1 The Extended Kalman Filter

Components

The Extended Kalman Filter is an algorithm that estimates the current state of a system based on the previous state and augments that prediction using a measurement. The Extended Kalman filter differs from the standard Kalman filter in that non-linear state update and measurement mapping equations can be used. The Extended Kalman filter has the following parts:

State vector ($\hat{x}$): a vector describing the state of the system at a given time. For this filter, the state vector is composed of the x and y position and velocity of the system in the world frame and the x and y velocity of the current affecting the vehicle.

$$\hat{x_k} = \begin{bmatrix} x_w \\ y_w \\ \dot{x_w} \\ \dot{y_w} \\ \dot{x_c} \\ \dot{y_c} \end{bmatrix}$$

State Transition Equations (F, G, u, w): a matrix describing the physical model of the system. In this case, the model combines simple linear motion, an input vector as a function of speed through the water (stw) and heading (hdg), and white noise:

$$x_{w_{k+1}} = x_{w_k} + stw_{k+1}\cos(hdg_{k+1})\Delta t + .5a_{w_k}\Delta t^2 + w_k$$
$$y_{w_{k+1}} = y_{w_k} + stw_{k+1}\sin(hdg_{k+1})\Delta t + .5a_{w_k}\Delta t^2 + w_k$$
$$x_{\dot{w_{k+1}}} = \dot{x_{c_k}} + stw_{k+1}\cos(hdg_{k+1}) + a_{w_k}\Delta t + w_k$$
$$y_{\dot{w_{k+1}}} = \dot{y_{c_k}} + stw_{k+1}\sin(hdg_{k+1}) + a_{w_k}\Delta t + w_k$$
$$x_{\dot{c_{k+1}}} = \dot{x_{c_k}} + a_{c_k}\Delta t + w_k$$
$$y_{\dot{c_{k+1}}} = \dot{y_{c_k}} + a_{c_k}\Delta t + w_k$$

Values for stw and hdg are specified in the initialization of the filter and the model is assumed to be constant velocity, therefore the acceleration terms are zero.

Rearranging into a matrix format results in:

$$\hat{x_{k+1}} = F\hat{x_k} + u_k + Ga_k + w_k$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, u_k = \begin{bmatrix} stw_{k+1}\cos(hdg_{k+1})\Delta t \\ stw_{k+1}\sin(hdg_{k+1})\Delta t \\ stw_{k+1}\cos(hdg_{k+1}) \\ stw_{k+1}\sin(hdg_{k+1}) \\ 0 \\ 0 \end{bmatrix}, G = \begin{bmatrix} .5\Delta t^2 \\ .5\Delta t^2 \\ \Delta t \\ \Delta t \\ \Delta t \\ \Delta t \end{bmatrix}$$

Process Covariance Matrix (Q): a matrix containing the variances and covariances for each state vector variable. This matrix characterizes the assumed white noise present in the environment.

$$Q = G\sigma_q^2 G^T$$

Measurement Covariance Matrix (R): a square, diagonal matrix containing the variances and covariances for each measurement vector variable. This matrix represents the confidence in the accuracy of the measurements.

$$R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

Error Covariance Matrix (P): a matrix containing the variances and covariances of the error between the estimated state vector and the calculated output vector. Initially, the vector components are assumed to be uncorrelated and the off-diagonal entries are zero. As the filter evolves, the covariances will change depending on the calculated correlation of the components

$$P = \begin{bmatrix} \sigma^2_{x_w x_w} & \sigma^2_{x_w y_w} & \sigma^2_{x_w \dot{x}_w} & \sigma^2_{x_w \dot{y}_w} & \sigma^2_{x_w \dot{x}_c} & \sigma^2_{x_w \dot{y}_c} \\ \sigma^2_{y_w x_w} & \sigma^2_{y_w y_w} & \sigma^2_{y_w \dot{x}_w} & \sigma^2_{y_w \dot{y}_w} & \sigma^2_{y_w \dot{x}_c} & \sigma^2_{y_w \dot{y}_c} \\ \sigma^2_{\dot{x}_w x_w} & \sigma^2_{\dot{x}_w y_w} & \sigma^2_{\dot{x}_w \dot{x}_w} & \sigma^2_{\dot{x}_w \dot{y}_w} & \sigma^2_{\dot{x}_w \dot{x}_c} & \sigma^2_{\dot{x}_w \dot{y}_c} \\ \sigma^2_{\dot{y}_w x_w} & \sigma^2_{\dot{y}_w y_w} & \sigma^2_{\dot{y}_w \dot{x}_w} & \sigma^2_{\dot{y}_w \dot{y}_w} & \sigma^2_{\dot{y}_w \dot{x}_c} & \sigma^2_{\dot{y}_w \dot{y}_c} \\ \sigma^2_{\dot{x}_c x_w} & \sigma^2_{\dot{x}_c y_w} & \sigma^2_{\dot{x}_c \dot{x}_w} & \sigma^2_{\dot{x}_c \dot{y}_w} & \sigma^2_{\dot{x}_c \dot{x}_c} & \sigma^2_{\dot{x}_c \dot{y}_c} \\ \sigma^2_{\dot{y}_c x_w} & \sigma^2_{\dot{y}_c y_w} & \sigma^2_{\dot{y}_c \dot{x}_w} & \sigma^2_{\dot{y}_c \dot{y}_w} & \sigma^2_{\dot{y}_c \dot{x}_c} & \sigma^2_{\dot{y}_c \dot{y}_c} \end{bmatrix}$$

Measurement Mapping Equation ($h(k, x_w, y_w, \dot{x_w}, \dot{y_w})$): a nonlinear equation to map the state vector to the measurements of range and angle of azimuth.

$$h = \begin{bmatrix} \sqrt{x_{w_k}^2 + y_{w_k}^2} & \arctan(\frac{\dot{y}_{w_k}}{\dot{x}_{w_k}}) \end{bmatrix}$$

Measurement Mapping Jacobian (H): a matrix containing the partial derivatives of $h(k, x_w, y_w, \dot{x}, \dot{y})$ with respect to the state vector components.

$$H = \begin{bmatrix} \frac{x_k}{\sqrt{x_k^2+\dot{y}_k^2}} & \frac{y_k}{\sqrt{x_k^2+y_k^2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\dot{y}_k}{\dot{x}_k^2+\dot{y}_k^2} & \frac{1}{\dot{x}_k+\frac{\dot{y}_k^2}{\dot{x}_k}} & 0 & 0 \end{bmatrix}$$

Kalman Gain Matrix (K): a matrix used to weight the contributions of the state estimate and measurement to produce the most accurate calculated state of the system.

$$K = \frac{PH^T}{HPH^T + R}$$

Measurement vector(z_k): a simulated measurement composed of the actual state of the system with Gaussian distributed noise added.

$$z_k = \begin{bmatrix} range \\ azimuth \end{bmatrix} + v_k$$

Execution

The filter is first initialized with the initial condition of the state vector.
For the simulation, I chose a speed of 10 m/s and a heading of 30 degrees with a 2 m/s current at an angle of 180 degrees which resulted in:

$$\hat{x_0} = \begin{bmatrix} 0 \\ 0 \\ 6.6603 \\ 5.0 \\ -2 \\ 0 \end{bmatrix}$$

Next, the covariance matrices are initialized. For P, I chose an initial covariance value of 10m squared. For R, I chose a an error of one meter for range measurement and an error of two degrees for the heading measurement. For Q, I chose an error of .5. The resulting matrices:

$$P_0 = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.0625 & 0.1250 & 0.0625 & 0.1250 & 0.1250 & 0.1250 \\ 0.0625 & 0.1250 & 0.0625 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.2500 & 0.1250 & 0.2500 & 0.1250 & 0.1250 \\ 0.1250 & 0.2500 & 0.1250 & 0.2500 & 0.1250 & 0.1250 \\ 0.1250 & 0.2500 & 0.1250 & 0.2500 & 0.1250 & 0.1250 \\ 0.1250 & 0.2500 & 0.1250 & 0.2500 & 0.1250 & 0.1250 \end{bmatrix}$$

Once the initial conditions are established, the state vector and error covariance matrix are estimated for the next step:

$$\hat{x_k}^- = F\hat{x_{k-1}}^+ + u_k + w_k$$

$$P_k^- = FP_{k-1}^+F^T + Q$$

Next, H is calculated using the previously determined variables and used to compute K. Then, the state vector is updated using the measurement.

$$\hat{x_k}^+ = \hat{x_k}^- + K(z_k - H\hat{x_k}^-)$$

Now, compute the updated state vector covariance. I chose this particular version, the Joesph stabilized equation, due to the added stability provided by the fact that the equation always results in a positive definite matrix:

$$P_k^+ = (I - KH)P_k^-(I - KH)^T + KRK^T$$

Now the filter is ready to begin the cycle again.