)$ is the complex frequency response of the transmit aperture at $\mathbf{r}_0$ with units of $\left(\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}\right)/\mathrm{m}^3$,

$$g_f\left(\mathbf{r} \,|\, \mathbf{r}_0\right) = -\frac{\exp\left(-jk\left|\mathbf{r}-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}-\mathbf{r}_0\right|} = -\frac{\exp(-jkR)}{4\pi R} \tag{3B-2}$$

is the time-independent, free-space, Green's function of an unbounded, ideal (nonviscous), homogeneous, fluid medium with units of inverse meters,

$$k = 2\pi f/c = 2\pi/\lambda \tag{3B-3}$$

is the wavenumber in radians per meter, $c = f\lambda$ is the constant speed of sound in the fluid medium in meters per second, λ is the wavelength in meters,

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{3B-4}$$

is the position vector to a field point,

$$\mathbf{r}_0 = x_0\hat{x} + y_0\hat{y} + z_0\hat{z} \tag{3B-5}$$

is the position vector to a source point, and

$$R = \left|\mathbf{r}-\mathbf{r}_0\right| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \tag{3B-6}$$

is the range in meters between a source point and a field point. If an identical input electrical signal is applied at all locations $\mathbf{r}_0$ of the transmit aperture, then (see Example 1.2-1)

$$x(t, \mathbf{r}_0) = x(t) \tag{3B-7}$$

and

$$X(f, \mathbf{r}_0) = X(f), \tag{3B-8}$$

where $X(f)$ is the complex frequency spectrum of $x(t)$. Substituting (3B-8) into (3B-1) yields

$$\varphi(t, \mathbf{r}) = \int_{-\infty}^{\infty} X(f) \int_{V_0} A_T(f, \mathbf{r}_0) g_f\left(\mathbf{r} \mid \mathbf{r}_0\right) dV_0 \exp(+j2\pi f t) df \,. \tag{3B-9}$$

For the case of a planar aperture lying in the XY plane,

$$\mathbf{r}_0 = (x_0, y_0, 0)\,, \tag{3B-10}$$

and as a result,

$$A_T(f, \mathbf{r}_0) = A_T(f, x_0, y_0, z_0) \rightarrow A_T(f, x_0, y_0) \delta(z_0)\,. \tag{3B-11}$$

Substituting (3B-11) into (3B-9) yields

$$\boxed{\varphi(t, \mathbf{r}) = \int_{-\infty}^{\infty} X(f) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_T(f, x_0, y_0) g_f\left(\mathbf{r} \mid \mathbf{r}_0\right) dx_0 \, dy_0 \exp(+j2\pi f t) df} \tag{3B-12}$$

where $g_f\left(\mathbf{r} \mid \mathbf{r}_0\right)$ is given by (3B-2) and

$$R = \left|\mathbf{r} - \mathbf{r}_0\right| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}\,. \tag{3B-13}$$

Equation (3B-12) is a general expression for the scalar velocity potential of the acoustic field radiated by a planar aperture lying in the XY plane. The corresponding solution for the radiated acoustic pressure $p(t, \mathbf{r})$ in pascals can be obtained by substituting (3B-12) into (1.2-8). Doing so yields

$$\boxed{p(t, \mathbf{r}) = -j2\pi\rho_0 \int_{-\infty}^{\infty} f X(f) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_T(f, x_0, y_0) g_f\left(\mathbf{r} \mid \mathbf{r}_0\right) dx_0 \, dy_0 \exp(+j2\pi f t) df} \tag{3B-14}$$

where ρ_0 is the constant ambient (equilibrium) density of the fluid medium in kilograms per cubic meter, and $g_f\left(\mathbf{r} \mid \mathbf{r}_0\right)$ is given by (3B-2) and (3B-13).

If the input electrical signal is time-harmonic, that is, if (see Example 1.2-1)

$$x(t) = A_x \exp(+j2\pi f_0 t), \tag{3B-15}$$

where A_x is the complex amplitude with units of volts, then

$$X(f) = A_x \delta(f - f_0), \tag{3B-16}$$

where the impulse function $\delta(f - f_0)$ has units of inverse hertz. Substituting (3B-16) into (3B-12) yields

$$\varphi(t,\mathbf{r}) = A_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f_0, x_0, y_0)\, g_{f_0}\left(\mathbf{r}\,|\,\mathbf{r}_0\right) dx_0\, dy_0 \exp(+j2\pi f_0 t), \tag{3B-17}$$

and by replacing f_0 with f,

$$\boxed{\varphi(t,\mathbf{r}) = A_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f, x_0, y_0)\, g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) dx_0\, dy_0 \exp(+j2\pi f t)} \tag{3B-18}$$

or

$$\varphi(t,\mathbf{r}) = \varphi_f(\mathbf{r}) \exp(+j2\pi f t), \tag{3B-19}$$

where

$$\varphi_f(\mathbf{r}) = A_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f, x_0, y_0)\, g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) dx_0\, dy_0 \tag{3B-20}$$

and $g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)$ is given by (3B-2) and (3B-13). Equation (3B-18) is the time-harmonic, scalar velocity potential of the acoustic field radiated by a planar aperture lying in the XY plane. The corresponding solution for the time-harmonic, radiated acoustic pressure $p(t,\mathbf{r})$ in pascals is given by

$$\boxed{p(t,\mathbf{r}) = -j2\pi f \rho_0 A_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f, x_0, y_0)\, g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) dx_0\, dy_0 \exp(+j2\pi f t)} \tag{3B-21}$$

or

$$p(t,\mathbf{r}) = p_f(\mathbf{r}) \exp(+j2\pi f t), \tag{3B-22}$$

where

$$p_f(\mathbf{r}) = -j2\pi f \rho_0 A_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f, x_0, y_0)\, g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) dx_0\, dy_0 \tag{3B-23}$$

and $g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)$ is given by (3B-2) and (3B-13). Note that

$$p_f(\mathbf{r}) = -j2\pi f \rho_0 \varphi_f(\mathbf{r}), \tag{3B-24}$$

where $\varphi_f(\mathbf{r})$ is given by (3B-20).

Example 3B-1 Transmitter Sensitivity Function and Source Strength of a Planar Transducer

Consider the case of a single electroacoustic transducer, rectangular in shape (an example of a planar aperture), with sides equal to L_X and L_Y meters in length, lying in the XY plane and centered at the origin of the coordinate system as shown in Fig. 3.2-1. The transducer extends from $-L_X/2$ to $L_X/2$ along the X axis, and from $-L_Y/2$ to $L_Y/2$ along the Y axis. Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f, x_0, y_0)\, g_f\big(\mathbf{r}\,|\,\mathbf{r}_0\big)\,dx_0\,dy_0 = \int_{-L_X/2}^{L_X/2}\int_{-L_Y/2}^{L_Y/2} A_T(f, x_0, y_0)\, g_f\big(\mathbf{r}\,|\,\mathbf{r}_0\big)\,dx_0\,dy_0\,, \tag{3B-25}$$

where $g_f\big(\mathbf{r}\,|\,\mathbf{r}_0\big)$ is given by (3B-2) and (3B-13). If the radiated acoustic field is measured at *broadside* at $\mathbf{r} = (0, 0, z)$, where $z \neq 0$, then (3B-13) reduces to

$$R = \left|\mathbf{r} - \mathbf{r}_0\right| = z\sqrt{1 + \left(r_0/z\right)^2}\,, \tag{3B-26}$$

where

$$r_0 = \sqrt{x_0^2 + y_0^2} \tag{3B-27}$$

is the range (polar radius) in meters to a source point in the XY plane. The maximum value of r_0, which is the maximum radial extent of the aperture R_A, is given by

$$R_A = \max r_0 = \sqrt{\left(\frac{L_X}{2}\right)^2 + \left(\frac{L_Y}{2}\right)^2}\,, \tag{3B-28}$$

and is obtained when x_0 and y_0 are equal to either the upper or lower limits of integration $\pm L_X/2$ and $\pm L_Y/2$, respectively. Setting r_0 equal to R_A in (3B-26) yields

$$R = \left|\mathbf{r} - \mathbf{r}_0\right| = z\sqrt{1 + \left(R_A/z\right)^2}\,, \tag{3B-29}$$

and if

$$\left(R_A/z\right)^2 < 0.1\,, \tag{3B-30}$$

or

$$R_A < 0.316\,z\,, \tag{3B-31}$$

then

$$R = \left|\mathbf{r} - \mathbf{r}_0\right| \approx z\,, \qquad R_A < 0.316\,z\,, \tag{3B-32}$$

for $\mathbf{r} = (0,0,z)$ and $\mathbf{r}_0 = (x_0, y_0, 0)$, where $\left|x_0\right| \le L_X/2$ and $\left|y_0\right| \le L_Y/2$. Therefore, substituting (3B-32) into (3B-2) yields

$$g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right) = g_f\left(0,0,z\,|\,x_0,y_0,0\right) \approx -\frac{\exp(-jkz)}{4\pi z}, \qquad R_A < 0.316\,z\,, \tag{3B-33}$$

and substituting (3B-33) into (3B-25) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_T(f,x_0,y_0)\,g_f\left(\mathbf{r}\,|\,\mathbf{r}_0\right)dx_0\,dy_0 \approx -\frac{\exp(-jkz)}{4\pi z}\mathcal{S}_T(f), \qquad R_A < 0.316\,z\,, \tag{3B-34}$$

where

$$\boxed{\mathcal{S}_T(f) = \int_{-L_X/2}^{L_X/2}\int_{-L_Y/2}^{L_Y/2} A_T(f,x_0,y_0)\,dx_0\,dy_0} \tag{3B-35}$$

is the *transmitter sensitivity function* in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$. Next we shall use (3B-34) to compute the time-harmonic, radiated acoustic pressure at $\mathbf{r} = (0,0,z)$.

The time-harmonic, radiated acoustic pressure at $\mathbf{r} = (0,0,z)$ is given by

$$p(t,0,0,z) = p_f(0,0,z)\exp(+j2\pi ft)\,, \tag{3B-36}$$

where, by substituting (3B-34) into (3B-23),

$$p_f(0,0,z) \approx j\frac{f\rho_0}{2z}A_x\mathcal{S}_T(f)\exp(-jkz), \qquad R_A < 0.316\,z\,, \tag{3B-37}$$

or

$$p_f(0,0,z) \approx j\frac{f\rho_0}{2z}S_0\exp(-jkz), \qquad R_A < 0.316\,z\,, \tag{3B-38}$$

where

$$\boxed{S_0 = A_x\mathcal{S}_T(f)} \tag{3B-39}$$

is the source strength of the planar transducer in cubic meters per second at frequency f hertz.

From (3B-36) it can be seen that

$$\left|p(t,0,0,z)\right| = \left|p_f(0,0,z)\right|, \tag{3B-40}$$

where from (3B-37)

$$\left|p_f(0,0,z)\right| \approx \frac{f\rho_0}{2z}\left|A_x\right|\left|\mathcal{S}_T(f)\right|, \qquad R_A < 0.316z\,. \tag{3B-41}$$

Therefore, using (3B-41) to solve for $\left|\mathcal{S}_T(f)\right|$ yields

$$\boxed{\left|\mathcal{S}_T(f)\right| \approx \frac{2z}{f\rho_0}\mathrm{TS}(f), \qquad R_A < 0.316z} \tag{3B-42}$$

where

$$\boxed{\mathrm{TS}(f) = \left|p_f(0,0,z)\right| \big/ \left|A_x\right|} \tag{3B-43}$$

is the *transmitter sensitivity* in pascals per volt. Recall that the field point at coordinates $(0,0,z)$ is at *broadside* relative to the planar transducer. Evaluating (3B-42) at $z = 1\text{ m}$ yields

$$\boxed{\left|\mathcal{S}_T(f)\right| \approx \left.\frac{2z}{f\rho_0}\mathrm{TS}(f)\right|_{z=1\text{ m}}, \qquad R_A < 0.316\text{ m}} \tag{3B-44}$$

where R_A is given by (3B-28). If the transducer is circular in shape, then R_A is equal to the radius of the transducer because r_0 given by (3B-27) is the polar radius in meters to a source point in the XY plane. Note that (3B-44) is valid only if the maximum *radial* extent of the aperture (transducer) R_A is less than 0.316 m or 12.4 in.

The magnitude of the source strength can be obtained from (3B-38) as follows:

$$\boxed{\left|S_0\right| \approx \frac{2z}{f\rho_0}\left|p_f(0,0,z)\right|, \qquad R_A < 0.316z} \tag{3B-45}$$

Evaluating (3B-45) at $z = 1\text{ m}$ yields

$$\boxed{|S_0| \approx \frac{2z}{f\rho_0}\left|p_f(0,0,z)\right|\Big|_{z=1\text{ m}}, \qquad R_A < 0.316\text{ m}} \tag{3B-46}$$

Note that (3B-46) is valid only if the maximum *radial* extent of the aperture (transducer) R_A is less than 0.316 m or 12.4 in. Also, from (3B-39),

$$\boxed{|S_0| = |A_x|\left|\mathcal{S}_T(f)\right|} \tag{3B-47}$$

where the transmitter sensitivity function $\mathcal{S}_T(f)$ is given by (3B-35). ■

Appendix 3C Computing the Normalization Factor

The far-field beam pattern of a planar aperture lying in the XY plane is given by [see (3.1-9)]

$$D(f, f_X, f_Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f, x_A, y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\,dy_A\,, \tag{3C-1}$$

where

$$A(f, x_A, y_A) = a(f, x_A, y_A)\exp[+j\theta(f, x_A, y_A)] \tag{3C-2}$$

is the complex frequency response (complex aperture function) of the planar aperture [see (3.1-10)]. Computing the magnitude of (3C-1) yields

$$\begin{aligned}
|D(f, f_X, f_Y)| &= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f, x_A, y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]dx_A\,dy_A\right| \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|A(f, x_A, y_A)\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]\right|dx_A\,dy_A \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A(f, x_A, y_A)|\left|\exp\left[+j2\pi(f_X x_A + f_Y y_A)\right]\right|dx_A\,dy_A \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a(f, x_A, y_A)|\,dx_A\,dy_A.
\end{aligned} \tag{3C-3}$$

Therefore,

$$\max|D(f, f_X, f_Y)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a(f, x_A, y_A)|\,dx_A\,dy_A\,, \tag{3C-4}$$

and by following the steps in (3C-3), it can also be shown that

$$\max|\mathcal{S}(f)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a(f, x_A, y_A)|\, dx_A\, dy_A \;, \tag{3C-5}$$

where $\mathcal{S}(f)$ is the element sensitivity function (see Appendix 3A). The absolute value of the real amplitude response $a(f, x_A, y_A)$ is required in (3C-4) and (3C-5) because $a(f, x_A, y_A)$ can have, in general, both positive and negative values as a function of x_A and y_A.

Equation (3C-4) is the absolute maximum value of $|D(f, f_X, f_Y)|$. The actual maximum value may be less than (3C-4) as indicated by (3C-3). However, if $a(f, x_A, y_A) \geq 0$, then the normalization factor $D_{\max}$ is

$$\boxed{D_{\max} = \max|D(f, f_X, f_Y)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A} \tag{3C-6}$$

or

$$\boxed{D_{\max} = \max|\mathcal{S}(f)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A} \tag{3C-7}$$

Note that if (3C-1) is evaluated at broadside (i.e., at $f_X = 0$ and $f_Y = 0$) and if the phase response $\theta(f, x_A, y_A) = 0$, then

$$D(f, 0, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A \tag{3C-8}$$

and

$$|D(f, 0, 0)| = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A\right| \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a(f, x_A, y_A)|\, dx_A\, dy_A \;. \tag{3C-9}$$

If in addition $a(f, x_A, y_A) \geq 0$, then

$$|D(f, 0, 0)| = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A\right| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(f, x_A, y_A)\, dx_A\, dy_A \;, \tag{3C-10}$$

and by comparing (3C-6) and (3C-10),

$$\boxed{D_{\max} = \max|D(f, f_X, f_Y)| = |D(f, 0, 0)|} \tag{3C-11}$$

If the planar aperture is circular in shape with radius a meters, then [see (3.3-9)]