Chapter 14

Underwater Acoustic Communication Signals

The transmission of digital information corresponds to the transmission of *binary words* which are encoded by the *alphabet* of a *symbol*. Binary words are created by processing sampled values of an analog (continuous-time) message signal by an analog-to-digital (A/D) converter. Binary words encode the output voltages from a quantizer in an A/D converter.

In this chapter we shall discuss the basic principles of three important digital modulation techniques that are not only used to transmit digital information in traditional communication systems, but are also used for underwater acoustic communication. The three digital modulation techniques are 1) M-ary Frequency-Shift Keying (MFSK), 2) M-ary Quadrature Amplitude Modulation (MQAM), and 3) Orthogonal Frequency-Division Multiplexing (OFDM). For each waveform we shall provide a time-domain description, derive its frequency spectrum, derive bandwidth and time-average power formulas, and discuss how to demodulate the waveform. In this way we can compare the advantages and disadvantages of each digital modulation technique.

14.1 M-ary Frequency-Shift Keying

14.1.1 Time-Domain Description

If digital information is being transmitted using M-ary frequency-shift-keying (MFSK), then a time-domain description of the transmitted signal $x(t)$ is given as follows:

$$x(t) = \sum_{n=1}^{N} x_n(t - t_n), \qquad 0 \le t \le T_d, \tag{14.1-1}$$

where N is the total number of transmitted pulses (symbols),

$$x_n(t) = A\cos\left(2\pi[f_c + \Delta f_n]t + \varepsilon_n\right)\mathrm{rect}\left[(t - 0.5T)/T\right] \tag{14.1-2}$$

is the nth pulse where A is the amplitude factor in volts, f_c is the carrier frequency in hertz,

$$\Delta f_n = k_n/T, \qquad k_n \in \left\{\pm 1, \pm 2, \ldots, \pm M/2\right\} \tag{14.1-3}$$

is the *frequency offset* in hertz where the *symbol* k_n is an *integer* (positive or negative),

$$T = T_{\text{sym}} = n_b T_b \tag{14.1-4}$$

is the pulse length of an individual pulse, which is equal to the *symbol duration* T_{sym} in seconds – also known as (a.k.a.) the duration of a binary word; n_b is the number of bits per symbol – a.k.a. the number of bits per binary word; T_b is the *bit duration* in seconds,

$$M = 2^{n_b} \tag{14.1-5}$$

is the total *even* number of *unique* symbol values – the different unique symbol values are known as the *alphabet*; ε_n is a possible, unwanted phase shift in radians at the transmitter,

$$t_n = (n-1)T \tag{14.1-6}$$

is the time instant in seconds when the nth pulse (symbol) begins, and

$$T_d = NT \tag{14.1-7}$$

is the total duration in seconds of the transmitted signal $x(t)$. Note that the total number of transmitted pulses (symbols) N is greater than or equal to the total even number of unique symbol values M, that is,

$$N \geq M \,. \tag{14.1-8}$$

In Subsection 14.1.4 it is shown that if the frequency offset Δf_n is given by (14.1-3), and the product $f_c T$ is equal to an *integer*, or $f_c T >> 1$ if $f_c T$ does not equal an integer, then the set of functions $x_n(t)$, $n = 1, 2, \ldots, N$, given by (14.1-2) is an *orthogonal* set of functions in the time interval $[0, T]$. The frequency offset Δf_n given by (14.1-3) is required for orthogonality when performing *noncoherent* demodulation and detection of MFSK signals[1]. In noncoherent demodulation and detection, no attempt is made to estimate the unknown phase of a received signal.

For the special case known as *binary frequency-shift keying* (BFSK), $n_b = 1$. Therefore, $M = 2$ and (14.1-3) reduces to

$$\Delta f_n = \frac{k_n}{T}, \qquad k_n = \begin{cases} +1, & \text{binary 1} \\ -1, & \text{binary 0.} \end{cases} \tag{14.1-9}$$

As can be seen from (14.1-1), the MFSK signal $x(t)$ is a *pulse train* where

$$\text{PRI} = T \tag{14.1-10}$$

[1] J. G. Proakis and M. Salehi, *Digital Communications*, 5th ed., McGraw-Hill, New York, 2008, pg. 215.

is the pulse-repetition interval (PRI) in seconds. Since the $\text{PRI} = T$, the time-shifted rectangle function in (14.1-2) is nonzero in the interval $[0, T)$ for the first $N-1$ pulses. Therefore, for $n = 1, 2, \ldots, N-1$,

$$\text{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise.} \end{cases} \tag{14.1-11}$$

For the last pulse $n = N$, the time-shifted rectangle function is nonzero in the interval $[0, T]$, that is,

$$\text{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases} \tag{14.1-12}$$

14.1.2 Frequency Spectrum and Bandwidth

In order to derive an equation for the bandwidth of the MFSK signal $x(t)$ given by (14.1-1), we first have to find its frequency spectrum. Taking the Fourier transform of (14.1-1) yields

$$X(f) = \sum_{n=1}^{N} X_n(f) \exp(-j2\pi f t_n), \tag{14.1-13}$$

and by substituting (14.1-6) into (14.1-13) we obtain

$$X(f) = \exp(+j2\pi f T) \sum_{n=1}^{N} X_n(f) \exp(-j2\pi f n T), \tag{14.1-14}$$

where

$$X(f) = F\{x(t)\} \tag{14.1-15}$$

and

$$X_n(f) = F\{x_n(t)\} \tag{14.1-16}$$

is the frequency spectrum of the *n*th pulse $x_n(t)$ given by (14.1-2). The Fourier transform of $x_n(t)$ can be obtained by using the Fourier-transform pair

$$\begin{aligned} A\cos(2\pi f_c t + \theta_0)\,\text{rect}\left(\frac{t-0.5T}{T}\right) \leftrightarrow\ & \frac{A}{2} T \,\text{sinc}\big[(f - f_c)T\big] \exp\big[-j\pi(f - f_c)T\big] \exp(+j\theta_0) + \\ & \frac{A}{2} T \,\text{sinc}\big[(f + f_c)T\big] \exp\big[-j\pi(f + f_c)T\big] \exp(-j\theta_0) \end{aligned} \tag{14.1-17}$$

and replacing f_c with $f_c + \Delta f_n$, and θ_0 with ε_n. Doing so and substituting the resulting expression into (14.1-14) yields the frequency spectrum of the MFSK signal $x(t)$ given by (14.1-1):

$$\boxed{\begin{aligned} X(f) = {} & \frac{A}{2} T c_1(f) \sum_{n=1}^{N} \operatorname{sinc}\{[f-(f_c+\Delta f_n)]T\} \exp(+j\pi \Delta f_n T) \exp(-j2\pi f n T) \exp(+j\varepsilon_n) + \\ & \frac{A}{2} T c_2(f) \sum_{n=1}^{N} \operatorname{sinc}\{[f+(f_c+\Delta f_n)]T\} \exp(-j\pi \Delta f_n T) \exp(-j2\pi f n T) \exp(-j\varepsilon_n) \end{aligned}}$$

(14.1-18)

where

$$c_1(f) = \exp[+j\pi(f+f_c)T] \tag{14.1-19}$$

and

$$c_2(f) = \exp[+j\pi(f-f_c)T]. \tag{14.1-20}$$

The units of $X(f)$ are volts per hertz.

Since bandwidth is always measured along the positive frequency axis, we shall work with the first term in (14.1-18), that is,

$$\begin{aligned} X(f) = \frac{A}{2} T c_1(f) \sum_{n=1}^{N} \operatorname{sinc}\{[f-(f_c+\Delta f_n)]T\} \exp(+j\pi \Delta f_n T) \times \\ \exp(-j2\pi f n T) \exp(+j\varepsilon_n), \qquad f \geq 0. \end{aligned}$$

(14.1-21)

By examining the argument of the sinc function in (14.1-21) and referring to (14.1-3), estimates of the maximum and minimum frequency components are given by

$$f_{\max} = f_c + \max \Delta f_n + \frac{\text{NZC}}{T} = f_c + \frac{M}{2}\frac{1}{T} + \frac{\text{NZC}}{T} \tag{14.1-22}$$

and

$$f_{\min} = f_c + \min \Delta f_n - \frac{\text{NZC}}{T} = f_c - \frac{M}{2}\frac{1}{T} - \frac{\text{NZC}}{T}, \tag{14.1-23}$$

where NZC is the integer number of zero-crossings of the sinc function that is used to estimate both the maximum and minimum frequency components $f_{\max}$ and $f_{\min}$, respectively. Therefore, the bandwidth BW_x (in hertz) of the MFSK signal is given by

$$\text{BW}_x = f_{\max} - f_{\min}, \tag{14.1-24}$$

and by substituting (14.1-22), (14.1-23), and (14.1-4) into (14.1-24), we obtain

$$\boxed{\mathrm{BW}_x = (M + 2\,\mathrm{NZC})\,D} \tag{14.1-25}$$

where

$$\boxed{D = \frac{1}{T_{\mathrm{sym}}} = \frac{1}{n_b T_b} = \frac{R_b}{n_b}} \tag{14.1-26}$$

is the *symbol rate (baud)* with units of symbols per second, and

$$\boxed{R_b = 1/T_b} \tag{14.1-27}$$

is the *bit rate* in bits per second. Since $\max\left|\mathrm{sinc}(fT)\right| = 1$ for $f = 0$; $\mathrm{sinc}(fT) = 0$ for $f = i/T$, where $i = \pm 1, \pm 2, \ldots$; and $\left|\mathrm{sinc}(fT)\right| < 0.1$ for $f > 3/T$; NZC should be at least 3, with 5 being a conservative choice.

14.1.3 Signal Energy and Time-Average Power

In this subsection we shall compute the energy and time-average power of the MFSK signal $x(t)$ given by (14.1-1). We begin by computing the energy of the *n*th pulse $x_n(t)$ given by (14.1-2). Note that $x_n(t)$ is an amplitude-and-angle-modulated carrier with amplitude and angle-modulating functions

$$a_n(t) = A\,\mathrm{rect}\big[(t - 0.5T)/T\big] \tag{14.1-28}$$

and

$$\theta_n(t) = 2\pi\Delta f_n t + \varepsilon_n\,, \tag{14.1-29}$$

respectively (see Section 11.2), where the time-shifted rectangle function is given by (14.1-11) and (14.1-12). In order to compute the energy of $x_n(t)$, we shall first compute the energy of its complex envelope $\tilde{x}_n(t)$.

Since the general form of the complex envelope of $x_n(t)$ is given by (see Section 11.2)

$$\tilde{x}_n(t) = a_n(t)\exp[+j\theta_n(t)]\,, \tag{14.1-30}$$

substituting (14.1-28) and (14.1-29) into (14.1-30) yields

$$\tilde{x}_n(t) = A\exp\big[+j(2\pi\Delta f_n t + \varepsilon_n)\big]\mathrm{rect}\big[(t - 0.5T)/T\big]. \tag{14.1-31}$$

The energy $E_{\tilde{x}_n}$ of $\tilde{x}_n(t)$ is given by

$$E_{\tilde{x}_n} = \int_{-\infty}^{\infty} a_n^2(t)\,dt \,, \tag{14.1-32}$$

and the energy E_{x_n} of $x_n(t)$ is given by (see Section 11.2)

$$E_{x_n} = E_{\tilde{x}_n} / 2 \,. \tag{14.1-33}$$

Therefore, substituting (14.1-28) into (14.1-32) yields

$$E_{\tilde{x}_n} = A^2 T \,, \tag{14.1-34}$$

which is the energy (in joules-ohms) of the complex envelope $\tilde{x}_n(t)$ given by (14.1-31) $\forall\ n$, and substituting (14.1-34) into (14.1-33) yields

$$\boxed{E_{x_n} = \frac{A^2}{2} T} \tag{14.1-35}$$

which is the energy (in joules-ohms) of $x_n(t)$ given by (14.1-2) $\forall\ n$. The energy E_{x_n} is also referred to as the energy per symbol because the pulse length T is equal to the symbol duration T_{sym} [see (14.1-4)]. The time-average power of $x_n(t)$ (in watts-ohms) $\forall\ n$ is given by

$$\boxed{P_{\text{avg},\,x_n,\,T} = \frac{E_{x_n}}{T} = \frac{A^2}{2}} \tag{14.1-36}$$

where $P_{\text{avg},\,x_n,\,T}$ is also referred to as the time-average power per symbol.

The energy per bit E_b can also be derived using the energy E_{x_n} given by (14.1-35). Substituting (14.1-4) into (14.1-35) yields

$$E_{x_n} = \frac{A^2}{2} T_b n_b \,, \tag{14.1-37}$$

which can be rewritten as

$$\boxed{E_{x_n} = E_b n_b} \tag{14.1-38}$$

where

$$\boxed{E_b = \frac{A^2}{2} T_b} \tag{14.1-39}$$

is the *energy per bit* in joules-ohms.

From signal theory, the energy E_x of a signal $x(t)$ is defined as

$$E_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt \ . \tag{14.1-40}$$

Using (14.1-1),

$$|x(t)|^2 = x(t)x^*(t) = \sum_{m=1}^{N}\sum_{n=1}^{N} x_m(t-t_m)x_n^*(t-t_n) \, , \tag{14.1-41}$$

and since

$$x_m(t-t_m)x_n^*(t-t_n) = \begin{cases} |x_n(t-t_n)|^2, & m=n \\ 0, & m \neq n, \end{cases} \tag{14.1-42}$$

(14.1-41) reduces to

$$|x(t)|^2 = \sum_{n=1}^{N} |x_n(t-t_n)|^2 \ . \tag{14.1-43}$$

Substituting (14.1-43) into (14.1-40) yields

$$E_x = \sum_{n=1}^{N} \int_{-\infty}^{\infty} |x_n(t-t_n)|^2 \, dt \, , \tag{14.1-44}$$

and by letting $\alpha = t - t_n$ in (14.1-44),

$$E_x = \sum_{n=1}^{N} \int_{-\infty}^{\infty} |x_n(\alpha)|^2 \, d\alpha = \sum_{n=1}^{N} E_{x_n} \ . \tag{14.1-45}$$

Substituting (14.1-35) into (14.1-45) yields

$$E_x = N\frac{A^2}{2}T \, , \tag{14.1-46}$$

or [see (14.1-7)]

$$\boxed{E_x = \frac{A^2}{2}T_d} \tag{14.1-47}$$

which is the energy (in joules-ohms) of the MFSK signal $x(t)$ given by (14.1-1). Therefore, the time-average power of $x(t)$ in watts-ohms is given by

$$\boxed{P_{\text{avg},x,T_d} = \frac{E_x}{T_d} = \frac{A^2}{2}} \tag{14.1-48}$$

14.1.4 Orthogonality Conditions

In this subsection we shall show that if the frequency offset Δf_n is given by (14.1-3), and the product $f_c T$ is equal to an *integer*, then the set of functions $x_n(t)$, $n = 1, 2, \ldots, N$, given by (14.1-2) is an *orthogonal* set of functions in the time interval $[0, T]$. We begin by computing the following *inner product*:

$$\langle x_m(t), x_n(t) \rangle = \int_0^T x_m(t) x_n^*(t)\, dt \,, \tag{14.1-49}$$

where

$$x_m(t) = A\cos\left(2\pi[f_c + \Delta f_m]t + \varepsilon_m\right)\text{rect}\left[(t - 0.5T)/T\right] \tag{14.1-50}$$

and

$$x_n(t) = A\cos\left(2\pi[f_c + \Delta f_n]t + \varepsilon_n\right)\text{rect}\left[(t - 0.5T)/T\right]. \tag{14.1-51}$$

Substituting (14.1-50) and (14.1-51) into (14.1-49) yields

$$\begin{aligned}\int_0^T x_m(t) x_n^*(t)\, dt = {} & \frac{A^2}{2} T \,\text{sinc}\left[2(\Delta f_m - \Delta f_n)T\right]\cos(\varepsilon_m - \varepsilon_n) - \\ & A^2 \frac{\pi(\Delta f_m - \Delta f_n)T^2}{2}\text{sinc}^2\left[(\Delta f_m - \Delta f_n)T\right]\sin(\varepsilon_m - \varepsilon_n) + \\ & \frac{A^2}{2} T \,\text{sinc}\left[4 f_c T + 2(\Delta f_m + \Delta f_n)T\right]\cos(\varepsilon_m + \varepsilon_n) - \\ & A^2 \frac{\left[2\pi f_c + \pi(\Delta f_m + \Delta f_n)\right]T^2}{2}\text{sinc}^2\left[2 f_c T + (\Delta f_m + \Delta f_n)T\right]\sin(\varepsilon_m + \varepsilon_n).\end{aligned} \tag{14.1-52}$$

If the inner product given by (14.1-52) is equal to *zero* when $m \neq n$, then $x_m(t)$ and $x_n(t)$ are said to be *orthogonal*.

By referring to (14.1-3), it can be seen that $\Delta f_m \pm \Delta f_n = i/T$, where i is either a positive or negative *integer*, including zero ($\Delta f_m - \Delta f_n = 0$ when $m = n$). If the product $f_c T$ is also equal to an *integer*, then (14.1-52) reduces to

$$\langle x_m(t), x_n(t) \rangle = \int_0^T x_m(t) x_n^*(t)\, dt = \begin{cases} \dfrac{A^2}{2} T, & m = n \\ 0, & m \neq n, \end{cases} \tag{14.1-53}$$

or

$$\boxed{\langle x_m(t), x_n(t)\rangle = \int_0^T x_m(t)x_n^*(t)\,dt = E_{x_n}\delta_{mn}, \qquad m,n = 1,2,\ldots,N} \tag{14.1-54}$$

where

$$E_{x_n} = \langle x_n(t), x_n(t)\rangle = \int_0^T |x_n(t)|^2\,dt = \frac{A^2}{2}T, \qquad n = 1,2,\ldots,N, \tag{14.1-55}$$

is the energy (in joules-ohms) of $x_n(t)$ given by (14.1-2) $\forall\ n$ [see (14.1-35)], and

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{14.1-56}$$

is the Kronecker delta. Therefore, the set of functions $x_n(t)$, $n = 1,2,\ldots,N$, given by (14.1-2) is an *orthogonal* set of functions in the time interval $[0,T]$. If the product f_cT does *not* equal an integer, but $f_cT >> 1$, then

$$\langle x_m(t), x_n(t)\rangle \approx E_{x_n}\delta_{mn}, \qquad m,n = 1,2,\ldots,N. \tag{14.1-57}$$

14.1.5 Demodulation

Demodulation is a procedure meant to retrieve a transmitted message at a receiver. In this subsection we shall briefly discuss two different methods to demodulate a MFSK signal at a receiver. The first method takes advantage of the orthogonality of the pulses $x_n(t)$, $n = 1,2,\ldots,N$, given by (14.1-2). The second method is based on computing the Fourier transform of each pulse in the received signal.

In order to discuss demodulation, we first need to model a MFSK signal at a receiver, which we shall designate as $y(t)$. We shall use the following simple model for $y(t)$:

$$y(t) = Kx(t-\tau), \qquad \tau \le t \le \tau + T_d, \tag{14.1-58}$$

where the transmitted signal $x(t)$ is given by (14.1-1). As can be seen from (14.1-58), $y(t)$ is modeled as an amplitude-scaled, time-delayed version of $x(t)$ (distortionless transmission) where $K > 0$ is a dimensionless constant. Substituting (14.1-1) into (14.1-58) yields

$$y(t) = \sum_{n=1}^{N} y_n(t - t_n), \qquad \tau \le t \le \tau + T_d, \tag{14.1-59}$$

where

$$y_n(t) = KA\cos\left(2\pi[f_c + \Delta f_n]t + \delta_n\right)\operatorname{rect}\left[\left(t - [0.5T + \tau]\right)/T\right] \tag{14.1-60}$$

is the nth pulse of the received signal,

$$\Delta f_n = k_n/T, \qquad k_n \in \{\pm 1, \pm 2, \ldots, \pm M/2\} \tag{14.1-61}$$

is the frequency offset in hertz,

$$\delta_n = \varepsilon_n - 2\pi(f_c + \Delta f_n)\tau \tag{14.1-62}$$

is a phase shift in radians that takes into account the time delay τ in seconds, the time-shifted rectangle function is given by (14.1-11) and (14.1-12), and [see (14.1-6)]

$$t_n = (n-1)T \ . \tag{14.1-63}$$

In order to demodulate $y(t)$, compute the following inner product for *each* received pulse $y_n(t - t_n)$, $n = 1, 2, \ldots, N$: For *each* value of n, compute

$$\left\langle y_n(t - t_n), g_m(t - t_n)\right\rangle = \int_{\tau + t_n}^{\tau + t_n + T} y_n(t - t_n)\, g_m^*(t - t_n)\, dt \tag{14.1-64}$$

$\forall\ m$, where

$$g_m(t) = A\cos\left(2\pi[f_c + \Delta f_m]t + \phi_m\right)\operatorname{rect}\left[\left(t - [0.5T + \tau]\right)/T\right], \tag{14.1-65}$$

$$\Delta f_m = m/T, \qquad m = \pm 1, \pm 2, \ldots, \pm M/2, \tag{14.1-66}$$

and ϕ_m is a possible, unwanted phase shift in radians. Evaluating the integral on the right-hand side of (14.1-64) is equivalent to evaluating the integral I where

$$\begin{aligned} I &= KA^2 \int_0^T \cos\left(2\pi[f_c + \Delta f_n]t + \delta_n\right)\cos\left(2\pi[f_c + \Delta f_m]t + \phi_m\right)dt \\ &= KA^2 \int_0^T \cos\left(2\pi[f_c + \Delta f_m]t + \phi_m\right)\cos\left(2\pi[f_c + \Delta f_n]t + \delta_n\right)dt. \end{aligned} \tag{14.1-67}$$

By referring to (14.1-52) and replacing ε_m and ε_n with ϕ_m and δ_n, respectively, it can be seen that

$$I = \begin{cases} K\dfrac{A^2}{2}T\cos(\phi_m - \delta_n), & \Delta f_m = \Delta f_n \\ 0, & \Delta f_m \neq \Delta f_n. \end{cases} \tag{14.1-68}$$

Therefore,

$$\boxed{\langle y_n(t-t_n), g_m(t-t_n)\rangle = \begin{cases} K\dfrac{A^2}{2}T\cos(\phi_m - \delta_n), & \Delta f_m = \Delta f_n \\ 0, & \Delta f_m \neq \Delta f_n \end{cases}} \tag{14.1-69}$$

For example, if $n = 4$ and $m = -2$ so that $\Delta f_{-2} = \Delta f_4$, then the symbol value that was transmitted by the 4th pulse is $k_4 = -2$ because

$$\left|\langle y_4(t-t_4), g_{-2}(t-t_4)\rangle\right| = K\frac{A^2}{2}T\left|\cos(\phi_{-2} - \delta_4)\right| \tag{14.1-70}$$

and

$$\left|\langle y_4(t-t_4), g_m(t-t_4)\rangle\right| = 0, \qquad \forall\, m \neq -2. \tag{14.1-71}$$

However, if $\phi_{-2} - \delta_4 = \pm\pi/2$, then $\left|\langle y_4(t-t_4), g_{-2}(t-t_4)\rangle\right| = 0$, and as a result, the symbol value that was transmitted by the 4th pulse will not be detected. Once all the transmitted symbol values have been determined, decode the symbol values back into binary words, and then decode the binary words back into quantized voltages. One can then use the quantized voltages to approximate the original analog (continuous-time) message using the reconstruction formula from the Sampling Theorem.

Another easier way to demodulate $y(t)$ is to compute the Fourier transform of each received pulse $y_n(t-t_n)$, $n = 1, 2, \ldots, N$. In other words, compute the Fourier transform of each T seconds worth of data. For example,

$$F\{y_n(t-t_n)\} = Y_n(f)\exp(-j2\pi f t_n), \tag{14.1-72}$$

where $Y_n(f)$ is the complex frequency spectrum of the *n*th received pulse $y_n(t)$ given by (14.1-60). The magnitude spectrum of $y_n(t)$ along the positive frequency axis is given by

$$|Y_n(f)| = K\frac{A}{2}T\left|\operatorname{sinc}\{[f - (f_c + \Delta f_n)]T\}\right|, \qquad f \geq 0, \tag{14.1-73}$$

which is equal to the magnitude spectrum of the *n*th transmitted pulse $x_n(t)$ along the positive frequency axis multiplied by the factor K. As can be seen from (14.1-73), the maximum value of $|Y_n(f)|$ is at $f = f_c + \Delta f_n$. The value of this frequency determines the binary word that was transmitted by the *n*th pulse (see Table 14.1-3 in Example 14.1-1). For all other frequencies such that the argument of the sinc function is equal to a *nonzero integer*, the sinc function is equal to zero. A numerical estimate of the Fourier transform can be obtained by using the algorithm discussed in Appendix 7B. This method of demodulation is demonstrated in Example 14.1-1.

Example 14.1-1

In this example we shall consider the problem of transmitting the following binary words

00 11 01 10 00 11 10 00 01 11

using MFSK where a binary word is equal to $n_b = 2$ bits. Since there are 10 binary words (20 bits), the total number of pulses (symbols) to be transmitted is $N = 10$. Also, since $n_b = 2$, the total even number of *unique* symbol values $M = 2^2 = 4$. Therefore, this example corresponds to 4-ary FSK or *quaternary FSK*. Substituting $M = 4$ into (14.1-3) yields

$$\Delta f_n = k_n / T, \qquad k_n \in \{\pm 1, \pm 2\}. \tag{14.1-74}$$

As can be seen from (14.1-74), there are 4 unique symbol values, where the set of numbers $\{\pm 1, \pm 2\}$ is the alphabet.

The next step is to assign a unique symbol value to each of the 4 *unique* binary words (4 possible combinations of 2 bits). For example, see Table 14.1-1 where the 4 unique binary words are arranged in a *Gray code*. In a Gray code, the leftmost or most-significant-bit (MSB) is the *sign bit*. If the MSB is 0, then a negative number is being represented. If the MSB is 1, then a positive number is being represented. Also, in a Gray code, adjacent binary words differ by only *one* bit.

In order to simulate a time-domain MFSK signal using the information given up to this point, we need values for the amplitude factor A, the carrier frequency f_c, and the pulse length T of an individual pulse. For example, if $A = 100\text{ V}$, $f_c = 1\text{ kHz}$, and $T = T_{\text{sym}} = n_b T_b = 20\text{ msec}$, then we obtain the parameter values shown in Table 14.1-2 for the MFSK signal. Note that the values for A and T were chosen so that the factor $AT/2 = 1$ in (14.1-18). Also note that $f_c T = 20$ is an integer (see Subsection 14.1.4). Using the symbol

Table 14.1-1 Four Unique Symbol Values Assigned to Four Unique Binary Words

Gray Code	Symbol k_n	Frequency Offset Δf_n (Hz)
00	-2	$-2/T$
01	-1	$-1/T$
11	1	$1/T$
10	2	$2/T$

Table 14.1-2 Parameter Values of the Transmitted MFSK Signal

Amplitude factor A	100 V
Carrier frequency f_c	1 kHz
Pulse length of an individual pulse T	20 msec
Frequency offset $\Delta f_n = k_n/T$	$50\,k_n$ Hz
Total number of transmitted pulses (symbols) N	10
Total duration of the transmitted signal $T_d = NT$	200 msec
Symbol duration T_{sym}	20 msec
Baud (symbol rate) $D = 1/T_{\text{sym}}$	50 symbols/sec
Number of bits per symbol (binary word) n_b	2
Number of *unique* symbol values $M = 2^{n_b}$	4
Bit duration $T_b = T_{\text{sym}}/n_b$	10 msec
Bit rate $R_b = 1/T_b$	100 bps
Time-average power $P_{\text{avg},x,T_d} = A^2/2$	5000 W-Ω

assignments for the 4 unique binary words in Table 14.1-1, and $\Delta f_n = 50\,k_n$ Hz from Table 14.1-2, Table 14.1-3 shows the frequencies that are assigned to the 4 unique binary words, and Table 14.1-4 shows the frequencies of the 10 transmitted pulses associated with the 10 binary words, where the values for the phase term ε_n were made up for example purposes.

Table 14.1-3 Frequencies Assigned to the Four Unique Binary Words

Binary Word	k_n	$\Delta f_n = 50k_n$ (Hz)	$f_c + \Delta f_n$ (Hz)
00	-2	-100	900
01	-1	-50	950
11	1	50	1050
10	2	100	1100

Table 14.1-4 Frequencies of the Ten Transmitted Pulses

n	Binary Word	k_n	$\Delta f_n = 50k_n$ (Hz)	$f_c + \Delta f_n$ (Hz)	ε_n (deg)
1	00	-2	-100	900	5
2	11	1	50	1050	10
3	01	-1	-50	950	15
4	10	2	100	1100	20
5	00	-2	-100	900	25
6	11	1	50	1050	30
7	10	2	100	1100	35
8	00	-2	-100	900	40
9	01	-1	-50	950	45
10	11	1	50	1050	50

Figure 14.1-1 is a plot of the MFSK signal $x(t)$ given by (14.1-1), and Fig. 14.1-2 is a plot of the magnitude of the MFSK frequency spectrum $X(f)$ given by (14.1-21), using the parameter values shown in Tables 14.1-2 and 14.1-4. The bandwidth of $x(t)$ can be computed using (14.1-25). Since $M=4$ and $D=50$ symbols/sec, if $\text{NZC}=1$, 3, and 5, then $\text{BW}_x = 300$ Hz, 500 Hz, and 700 Hz, respectively. As can be seen from Fig. 14.1-2, a bandwidth of 500 Hz is a good estimate, whereas a bandwidth of 700 Hz is more conservative.

As was previously mentioned, an easy way to demodulate the received MFSK signal $y(t)$ is to compute the Fourier transform of each received pulse. The magnitude spectrum of each received pulse was shown to be directly proportional to the magnitude spectrum of each transmitted pulse [see (14.1-73)]. Therefore, for example purposes, we shall compute the Fourier transforms of the first four transmitted pulses shown in Fig. 14.1-1 using the algorithm discussed in Appendix 7B.

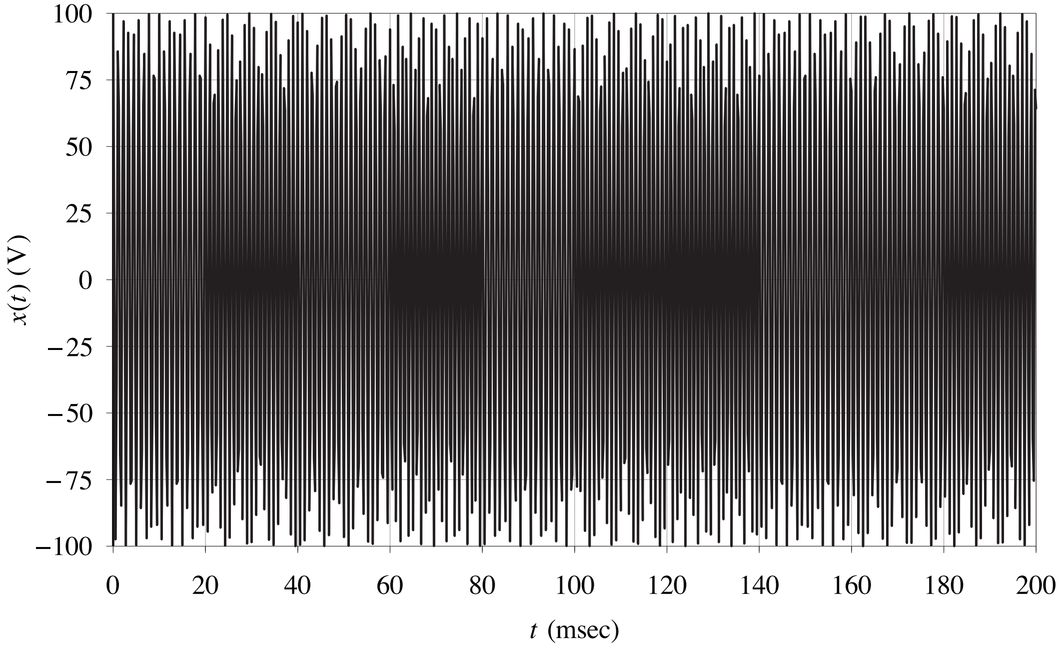


Figure 14.1-1 MFSK signal $x(t)$ given by (14.1-1) using the parameter values shown in Tables 14.1-2 and 14.1-4.

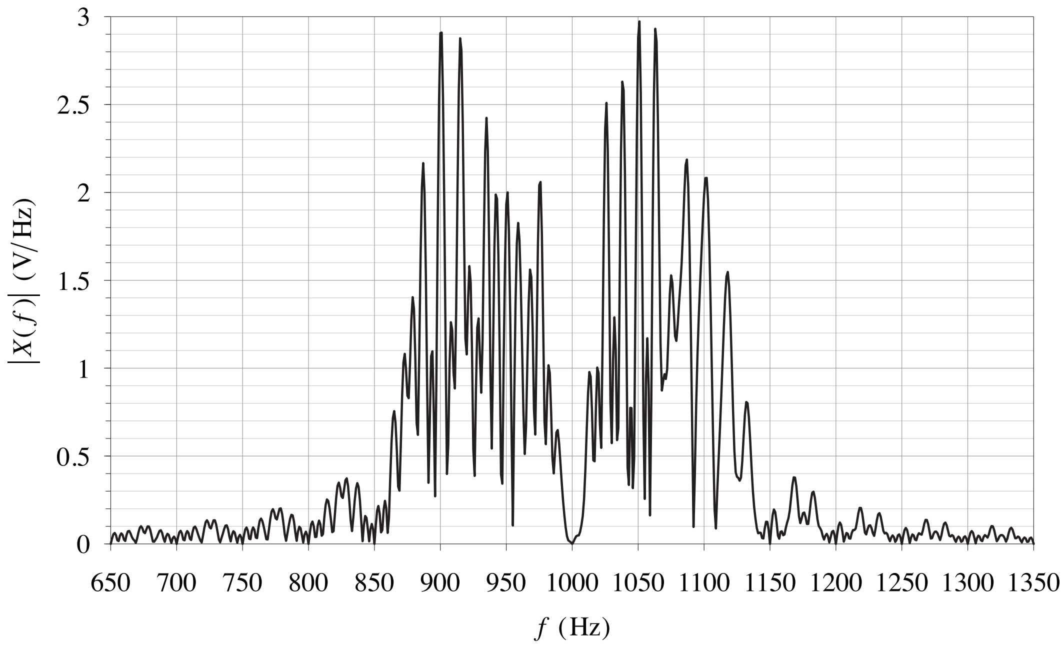


Figure 14.1-2 Magnitude of the MFSK frequency spectrum $X(f)$ given by (14.1-21) using the parameter values shown in Tables 14.1-2 and 14.1-4.

In order to compute the Fourier transforms, a sampling frequency of 4 kHz was used. Since each pulse has a pulse length of 20 msec, the total number of samples taken in a time interval of 20 msec was

$$N = f_S T = 4000\text{ Hz} \times 0.02\text{ sec} = 80\,. \tag{14.1-75}$$

If padding-with-zeros is not done, then $Z = 0$, the fundamental period $T_0 = T$, and the DFT bin spacing is

$$f_0 = \frac{1}{T_0} = \frac{1}{T} = \frac{1}{0.02\text{ sec}} = 50\text{ Hz}\,. \tag{14.1-76}$$

Since the four frequencies of interest, 900 Hz, 950 Hz, 1050 Hz, and 1100 Hz, are located at DFT bins when $Z = 0$, padding-with-zeros was not required. From (14.1-21), it can be seen that at $f = f_c + \Delta f_n$,

$$\left|X_n(f)\right| = \frac{A}{2}T = \frac{100\text{ V}}{2}(0.02\text{ sec}) = 1\text{ V/Hz}\,. \tag{14.1-77}$$

Table 14.1-5 Estimates of the Magnitude Spectra of the First Four Transmitted Pulses

		$n=1$	$n=2$	$n=3$	$n=4$
q	f (Hz)	$\lvert\hat{X}_1(f)\rvert$ (V/Hz)	$\lvert\hat{X}_2(f)\rvert$ (V/Hz)	$\lvert\hat{X}_3(f)\rvert$ (V/Hz)	$\lvert\hat{X}_4(f)\rvert$ (V/Hz)
18	900	1	0	0	0
19	950	0	0	1	0
20	1000	0	0	0	0
21	1050	0	1	0	0
22	1100	0	0	0	1

Estimates of the magnitude spectra of the first four transmitted pulses are shown in Table 14.1-5. In Table 14.1-5, integer q is the DFT bin number, frequency $f = qf_0$ Hz where the DFT bin spacing f_0 is given by (14.1-76), and $\hat{X}_n(f)$ is the estimate of the Fourier transform of $x_n(t)$ using the algorithm discussed in Appendix 7B. As can be seen in Table 14.1-5, $\left|\hat{X}_n(f)\right| = 1\text{ V/Hz}$, $n = 1, 2, 3, 4$, which agrees with the theoretical value given by (14.1-77). Since frequency 900 Hz is present in pulse 1, binary word 00 was transmitted (see

Table 14.1-3). Since frequency 1050 Hz is present in pulse 2, binary word 11 was transmitted. Since frequency 950 Hz is present in pulse 3, binary word 01 was transmitted. And since frequency 1100 Hz is present in pulse 4, binary word 10 was transmitted.

Once all the transmitted binary words have been determined, decode the binary words back into quantized voltages. One can then use the quantized voltages to approximate the original analog (continuous-time) message using the reconstruction formula from the Sampling Theorem. ■

14.2 *M*-ary Quadrature Amplitude Modulation

14.2.1 Time-Domain Description

If digital information is being transmitted using *M*-ary quadrature amplitude-modulation (MQAM), then a time-domain description of the transmitted signal $x(t)$ is given as follows:

$$x(t) = \sum_{n=1}^{N} x_n(t), \qquad 0 \le t \le T_d, \tag{14.2-1}$$

where N is the total number of transmitted pulses (symbols),

$$x_n(t) = A|w_n|\cos\left(2\pi f_c t + \angle w_n\right) p(t - t_n) \tag{14.2-2}$$

is the nth pulse where A is the amplitude factor in volts, $|w_n|$ and $\angle w_n$ are the magnitude and phase of the *complex symbol* (see Fig. 14.2-1)

$$w_n = |w_n|\exp(+j\angle w_n), \tag{14.2-3}$$

f_c is the carrier frequency in hertz, $p(t)$ is the pulse-shape function where

$$T = T_{\text{sym}} = n_b T_b \tag{14.2-4}$$

is the pulse length of an individual pulse, which is equal to the *symbol duration* T_{sym} in seconds – also known as (a.k.a.) the duration of a binary word; n_b is the number of bits per symbol – a.k.a. the number of bits per binary word; T_b is the *bit duration* in seconds,

$$M = 2^{n_b} \tag{14.2-5}$$

is the total *even* number of *unique* symbol values – the different unique symbol

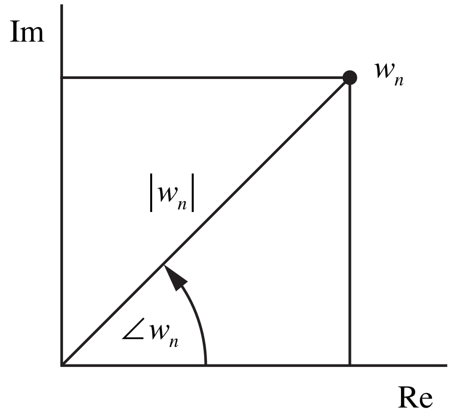


Figure 14.2-1 Complex symbol w_n in the complex plane.

values are known as the *alphabet*;

$$t_n = (n-1)T \tag{14.2-6}$$

is the time instant in seconds when the *n*th pulse (symbol) begins, and

$$T_d = NT \tag{14.2-7}$$

is the total duration in seconds of the transmitted signal $x(t)$. Note that the total number of transmitted pulses (symbols) N is greater than or equal to the total even number of unique symbol values M, that is,

$$N \geq M\ . \tag{14.2-8}$$

Because a MQAM signal uses a complex symbol in general, MQAM is a combination of amplitude-shift keying (ASK) via $|w_n|$, and M-ary phase-shift keying (MPSK) via $\angle w_n$.

By referring to (14.2-2), it can be seen that there is *no* frequency offset associated with the *n*th pulse. Also, the MQAM signal $x(t)$ is a *pulse train* where

$$\mathrm{PRI} = T \tag{14.2-9}$$

is the pulse-repetition interval (PRI) in seconds. In this section, the pulse-shape function $p(t)$ is equal to the time-shifted rectangle function. Since the $\mathrm{PRI} = T$, $p(t)$ is nonzero in the interval $[0, T)$ for the first $N-1$ pulses. Therefore, for $n = 1, 2, \ldots, N-1$,

$$p(t) = \mathrm{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise.} \end{cases} \tag{14.2-10}$$

For the last pulse $n = N$, $p(t)$ is nonzero in the interval $[0, T]$, that is,

$$p(t) = \text{rect}\left(\frac{t - 0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases} \tag{14.2-11}$$

By properly choosing values for $|w_n|$ and $\angle w_n$, several well known special cases can be obtained as shall be discussed next.

Amplitude-Shift Keying (ASK)

If we set $\angle w_n = 0 \;\; \forall \;\; n$, and replace $|w_n|$ with w_n, then (14.2-2) reduces to

$$x_n(t) = A w_n \cos(2\pi f_c t)\, p(t - t_n), \tag{14.2-12}$$

where

$$w_n \in \{2i - 1 - M\}_{i=1}^{M}. \tag{14.2-13}$$

For example, if $n_b = 2$, then $M = 2^{n_b} = 4$ and $w_n \in \{\pm 1, \pm 3\}$. For the special case known as *binary amplitude-shift keying* (BASK), $n_b = 1$. Therefore, $M = 2$ and

$$w_n = \begin{cases} 1, & \text{binary 1} \\ -1, & \text{binary 0.} \end{cases} \tag{14.2-14}$$

For *on-off keying* (OOK), use

$$w_n = \begin{cases} 1, & \text{binary 1} \\ 0, & \text{binary 0.} \end{cases} \tag{14.2-15}$$

M-ary Phase-Shift Keying (MPSK)

If we set $|w_n| = 1 \;\; \forall \;\; n$ so that $w_n = \exp(+j\angle w_n)$, then (14.2-2) reduces to

$$x_n(t) = A\cos\big(2\pi f_c t + \angle w_n\big)\, p(t - t_n), \tag{14.2-16}$$

where

$$\angle w_n \in \left\{\frac{2\pi}{M}(i-1)\right\}_{i=1}^{M} \tag{14.2-17}$$

or

$$\angle w_n \in \left\{\frac{2\pi}{M}(i-1) + \frac{\pi}{M}\right\}_{i=1}^{M}. \tag{14.2-18}$$

For example, for the special case known as 4-ary PSK, also known as *quadrature phase-shift keying* (QPSK), $n_b = 2$. Therefore, $M = 4$ and

$$\angle w_n \in \left\{ 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\}, \tag{14.2-19}$$

or

$$\angle w_n \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}. \tag{14.2-20}$$

For the special case known as *binary phase-shift keying* (BPSK), $n_b = 1$. Therefore, $M = 2$ and

$$\angle w_n = \begin{cases} 0, & \text{binary 1} \\ \pi, & \text{binary 0} \end{cases} \tag{14.2-21}$$

or

$$\angle w_n = \begin{cases} \pi/2, & \text{binary 1} \\ 3\pi/2, & \text{binary 0.} \end{cases} \tag{14.2-22}$$

Equations (14.2-1) and (14.2-2) represent one way to describe a MQAM signal in the time domain. An alternate time-domain description can be obtained by rewriting the equation for the nth transmitted pulse $x_n(t)$ given by (14.2-2) in terms of its cosine and sine components. Since $x_n(t)$ is an amplitude-and-angle-modulated carrier with amplitude-modulating function

$$a_n(t) = A|w_n|\, p(t - t_n), \tag{14.2-23}$$

and angle-modulating function

$$\theta_n(t) = \angle w_n, \tag{14.2-24}$$

it can be rewritten as

$$x_n(t) = x_{cn}(t)\cos(2\pi f_c t) - x_{sn}(t)\sin(2\pi f_c t), \tag{14.2-25}$$

where

$$x_{cn}(t) = a_n(t)\cos\theta_n(t) = A|w_n|\cos(\angle w_n)\, p(t - t_n) \tag{14.2-26}$$

is the cosine component of $x_n(t)$, and

$$x_{sn}(t) = a_n(t)\sin\theta_n(t) = A|w_n|\sin(\angle w_n)\, p(t - t_n) \tag{14.2-27}$$

is the sine component of $x_n(t)$ (see Section 11.2). By referring to (14.2-3), it can be seen that

$$\mathrm{Re}\{w_n\} = |w_n|\cos(\angle w_n) \tag{14.2-28}$$

and

$$\mathrm{Im}\{w_n\} = |w_n|\sin(\angle w_n)\,. \tag{14.2-29}$$

Substituting (14.2-28) into (14.2-26), and (14.2-29) into (14.2-27) yields

$$x_{cn}(t) = A\,\mathrm{Re}\{w_n\}\,p(t-t_n) \tag{14.2-30}$$

and

$$x_{sn}(t) = A\,\mathrm{Im}\{w_n\}\,p(t-t_n)\,, \tag{14.2-31}$$

respectively, and by substituting (14.2-30) and (14.2-31) into (14.2-25), we obtain

$$x_n(t) = A\,\mathrm{Re}\{w_n\}\,p(t-t_n)\cos(2\pi f_c t) - A\,\mathrm{Im}\{w_n\}\,p(t-t_n)\sin(2\pi f_c t)\,. \tag{14.2-32}$$

Therefore, substituting (14.2-32) into (14.2-1) yields

$$x(t) = x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t), \qquad 0 \le t \le T_d\,, \tag{14.2-33}$$

where

$$x_c(t) = A\sum_{n=1}^{N}\mathrm{Re}\{w_n\}\,p(t-t_n) \tag{14.2-34}$$

is the cosine component of $x(t)$, and

$$x_s(t) = A\sum_{n=1}^{N}\mathrm{Im}\{w_n\}\,p(t-t_n) \tag{14.2-35}$$

is the sine component of $x(t)$. Equations (14.2-33) through (14.2-35) represent an alternative way of describing a MQAM signal in the time domain.

Example 14.2-1

In this example we shall consider the problem of transmitting binary words using MQAM where a binary word is equal to $n_b = 3$ bits. Since $n_b = 3$, there are $M = 2^3 = 8$ unique *pairs* of symbol values $\left(\mathrm{Re}\{w_n\}, \mathrm{Im}\{w_n\}\right)$, and 8 unique binary words (8 possible combinations of 3 bits).

The next step is to assign a unique pair of symbol values to each of the 8 unique binary words. For example, see Table 14.2-1 where the 8 unique binary

Table 14.2-1 Eight Unique Pairs of Symbol Values Assigned to Eight Unique Binary Words for a Rectangular Signal-Space Constellation

Gray Code	$\mathrm{Re}\{w_n\}$	$\mathrm{Im}\{w_n\}$
000	-3	-1
001	-1	-1
011	1	-1
010	3	-1
100	-3	1
101	-1	1
111	1	1
110	3	1

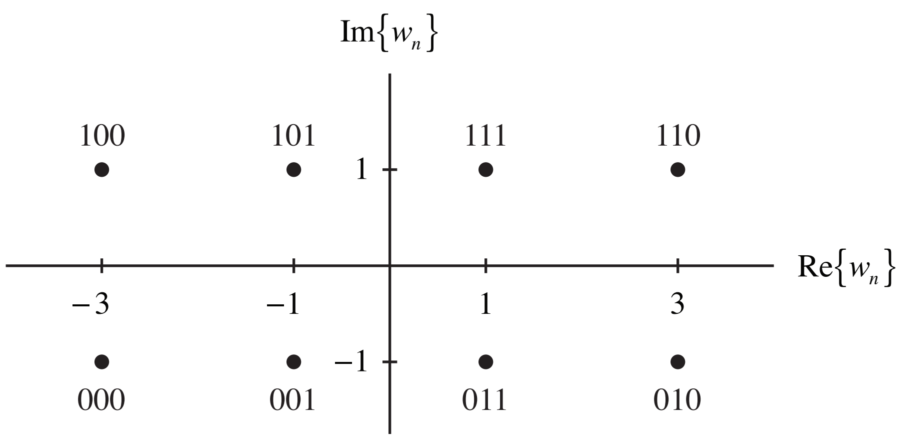


Figure 14.2-2 Rectangular signal-space constellation for Gray-encoded 8-QAM.

words are arranged in a Gray code to form a rectangular signal-space constellation. Figure 14.2-2 is a pictorial representation of Table 14.2-1. As can be seen from Fig. 14.2-2, only one bit is changed going from one signal-point to an adjacent signal-point in the constellation, either in the horizontal or vertical direction. Also, the distance between adjacent signal-points, either in the horizontal or vertical direction, is 2.

The 8 unique binary words can also be encoded as shown in Table 14.2-2. Figure 14.2-3 is a pictorial representation of Table 14.2-2. As can be seen from Fig. 14.2-3, only one bit is changed going from one signal-point to an adjacent signal-point on the inner circle in the constellation, and from one signal-point to an adjacent signal-point on the outer circle in the constellation. The distance between adjacent signal-points on the inner circle, either in the horizontal or

Table 14.2-2 Eight Unique Pairs of Symbol Values Assigned to Eight Unique Binary Words for a Concentrically-Circular Signal-Space Constellation

Inner Circle				Outer Circle		
Gray Code	$\mathrm{Re}\{w_n\}$	$\mathrm{Im}\{w_n\}$		Gray Code	$\mathrm{Re}\{w_n\}$	$\mathrm{Im}\{w_n\}$
001	-1	-1		000	$-\left(1+\sqrt{3}\right)$	0
011	1	-1		010	0	$-\left(1+\sqrt{3}\right)$
111	1	1		110	$1+\sqrt{3}$	0
101	-1	1		100	0	$1+\sqrt{3}$

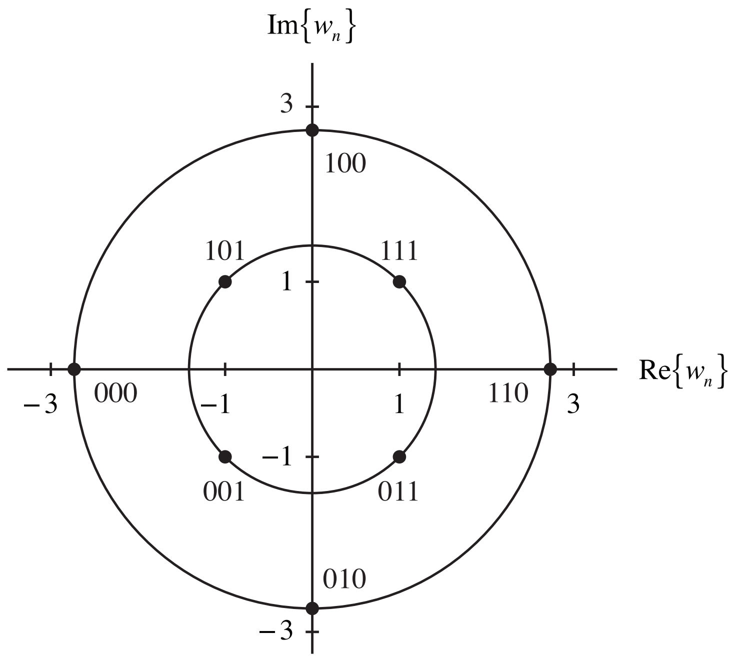


Figure 14.2-3 Concentrically-circular signal-space constellation for Gray-encoded 8-QAM.

vertical direction, is 2. The distance between adjacent signal-points on the inner and outer circles, measured diagonally, is also 2 (see Fig. 14.2-4). Figure 14.2-3 is the *optimum* 8-QAM signal-space constellation because it requires the smallest time-average power for a given distance between adjacent signal points[1] (see Example 14.2-2).

[1] J. G. Proakis and M. Salehi, *Digital Communications*, 5th ed., McGraw-Hill, New York, 2008, pg. 197.

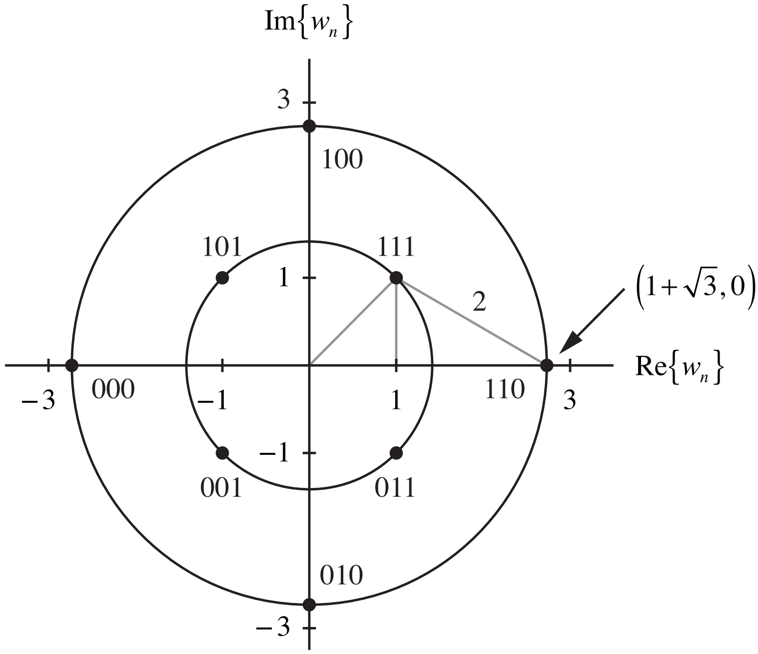


Figure 14.2-4 Right triangles used to compute the radii of the inner and outer circles.

Figure 14.2-4 shows the right triangles used to compute the radii of the inner and outer circles. By referring to Fig. 14.2-4, it can be seen that the radius of the inner circle is

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}\,. \tag{14.2-36}$$

Since the base of the right triangle with hypotenuse 2 is

$$b = \sqrt{2^2 - 1^2} = \sqrt{3}\,, \tag{14.2-37}$$

the radius of the outer circle is

$$r = 1 + \sqrt{3}\,. \tag{14.2-38}$$

And finally, by referring to Fig. 14.2-1:

$$|w_n| = \sqrt{\left(\mathrm{Re}\{w_n\}\right)^2 + \left(\mathrm{Im}\{w_n\}\right)^2} \tag{14.2-39}$$

and

$$\angle w_n = \tan^{-1}\left[\mathrm{Im}\{w_n\}/\mathrm{Re}\{w_n\}\right]. \tag{14.2-40}$$

So although Tables 14.2-1 and 14.2-2 specify values for the real and imaginary parts of the complex symbol w_n, with the use of (14.2-39) and (14.2-40), the magnitude and phase of w_n can also be computed. ■

14.2.2 Frequency Spectrum and Bandwidth

In order to derive an equation for the bandwidth of the MQAM signal $x(t)$ given by (14.2-1), we first have to find its frequency spectrum. Taking the Fourier transform of (14.2-1) yields

$$X(f) = \sum_{n=1}^{N} X_n(f)\,, \tag{14.2-41}$$

where

$$X(f) = F\{x(t)\} \tag{14.2-42}$$

and

$$X_n(f) = F\{x_n(t)\} \tag{14.2-43}$$

is the frequency spectrum of the *n*th pulse $x_n(t)$ given by (14.2-2). The Fourier transform of $x_n(t)$ can be obtained by using the Fourier-transform pair

$$a(t)\cos(2\pi f_c t + \theta_0) \leftrightarrow \frac{1}{2}A(f - f_c)\exp(+j\theta_0) + \frac{1}{2}A(f + f_c)\exp(-j\theta_0)\,, \tag{14.2-44}$$

where

$$a(t) = A\left|w_n\right| p(t - t_n) \tag{14.2-45}$$

and $\theta_0 = \angle w_n$. Taking the Fourier transform of (14.2-45) yields

$$A(f) = A\left|w_n\right| P(f)\exp(-j2\pi f t_n)\,, \tag{14.2-46}$$

and by substituting (14.2-6) into (14.2-46), we obtain

$$A(f) = A\left|w_n\right| P(f)\exp(-j2\pi f nT)\exp(+j2\pi f T)\,. \tag{14.2-47}$$

Since the Fourier transform of the time-shifted rectangle function $p(t)$ is given by

$$P(f) = T\,\mathrm{sinc}(fT)\exp(-j\pi f T)\,, \tag{14.2-48}$$

substituting (14.2-48) into (14.2-47) yields

$$A(f) = A\left|w_n\right|T\,\mathrm{sinc}(fT)\exp(-j2\pi fnT)\exp(+j\pi fT). \qquad (14.2\text{-}49)$$

Therefore, substituting (14.2-49), $\theta_0 = \angle w_n$, and (14.2-3) into the right-hand side of (14.2-44) yields

$$X_n(f) = \frac{A}{2}T w_n \mathrm{sinc}\big[(f-f_c)T\big]\exp\big[-j2\pi(f-f_c)nT\big]\exp\big[+j\pi(f-f_c)T\big] + \frac{A}{2}T w_n^* \mathrm{sinc}\big[(f+f_c)T\big]\exp\big[-j2\pi(f+f_c)nT\big]\exp\big[+j\pi(f+f_c)T\big], \qquad (14.2\text{-}50)$$

and by substituting (14.2-50) into (14.2-41), we finally obtain the frequency spectrum of the MQAM signal $x(t)$ given by (14.2-1):

$$\boxed{X(f) = \frac{A}{2}T\left[\sum_{n=1}^{N} w_n \exp\big[-j2\pi(f-f_c)nT\big]\right]\mathrm{sinc}\big[(f-f_c)T\big]\exp\big[+j\pi(f-f_c)T\big] + \frac{A}{2}T\left[\sum_{n=1}^{N} w_n^* \exp\big[-j2\pi(f+f_c)nT\big]\right]\mathrm{sinc}\big[(f+f_c)T\big]\exp\big[+j\pi(f+f_c)T\big]} \qquad (14.2\text{-}51)$$

The units of $X(f)$ are volts per hertz.

Since bandwidth is always measured along the positive frequency axis, we shall work with the first term in (14.2-51), that is,

$$X(f) = \frac{A}{2}T\left[\sum_{n=1}^{N} w_n \exp\big[-j2\pi(f-f_c)nT\big]\right]\mathrm{sinc}\big[(f-f_c)T\big]\times \exp\big[+j\pi(f-f_c)T\big], \qquad f \ge 0. \qquad (14.2\text{-}52)$$

By examining the argument of the sinc function in (14.2-52), estimates of the maximum and minimum frequency components are given by

$$f_{\max} = f_c + \frac{\mathrm{NZC}}{T} \qquad (14.2\text{-}53)$$

and

$$f_{\min} = f_c - \frac{\mathrm{NZC}}{T}, \qquad (14.2\text{-}54)$$

where NZC is the integer number of zero-crossings of the sinc function that is used to estimate both the maximum and minimum frequency components $f_{\max}$ and $f_{\min}$, respectively. Therefore, the bandwidth BW_x (in hertz) of the MQAM signal is given by

$$\mathrm{BW}_x = f_{\max} - f_{\min}, \tag{14.2-55}$$

and by substituting (14.2-53), (14.2-54), and (14.2-4) into (14.2-55), we obtain

$$\boxed{\mathrm{BW}_x = 2\,\mathrm{NZC} \times D} \tag{14.2-56}$$

where

$$\boxed{D = \frac{1}{T_{\mathrm{sym}}} = \frac{1}{n_b T_b} = \frac{R_b}{n_b}} \tag{14.2-57}$$

is the *symbol rate (baud)* with units of symbols per second, and

$$\boxed{R_b = 1/T_b} \tag{14.2-58}$$

is the *bit rate* in bits per second. Since $\max|\mathrm{sinc}(fT)| = 1$ for $f = 0$; $\mathrm{sinc}(fT) = 0$ for $f = i/T$, where $i = \pm 1, \pm 2, \ldots$; and $|\mathrm{sinc}(fT)| < 0.1$ for $f > 3/T$; NZC should be at least 3, with 5 being a conservative choice.

14.2.3 Signal Energy and Time-Average Power

In this subsection we shall compute the energy and time-average power of the MQAM signal $x(t)$ given by (14.2-1). We begin by computing the energy of the nth pulse $x_n(t)$ given by (14.2-2). Note that $x_n(t)$ is an amplitude-and-angle-modulated carrier with amplitude and angle-modulating functions

$$a_n(t) = A|w_n|\,p(t - t_n) \tag{14.2-59}$$

and

$$\theta_n(t) = \angle w_n, \tag{14.2-60}$$

respectively (see Section 11.2), where the pulse-shape function $p(t)$ is given by (14.2-10) and (14.2-11). In order to compute the energy of $x_n(t)$, we shall first compute the energy of its complex envelope $\tilde{x}_n(t)$.

Since the general form of the complex envelope of $x_n(t)$ is given by (see Section 11.2)

$$\tilde{x}_n(t) = a_n(t)\exp[+j\theta_n(t)], \tag{14.2-61}$$

substituting (14.2-59) and (14.2-60) into (14.2-61) yields

$$\tilde{x}_n(t) = A\left|w_n\right|\exp(+j\angle w_n)\,p(t-t_n) = Aw_n\,p(t-t_n)\,. \tag{14.2-62}$$

The energy $E_{\tilde{x}_n}$ of $\tilde{x}_n(t)$ is given by

$$E_{\tilde{x}_n} = \int_{-\infty}^{\infty} a_n^2(t)\,dt\,, \tag{14.2-63}$$

and the energy E_{x_n} of $x_n(t)$ is given by (see Section 11.2)

$$E_{x_n} = E_{\tilde{x}_n}/2\,. \tag{14.2-64}$$

Therefore, substituting (14.2-59) into (14.2-63) yields

$$E_{\tilde{x}_n} = A^2\left|w_n\right|^2 T\,, \tag{14.2-65}$$

which is the energy (in joules-ohms) of the complex envelope $\tilde{x}_n(t)$ given by (14.2-62), and substituting (14.2-65) into (14.2-64) yields

$$\boxed{E_{x_n} = \frac{A^2}{2}\left|w_n\right|^2 T} \tag{14.2-66}$$

which is the energy (in joules-ohms) of $x_n(t)$ given by (14.2-2), where $\left|w_n\right|$ is given by (14.2-39). The time-average power of $x_n(t)$ in watts-ohms is given by

$$\boxed{P_{\text{avg},x_n,T} = \frac{E_{x_n}}{T} = \frac{A^2}{2}\left|w_n\right|^2} \tag{14.2-67}$$

From signal theory, the energy E_x of a signal $x(t)$ is defined as

$$E_x \triangleq \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt\,. \tag{14.2-68}$$

Using (14.2-1),

$$\left|x(t)\right|^2 = x(t)x^*(t) = \sum_{m=1}^{N}\sum_{n=1}^{N} x_m(t)x_n^*(t)\,, \tag{14.2-69}$$

and since

$$x_m(t)\,x_n^*(t) = \begin{cases} |x_n(t)|^2, & m = n \\ 0, & m \neq n \end{cases} \tag{14.2-70}$$

because $p(t-t_m)\,p(t-t_n) = 0$, $m \neq n$; (14.2-69) reduces to

$$|x(t)|^2 = \sum_{n=1}^{N} |x_n(t)|^2 . \tag{14.2-71}$$

Substituting (14.2-71) into (14.2-68) yields

$$E_x = \sum_{n=1}^{N} \int_{-\infty}^{\infty} |x_n(t)|^2\, dt = \sum_{n=1}^{N} E_{x_n} . \tag{14.2-72}$$

Substituting (14.2-66) into (14.2-72) yields

$$\boxed{E_x = \frac{A^2}{2} T \sum_{n=1}^{N} |w_n|^2} \tag{14.2-73}$$

which is the energy (in joules-ohms) of the MQAM signal $x(t)$ given by (14.2-1), where $|w_n|$ is given by (14.2-39). Therefore, the time-average power of $x(t)$ (in watts-ohms) in the time interval $[0, T_d]$ is given by

$$\boxed{P_{\text{avg},x,T_d} = \frac{E_x}{T_d} = \frac{A^2}{2} \frac{1}{N} \sum_{n=1}^{N} |w_n|^2} \tag{14.2-74}$$

where T_d is given by (14.2-7). Table 14.2-3 compares the signal bandwidths and time-average powers of MFSK and MQAM signals. As can be seen from Table 14.2-3, MQAM has a smaller bandwidth but a larger time-average power, in general, compared to MFSK. For the special case of MPSK where $|w_n| = 1 \ \forall \ n$, the time-average powers of MFSK and MQAM signals are equal. In addition, both signals have the same total duration T_d. Recall that the pulse length T is equal to the symbol duration T_{sym} in both MFSK and MQAM signals [see (14.1-4) and (14.2-4)].

Table 14.2-3 Signal Bandwidth and Time-Average Power of MFSK and MQAM Signals

	Signal Bandwidth BW_x (Hz)	Time-Average Power (W-Ω)
MFSK	$(M+2\,\mathrm{NZC})D$	$P_{\mathrm{avg},x,T_d}=\dfrac{A^2}{2}$ $T_d = NT_{\mathrm{sym}}$
MQAM	$2\,\mathrm{NZC}\times D$	$P_{\mathrm{avg},x,T_d}=\dfrac{A^2}{2}\dfrac{1}{N}\sum_{n=1}^{N}\lvert w_n\rvert^2$ $T_d = NT_{\mathrm{sym}}$

Example 14.2-2

In this example we shall compute the time-average power of a MQAM signal using the 8-QAM rectangular signal-space constellation shown in Table 14.2-1 and Fig. 14.2-2, and the optimum 8-QAM concentrically-circular signal-space constellation shown in Table 14.2-2 and Fig. 14.2-3. By substituting (14.2-39) into (14.2-74), the time-average power of a MQAM signal can be expressed as

$$P_{\mathrm{avg},x,T_d}=\frac{A^2}{2}\frac{1}{N}\sum_{n=1}^{N}\left[\left(\mathrm{Re}\{w_n\}\right)^2+\left(\mathrm{Im}\{w_n\}\right)^2\right]. \tag{14.2-75}$$

For example purposes, let the total number of transmitted pulses (symbols) $N = M = 8$. If the 8 transmitted symbols are those shown in Table 14.2-1, then

$$P_{\mathrm{avg},x,T_d} = 3A^2 \text{ W-}\Omega\,, \tag{14.2-76}$$

and if the 8 transmitted symbols are those shown in Table 14.2-2, then

$$P_{\mathrm{avg},x,T_d} = 2.37A^2 \text{ W-}\Omega\,. \tag{14.2-77}$$

Therefore, the 8-QAM concentrically-circular signal-space constellation requires less time-average power than the 8-QAM rectangular signal-space constellation. ■

Example 14.2-3

In this example we shall consider the problem of transmitting the following binary words

00 11 01 10 00 11 10 00 01 11

using MQAM where a binary word is equal to $n_b = 2$ bits. This is the same problem that was considered in Example 14.1-1 using MFSK. Since there are 10 binary words (20 bits), the total number of pulses (symbols) to be transmitted is $N = 10$. Also, since $n_b = 2$, there are $M = 2^2 = 4$ unique *pairs* of symbol values $(\text{Re}\{w_n\}, \text{Im}\{w_n\})$, and 4 unique binary words (4 possible combinations of 2 bits).

The next step is to assign a unique pair of symbol values to each of the 4 unique binary words. For example, see Table 14.2-4 where the 4 unique binary words are arranged in a Gray code to form a rectangular signal-space constellation. The set of pairs of numbers $\{(-1,-1),(1,-1),(1,1),(-1,1)\}$ is the alphabet. Figure 14.2-5 is a pictorial representation of Table 14.2-4. As can be seen from Fig. 14.2-5, only one bit is changed going from one signal-point to an adjacent signal-point in the constellation, either in the horizontal or vertical direction. Also, the distance between adjacent signal-points, either in the horizontal or vertical direction, is 2.

Table 14.2-4 Four Unique Pairs of Symbol Values Assigned to Four Unique Binary Words for a Rectangular Signal-Space Constellation

Gray Code	$\text{Re}\{w_n\}$	$\text{Im}\{w_n\}$
00	-1	-1
01	1	-1
11	1	1
10	-1	1

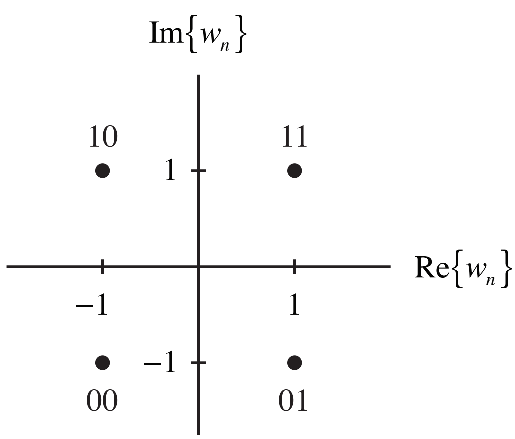


Figure 14.2-5 Rectangular signal-space constellation for Gray-encoded 4-QAM.

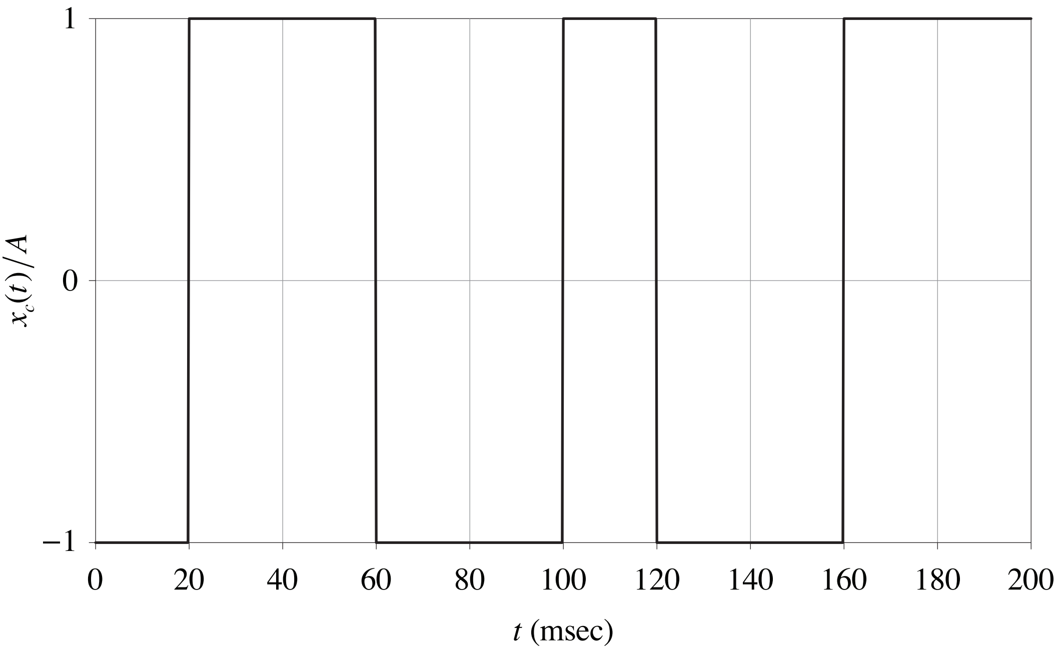
In order to simulate a time-domain MQAM signal using the information given up to this point, we need values for the amplitude factor A, the carrier frequency f_c, and the pulse length T of an individual pulse. For example, if $A = 100\text{ V}$, $f_c = 1\text{ kHz}$, and $T = T_{\text{sym}} = n_b T_b = 20\text{ msec}$, then we obtain the parameter values shown in Table 14.2-5 for the MQAM signal. Note that the values for A and T were chosen so that the factor $AT/2 = 1$ in (14.2-51). Using the pairs of symbol values assigned to the 4 unique binary words in Table 14.2-4, Table 14.2-6 shows the pairs of symbol values for the 10 transmitted pulses associated with the 10 binary words.

Table 14.2-5 Parameter Values of the Transmitted MQAM Signal

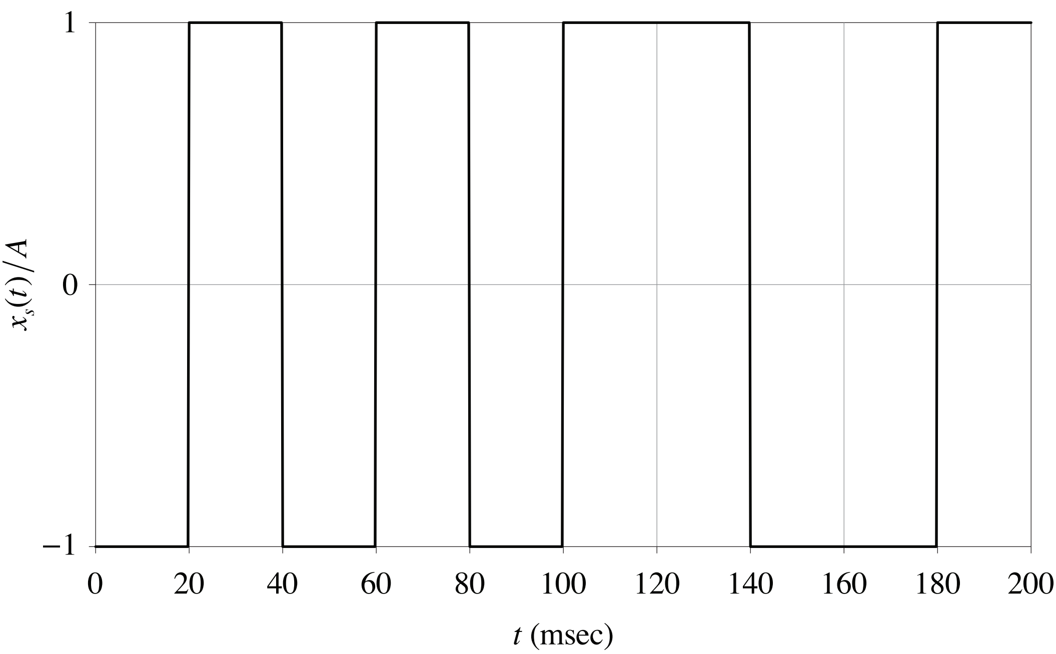
Amplitude factor A	100 V
Carrier frequency f_c	1 kHz
Pulse length of an individual pulse T	20 msec
Total number of transmitted pulses (symbols) N	10
Total duration of the transmitted signal $T_d = NT$	200 msec
Symbol duration T_{sym}	20 msec
Baud (symbol rate) $D = 1/T_{\text{sym}}$	50 symbols/sec
Number of bits per symbol (binary word) n_b	2
Number of *unique* symbol values $M = 2^{n_b}$	4
Bit duration $T_b = T_{\text{sym}}/n_b$	10 msec
Bit rate $R_b = 1/T_b$	100 bps
Time-average power P_{avg,x,T_d}	10,000 W-Ω

Table 14.2-6 Pairs of Symbol Values for the Ten Transmitted Pulses

n	Binary Word	$\text{Re}\{w_n\}$	$\text{Im}\{w_n\}$
1	00	−1	−1
2	11	1	1
3	01	1	−1
4	10	−1	1
5	00	−1	−1
6	11	1	1
7	10	−1	1
8	00	−1	−1
9	01	1	−1
10	11	1	1



(a)



(b)

Figure 14.2-6 (a) Normalized MQAM cosine component, and (b) normalized MQAM sine component using the parameter values shown in Tables 14.2-5 and 14.2-6.

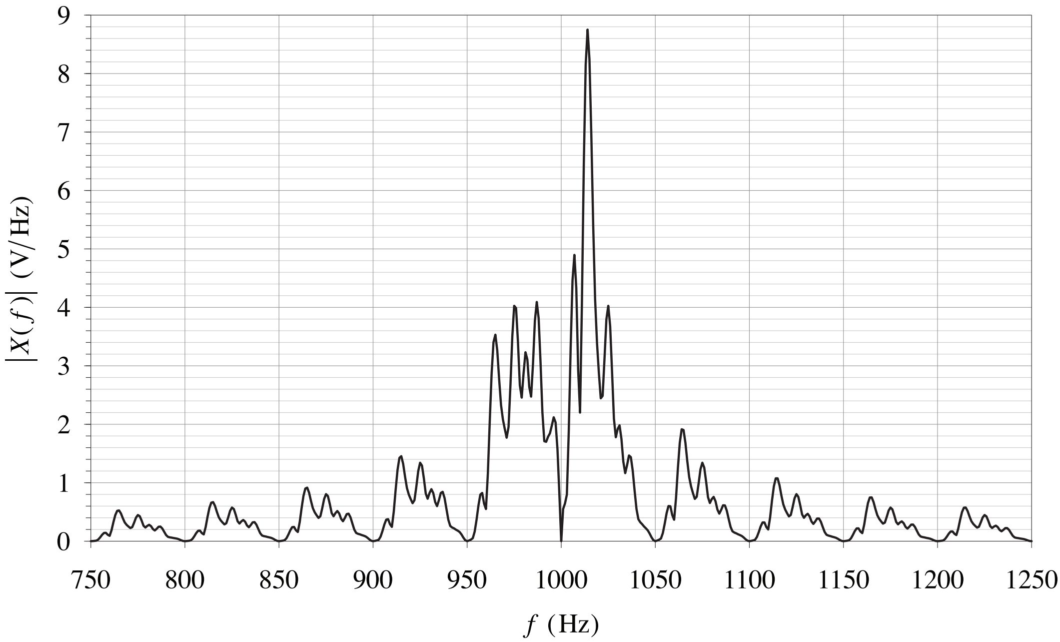


Figure 14.2-7 Magnitude of the MQAM frequency spectrum $X(f)$ given by (14.2-52) using the parameter values shown in Tables 14.2-5 and 14.2-6.

Figure 14.2-6 (a) is a plot of the normalized MQAM cosine component $x_c(t)/A$, Fig. 14.2-6 (b) is a plot of the normalized MQAM sine component $x_s(t)/A$, and Fig. 14.2-7 is a plot of the magnitude of the MQAM frequency spectrum $X(f)$ given by (14.2-52), using the parameter values shown in Tables 14.2-5 and 14.2-6. The bandwidth of $x(t)$ can be computed using (14.2-56). Since $D = 50$ symbols/sec, if $\text{NZC} = 1$, 3, and 5, then $\text{BW}_x = 100$ Hz, 300 Hz, and 500 Hz, respectively. As can be seen from Fig. 14.2-7, a bandwidth of 300 Hz is a good estimate, whereas a bandwidth of 500 Hz is more conservative. ■

14.2.4 Demodulation

As was mentioned in Subsection 14.1.5, demodulation is a procedure meant to retrieve a transmitted message at a receiver. In order to discuss demodulation, we first need to model a MQAM signal at a receiver, which we shall designate as $y(t)$. We shall use the following simple model for $y(t)$:

$$y(t) = Kx(t-\tau), \qquad \tau \le t \le \tau + T_d, \tag{14.2-78}$$

where the transmitted signal $x(t)$ is given by (14.2-33). As can be seen from

(14.2-78), $y(t)$ is modeled as an amplitude-scaled, time-delayed version of $x(t)$ (distortionless transmission) as was done for a MFSK signal at a receiver (see Subsection 14.1.5) where $K > 0$ is a dimensionless constant. Substituting (14.2-33) into (14.2-78) yields

$$y(t) = Kx_c(t-\tau)\cos(2\pi f_c t + \delta) - Kx_s(t-\tau)\sin(2\pi f_c t + \delta), \qquad \tau \le t \le \tau + T_d, \tag{14.2-79}$$

where [see (14.2-34) and (14.2-35)]

$$x_c(t-\tau) = A\sum_{n=1}^{N} \mathrm{Re}\{w_n\}\, p\big(t - [\tau + t_n]\big), \tag{14.2-80}$$

$$x_s(t-\tau) = A\sum_{n=1}^{N} \mathrm{Im}\{w_n\}\, p\big(t - [\tau + t_n]\big), \tag{14.2-81}$$

$p(t)$ is given by (14.2-10) and (14.2-11),

$$t_n = (n-1)T, \tag{14.2-82}$$

and

$$\delta = -2\pi f_c \tau \tag{14.2-83}$$

is a phase shift in radians that takes into account the time delay τ in seconds.

The received signal $y(t)$ can be demodulated by passing it through a quadrature-demodulator (QD) (see Section 11.3). As shown in Section 11.3, if there are *no* frequency and phase offsets, the outputs from the QD are

$$y_1(t) = GKx_c\big(t - [\tau + t_0]\big) \tag{14.2-84}$$

and

$$y_2(t) = GKx_s\big(t - [\tau + t_0]\big), \tag{14.2-85}$$

where $G > 0$ is the dimensionless gain and t_0 is the constant time delay of the ideal, lowpass filters in the QD. Substituting (14.2-80) into (14.2-84), and (14.2-81) into (14.2-85) yields

$$y_1(t) = GKA\sum_{n=1}^{N} \mathrm{Re}\{w_n\}\, p\big(t - [\tau + t_0 + t_n]\big) \tag{14.2-86}$$

and

$$y_2(t) = GKA\sum_{n=1}^{N} \mathrm{Im}\{w_n\}\, p\big(t - [\tau + t_0 + t_n]\big), \tag{14.2-87}$$

respectively. Each T seconds worth of data from $y_1(t)$ and $y_2(t)$ contains the values of $\text{Re}\{w_n\}$ and $\text{Im}\{w_n\}$ in the *n*th transmitted pulse, respectively. Once all the transmitted pairs of symbol values $(\text{Re}\{w_n\}, \text{Im}\{w_n\})$ have been determined, decode the pairs of symbol values back into binary words, and then decode the binary words back into quantized voltages. One can then use the quantized voltages to approximate the original analog (continuous-time) message using the reconstruction formula from the Sampling Theorem.

14.3 Orthogonal Frequency-Division Multiplexing

14.3.1 Time-Domain Description

If digital information is being transmitted using orthogonal frequency-division multiplexing (OFDM), then a time-domain description of the transmitted signal $x(t)$ is given as follows:

$$x(t) = \sum_{n=0}^{N-1} x_n(t), \qquad 0 \le t \le T , \tag{14.3-1}$$

where N is the total number of pulses (symbols) that are transmitted *simultaneously*,

$$x_n(t) = A\left|w_n\right|\cos\left(2\pi[f_c + \Delta f_n]t + \angle w_n + \varepsilon_n\right)\text{rect}\left[(t - 0.5T)/T\right] \tag{14.3-2}$$

is the *n*th pulse, also known as (a.k.a.) the *n*th *subcarrier*, where A is the amplitude factor in volts; $\left|w_n\right|$ and $\angle w_n$ are the magnitude and phase of the *complex symbol* (see Fig. 14.2-1)

$$w_n = \left|w_n\right|\exp(+j\angle w_n) , \tag{14.3-3}$$

f_c is the carrier frequency in hertz, the *frequency offset* in hertz is given by either

$$\Delta f_n = \frac{n - N'}{T}, \qquad N' = \begin{cases} N/2, & N \text{ even} \\ (N-1)/2, & N \text{ odd} \end{cases} \tag{14.3-4}$$

or

$$\Delta f_n = n/T , \tag{14.3-5}$$

$$T = NT_{\text{sym}} \tag{14.3-6}$$

is the pulse length of an individual pulse, which is equal to the total duration of the transmitted signal in seconds;

$$T_{\text{sym}} = n_b T_b \tag{14.3-7}$$

is the *input symbol duration* in seconds – a.k.a. the duration of a binary word; n_b is the number of bits per symbol – a.k.a. the number of bits per binary word; T_b is the *bit duration* in seconds,

$$M = 2^{n_b} \tag{14.3-8}$$

is the total *even* number of *unique* symbol values – the different unique symbol values are known as the *alphabet*; ε_n is a possible, unwanted phase shift in radians at the transmitter, and

$$\text{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{14.3-9}$$

is the time-shifted rectangle function. Note that the total number of transmitted pulses (symbols) N is greater than or equal to the total even number of unique symbol values M , that is,

$$N \ge M \,. \tag{14.3-10}$$

Because the numerator of the frequency offset Δf_n given by either (14.3-4) or (14.3-5) is an *integer*, if the product $f_c T$ is also equal to an *integer*, or $f_c T >> 1$ if $f_c T$ does not equal an integer, then the set of functions $x_n(t)$, $n = 0, 1, \ldots, N-1$, given by (14.3-2) is an *orthogonal* set of functions in the time interval $[0, T]$ (see Subsection 14.1.4). Whereas MFSK and MQAM signals are pulse trains, OFDM signals are not. Because an OFDM signal uses a complex symbol in general, OFDM is a combination of amplitude-shift keying (ASK) via $|w_n|$, and *M*-ary phase-shift keying (MPSK) via $\angle w_n$. See Subsection 14.2.1 for how to choose w_n for the special cases of amplitude-shift keying (ASK) and *M*-ary Phase-Shift Keying (MPSK).

14.3.2 Frequency Spectrum and Bandwidth

In order to derive an equation for the bandwidth of the OFDM signal $x(t)$ given by (14.3-1), we first have to find its frequency spectrum. Taking the Fourier transform of (14.3-1) yields

$$X(f) = \sum_{n=0}^{N-1} X_n(f) \,, \tag{14.3-11}$$

where

$$X(f) = F\{x(t)\} \tag{14.3-12}$$

and

$$X_n(f) = F\{x_n(t)\} \tag{14.3-13}$$

is the frequency spectrum of the nth pulse $x_n(t)$ given by (14.3-2). The Fourier transform of $x_n(t)$ can be obtained by using the Fourier-transform pair

$$\begin{aligned} A\cos(2\pi f_c t + \theta_0)\operatorname{rect}\left(\frac{t - 0.5T}{T}\right) \leftrightarrow \frac{A}{2}T\operatorname{sinc}[(f - f_c)T]\exp[-j\pi(f - f_c)T]\exp(+j\theta_0) + \\ \frac{A}{2}T\operatorname{sinc}[(f + f_c)T]\exp[-j\pi(f + f_c)T]\exp(-j\theta_0) \end{aligned} \tag{14.3-14}$$

and replacing A with $A|w_n|$, f_c with $f_c + \Delta f_n$, and θ_0 with $\angle w_n + \varepsilon_n$. Doing so and substituting the resulting expression into (14.3-11) yields the frequency spectrum of the OFDM signal $x(t)$ given by (14.3-1):

$$\boxed{\begin{aligned} X(f) = \frac{A}{2}T c_1(f)\sum_{n=0}^{N-1}|w_n|\operatorname{sinc}\{[f - (f_c + \Delta f_n)]T\}\exp(+j\pi\Delta f_n T)\exp(+j\angle w_n)\exp(+j\varepsilon_n) + \\ \frac{A}{2}T c_2(f)\sum_{n=0}^{N-1}|w_n|\operatorname{sinc}\{[f + (f_c + \Delta f_n)]T\}\exp(-j\pi\Delta f_n T)\exp(-j\angle w_n)\exp(-j\varepsilon_n) \end{aligned}} \tag{14.3-15}$$

where

$$c_1(f) = \exp[-j\pi(f - f_c)T] \tag{14.3-16}$$

and

$$c_2(f) = \exp[-j\pi(f + f_c)T]. \tag{14.3-17}$$

The units of $X(f)$ are volts per hertz.

Since bandwidth is always measured along the positive frequency axis, we shall work with the first term in (14.3-15), that is,

$$\begin{aligned} X(f) = \frac{A}{2}T c_1(f)\sum_{n=0}^{N-1}|w_n|\operatorname{sinc}\{[f - (f_c + \Delta f_n)]T\}\exp(+j\pi\Delta f_n T)\exp(+j\angle w_n)\times \\ \exp(+j\varepsilon_n), \qquad f \geq 0. \end{aligned} \tag{14.3-18}$$

By examining the argument of the sinc function in (14.3-18), estimates of the maximum and minimum frequency components are given by

$$f_{\max} = f_c + \max \Delta f_n + \frac{\text{NZC}}{T} \tag{14.3-19}$$

and

$$f_{\min} = f_c + \min \Delta f_n - \frac{\text{NZC}}{T}, \tag{14.3-20}$$

where NZC is the integer number of zero-crossings of the sinc function that is used to estimate both the maximum and minimum frequency components $f_{\max}$ and $f_{\min}$, respectively. Since the frequency offset Δf_n is given by either (14.3-4) or (14.3-5), we shall use (14.3-4) first in order to estimate $f_{\max}$ and $f_{\min}$.

By referring to (14.3-4), it can be seen that for N even,

$$\max \Delta f_n = \Delta f_{N-1} = \left(\frac{N}{2} - 1 \right) \frac{1}{T} \tag{14.3-21}$$

and

$$\min \Delta f_n = \Delta f_0 = -\frac{N}{2}\frac{1}{T}, \tag{14.3-22}$$

and for N odd,

$$\max \Delta f_n = \Delta f_{N-1} = \frac{N-1}{2}\frac{1}{T} \tag{14.3-23}$$

and

$$\min \Delta f_n = \Delta f_0 = -\frac{N-1}{2}\frac{1}{T}. \tag{14.3-24}$$

Therefore, substituting (14.3-21) into (14.3-19), and (14.3-22) into (14.3-20) yields for N even,

$$f_{\max} = f_c + \left(\frac{N}{2} - 1 + \text{NZC} \right) \frac{1}{T} \tag{14.3-25}$$

and

$$f_{\min} = f_c - \left(\frac{N}{2} + \text{NZC} \right) \frac{1}{T}, \tag{14.3-26}$$

and substituting (14.3-23) into (14.3-19), and (14.3-24) into (14.3-20) yields for N odd,

$$f_{\max} = f_c + \left(\frac{N-1}{2} + \text{NZC} \right) \frac{1}{T} \tag{14.3-27}$$

and

$$f_{\min} = f_c - \left(\frac{N-1}{2} + \mathrm{NZC} \right) \frac{1}{T} . \tag{14.3-28}$$

Next we shall use (14.3-5) to estimate $f_{\max}$ and $f_{\min}$.

By referring to (14.3-5), it can be seen that

$$\max \Delta f_n = \Delta f_{N-1} = \frac{N-1}{T} \tag{14.3-29}$$

and

$$\min \Delta f_n = \Delta f_0 = 0 . \tag{14.3-30}$$

Therefore, substituting (14.3-29) into (14.3-19), and (14.3-30) into (14.3-20) yields

$$f_{\max} = f_c + (N + \mathrm{NZC} - 1) \frac{1}{T} \tag{14.3-31}$$

and

$$f_{\min} = f_c - \frac{\mathrm{NZC}}{T} . \tag{14.3-32}$$

The bandwidth BW_x (in hertz) of the OFDM signal is given by

$$\mathrm{BW}_x = f_{\max} - f_{\min} . \tag{14.3-33}$$

Substituting (14.3-25) and (14.3-26), or (14.3-27) and (14.3-28), or (14.3-31) and (14.3-32), and (14.3-6) into (14.3-33) yields the same result

$$\boxed{\mathrm{BW}_x = \left(1 + \frac{2\,\mathrm{NZC} - 1}{N} \right) D} \tag{14.3-34}$$

where

$$\boxed{D = \frac{1}{T_{\text{sym}}} = \frac{1}{n_b T_b} = \frac{R_b}{n_b}} \tag{14.3-35}$$

is the *input symbol rate (input baud)* with units of symbols per second, and

$$\boxed{R_b = 1/T_b} \tag{14.3-36}$$

is the *bit rate* in bits per second. Since $\max|\mathrm{sinc}(fT)| = 1$ for $f = 0$; $\mathrm{sinc}(fT) = 0$ for $f = i/T$, where $i = \pm 1, \pm 2, \ldots$; and $|\mathrm{sinc}(fT)| < 0.1$ for

$f > 3/T$; NZC should be at least 3, with 5 being a conservative choice. Substituting $\text{NZC} = 5$ into (14.3-34) yields

$$\text{BW}_x = \left(1 + \frac{9}{N}\right)D \tag{14.3-37}$$

or

$$\text{BW}_x \approx D, \quad N > 90\,. \tag{14.3-38}$$

Therefore, if the total number of transmitted pulses (symbols) $N > 90$, then the bandwidth of an OFDM signal is approximately equal to the input baud D.

14.3.3 Signal Energy and Time-Average Power

In this subsection we shall compute the energy and time-average power of the OFDM signal $x(t)$ given by (14.3-1). We begin by computing the energy of the nth pulse $x_n(t)$ given by (14.3-2). Note that $x_n(t)$ is an amplitude-and-angle-modulated carrier with amplitude and angle-modulating functions

$$a_n(t) = A|w_n|\,\text{rect}[(t - 0.5T)/T] \tag{14.3-39}$$

and

$$\theta_n(t) = 2\pi\Delta f_n t + \angle w_n + \varepsilon_n\,, \tag{14.3-40}$$

respectively (see Section 11.2), where the time-shifted rectangle function is given by (14.3-9). In order to compute the energy of $x_n(t)$, we shall first compute the energy of its complex envelope $\tilde{x}_n(t)$.

Since the general form of the complex envelope of $x_n(t)$ is given by (see Section 11.2)

$$\tilde{x}_n(t) = a_n(t)\exp[+j\theta_n(t)]\,, \tag{14.3-41}$$

substituting (14.3-39) and (14.3-40) into (14.3-41) yields

$$\begin{aligned}\tilde{x}_n(t) &= A|w_n|\exp[+j(2\pi\Delta f_n t + \angle w_n + \varepsilon_n)]\,\text{rect}[(t - 0.5T)/T] \\ &= A w_n \exp[+j(2\pi\Delta f_n t + \varepsilon_n)]\,\text{rect}[(t - 0.5T)/T].\end{aligned} \tag{14.3-42}$$

The energy $E_{\tilde{x}_n}$ of $\tilde{x}_n(t)$ is given by

$$E_{\tilde{x}_n} = \int_{-\infty}^{\infty} a_n^2(t)\,dt\,, \tag{14.3-43}$$

and the energy E_{x_n} of $x_n(t)$ is given by (see Section 11.2)

$$E_{x_n} = E_{\tilde{x}_n} / 2 . \tag{14.3-44}$$

Therefore, substituting (14.3-39) into (14.3-43) yields

$$E_{\tilde{x}_n} = A^2 \left| w_n \right|^2 T , \tag{14.3-45}$$

which is the energy (in joules-ohms) of the complex envelope $\tilde{x}_n(t)$ given by (14.3-42), and substituting (14.3-45) into (14.3-44) yields

$$\boxed{E_{x_n} = \frac{A^2}{2} \left| w_n \right|^2 T} \tag{14.3-46}$$

which is the energy (in joules-ohms) of $x_n(t)$ given by (14.3-2). The time-average power of $x_n(t)$ in watts-ohms is given by

$$\boxed{P_{\text{avg}, x_n, T} = \frac{E_{x_n}}{T} = \frac{A^2}{2} \left| w_n \right|^2} \tag{14.3-47}$$

From signal theory, the energy E_x of a signal $x(t)$ is defined as

$$E_x \triangleq \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt . \tag{14.3-48}$$

Using (14.3-1),

$$\left| x(t) \right|^2 = x(t) x^*(t) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x_m(t) x_n^*(t), \qquad 0 \le t \le T , \tag{14.3-49}$$

and by substituting (14.3-49) into (14.3-48), we obtain

$$E_x = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \int_0^T x_m(t) x_n^*(t) dt . \tag{14.3-50}$$

As was mentioned in Subsection 14.3.1, because the numerator of the frequency offset Δf_n given by either (14.3-4) or (14.3-5) is an *integer*, if the product $f_c T$ is also equal to an *integer*, or $f_c T >> 1$ if $f_c T$ does not equal an integer, then the set of functions $x_n(t)$, $n = 0, 1, \ldots, N-1$, given by (14.3-2) is an *orthogonal* set of functions in the time interval $[0,T]$. Therefore,

$$\int_0^T x_m(t)x_n^*(t)\,dt = E_{x_n}\delta_{mn}\,, \tag{14.3-51}$$

where the energy of the nth pulse E_{x_n} is given by (14.3-46), and δ_{mn} is the Kronecker delta given by (14.1-56). Substituting (14.3-51) into (14.3-50) yields

$$E_x = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E_{x_n}\delta_{mn} = \sum_{n=0}^{N-1} E_{x_n}\sum_{m=0}^{N-1}\delta_{mn} = \sum_{n=0}^{N-1} E_{x_n}\,, \tag{14.3-52}$$

or

$$\boxed{E_x = \frac{A^2}{2}T\sum_{n=0}^{N-1}|w_n|^2} \tag{14.3-53}$$

which is the energy (in joules-ohms) of the OFDM signal $x(t)$ given by (14.3-1). Therefore, the time-average power of $x(t)$ in watts-ohms is given by

$$\boxed{P_{\text{avg},x,T} = \frac{E_x}{T} = \frac{A^2}{2}\sum_{n=0}^{N-1}|w_n|^2} \tag{14.3-54}$$

Table 14.3-1 Signal Bandwidth and Time-Average Power of MFSK, MQAM, and OFDM Signals

	Signal Bandwidth BW_x (Hz)	Time-Average Power (W-Ω)
MFSK	$(M + 2\,\mathrm{NZC})D$	$P_{\text{avg},x,T_d} = \dfrac{A^2}{2}$ $T_d = NT_{\text{sym}}$
MQAM	$2\,\mathrm{NZC}\times D$	$P_{\text{avg},x,T_d} = \dfrac{A^2}{2}\dfrac{1}{N}\sum_{n=1}^{N}\lvert w_n\rvert^2$ $T_d = NT_{\text{sym}}$
OFDM	$\left(1+\dfrac{2\,\mathrm{NZC}-1}{N}\right)D$	$P_{\text{avg},x,T} = \dfrac{A^2}{2}\sum_{n=0}^{N-1}\lvert w_n\rvert^2$ $T = NT_{\text{sym}}$

Table 14.3-1 compares the signal bandwidths and time-average powers of MFSK, MQAM, and OFDM signals. As can be seen from Table 14.3-1, MFSK has the largest bandwidth and smallest time-average power, whereas OFDM has the smallest bandwidth and largest time-average power. All three signals have the same total duration of NT_{sym} sec. Recall that the pulse length T is equal to the symbol duration T_{sym} in both MFSK and MQAM signals [see (14.1-4) and (14.2-4)], whereas the pulse length T is equal to the total duration of an OFDM signal, that is, $T = NT_{\text{sym}}$.

14.3.4 Demodulation

As was mentioned in Subsections 14.1.5 and 14.2.4, demodulation is a procedure meant to retrieve a transmitted message at a receiver. In order to discuss demodulation, we first need to model an OFDM signal at a receiver, which we shall designate as $y(t)$. We shall use the following simple model for $y(t)$:

$$y(t) = Kx(t-\tau), \qquad \tau \le t \le \tau + T, \tag{14.3-55}$$

where the transmitted signal $x(t)$ is given by (14.3-1). As can be seen from (14.3-55), $y(t)$ is modeled as an amplitude-scaled, time-delayed version of $x(t)$ (distortionless transmission) as was done for MFSK and MQAM signals at a receiver (see Subsections 14.1.5 and 14.2.4, respectively) where $K > 0$ is a dimensionless constant. Substituting (14.3-1) into (14.3-55) yields

$$y(t) = \sum_{n=0}^{N-1} y_n(t), \qquad \tau \le t \le \tau + T, \tag{14.3-56}$$

where

$$y_n(t) = KA\left|w_n\right|\cos\left(2\pi[f_c + \Delta f_n](t-\tau) + \angle w_n + \varepsilon_n\right)\operatorname{rect}\left[\left(t - [0.5T + \tau]\right)/T\right] \tag{14.3-57}$$

is the nth pulse of the received signal, and

$$\operatorname{rect}\left(\frac{t - [0.5T + \tau]}{T}\right) = \begin{cases} 1, & \tau \le t \le \tau + T \\ 0, & \text{otherwise.} \end{cases} \tag{14.3-58}$$

The received OFDM signal $y(t)$ is demodulated by computing its Fourier transform. Taking the Fourier transform of (14.3-55) yields

$$Y(f) = KX(f)\exp(-j2\pi f\tau), \tag{14.3-59}$$

where $X(f)$ is the Fourier transform of the transmitted signal $x(t)$ given by (14.3-15). Substituting (14.3-18) for $X(f)$ along the positive frequency axis into (14.3-59) yields

$$Y(f) = K\frac{A}{2}T\,c_1(f)\exp(-j2\pi f\tau)\times$$
$$\sum_{n=0}^{N-1}|w_n|\operatorname{sinc}\{[f-(f_c+\Delta f_n)]T\}\exp[+j(\pi\Delta f_n T+\angle w_n+\varepsilon_n)], \qquad f\geq 0. \tag{14.3-60}$$

The demodulation of $y(t)$ is demonstrated in Example 14.3-1.

Example 14.3-1

In this example we shall consider the problem of transmitting the following binary words

00 11 01 10 00 11 10 00 01 11

using OFDM where a binary word is equal to $n_b = 2$ bits. This is the same problem that was considered in Example 14.1-1 using MFSK, and Example 14.2-3 using MQAM. Since there are 10 binary words (20 bits), the total number of pulses (symbols) to be transmitted is $N = 10$. Also, since $n_b = 2$, the total even number of *unique* symbol values $M = 2^2 = 4$.

Table 14.3-2 Four Unique Symbol Values Assigned to Four Unique Binary Words

Gray Code	Symbol w_n
00	$1\exp(+j225°)$
01	$1\exp(+j315°)$
11	$1\exp(+j45°)$
10	$1\exp(+j135°)$

The next step is to assign a unique symbol value to each of the 4 *unique* binary words (4 possible combinations of 2 bits). For example, see Table 14.3-2 where the 4 unique binary words are arranged in a Gray code using 4-ary PSK, also known as quadrature PSK (QPSK). The set of complex numbers $\{1\exp(+j45°), 1\exp(+j135°), 1\exp(+j225°), 1\exp(+j315°)\}$ is the alphabet. Note that $\pi/4 = 45°$, $3\pi/4 = 135°$, $5\pi/4 = 225°$, and $7\pi/4 = 315°$ [see (14.2-20)]. Since QPSK is being used, $|w_n| = 1$ is not important, only the values of $\angle w_n$ encode the 4 unique binary words. Figure 14.3-1 is a pictorial representation of Table 14.3-2. As can be seen from Fig. 14.3-1, only one bit is changed going from one signal-point to an adjacent signal-point in the constellation.

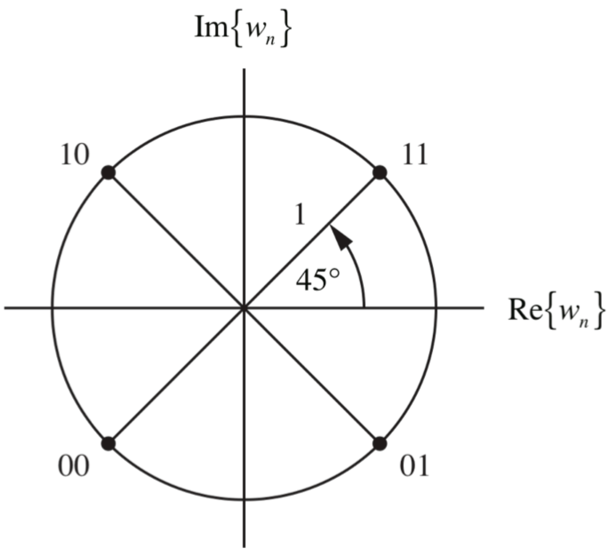


Figure 14.3-1 Signal-space constellation for Gray-encoded QPSK.

In order to simulate a time-domain OFDM signal using the information given up to this point, we need values for the amplitude factor A, the carrier frequency f_c, the input symbol duration T_{sym}, and to decide which equation to use for the frequency offset Δf_n. For example, if $A = 10\text{ V}$, $f_c = 1\text{ kHz}$, and $T_{\text{sym}} = n_b T_b = 20\text{ msec}$, then $T = NT_{\text{sym}} = 200\text{ msec}$, the factor $AT/2 = 1$ in (14.3-15), and $f_c T = 200$ is an integer (see Subsection 14.1.4). In this example we shall use the frequency offset given by (14.3-4). Substituting $N = 10$ and $T = 200\text{ msec}$ into (14.3-4) yields

$$\Delta f_n = 5(n-5)\text{ Hz}. \tag{14.3-61}$$

Table 14.3-3 summarizes the parameter values of the OFDM signal. Using the symbol values for the 4 unique binary words in Table 14.3-2, and Δf_n given by

(14.3-61), Table 14.3-4 shows the symbol values and frequencies of the 10 transmitted pulses associated with the 10 binary words, where the phase term $\varepsilon_n = 0 \;\; \forall \;\; n$ for example purposes.

Table 14.3-3 Parameter Values of the Transmitted OFDM Signal

Amplitude factor A	10 V
Carrier frequency f_c	1 kHz
Input symbol duration T_{sym}	20 msec
Total number of transmitted pulses (symbols) N	10
Pulse length of an individual pulse $T = NT_{\text{sym}}$	200 msec
Total duration of the transmitted signal $T = NT_{\text{sym}}$	200 msec
Frequency offset Δf_n	$5(n-5)$ Hz
Input baud (input symbol rate) $D = 1/T_{\text{sym}}$	50 symbols/sec
Number of bits per symbol (binary word) n_b	2
Number of *unique* symbol values $M = 2^{n_b}$	4
Bit duration $T_b = T_{\text{sym}}/n_b$	10 msec
Bit rate $R_b = 1/T_b$	100 bps
Time-average power $P_{\text{avg},x,T}$	500 W-Ω

Table 14.3-4 Symbol Values and Frequencies of the Ten Transmitted Pulses

n	Binary Word	w_n	$\Delta f_n = 5(n-5)$ (Hz)	$f_c + \Delta f_n$ (Hz)	ε_n (deg)
0	00	$1\exp(+j225°)$	-25	975	0
1	11	$1\exp(+j45°)$	-20	980	0
2	01	$1\exp(+j315°)$	-15	985	0
3	10	$1\exp(+j135°)$	-10	990	0
4	00	$1\exp(+j225°)$	-5	995	0
5	11	$1\exp(+j45°)$	0	1000	0
6	10	$1\exp(+j135°)$	5	1005	0
7	00	$1\exp(+j225°)$	10	1010	0
8	01	$1\exp(+j315°)$	15	1015	0
9	11	$1\exp(+j45°)$	20	1020	0

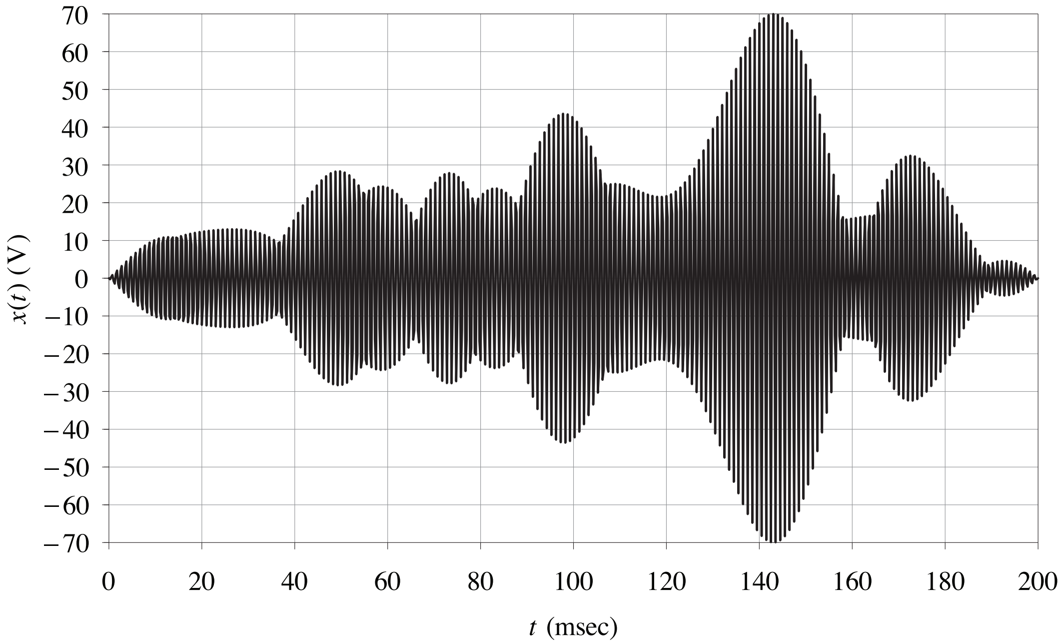


Figure 14.3-2 OFDM signal $x(t)$ given by (14.3-1) using the parameter values shown in Tables 14.3-3 and 14.3-4.

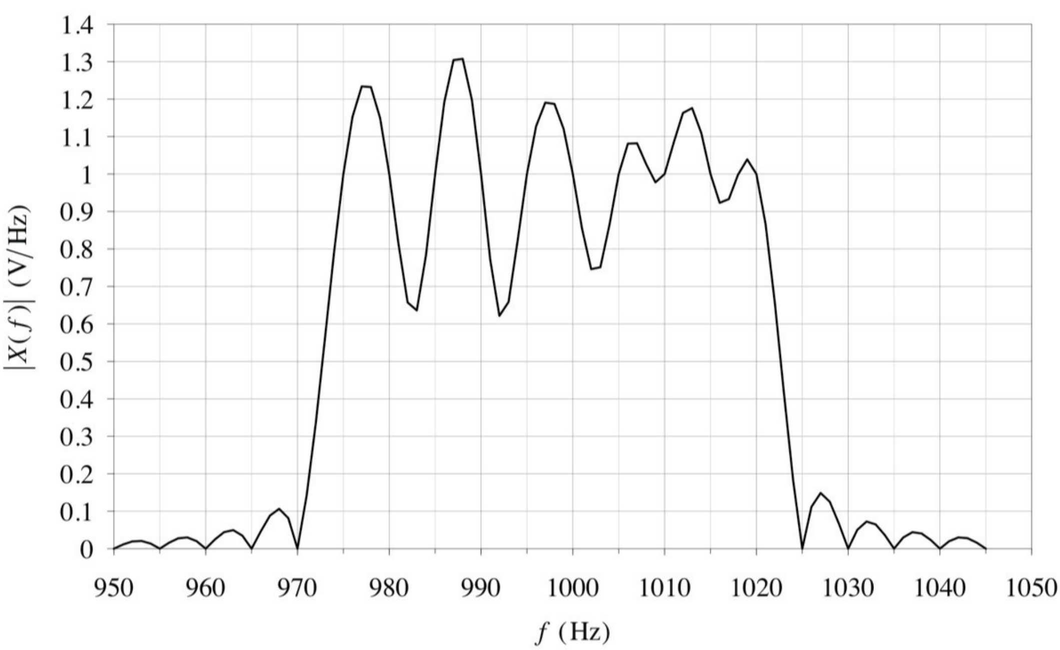


Figure 14.3-3 Magnitude of the OFDM frequency spectrum $X(f)$ given by (14.3-18) using the parameter values shown in Tables 14.3-3 and 14.3-4.

Figure 14.3-2 is a plot of the OFDM signal $x(t)$ given by (14.3-1), and Fig. 14.3-3 is a plot of the magnitude of the OFDM frequency spectrum $X(f)$ given by (14.3-18), using the parameter values shown in Tables 14.3-3 and 14.3-4. Although the amplitude factor $A = 10\text{ V}$, by inspecting Fig. 14.3-2, it can be seen that the amplitude of $x(t)$ grows to 70 V as a result of all the pulses being transmitted simultaneously and constructive interference. This is a disadvantage of OFDM. The bandwidth of $x(t)$ can be computed using (14.3-34). Since $N = 10$ and $D = 50\text{ symbols/sec}$, if $\text{NZC} = 1$, 3, and 5, then $\text{BW}_x = 55\text{ Hz}$, 75 Hz, and 95 Hz, respectively. As can be seen from Fig. 14.3-3, a bandwidth of 75 Hz is a good estimate, whereas a bandwidth of 95 Hz is more conservative.

As was previously mentioned, the received OFDM signal $y(t)$ is demodulated by computing its Fourier transform. Since $Y(f)$ was shown to be directly proportional to $X(f)$ [see (14.3-59)], for example purposes, we shall compute the Fourier transform of the transmitted signal $x(t)$ shown in Fig. 14.3-2 using the algorithm discussed in Appendix 7B.

In order to compute the Fourier transform of $x(t)$, a sampling frequency of 4 kHz was used. Since the total duration of $x(t)$ is $T = 200\text{ msec}$, the total number of samples taken was

$$N = f_S T = 4000\text{ Hz} \times 0.2\text{ sec} = 800\,. \tag{14.3-62}$$

If padding-with-zeros is not done, then $Z = 0$, the fundamental period $T_0 = T$, and the DFT bin spacing is

$$f_0 = \frac{1}{T_0} = \frac{1}{T} = \frac{1}{0.2\text{ sec}} = 5\text{ Hz}\,. \tag{14.3-63}$$

Since the frequencies of interest $\{975\text{ Hz}, 980\text{ Hz}, \ldots, 1015\text{ Hz}, 1020\text{ Hz}\}$ are located at DFT bins when $Z = 0$, padding-with-zeros was not required. Note that the frequency offset Δf_n, given by either (14.3-4) or (14.3-5), is an integer multiple of the DFT bin spacing f_0 given by (14.3-63).

Estimates of the magnitude and phase spectra of $x(t)$ are shown in Table 14.3-5 where integer n is the pulse (subcarrier) number, integer q is the DFT bin number, frequency $f = qf_0$ Hz where the DFT bin spacing f_0 is given by (14.3-63), and $\hat{X}(f)$ is the estimate of the Fourier transform of $x(t)$ using the algorithm discussed in Appendix 7B. Each frequency component in Table 14.3-5 corresponds to an individual transmitted pulse (subcarrier) where $\left|\hat{X}(f)\right| = |w_n|$ because $AT/2 = 1$ and $\angle\hat{X}(f) = \angle w_n$ because $\varepsilon_n = 0 \;\; \forall \;\; n$. Note that the phase angles $225°$ and $315°$ are equivalent to $-135°$ and $-45°$, respectively. The magnitude and phase values in Table 14.3-5 match exactly the symbol values in Table 14.3-4. As was mentioned earlier, since QPSK is used in this example, the value of $|w_n|$ for each pulse is not important, only the value of $\angle w_n$ encodes a binary word.

Table 14.3-5 Estimates of the Magnitude and Phase Spectra of the Transmitted OFDM Signal

n	q	f (Hz)	$\lvert\hat{X}(f)\rvert$ (V/Hz)	$\angle\hat{X}(f)$ (deg)
0	195	975	1	225
1	196	980	1	45
2	197	985	1	315
3	198	990	1	135
4	199	995	1	225
5	200	1000	1	45
6	201	1005	1	135
7	202	1010	1	225
8	203	1015	1	315
9	204	1020	1	45

By computing the Fourier transform of $x(t)$, we were able to demonstrate that values of the symbol w_n, that is, $|w_n|$ and $\angle w_n$ where $w_n = |w_n|\exp(+j\angle w_n)$, are located at DFT bins corresponding to frequencies $f = f_c + \Delta f_n$, $n = 0, 1, \ldots, N-1$. That is why the received OFDM signal $y(t)$ is demodulated by computing its Fourier transform using the algorithm discussed in Appendix 7B. Once all the transmitted symbol values have been determined, decode the symbol values back into binary words, and then decode the binary words back into quantized voltages. One can then use the quantized voltages to approximate the original analog (continuous-time) message using the reconstruction formula from the Sampling Theorem. ■

Problems

Section 14.1

14-1 If a total of 50 symbols are transmitted using MFSK, where each symbol