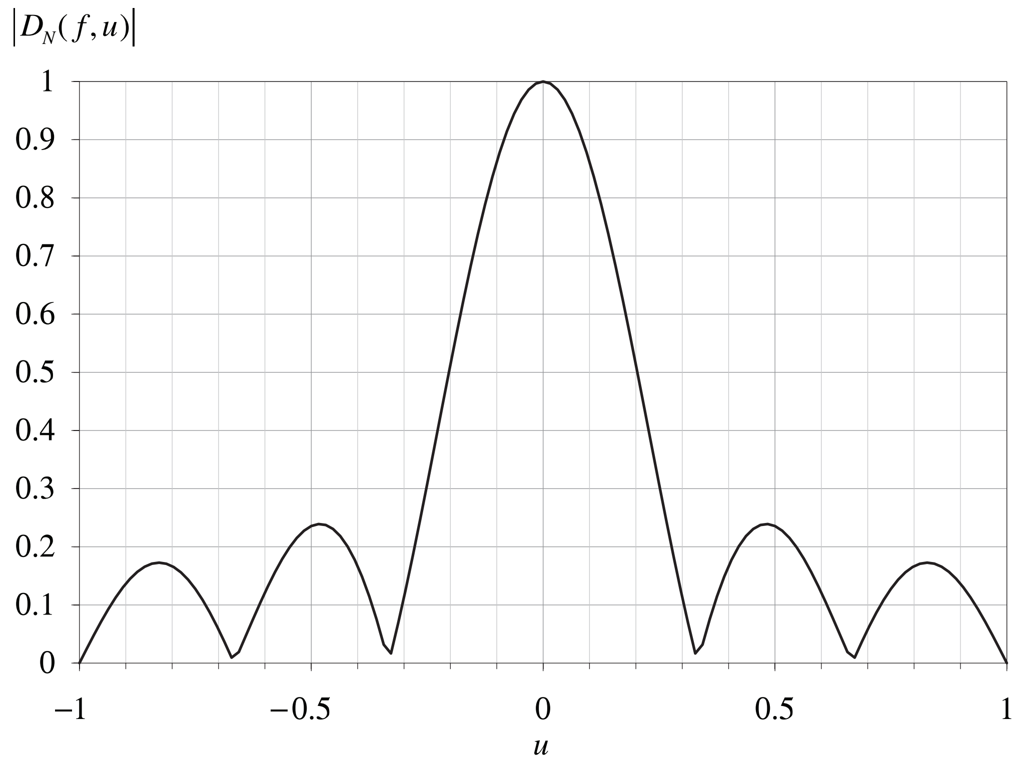

Figure 6E-1 Magnitude of the normalized, far-field beam pattern of a linear array of 6 identical, equally-spaced, rectangular amplitude-weighted, omni-directional point-elements lying along the X axis using a spatial DFT and plotted as a function of direction cosine u.

Chapter 7

Array Gain

Using an array of electroacoustic transducers (elements) in the passive mode as a receiver can increase the probability of detecting very weak signals corrupted by additive noise. This is especially important in underwater acoustics when trying to detect sound fields radiated by potential targets of interest such as submarines, surface ships, and marine mammals (e.g., whales and dolphins). Sound radiated by marine life is referred to as biologics. Besides detection, it is also very important to be able to classify (identify) sound-sources (targets).

An array has its own output signal-to-noise ratio (SNR) that can be very large compared to the SNR at the output from a single element in an array. Array gain (AG) is simply a decibel measure of the increase in the output SNR that is obtained by using an array of elements versus a single element. Increasing the AG increases the chances of both detection and classification.

7.1 General Definition of Array Gain for a Linear Array

In this section we shall define array gain (AG) for a linear array composed of an odd number N of identical, equally-spaced, complex-weighted, omnidirectional point-elements lying along the X axis. The array is being used in the passive mode as a receiver. The received signal at the output of each element in the array is a *random process* and is equal to the sum of the output signals due to the acoustic field radiated by a target, the ambient (background) noise in the ocean, and receiver noise. Therefore, the received electrical signal (in volts) at the output of element i in the array *before* and *after* complex weighting can be expressed as

$$r'(t, x_i) = y'_{Trgt}(t, x_i) + y'_{n_a}(t, x_i) + n'_r(t, x_i) \tag{7.1-1}$$

and

$$r(t, x_i) = y_{Trgt}(t, x_i) + y_{n_a}(t, x_i) + n_r(t, x_i)\,, \tag{7.1-2}$$

respectively, where $y'_{Trgt}(t, x_i)$, $y'_{n_a}(t, x_i)$, and $n'_r(t, x_i)$ are the output electrical signals from element i due to the target, ambient noise, and receiver noise, respectively, *before* complex weighting; and $y_{Trgt}(t, x_i)$, $y_{n_a}(t, x_i)$, and $n_r(t, x_i)$ are the output electrical signals from element i due to the target, ambient noise, and receiver noise, respectively, *after* complex weighting, where x_i is the x coordinate of the center of element i (see Fig. 7.1-1). As was discussed in Section 6.4, the procedure illustrated in Fig. 7.1-1 is known as beamforming. In practical

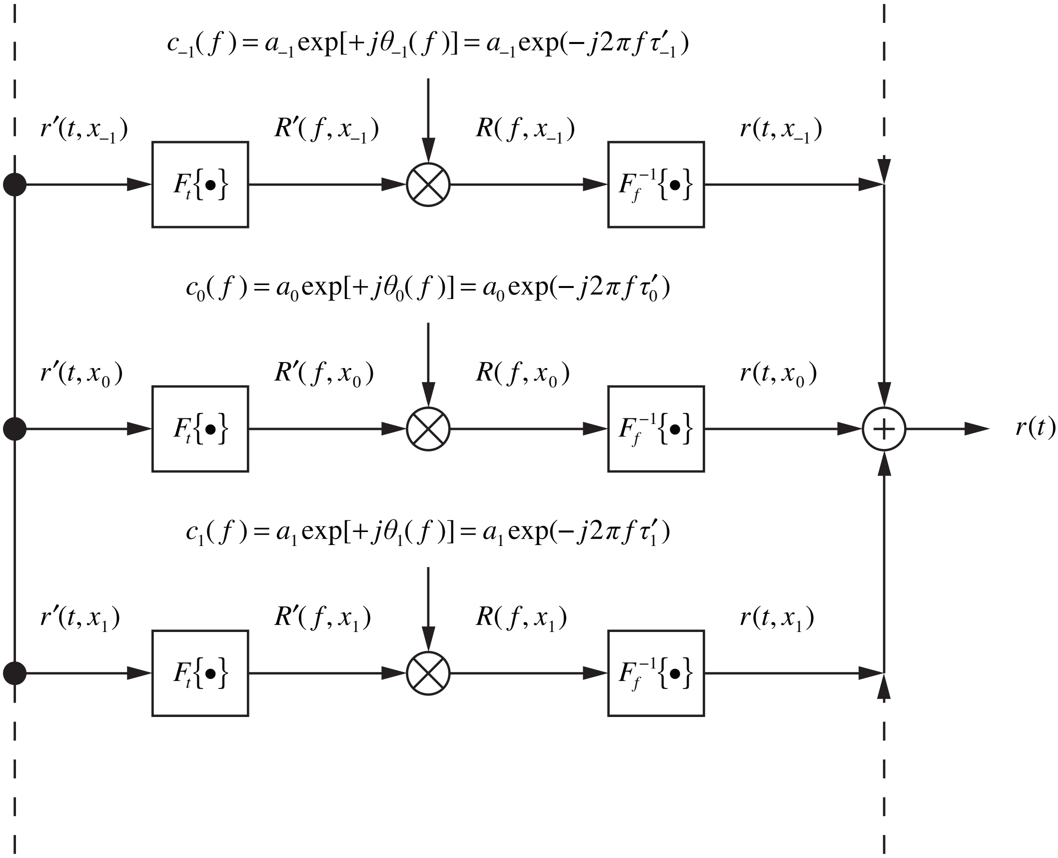

Figure 7.1-1 Beamforming for a linear array. Implementing complex weights (amplitude and phase weights) using forward and inverse Fourier transforms.

applications, the forward and inverse Fourier transforms are computed using forward and inverse *discrete Fourier transforms* (DFTs), which can be evaluated very quickly by using forward and inverse *fast-Fourier-transform* (FFT) algorithms. This is known as digital beamforming or FFT beamforming.

The *total* output electrical signal from the array (in volts) is given by

$$r(t) = \sum_{i=-N'}^{N'} r(t, x_i), \tag{7.1-3}$$

and by substituting (7.1-2) into (7.1-3), we obtain

$$r(t) = y_{Trgt}(t) + y_{n_a}(t) + n_r(t), \tag{7.1-4}$$

where

$$y_{Trgt}(t) = \sum_{i=-N'}^{N'} y_{Trgt}(t, x_i), \tag{7.1-5}$$

$$y_{n_a}(t) = \sum_{i=-N'}^{N'} y_{n_a}(t, x_i), \tag{7.1-6}$$

and

$$n_r(t) = \sum_{i=-N'}^{N'} n_r(t, x_i) \tag{7.1-7}$$

are the *total* output electrical signals from the array due to the target, ambient noise, and receiver noise, respectively, where

$$N' = (N-1)/2 \,. \tag{7.1-8}$$

If we let

$$z(t) = y_{n_a}(t) + n_r(t), \tag{7.1-9}$$

then the *output signal-to-noise ratio of the array* is defined as follows:

$$\boxed{\mathrm{SNR_A} \triangleq \frac{E\left\{\left|y_{Trgt}(t)\right|^2\right\}}{E\left\{\left|z(t)\right|^2\right\}}} \tag{7.1-10}$$

where the second moment (mean-squared value)

$$E\left\{\left|y_{Trgt}(t)\right|^2\right\} = \sum_{i=-N'}^{N'} \sum_{j=-N'}^{N'} E\left\{y_{Trgt}(t, x_i)\, y^*_{Trgt}(t, x_j)\right\} \tag{7.1-11}$$

is the average signal power at the output of the array, and

$$E\left\{\left|z(t)\right|^2\right\} = E\left\{\left|y_{n_a}(t)\right|^2\right\} + 2\,\mathrm{Re}\left\{E\left\{y_{n_a}(t)\, n_r^*(t)\right\}\right\} + E\left\{\left|n_r(t)\right|^2\right\} \tag{7.1-12}$$

is the average noise power at the output of the array, where

$$E\left\{\left|y_{n_a}(t)\right|^2\right\} = \sum_{i=-N'}^{N'} \sum_{j=-N'}^{N'} E\left\{y_{n_a}(t, x_i)\, y^*_{n_a}(t, x_j)\right\} \tag{7.1-13}$$

and

$$E\left\{\left|n_r(t)\right|^2\right\} = \sum_{i=-N'}^{N'} \sum_{j=-N'}^{N'} E\left\{n_r(t, x_i)\, n_r^*(t, x_j)\right\} \tag{7.1-14}$$

are the average noise powers at the output of the array due to ambient noise and receiver noise, respectively. Assuming that $y_{n_a}(t)$ and $n_r(t)$ are *statistically* *independent*, *zero-mean* random processes,

$$E\left\{y_{n_a}(t)n_r^*(t)\right\} = E\left\{y_{n_a}(t)\right\}E\left\{n_r^*(t)\right\} = 0\,, \tag{7.1-15}$$

and as a result, (7.1-12) reduces to

$$E\left\{\left|z(t)\right|^2\right\} = E\left\{\left|y_{n_a}(t)\right|^2\right\} + E\left\{\left|n_r(t)\right|^2\right\}. \tag{7.1-16}$$

The average powers are functions of time in general. However, if $y_{Trgt}(t)$, $y_{n_a}(t)$, and $n_r(t)$ are *wide-sense stationary* (WSS) random processes, then the average powers are *constants*. The expected value (ensemble average) $E\left\{y_{Trgt}(t,x_i)y_{Trgt}^*(t,x_j)\right\}$ is the *cross-correlation* of $y_{Trgt}(t,x_i)$ and $y_{Trgt}(t,x_j)$, $E\left\{y_{n_a}(t,x_i)y_{n_a}^*(t,x_j)\right\}$ is the *cross-correlation* of $y_{n_a}(t,x_i)$ and $y_{n_a}(t,x_j)$, and $E\left\{n_r(t,x_i)n_r^*(t,x_j)\right\}$ is the *cross-correlation* of $n_r(t,x_i)$ and $n_r(t,x_j)$. The SNR_A given by (7.1-10) is dimensionless.

In a similar manner, the *output signal-to-noise ratio at the center element in the array* (element $i=0$) is defined as follows:

$$\boxed{\text{SNR}_0 \triangleq \frac{E\left\{\left|y_{Trgt}(t,x_0)\right|^2\right\}}{E\left\{\left|z(t,x_0)\right|^2\right\}}} \tag{7.1-17}$$

where $E\left\{\left|y_{Trgt}(t,x_0)\right|^2\right\}$ is the average signal power, and assuming that $y_{n_a}(t,x_0)$ and $n_r(t,x_0)$ are statistically independent, zero-mean random processes,

$$E\left\{\left|z(t,x_0)\right|^2\right\} = E\left\{\left|y_{n_a}(t,x_0)\right|^2\right\} + E\left\{\left|n_r(t,x_0)\right|^2\right\} \tag{7.1-18}$$

is the average noise power, where $E\left\{\left|y_{n_a}(t,x_0)\right|^2\right\}$ and $E\left\{\left|n_r(t,x_0)\right|^2\right\}$ are the average noise powers due to ambient noise and receiver noise, respectively, all at the output of element $i=0$ *after* complex weighting. The average powers are functions of time in general. However, if $y_{Trgt}(t,x_0)$, $y_{n_a}(t,x_0)$, and $n_r(t,x_0)$ are WSS in time, then the average powers are constants. The SNR_0 given by (7.1-17) is dimensionless.

Array gain (AG) is defined as follows:

$$\boxed{\mathrm{AG} \triangleq 10\log_{10}\frac{\mathrm{SNR_A}}{\mathrm{SNR_0}}\ \mathrm{dB}} \tag{7.1-19}$$

where $\mathrm{SNR_A}$ is given by (7.1-10) and $\mathrm{SNR_0}$ is given by (7.1-17). As can be seen from (7.1-19), AG is a decibel measure of the increase in the output signal-to-noise ratio that is obtained by using an array of elements versus a single element. Array gain depends on the statistical properties of the signal (the acoustic field radiated by a target), ambient noise, and receiver noise. Therefore, in general, the same array will have different AG values when placed in different signal and ambient noise sound fields. In order to evaluate the equation for AG as given by (7.1-19) (see Section 7.5), we need mathematical models for the acoustic field radiated by a target (see Section 7.2), the total output signal from a linear array due to the target (see Section 7.3), and the total output signal from a linear array due to ambient noise in the ocean and receiver noise (see Section 7.4).

7.2 Acoustic Field Radiated by a Target

A mathematical model for the acoustic field radiated by a sound-source (target) depends on the models used for both the target and the fluid medium. The model that we shall use for the target is the same as that used in Subsection 1.2.2 – the target is modeled as an omnidirectional point-source located at $\mathbf{r}_S = (x_S, y_S, z_S)$ with arbitrary time dependence $s_0(t)$, where $s_0(t)$, $t \geq 0$, is the source strength of the target in cubic meters per second (see Fig. 1.2-5). In this chapter, $s_0(t)$ is assumed to be a *wide-sense stationary* (WSS) *random process*. The model that we shall *initially* use for the fluid medium is also the same as that used in Subsection 1.2.2 – the fluid medium is unbounded, ideal (nonviscous), and homogeneous (constant speed of sound). However, later in this section we shall generalize the fluid-medium model to include viscosity (resistance to fluid flow).

The receive aperture in Subsection 1.2.2 was a closed-surface, volume aperture. The receive aperture in this chapter is a linear array, as described in Section 7.1. Since a linear array is an example of a linear aperture, we shall first obtain results for a receive linear aperture lying along the X axis. By applying the receive volume aperture results obtained in Subsection 1.2.2 to a receive linear aperture lying along the X axis, the position vector to an aperture point, $\mathbf{r}_R$, is given by

$$\mathbf{r}_R = x_R\hat{x}, \tag{7.2-1}$$

and the velocity potential in squared meters per second of the output acoustic signal from the fluid medium due to the target and incident upon the aperture at x_R is given by

$$y_{M,Trgt}(t, x_R) = -\frac{1}{4\pi R} s_0(t-\tau)\,, \tag{7.2-2}$$

for $t \ge \tau$, where

$$\begin{aligned} R &= \left|\mathbf{r}_R - \mathbf{r}_S\right| \\ &= \sqrt{(x_R - x_S)^2 + (y_R - y_S)^2 + (z_R - z_S)^2} \\ &= \sqrt{(x_R - x_S)^2 + y_S^2 + z_S^2} \end{aligned} \tag{7.2-3}$$

is the range in meters between the target and an aperture point,

$$\tau = R/c \tag{7.2-4}$$

is the corresponding *one-way* travel time or *one-way* time delay in seconds, and c is the constant speed of sound in the fluid medium in meters per second. Equation (7.2-2) is the velocity potential of a spherical wave with arbitrary time dependence propagating in an unbounded, ideal (nonviscous), homogeneous, fluid medium.

The position vector to the sound-source (target) is given by either

$$\mathbf{r}_S = x_S\hat{x} + y_S\hat{y} + z_S\hat{z} \tag{7.2-5}$$

or

$$\mathbf{r}_S = r_S\hat{r}_S\,, \tag{7.2-6}$$

where

$$r_S = \left|\mathbf{r}_S\right| = \sqrt{x_S^2 + y_S^2 + z_S^2} \tag{7.2-7}$$

is the range to the target in meters as measured from the origin of the coordinate system (the center of the array) (see Fig. 1.2-5),

$$\hat{r}_S = u_S\hat{x} + v_S\hat{y} + w_S\hat{z} \tag{7.2-8}$$

is the unit vector in the direction of $\mathbf{r}_S$, and

$$u_S = \sin\theta_S \cos\psi_S\,, \tag{7.2-9}$$

$$v_S = \sin\theta_S \sin\psi_S\,, \tag{7.2-10}$$

$$w_S = \cos\theta_S \tag{7.2-11}$$

are dimensionless direction cosines with respect to the X, Y, and Z axes, respectively (see Fig. 1.2-6). The spherical coordinates of the target's location

(r_S, θ_S, ψ_S) are *unknown* a priori.

All real fluids possess viscosity (resistance to fluid flow). For example, in ocean acoustics, seawater is correctly modeled as a viscous fluid with a frequency-dependent attenuation coefficient. Because of viscosity, sound is attenuated by conversion of its acoustic energy into heat. In order to take viscosity into account, we need to work in the frequency domain. Since both the target and aperture are not in motion, output (received) frequencies η are equal to input (transmitted) frequencies f since there is no Doppler shift nor bandwidth compression or expansion (see Subsection 1.1.2). Therefore, taking the Fourier transform of (7.2-2) with respect to time t yields

$$Y_{M,Trgt}(f, x_R) = -\frac{1}{4\pi R} S_0(f) \exp(-j2\pi f \tau), \tag{7.2-12}$$

and by substituting (7.2-3) and (7.2-4) into (7.2-12), we obtain

$$Y_{M,Trgt}(f, x_R) = S_0(f) g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right), \tag{7.2-13}$$

where $Y_{M,Trgt}(f, x_R)$ is the complex frequency spectrum of the acoustic field radiated by the target and incident upon the receive aperture at x_R, with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$; $S_0(f)$ is the complex frequency spectrum of the target's source strength in $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{Hz}$,

$$g_f\left(\mathbf{r}_R \,|\, \mathbf{r}_S\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_S\right|\right)}{4\pi\left|\mathbf{r}_R - \mathbf{r}_S\right|} \tag{7.2-14}$$

is the time-independent, free-space, Green's function of an unbounded, ideal (nonviscous), homogeneous, fluid medium with units of inverse meters,

$$k = 2\pi f / c = 2\pi / \lambda \tag{7.2-15}$$

is the wavenumber in radians per meter, and $c = f\lambda$.

If we next replace the real wavenumber k in (7.2-14) with the *complex wavenumber*

$$\boxed{K = k - j\alpha(f)} \tag{7.2-16}$$

where K has units of inverse meters and $\alpha(f)$ is the real, nonnegative, frequency-dependent, *attenuation coefficient* of the fluid medium in nepers (Np) per meter, then

$$\boxed{g_f\left(\mathbf{r}_R \mid \mathbf{r}_S\right) = -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}_R - \mathbf{r}_S\right|\right]}{4\pi\left|\mathbf{r}_R - \mathbf{r}_S\right|}\exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_S\right|\right)} \tag{7.2-17}$$

Equation (7.2-17) is the time-independent, free-space, Green's function of an unbounded, *viscous*, homogeneous, fluid medium with units of inverse meters. The Green's function given by (7.2-17) is the complex frequency response of the fluid medium at frequency f hertz and location $\mathbf{r}_R$ due to the application of a unit-amplitude impulse (i.e., a unit-amplitude, omnidirectional point-source) at location $\mathbf{r}_S$. Substituting (7.2-17) into (7.2-13) yields

$$Y_{M,Trgt}(f, x_R) = -\frac{1}{4\pi\left|\mathbf{r}_R - \mathbf{r}_S\right|} S_0(f)\exp\left[-\alpha(f)\left|\mathbf{r}_R - \mathbf{r}_S\right|\right]\exp\left(-jk\left|\mathbf{r}_R - \mathbf{r}_S\right|\right), \tag{7.2-18}$$

or

$$Y_{M,Trgt}(f, x_R) = -\frac{1}{4\pi R} S_0(f)\exp[-\alpha(f)R]\exp(-j2\pi f\tau), \tag{7.2-19}$$

where the range R and time delay τ are given by (7.2-3) and (7.2-4), respectively. If we let

$$\boxed{S_0'(f, x_R) = S_0(f)\exp[-\alpha(f)R]} \tag{7.2-20}$$

then (7.2-19) can be rewritten as

$$Y_{M,Trgt}(f, x_R) = -\frac{1}{4\pi R} S_0'(f, x_R)\exp(-j2\pi f\tau). \tag{7.2-21}$$

Taking the inverse Fourier transform of (7.2-21) yields

$$\boxed{y_{M,Trgt}(t, x_R) = -\frac{1}{4\pi R} s_0'(t-\tau, x_R)} \tag{7.2-22}$$

for $t \geq \tau$, where

$$\boxed{s_0'(t, x_R) = F_f^{-1}\left\{S_0'(f, x_R)\right\}} \tag{7.2-23}$$

Note that $S_0'(f, x_R)$ and, hence, $s_0'(t, x_R)$ are functions of x_R because the range R is a function of x_R [see (7.2-3)]. Equation (7.2-22) is a mathematical model of the velocity potential (in squared meters per second) of the output acoustic signal from the fluid medium due to the target and incident upon the receive aperture at x_R that takes into account the viscosity of the fluid medium. It is the velocity potential of a spherical wave with arbitrary time dependence propagating in an unbounded, *viscous*, homogeneous, fluid medium. If there is no viscosity, that is, if $\alpha(f)=0$, then $s_0'(t,x_R)=s_0(t)$ and (7.2-22) reduces to (7.2-2). The mathematical model for the incident acoustic field given by (7.2-22), although simple, will allow us to discuss both near-field and far-field problems. As was discussed in Section 6.6, with the advent of very long, linear towed arrays, near-field issues cannot be ignored.

In the ocean acoustics literature, equations and values for the attenuation coefficient of seawater are usually given in decibels (dB) per meter or decibels per kilometer. For example, an accurate equation for the attenuation coefficient of seawater in decibels per kilometer is given in Appendix 7A. However, in order to evaluate decaying exponentials such as $\exp[-\alpha(f)R]$, only $\alpha(f)$ in nepers per meter can be used. Therefore, in order to convert the attenuation coefficient $\alpha'(f)$ in decibels per meter to $\alpha(f)$ in nepers per meter, use

$$\boxed{\alpha(f)=\frac{\alpha'(f)}{8.686}\,\frac{\text{Np}}{\text{m}}} \tag{7.2-24}$$

and to convert $\alpha(f)$ in nepers per meter to $\alpha'(f)$ in decibels per meter, use

$$\boxed{\alpha'(f)=8.686\,\alpha(f)\,\frac{\text{dB}}{\text{m}}} \tag{7.2-25}$$

Figure 7.2-1 is a plot of the attenuation coefficient of seawater $\alpha'(f)$ in decibels per kilometer (dB/km) given by (7A-1) versus frequency f in kilohertz (kHz) for temperatures $T=14°\,\text{C}$ ($57.2°\,\text{F}$) and $T=20°\,\text{C}$ ($68°\,\text{F}$) at a depth of 50 m. A salinity of 35 ppt and an acidity $p\text{H}=8$ was used.

With the use of (7.2-7), the range R between the target and an aperture point [see (7.2-3)]

$$R=\sqrt{(x_R-x_S)^2+y_S^2+z_S^2} \tag{7.2-26}$$

can be rewritten as

$$R=\sqrt{r_S^2-2x_Sx_R+x_R^2}\,. \tag{7.2-27}$$

Since

$$x_S=r_Su_S\,, \tag{7.2-28}$$

substituting (7.2-28) into (7.2-27) yields

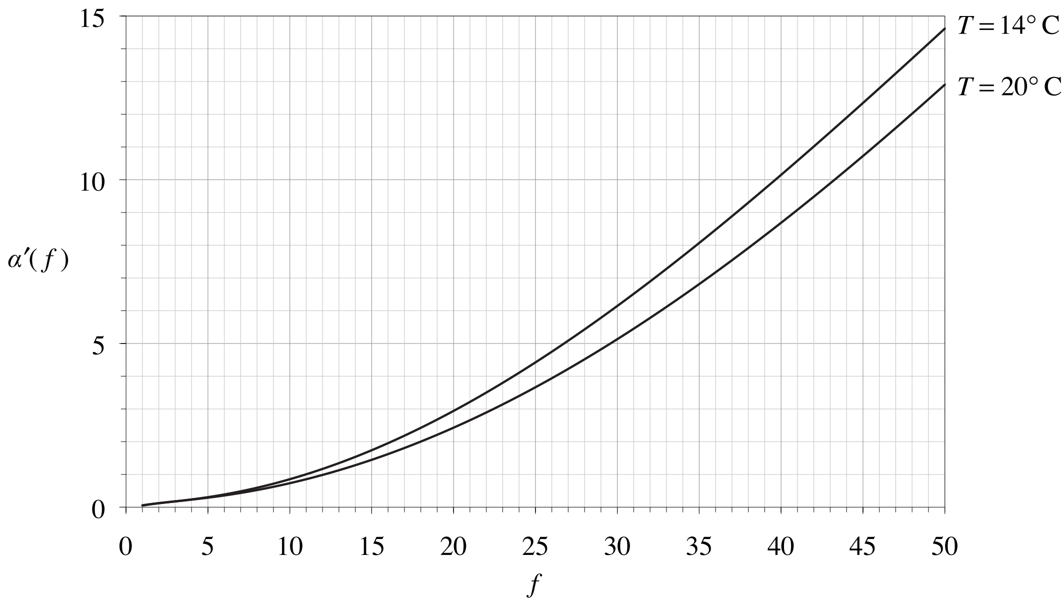

Figure 7.2-1 Attenuation coefficient of seawater $\alpha'(f)$ in decibels per kilometer (dB/km) given by (7A-1) versus frequency f in kilohertz (kHz) for temperatures $T = 14°$ C (57.2° F) and $T = 20°$ C (68° F) at a depth of 50 m. A salinity of 35 ppt and an acidity pH = 8 was used.

$$R = \sqrt{r_S^2 - 2r_S u_S x_R + x_R^2}\,, \tag{7.2-29}$$

where r_S is given by (7.2-7) and u_S is given by (7.2-9). Equations (7.2-26), (7.2-27), and (7.2-29) are all *exact* expressions for the range R that can be used to compute both the amplitude factor $1/R$ and time delay $\tau = R/c$ in (7.2-22). For example, if (7.2-29) is used, then

$$\frac{1}{R} = \frac{1}{\sqrt{r_S^2 - 2r_S u_S x_R + x_R^2}} \tag{7.2-30}$$

and

$$\tau = \frac{1}{c}\sqrt{r_S^2 - 2r_S u_S x_R + x_R^2}\,. \tag{7.2-31}$$

If the target is in the Fresnel region of the aperture, then

$$\frac{1}{R} \approx \frac{1}{r_S} \tag{7.2-32}$$

is a good approximation of the amplitude factor in (7.2-22) (see Subsection 1.2.2).

However, in order to obtain a good approximation of the time delay τ in (7.2-22), use (see Subsection 1.2.2)

$$R \approx r_S - \hat{r}_S \bullet \mathbf{r}_R + \frac{1}{2r_S} r_R^2 . \tag{7.2-33}$$

Substituting (7.2-1) and (7.2-8) into (7.2-33), and since $r_R^2 = \left|\mathbf{r}_R\right|^2 = x_R^2$, (7.2-33) reduces to

$$R \approx r_S - u_S x_R + \frac{1}{2r_S} x_R^2 , \tag{7.2-34}$$

so that

$$\tau \approx \tau_S - \frac{u_S}{c} x_R + \frac{1}{2 r_S c} x_R^2 , \tag{7.2-35}$$

where

$$\tau_S = r_S / c \tag{7.2-36}$$

is the *one-way* time delay in seconds associated with the path length between the sound-source (target) and the center of the aperture. The time delay τ_S is also *unknown* a priori because the range r_S is unknown a priori.

If the target is in the far-field region of the aperture, then (7.2-32) is also a good approximation of the amplitude factor in (7.2-22) (see Subsection 1.3.2). However, in order to obtain a good approximation of the time delay τ in (7.2-22), use (see Subsection 1.3.2)

$$R \approx r_S - \hat{r}_S \bullet \mathbf{r}_R , \tag{7.2-37}$$

or

$$R \approx r_S - u_S x_R , \tag{7.2-38}$$

so that

$$\tau \approx \tau_S - \frac{u_S}{c} x_R , \tag{7.2-39}$$

where τ_S is given by (7.2-36).

7.3 Total Output Signal from a Linear Array due to the Target

By applying the receive volume aperture results obtained in Subsection 1.1.2 to a receive linear aperture lying along the X axis, and since the target and aperture are not in motion (output (received) frequencies η are equal to input (transmitted) frequencies f since there is no Doppler shift nor bandwidth compression or expansion), we can write that

$$Y_{Trgt}(f, x_R) = Y_{M,Trgt}(f, x_R) A_R(f, x_R), \tag{7.3-1}$$

where $Y_{Trgt}(f, x_R)$ is the complex frequency spectrum of the output electrical signal from the receive aperture at x_R due to the target, with units of $(\mathrm{V/Hz})/\mathrm{m}$; $Y_{M,Trgt}(f, x_R)$ is the complex frequency spectrum of the acoustic field radiated by the target and incident upon the receive aperture at x_R, with units of $(\mathrm{m^2/sec})/\mathrm{Hz}$, and is given by (7.2-21); and $A_R(f, x_R)$ is the complex frequency response of the receive aperture at x_R (a.k.a. the complex receive aperture function), with units of $(\mathrm{V}/(\mathrm{m^2/sec}))/\mathrm{m}$, where f represents frequencies in hertz transmitted by the target.

Since the receive linear aperture is a linear array composed of an odd number N of identical, equally-spaced, complex-weighted, omnidirectional point-elements lying along the X axis, the complex frequency response of the array is given by (see Subsection 6.1.2)

$$A_R(f, x_R) = \mathcal{S}_R(f) \sum_{i=-N'}^{N'} c_i(f)\,\delta(x_R - x_i), \tag{7.3-2}$$

where $\mathcal{S}_R(f)$ is the complex, receiver sensitivity function with units of $\mathrm{V}/(\mathrm{m^2/sec})$ (see Table 6.1-2 and Appendix 6B),

$$c_i(f) = a_i \exp[+j\theta_i(f)] = a_i \exp(-j2\pi f \tau_i') \tag{7.3-3}$$

is the dimensionless complex weight associated with element i in the array, the impulse function $\delta(x_R - x_i)$ has units of inverse meters,

$$x_i = id, \qquad i = -N', \ldots, 0, \ldots, N', \tag{7.3-4}$$

is the x coordinate of the center of element i, d is the interelement spacing in meters, and

$$N' = (N-1)/2\,. \tag{7.3-5}$$

Note that the amplitude weight a_i is not a function of frequency. The different equations that can be used for the time delay τ_i' for purposes of beam steering and array focusing shall be discussed later in this section.

Substituting (7.3-2) into (7.3-1) yields the following expression for the complex frequency spectrum of the output electrical signal from the array at x_R due to the target, with units of $(\mathrm{V/Hz})/\mathrm{m}$:

$$Y_{Trgt}(f, x_R) = Y_{M,Trgt}(f, x_R)\mathcal{S}_R(f) \sum_{i=-N'}^{N'} c_i(f)\delta(x_R - x_i)\,. \tag{7.3-6}$$

The complex frequency spectrum (in volts per hertz) of the *total* output electrical signal from the array due to the target is given by [see (1.2-99)]

$$Y_{Trgt}(f) = \int_{-\infty}^{\infty} Y_{Trgt}(f, x_R)\,dx_R\,. \tag{7.3-7}$$

Substituting (7.3-6) into (7.3-7) yields

$$Y_{Trgt}(f) = \mathcal{S}_R(f) \sum_{i=-N'}^{N'} c_i(f) \int_{-\infty}^{\infty} Y_{M,Trgt}(f, x_R)\delta(x_R - x_i)\,dx_R\,, \tag{7.3-8}$$

and since the impulse function will be nonzero only when $x_R = x_i$, by using the sifting property of impulse functions, (7.3-8) reduces to

$$Y_{Trgt}(f) = \mathcal{S}_R(f) \sum_{i=-N'}^{N'} c_i(f) Y_{M,Trgt}(f, x_i)\,, \tag{7.3-9}$$

or

$$Y_{Trgt}(f) = \sum_{i=-N'}^{N'} Y_{M,Trgt}(f, x_i)\mathcal{S}_R(f)c_i(f)\,. \tag{7.3-10}$$

Before proceeding further with our analysis, let us summarize our results up to this point.

Since $x_R = x_i = id$ [see (7.3-4)], the position vector to an aperture point, $\mathbf{r}_R$, given by (7.2-1) becomes

$$\mathbf{r}_R = \mathbf{r}_i = x_i\hat{x} = id\hat{x}, \qquad i = -N', \ldots, 0, \ldots, N'\,, \tag{7.3-11}$$

where $\mathbf{r}_i$ is the position vector to element i in the array. Therefore, the velocity potential in squared meters per second of the output acoustic signal from the fluid medium due to the target and incident upon element i in the array is given by [see (7.2-22)]

$$\boxed{y_{M,Trgt}(t, x_i) = -\frac{1}{4\pi R_i} s_0'(t - \tau_i, x_i)} \tag{7.3-12}$$

for $t \geq \tau_i$, where

$$\boxed{s_0'(t,x_i) = F_f^{-1}\{S_0'(f,x_i)\}} \tag{7.3-13}$$

has units of cubic meters per second [see (7.2-23)],

$$\boxed{S_0'(f,x_i) = S_0(f)\exp[-\alpha(f)R_i]} \tag{7.3-14}$$

has units of $\left(\text{m}^3/\text{sec}\right)/\text{Hz}$ [see (7.2-20)], $S_0(f)$ is the complex frequency spectrum of the target's source strength in $\left(\text{m}^3/\text{sec}\right)/\text{Hz}$, $\alpha(f)$ is the frequency-dependent attenuation coefficient in nepers per meter,

$$R_i = \left|\mathbf{r}_i - \mathbf{r}_S\right| = \sqrt{r_S^2 - 2r_S u_S x_i + x_i^2} \tag{7.3-15}$$

is the range in meters between the target and the center of element i [see (7.2-29)], and

$$\tau_i = R_i/c \tag{7.3-16}$$

is the corresponding *one-way* travel time or *one-way* time delay in seconds [see (7.2-4)]. The exact equations for the amplitude factor $1/R_i$ and time delay τ_i are [see (7.2-30) and (7.2-31)]

$$\frac{1}{R_i} = \frac{1}{\sqrt{r_S^2 - 2r_S u_S x_i + x_i^2}} \tag{7.3-17}$$

and

$$\tau_i = \frac{1}{c}\sqrt{r_S^2 - 2r_S u_S x_i + x_i^2} \; . \tag{7.3-18}$$

If the target is in the Fresnel region of the array, then [see (7.2-32) and (7.2-35)]

$$\frac{1}{R_i} \approx \frac{1}{r_S} \tag{7.3-19}$$

and

$$\tau_i \approx \tau_S - \frac{u_S}{c}x_i + \frac{1}{2r_S c}x_i^2 \; , \tag{7.3-20}$$

where τ_S is given by (7.2-36). If the target is in the far-field region of the array, then the amplitude factor given by (7.3-19) can also be used and [see (7.2-39)]

$$\tau_i \approx \tau_S - \frac{u_S}{c} x_i \,, \tag{7.3-21}$$

where τ_S is given by (7.2-36). And finally, the complex frequency spectrum of the acoustic field radiated by the target and incident upon element i in the array, with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$, is given by [see (7.2-21)]

$$Y_{M,Trgt}(f, x_i) = -\frac{1}{4\pi R_i} S_0'(f, x_i) \exp(-j2\pi f \tau_i) \,, \tag{7.3-22}$$

or

$$Y_{M,Trgt}(f, x_i) = S_0(f)\, g_f\left(\mathbf{r}_i \,\middle|\, \mathbf{r}_S\right), \tag{7.3-23}$$

where

$$\boxed{g_f\left(\mathbf{r}_i \,\middle|\, \mathbf{r}_S\right) = -\frac{\exp\left[-\alpha(f)\left|\mathbf{r}_i - \mathbf{r}_S\right|\right]}{4\pi\left|\mathbf{r}_i - \mathbf{r}_S\right|} \exp\left(-jk\left|\mathbf{r}_i - \mathbf{r}_S\right|\right)} \tag{7.3-24}$$

is the time-independent, free-space, Green's function of an unbounded, viscous, homogeneous, fluid medium with units of inverse meters, and k is the wavenumber in radians per meter given by (7.2-15). The Green's function given by (7.3-24) is the complex frequency response of the fluid medium at frequency f hertz and location $\mathbf{r}_i$ due to the application of a unit-amplitude impulse (i.e., a unit-amplitude, omnidirectional point-source) at location $\mathbf{r}_S$.

Let us now continue with the derivation of $Y_{Trgt}(f)$ by rewriting (7.3-10) as follows:

$$Y_{Trgt}(f) = \sum_{i=-N'}^{N'} Y_{Trgt}(f, x_i) \,, \tag{7.3-25}$$

where

$$Y_{Trgt}(f, x_i) = Y'_{Trgt}(f, x_i)\, c_i(f) \tag{7.3-26}$$

and

$$Y'_{Trgt}(f, x_i) = Y_{M,Trgt}(f, x_i)\, \mathcal{S}_R(f) \tag{7.3-27}$$

are the complex frequency spectra (in volts per hertz) of the output electrical signal from element i in the array due to the target *after* and *before* the application of the complex weight $c_i(f)$, respectively (see Fig. 7.3-1). Substituting (7.3-22) and (7.3-14) into (7.3-27) yields

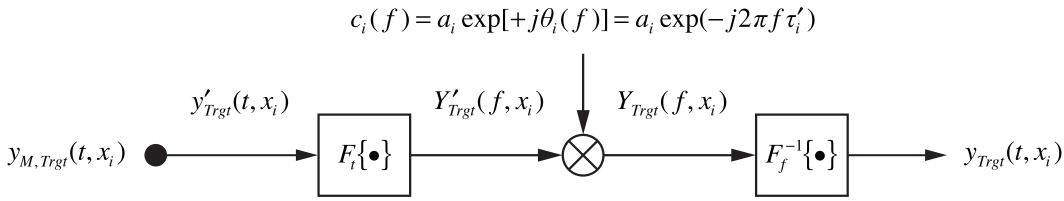

Figure 7.3-1 Beamforming. Implementing a complex weight (amplitude and phase weights) using forward and inverse Fourier transforms.

$$Y'_{Trgt}(f, x_i) = S(f, x_i)\exp(-j2\pi f \tau_i), \tag{7.3-28}$$

where

$$\boxed{S(f, x_i) = -\frac{1}{4\pi R_i} S_0(f)\exp[-\alpha(f)R_i]\mathcal{S}_R(f)} \tag{7.3-29}$$

has units of volts per hertz. Note that $S(f, x_i)$ is a function of x_i because the exact range R_i between the target and the center of each element in the array is a function of x_i [see (7.3-15)]. Substituting (7.3-3) and (7.3-28) into (7.3-26) yields

$$Y_{Trgt}(f, x_i) = a_i S(f, x_i)\exp\left[-j2\pi f(\tau_i + \tau_i')\right], \tag{7.3-30}$$

and by substituting (7.3-30) into (7.3-25), we finally obtain

$$Y_{Trgt}(f) = \sum_{i=-N'}^{N'} a_i S(f, x_i)\exp\left[-j2\pi f(\tau_i + \tau_i')\right]. \tag{7.3-31}$$

Equation (7.3-31) is the complex frequency spectrum (in volts per hertz) of the *total* output electrical signal from the array due to the target. We shall obtain a time-domain model next.

Taking the inverse Fourier transform of (7.3-28) yields

$$y'_{Trgt}(t, x_i) = s(t - \tau_i, x_i), \tag{7.3-32}$$

which is the output electrical signal in volts from element i in the array due to the target, *before* complex weighting (see Fig. 7.3-1), for $t \geq \tau_i$, where

$$\boxed{s(t, x_i) = F_f^{-1}\{S(f, x_i)\} = -\frac{1}{4\pi R_i} F_f^{-1}\left\{S_0(f)\exp[-\alpha(f)R_i]\mathcal{S}_R(f)\right\}}$$

(7.3-33)

The function $s(t, x_i)$ is a *time-domain*, *WSS random process* because it is the result of linear, time-invariant, space-invariant, deterministic filtering of the target's source strength $s_0(t)$, which was assumed to be WSS. Although we are working with a simple spherical-wave model for the acoustic field incident upon the array [see (7.3-12)], (7.3-33) provides some insight as to why it is so difficult to detect quiet submarines. If a target is radiating a very weak acoustic field, then the magnitudes of the frequency components contained in the magnitude spectrum of the target's source strength, $|S_0(f)|$, will be very small. By the time the radiated acoustic field reaches the array, the magnitude spectrum $|S_0(f)|$ is attenuated by the decaying exponential $\exp[-\alpha(f)R_i]$ and the amplitude factor $1/R_i$. The magnitude spectrum $|S_0(f)|$ is further attenuated by the filtering action of the elements in the array as modeled by the receiver sensitivity function $\mathcal{S}_R(f)$ and, in particular, its magnitude response $|\mathcal{S}_R(f)|$.

The next step is to inverse Fourier transform (7.3-25) and (7.3-30). Doing so yields

$$y_{Trgt}(t) = \sum_{i=-N'}^{N'} y_{Trgt}(t, x_i) \tag{7.3-34}$$

and

$$\boxed{y_{Trgt}(t, x_i) = a_i\, s(t - [\tau_i + \tau_i'], x_i)} \tag{7.3-35}$$

respectively, and by substituting (7.3-35) into (7.3-34), we obtain

$$\boxed{y_{Trgt}(t) = \sum_{i=-N'}^{N'} a_i\, s(t - [\tau_i + \tau_i'], x_i)} \tag{7.3-36}$$

where $s(t, x_i)$ is given by (7.3-33). Equation (7.3-35) is the output electrical signal (in volts) from element i in the array due to the target, *after* complex weighting (see Fig. 7.3-1), for $t \geq \tau_i + \tau_i'$. Equation (7.3-36) is the *total* output electrical signal from the array (in volts) due to the target. Both $y_{Trgt}(t, x_i)$ and $y_{Trgt}(t)$ are WSS in time because $s(t, x_i)$ is WSS in time.

From (7.3-35) it can be seen that the individual output electrical signals $y_{Trgt}(t, x_i)$, $i = -N', \ldots, 0, \ldots, N'$, are *out-of-phase* with each other because $\tau_i + \tau_i'$ is different at each element i in the array. As a result, the time-average power of the total output electrical signal from the array, $y_{Trgt}(t)$, given by (7.3-36) will *not* be maximized because out-of-phase signals add destructively (destructive interference). Therefore, in order to maximize the time-average power of $y_{Trgt}(t)$, all the individual signals $y_{Trgt}(t,x_i)$, $i=-N',\ldots,0,\ldots,N'$, must be *in-phase*, which requires that $\tau_i+\tau_i'$ be equal to either *zero* or a *constant* $\forall\ i$.

In order to cophase all the individual signals $y_{Trgt}(t,x_i)$, $i=-N',\ldots,0,\ldots,N'$, the correct equation for the time delay τ_i' must be used, which depends on the model used for τ_i. For example, if you want to be able to cophase $y_{Trgt}(t,x_i)$ $\forall\ i$ regardless of where the target is relative to the array – near-field, Fresnel, or far-field region – then [see (7.3-18)]

$$\tau_i=\frac{1}{c}\sqrt{r_S^2-2r_S u_S x_i+x_i^2}\ , \tag{7.3-37}$$

where [see (7.2-9)]

$$u_S=\sin\theta_S\cos\psi_S\ . \tag{7.3-38}$$

However, since the spherical coordinates of the target's location (r_S,θ_S,ψ_S) are unknown a priori,

$$\boxed{\tau_i'=-\frac{1}{c}\sqrt{(r')^2-2r'u'x_i+x_i^2}} \tag{7.3-39}$$

where

$$u'=\sin\theta'\cos\psi'\ . \tag{7.3-40}$$

Adding (7.3-37) and (7.3-39) yields

$$\tau_i+\tau_i'=\frac{1}{c}\left[\sqrt{r_S^2-2r_S u_S x_i+x_i^2}-\sqrt{(r')^2-2r'u'x_i+x_i^2}\ \right]. \tag{7.3-41}$$

Therefore, if the far-field beam pattern of the array is *focused* at the range to the target ($r'=r_S$) and *steered* in the direction of the acoustic field incident upon the array (in this case, $\theta'=\theta_S$ and $\psi'=\psi_S$ so that $u'=u_S$), then (7.3-41) reduces to

$$\tau_i+\tau_i'=0,\qquad i=-N',\ldots,0,\ldots,N'\ , \tag{7.3-42}$$

and, as a result, all the individual output electrical signals due to the target will be in-phase. Substituting (7.3-42) into (7.3-35) and (7.3-36) yields

$$\boxed{y_{Trgt}(t,x_i)=a_i\,s(t,x_i)} \tag{7.3-43}$$

and

$$\boxed{y_{Trgt}(t)=\sum_{i=-N'}^{N'}a_i\,s(t,x_i)} \tag{7.3-44}$$

respectively, where $s(t, x_i)$ is given by (7.3-33). If rectangular amplitude weights are used, then $a_i = 1 \;\; \forall \;\; i$, and (7.3-44) reduces to

$$\boxed{y_{Trgt}(t) = \sum_{i=-N'}^{N'} s(t, x_i)} \tag{7.3-45}$$

Equations (7.3-44) and (7.3-45) both indicate that $y_{Trgt}(t)$ is equal to the sum of N signals that are in-phase (constructive interference).

If the target is in the Fresnel region of the array, then a good approximation of τ_i is given by [see (7.3-20)]

$$\tau_i \approx \tau_S - \frac{u_S}{c} x_i + \frac{1}{2 r_S c} x_i^2 , \tag{7.3-46}$$

where the time delay τ_S is given by (7.2-36). Based on the model for τ_i given by (7.3-46), and ignoring τ_S (see Subsection 6.6.1),

$$\boxed{\tau_i' = \frac{u'}{c} x_i - \frac{1}{2r'c} x_i^2} \tag{7.3-47}$$

Adding (7.3-46) and (7.3-47) yields

$$\tau_i + \tau_i' = \tau_S + \frac{1}{c}(u' - u_S) x_i + \frac{1}{2c} \frac{r' - r_S}{r' r_S} x_i^2 . \tag{7.3-48}$$

If the far-field beam pattern of the array is *focused* at the range to the target ($r' = r_S$) and *steered* in the direction of the acoustic field incident upon the array (in this case, $\theta' = \theta_S$ and $\psi' = \psi_S$ so that $u' = u_S$), then (7.3-48) reduces to

$$\tau_i + \tau_i' = \tau_S , \qquad i = -N', \ldots, 0, \ldots, N' . \tag{7.3-49}$$

Since τ_S is a constant, all the individual output electrical signals due to the target will be in-phase. Substituting (7.3-49) into (7.3-35) and (7.3-36), and setting $R_i = r_S$ in (7.3-29) and (7.3-33) yields

$$\boxed{y_{Trgt}(t, x_i) = a_i s(t - \tau_S)} \tag{7.3-50}$$

and

$$\boxed{y_{Trgt}(t) = s(t - \tau_S) \sum_{i=-N'}^{N'} a_i} \tag{7.3-51}$$

respectively, for $t \geq \tau_S$, $S(f, x_i) \approx S(f)$ where [see (7.3-29)]

$$\boxed{S(f) = -\frac{1}{4\pi r_S} S_0(f) \exp[-\alpha(f) r_S] \mathcal{S}_R(f)} \tag{7.3-52}$$

and $s(t, x_i) \approx s(t)$ where [see (7.3-33)]

$$\boxed{s(t) = F_f^{-1}\{S(f)\} = -\frac{1}{4\pi r_S} F_f^{-1}\{S_0(f) \exp[-\alpha(f) r_S] \mathcal{S}_R(f)\}} \tag{7.3-53}$$

The function $s(t)$ is a WSS random process because it is the result of linear, time-invariant, space-invariant, deterministic filtering of the target's source strength $s_0(t)$, which was assumed to be WSS. If rectangular amplitude weights are used, then $a_i = 1 \;\; \forall \;\; i$, and (7.3-51) reduces to

$$\boxed{y_{Trgt}(t) = N s(t - \tau_S)} \tag{7.3-54}$$

for $t \geq \tau_S$. Equation (7.3-54) is the result of N identical signals adding in-phase (constructive interference).

If the target is in the far-field region of the array, then a good approximation of τ_i is given by [see (7.3-21)]

$$\tau_i \approx \tau_S - \frac{u_S}{c} x_i \,, \tag{7.3-55}$$

where the time delay τ_S is given by (7.2-36). Based on the model for τ_i given by (7.3-55), and ignoring τ_S (see Section 6.4),

$$\boxed{\tau_i' = \frac{u'}{c} x_i} \tag{7.3-56}$$

Adding (7.3-55) and (7.3-56) yields

$$\tau_i + \tau_i' = \tau_S + \frac{1}{c}(u' - u_S) x_i \,. \tag{7.3-57}$$

If the far-field beam pattern of the array is *steered* in the direction of the acoustic field incident upon the array (in this case, $\theta' = \theta_S$ and $\psi' = \psi_S$ so that $u' = u_S$), then (7.3-57) reduces to

$$\tau_i + \tau_i' = \tau_S, \qquad i = -N', \ldots, 0, \ldots, N' . \tag{7.3-58}$$

Since τ_S is a constant, all the individual output electrical signals due to the target will be in-phase. And since (7.3-58) is identical to (7.3-49), (7.3-50) through (7.3-54) are also applicable to the far-field case.

At this point in our discussion, we need to answer the following practical question: How do we determine when all the individual output electrical signals due to the target are in-phase when using the beamformer shown in Fig. 7.1-1? The answer is to compute the time-average power of the total output electrical signal from the array, $r(t)$, given by (7.1-4), for every set of phase weights used. One set of phase weights corresponds to evaluating

$$\theta_i(f) = -2\pi f \tau_i', \qquad i = -N', \ldots, 0, \ldots, N' , \tag{7.3-59}$$

$\forall\ i$, using one set of values for (r', θ', ψ') to evaluate τ_i' given by (7.3-39) or (7.3-47), or using one set of values for (θ', ψ') to evaluate τ_i' given by (7.3-56). The phase weights given by (7.3-59) are then used to form the set of complex weights [see (7.3-3) and Fig. 7.1-1]

$$c_i(f) = a_i \exp[+j\theta_i(f)] = a_i \exp(-j2\pi f \tau_i'), \qquad i = -N', \ldots, 0, \ldots, N' . \tag{7.3-60}$$

Using one set of values for (r', θ', ψ') or (θ', ψ') corresponds to forming *one beam*. Forming *multiple beams* with multiple "look directions" means using multiple sets of values for (r', θ', ψ') or (θ', ψ'). Note that the set of complex frequency spectra $R'(f, x_i)$, $i = -N', \ldots, 0, \ldots, N'$, only needs to be computed *once* in order to form multiple beams (see Fig. 7.1-1). By using *parallel signal processing*, multiple beams can be formed *simultaneously*. The set of values (r', θ', ψ') or (θ', ψ') that maximizes the time-average power of $r(t)$ is the best *estimate* of (r_S, θ_S, ψ_S) or (θ_S, ψ_S) for an omnidirectional, point-source target in an unbounded, viscous, homogeneous, fluid medium. A signal processing algorithm that can be used for computing the time-average power of $r(t)$ is discussed in Appendix 7B.

After the time-average power of $r(t)$ has been maximized, a *target signature analysis* can be performed by computing the power spectrum of $r(t)$ (see Appendix 7B). Estimating the frequency components (spectral lines) of the sound-field radiated by a target via a power spectrum calculation is referred to as performing a target signature analysis because different sound-sources (targets) such as submarines, surface ships, and marine mammals are known to radiate energy at different sets of frequencies. A target's unique set of radiated frequencies is known as a *target signature*. The information obtained from a target signature analysis can then be used to classify (identify) the target. This is known as *target classification*.

7.3.1 FFT Beamforming for Linear Arrays

In this subsection we shall show how to implement complex weights in a linear array using the method of *FFT beamforming*. We shall also show how to compute the beam pattern of the linear array by using frequency-domain data already obtained from the method of FFT beamforming. Although the output electrical signals from the individual elements in the array are equal to the sum of the output electrical signals due to the target, ambient noise, and receiver noise; for example purposes, we shall only work with the output electrical signals due to the target.

The first step is to sample $y'_{Trgt}(t, x_i)$, which is the output electrical signal from element i in the array due to the target *before* complex weighting given by [see (7.3-32) and Fig. 7.3-1]

$$y'_{Trgt}(t, x_i) = s(t - \tau_i, x_i) , \tag{7.3-61}$$

for $t \geq \tau_i$, where $s(t, x_i)$ is given by (7.3-33) and τ_i is given by (7.3-16). If we sample $y'_{Trgt}(t, x_i)$ at a rate of f_S samples per second, then

$$y'_{Trgt}(t_l, x_i) = s(t_l - \tau_i, x_i) , \tag{7.3-62}$$

or

$$y'_{Trgt}(lT_S, id) = s(lT_S - \tau_i, id) , \tag{7.3-63}$$

where

$$t_l = lT_S, \qquad l = 0, 1, \ldots, L-1 , \tag{7.3-64}$$

is the sampling time instant in seconds, $T_S = 1/f_S$ is the sampling period in seconds, L is the total number of time samples taken,

$$x_i = id, \qquad i = -N', \ldots, 0, \ldots, N' , \tag{7.3-65}$$

is the x coordinate of the center of element i [see (7.3-4)], d is the interelement spacing in meters, and [see (7.3-5)]

$$N' = (N-1)/2 , \tag{7.3-66}$$

where N is the total odd number of identical, equally-spaced, complex-weighted, omnidirectional point-elements in the linear array. Note that

$$y'_{Trgt}(l,i) \equiv y'_{Trgt}(t_l, x_i) = y'_{Trgt}(lT_S, id). \tag{7.3-67}$$

By referring to Fig. 7.3-1, the second step is to compute the time-domain Fourier transform of $y'_{Trgt}(l,i)$. Using a modified version of the algorithm discussed in Appendix 7B for computing time-domain Fourier transforms yields

$$\hat{Y}'_{Trgt}(q,i) = T_S \, DFT_l\left\{y'_{Trgt}(l,i)\right\}, \qquad q = -L'', \ldots, 0, \ldots, L'', \tag{7.3-68}$$

or

$$\hat{Y}'_{Trgt}(q,i) = T_S \sum_{l=0}^{L-1} y'_{Trgt}(l,i) W_{L+Z}^{-ql}, \qquad q = -L'', \ldots, 0, \ldots, L'', \tag{7.3-69}$$

where

$$\hat{Y}'_{Trgt}(q,i) \equiv \hat{Y}'_{Trgt}(qf_0, id) = \hat{Y}'_{Trgt}(f, x_i)\Big|_{f=qf_0} \tag{7.3-70}$$

is an estimate of the theoretical frequency spectrum $Y'_{Trgt}(f, x_i)$ given by (7.3-28) at discrete frequencies $f = qf_0$,

$$W_{L+Z} = \exp\left(+j\frac{2\pi}{L+Z}\right), \tag{7.3-71}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{(L+Z)T_S} = \frac{f_S}{L+Z} \tag{7.3-72}$$

is the frequency-domain DFT bin spacing [the fundamental frequency of $y'_{Trgt}(t, x_i)$] in hertz, T_0 is the fundamental period of $y'_{Trgt}(t, x_i)$ in seconds,

$$L'' = \begin{cases} (L+Z)/2, & L+Z \text{ even} \\ (L+Z-1)/2, & L+Z \text{ odd,} \end{cases} \tag{7.3-73}$$

and Z is the total number of "zeros" used for "padding-with-zeros". See Appendix 7B for an explanation of "padding-with-zeros". Substituting (7.3-67) and (7.3-63) into (7.3-68) and using the time-shifting property of Fourier transforms yields

$$\begin{aligned} \hat{Y}'_{Trgt}(q,i) &= T_S \, DFT_l\left\{s(lT_S - \tau_i, id)\right\} \\ &= T_S \, DFT_l\left\{s(l,i)\right\} \exp(-j2\pi q f_0 \tau_i), \qquad q = -L'', \ldots, 0, \ldots, L'', \end{aligned} \tag{7.3-74}$$

or

$$\hat{Y}'_{Trgt}(q,i) = \hat{S}(q,i)\exp(-j2\pi q f_0 \tau_i)\,, \tag{7.3-75}$$

where

$$\hat{S}(q,i) = T_S\, DFT_l\{s(l,i)\}, \qquad q = -L'',\ldots,0,\ldots,L''\,, \tag{7.3-76}$$

is an estimate of the theoretical frequency spectrum $S(f,x_i)$ given by (7.3-29) at discrete frequencies $f = qf_0$, and

$$s(l,i) \equiv s(t_l, x_i) = s(lT_S, id)\,. \tag{7.3-77}$$

As can be seen from (7.3-29), $S(f,x_i)$ is related to the complex frequency spectrum of the target's source strength $S_0(f)$.

By referring to Fig. 7.3-1, the third step is to multiply $\hat{Y}'_{Trgt}(q,i)$ by the complex weight [see (7.3-3)]

$$c_i(qf_0) = a_i \exp[+j\theta_i(qf_0)] = a_i \exp(-j2\pi q f_0 \tau'_i)\,. \tag{7.3-78}$$

Doing so yields

$$\hat{Y}_{Trgt}(q,i) = \hat{Y}'_{Trgt}(q,i)\, c_i(qf_0)\,, \tag{7.3-79}$$

and by substituting (7.3-75) and (7.3-78) into (7.3-79), we obtain

$$\hat{Y}_{Trgt}(q,i) = a_i \hat{S}(q,i)\exp\left[-j2\pi q f_0(\tau_i + \tau'_i)\right], \tag{7.3-80}$$

which is an estimate of the theoretical frequency spectrum $Y_{Trgt}(f,x_i)$ given by (7.3-30) at discrete frequencies $f = qf_0$.

By referring to Fig. 7.3-1, the fourth and final step is to compute the inverse Fourier transform of $\hat{Y}_{Trgt}(q,i)$. The inverse Fourier transform of $\hat{Y}_{Trgt}(q,i)$ is computed as follows:

$$\hat{y}_{Trgt}(l,i) = \frac{1}{T_S}\, IDFT_q\left\{\hat{Y}_{Trgt}(q,i)\right\}, \qquad l = 0,1,\ldots,L+Z-1\,, \tag{7.3-81}$$

or

$$\hat{y}_{Trgt}(l,i) = \frac{1}{T_S}\frac{1}{L+Z}\sum_{q=-L''}^{L''} \hat{Y}_{Trgt}(q,i) W_{L+Z}^{ql}\,, \qquad l = 0,1,\ldots,L+Z-1\,, \tag{7.3-82}$$

where IDFT is the abbreviation for inverse discrete Fourier transform.

Substituting (7.3-80) into (7.3-81) yields

$$\hat{y}_{Trgt}(l,i) = a_i \frac{1}{T_S} IDFT_q\left\{\hat{S}(q,i)\exp\left[-j2\pi q f_0(\tau_i + \tau_i')\right]\right\}, \qquad l = 0,1,\ldots,L+Z-1, \tag{7.3-83}$$

and by using the time-shifting property of Fourier transforms, we obtain

$$\hat{y}_{Trgt}(l,i) = a_i \hat{s}(lT_S - [\tau_i + \tau_i'], i), \qquad l = 0,1,\ldots,L+Z-1, \tag{7.3-84}$$

which is an estimate of time samples of the output electrical signal $y_{Trgt}(t, x_i)$ from element i in the array due to the target *after* complex weighting [see (7.3-35) and Fig. 7.3-1], where

$$\hat{s}(l,i) = \frac{1}{T_S} IDFT_q\left\{\hat{S}(q,i)\right\} \tag{7.3-85}$$

is an *estimate* of time samples of $s(t, x_i)$ because $\hat{Y}_{Trgt}(q,i)$ was obtained by multiplying the *estimate* $\hat{Y}'_{Trgt}(q,i)$ by the complex weight $c_i(qf_0)$ [see (7.3-79)].

In order to compute the beam pattern of the linear array, compute the *spatial* DFT of the frequency-domain data $\hat{Y}_{Trgt}(q,i)$ already obtained from the method of FFT beamforming as follows (see Section 6.5):

$$\hat{\mathrm{Y}}_{Trgt}(q,m) = DFT_i\left\{\hat{Y}_{Trgt}(q,i)\right\}, \qquad m = -N'',\ldots,0,\ldots,N'', \tag{7.3-86}$$

or

$$\hat{\mathrm{Y}}_{Trgt}(q,m) = \sum_{i=-N'}^{N'} \hat{Y}_{Trgt}(q,i) W_{N+Z_X}^{mi}, \qquad m = -N'',\ldots,0,\ldots,N'', \tag{7.3-87}$$

where

$$\hat{\mathrm{Y}}_{Trgt}(q,m) \equiv \hat{\mathrm{Y}}_{Trgt}(qf_0, m\Delta f_X) = \hat{\mathrm{Y}}_{Trgt}(f, f_X)\Big|_{\substack{f = qf_0 \\ f_X = m\Delta f_X}} \tag{7.3-88}$$

is an estimate of the frequency-and-angular spectrum of $y_{Trgt}(t, x_i)$ at discrete frequencies $f = qf_0$, and discrete spatial frequencies $f_X = m\Delta f_X$, or equivalently, direction cosine values u_m;

$$W_{N+Z_X} = \exp\left(+j\frac{2\pi}{N+Z_X}\right), \tag{7.3-89}$$

$$\Delta f_X = \frac{1}{(N+Z_X)d} \tag{7.3-90}$$

is the spatial-frequency spacing in the X direction with units of cycles per meter,

$$N'' = \begin{cases} (N+Z_X)/2, & N+Z_X \text{ even} \\ (N+Z_X-1)/2, & N+Z_X \text{ odd}, \end{cases} \tag{7.3-91}$$

$$u_m = \frac{m}{N+Z_X}\frac{\lambda}{d}, \qquad m = -N'', \ldots, 0, \ldots, N'', \tag{7.3-92}$$

is the value of direction cosine u at DFT bin m, and Z_X is the total number of "zeros" used for "padding-with-zeros" in the X direction. Next we shall show that $\hat{\mathrm{Y}}_{Trgt}(q,m)$ is related to the beam pattern of the linear array.

For example, if the target is in the far-field region of the array, then substituting (7.3-57) and (7.3-65) into (7.3-80), and setting $R_i = r_S$ in (7.3-29) yields

$$\hat{Y}_{Trgt}(q,i) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S)a_i \exp\left[-j2\pi q f_0 \frac{(u'-u_S)}{c} id\right], \tag{7.3-93}$$

where $\hat{S}(q)$ is an estimate of the theoretical frequency spectrum $S(f)$ given by (7.3-52) at discrete frequencies $f = qf_0$, and τ_S is the one-way time delay in seconds associated with the path length between the sound-source (target) and the center of the array [see (7.2-36)]. As can be seen from (7.3-52), $S(f)$ is related to the complex frequency spectrum of the target's source strength $S_0(f)$. Note that when $u' \neq u_S$, the *phase* of $\hat{Y}_{Trgt}(q,i)$ given by (7.3-93) is *different* at each element i. As a result, the output electrical signals from each element in the array will be *out-of-phase*.

If the far-field beam pattern of the array is steered in the direction of the acoustic field incident upon the array (in this case, $\theta' = \theta_S$ and $\psi' = \psi_S$ so that $u' = u_S$), then (7.3-93) reduces to

$$\hat{Y}_{Trgt}(q,i) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S)a_i . \tag{7.3-94}$$

The *phase* of $\hat{Y}_{Trgt}(q,i)$ given by (7.3-94) is now the *same* at each element i. Therefore, the output electrical signals from each element in the array will be *in-phase* and, as a result, the array gain will be *maximized*. Substituting (7.3-94) into (7.3-87) yields

$$\hat{\mathrm{Y}}_{Trgt}(q,m) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S) \sum_{i=-N'}^{N'} a_i W_{N+Z_X}^{mi}, \qquad m = -N'',\ldots,0,\ldots,N''. \tag{7.3-95}$$

The summation in (7.3-95) is the *array factor* for the set of amplitude weights a_i. The array factor is directly proportional to the far-field beam pattern of the array. At broadside, that is, at $m = 0$, (7.3-95) reduces to

$$\hat{\mathrm{Y}}_{Trgt}(q,0) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S) \sum_{i=-N'}^{N'} a_i . \tag{7.3-96}$$

When $u' \neq u_S$, substituting (7.3-93) into (7.3-87) yields

$$\hat{\mathrm{Y}}_{Trgt}(q,m) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S) \times \\ \sum_{i=-N'}^{N'} a_i \exp\left[-j2\pi q f_0 \frac{(u'-u_S)}{c} id\right] W_{N+Z_X}^{mi}, \qquad m = -N'',\ldots,0,\ldots,N''. \tag{7.3-97}$$

The summation in (7.3-97) is the array factor for the set of amplitude weights a_i steered in the direction $u = u' - u_S$ in direction-cosine space.

In order to prove the above statement, we need to find the value of DFT bin m, or equivalently, direction cosine $u = u_m$, so that (7.3-97) reduces to (7.3-96). Start by substituting (7.3-89) into (7.3-97). Doing so yields

$$\hat{\mathrm{Y}}_{Trgt}(q,m) = \hat{S}(q)\exp(-j2\pi q f_0 \tau_S) \sum_{i=-N'}^{N'} a_i \exp\left\{-j2\pi \left[q f_0 \frac{(u'-u_S)}{c} d - \frac{m}{N+Z_X} \right] i \right\}. \tag{7.3-98}$$

As can be seen from (7.3-98), the value of m that will reduce (7.3-98) to (7.3-96) is given by

$$m = m' = (N + Z_X) q f_0 \frac{(u'-u_S)}{c} d \tag{7.3-99}$$

since this value of m will cause the exponent inside the summation to be equal to 0. Although the parameter values on the right-hand side of (7.3-99) are such that m' is not an integer in general, for example purposes, we shall assume it is. Therefore, substituting (7.3-99) into (7.3-92), and since

$$c = f\lambda = q f_0 \lambda , \tag{7.3-100}$$

$$u_{m'} = u' - u_S \,. \tag{7.3-101}$$

In other words, when $u' \neq u_S$, the value of the array factor at broadside will be located at $u = u' - u_S$ in direction-cosine space, that is, the array factor will be steered to $u = u' - u_S$.

If *no* beam steering is done, that is, if $u' = 0$, then $u_{m'} = -u_S$, or $-u_{m'} = u_S$. If the sound-source (target) lies in the XY plane, then $u_S = \cos\psi_S$ since $\theta_S = 90°$. Therefore,

$$-u_{m'} = \cos\psi_S \,, \tag{7.3-102}$$

or

$$\psi_S = \cos^{-1}(-u_{m'}) \,, \tag{7.3-103}$$

which is the azimuthal (bearing) angle in the direction of the acoustic field incident upon the array. In order to obtain the correct value for ψ_S, the *port/starboard (left/right) ambiguity problem* associated with linear apertures (linear arrays) has to be resolved (see Examples 8.2-1 and 9.1-2).

7.4 Total Output Signal from a Linear Array due to Ambient Noise and Receiver Noise

In addition to an output electrical signal from each element in the array due to the sound-field radiated by the target, there is also present an output electrical signal due to ambient (background) noise in the ocean. We shall assume that the ambient-noise sound-field is *isotropic* (omnidirectional). Sources of ambient noise in the ocean are, for example, shipping, seismic activity, waves, rain, wind, ocean thermal noise, and biological.

Let $y_{M,n_a}(t, x_i)$ be the velocity potential in squared meters per second of the output acoustic signal from the fluid medium due to ambient noise and incident upon element i in the array. The velocity potential $y_{M,n_a}(t, x_i)$ is assumed to be a *zero-mean*, *time-domain* and *spatial-domain*, *wide-sense stationary* (WSS) *random process* $\forall\ i$. A spatial-domain WSS random process is also referred to as being *statistically homogeneous*. It is also assumed that $y_{M,n_a}(t, x_i)$ and $y_{M,n_a}(t, x_j)$ are *uncorrelated* for $i \neq j$. The complex frequency spectrum (in volts per hertz) of the *total* output electrical signal from the array due to ambient noise, $Y_{n_a}(f)$, can be expressed as follows [see (7.3-25) through (7.3-27)]:

$$Y_{n_a}(f) = \sum_{i=-N'}^{N'} Y_{n_a}(f, x_i) \,, \tag{7.4-1}$$

where

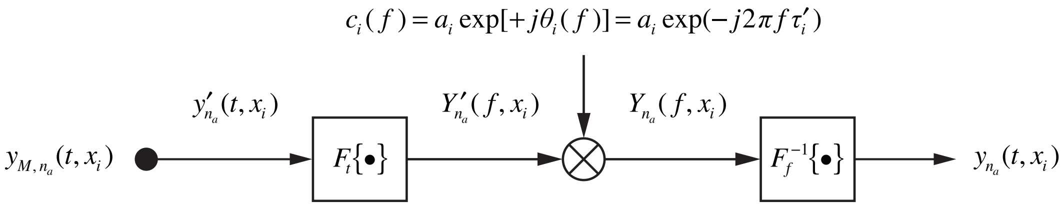

Figure 7.4-1 Beamforming. Implementing a complex weight (amplitude and phase weights) using forward and inverse Fourier transforms.

$$Y_{n_a}(f,x_i)=Y'_{n_a}(f,x_i)c_i(f) \tag{7.4-2}$$

and

$$Y'_{n_a}(f,x_i)=N_a(f,x_i)=Y_{M,n_a}(f,x_i)\mathcal{S}_R(f) \tag{7.4-3}$$

are the complex frequency spectra (in volts per hertz) of the output electrical signal from element i in the array due to ambient noise, *after* and *before* the application of the complex weight $c_i(f)$, respectively (see Fig. 7.4-1), and

$$Y_{M,n_a}(f,x_i)=F_t\left\{y_{M,n_a}(t,x_i)\right\} \tag{7.4-4}$$

is the complex frequency spectrum of the ambient noise incident upon element i in the array with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$. Substituting $Y'_{n_a}(f,x_i)=N_a(f,x_i)$ [see (7.4-3)] and (7.3-3) into (7.4-2) yields

$$Y_{n_a}(f,x_i)=a_i N_a(f,x_i)\exp(-j2\pi f\tau_i'), \tag{7.4-5}$$

and by substituting (7.4-5) into (7.4-1), we obtain

$$Y_{n_a}(f)=\sum_{i=-N'}^{N'} a_i N_a(f,x_i)\exp(-j2\pi f\tau_i'). \tag{7.4-6}$$

Equation (7.4-6) is the complex frequency spectrum (in volts per hertz) of the *total* output electrical signal from the array due to ambient noise, where $N_a(f,x_i)$ is given (7.4-3). We shall obtain a time-domain model next.

Taking the inverse Fourier transform of (7.4-3) yields

$$y'_{n_a}(t,x_i)=n_a(t,x_i)=F_f^{-1}\left\{Y_{M,n_a}(f,x_i)\mathcal{S}_R(f)\right\}, \tag{7.4-7}$$

which is the output electrical signal in volts from element i in the array due to ambient noise, *before* complex weighting (see Fig. 7.4-1). The signal $n_a(t,x_i)$ is a zero-mean, time-domain and spatial-domain, WSS random process $\forall\ i$, and $n_a(t,x_i)$ and $n_a(t,x_j)$ are uncorrelated for $i \neq j$, because $n_a(t,x_i)$ is the result of linear, time-invariant, space-invariant, deterministic filtering of the velocity potential $y_{M,n_a}(t,x_i)$ $\forall\ i$. Taking the inverse Fourier transform of (7.4-1) and (7.4-5) yields

$$y_{n_a}(t) = \sum_{i=-N'}^{N'} y_{n_a}(t,x_i) \tag{7.4-8}$$

and

$$\boxed{y_{n_a}(t,x_i) = a_i n_a(t-\tau_i', x_i)} \tag{7.4-9}$$

respectively, and by substituting (7.4-9) into (7.4-8), we obtain

$$\boxed{y_{n_a}(t) = \sum_{i=-N'}^{N'} a_i n_a(t-\tau_i', x_i)} \tag{7.4-10}$$

where $n_a(t,x_i)$ is given by (7.4-7). Equation (7.4-9) is the output electrical signal (in volts) from element i in the array due to ambient noise, *after* complex weighting (see Fig. 7.4-1). The signal $y_{n_a}(t,x_i)$ is a zero-mean, time-domain and spatial-domain, WSS random process $\forall\ i$, and $y_{n_a}(t,x_i)$ and $y_{n_a}(t,x_j)$ are uncorrelated for $i \neq j$. Equation (7.4-10) is the *total* output electrical signal from the array (in volts) due to ambient noise, and is a zero-mean, WSS random process.

In addition to ambient noise, there is also receiver noise present at the output of each element in the array. The output electrical signals in volts from element i in the array due to receiver noise, *before* and *after* complex weighting (see Fig. 7.4-2), are given by

$$n_r'(t,x_i) = n(t,x_i) \tag{7.4-11}$$

and

$$\boxed{n_r(t,x_i) = a_i n(t-\tau_i', x_i)} \tag{7.4-12}$$

respectively, where $n(t,x_i)$ is assumed to be a *zero-mean*, *time-domain* and *spatial-domain*, *WSS random process* $\forall\ i$, and $n(t,x_i)$ and $n(t,x_j)$ are *uncorrelated* for $i \neq j$. As a result, $n_r(t,x_i)$ is a zero-mean, time-domain and spatial-domain, WSS random process $\forall\ i$, and $n_r(t,x_i)$ and $n_r(t,x_j)$ are uncorrelated for $i \neq j$. The *total* output electrical signal from the array (in volts) due to receiver noise is given by

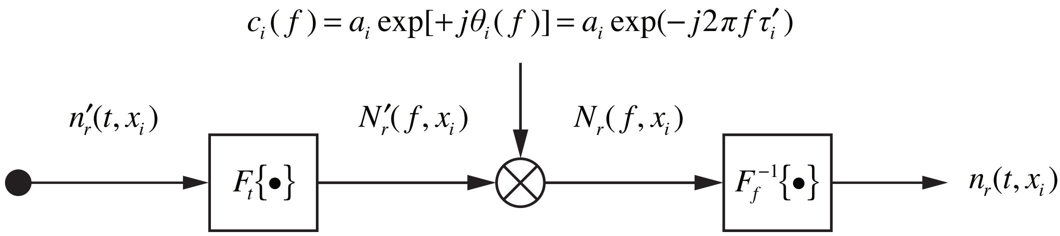

Figure 7.4-2 Beamforming. Implementing a complex weight (amplitude and phase weights) using forward and inverse Fourier transforms.

$$n_r(t) = \sum_{i=-N'}^{N'} n_r(t, x_i) \tag{7.4-13}$$

or

$$\boxed{n_r(t) = \sum_{i=-N'}^{N'} a_i n(t - \tau_i', x_i)} \tag{7.4-14}$$

where $n_r(t)$ is a zero-mean, WSS random process.

7.5 Evaluation of the Equation for Array Gain

In order to evaluate the equation for AG as given by (7.1-19), we need to decide on which model to use for $y_{Trgt}(t, x_i)$ – the output electrical signal from element i in the array due to the target, after complex weighting. There are three different models to choose from, namely, (7.3-35), (7.3-43), and (7.3-50). We shall use (7.3-50) in this section. In Section 7.3 it was shown that if the target is in the Fresnel or far-field region of the array, and if the far-field beam pattern of the array is focused at the range to the target and steered in the direction of the acoustic field incident upon the array (Fresnel region case), or if the far-field beam pattern is simply steered in the direction of the incident acoustic field (far-field region case), then [see (7.3-50)]

$$y_{Trgt}(t, x_i) = a_i s(t - \tau_S), \qquad i = -N', \ldots, 0, \ldots, N', \tag{7.5-1}$$

for $t \geq \tau_S$, where $s(t)$ is given by (7.3-53). As was discussed in Section 7.3, since all the individual signals $y_{Trgt}(t, x_i)$ given by (7.5-1) are *in-phase*, the time-average power of $y_{Trgt}(t)$ given by (7.1-5) will be maximized. As a result, the time-average power of $r(t)$ given by (7.1-4) will also be maximized.

We begin the evaluation of the equation for AG by first computing the output signal-to-noise ratio of the array SNR_A given by (7.1-10). With the use of (7.5-1), the cross-correlation of $y_{Trgt}(t, x_i)$ and $y_{Trgt}(t, x_j)$ is

$$E\left\{y_{Trgt}(t, x_i)\, y^*_{Trgt}(t, x_j)\right\} = a_i a_j E\left\{\left|s(t-\tau_S)\right|^2\right\} = a_i a_j E\left\{\left|s(t)\right|^2\right\}, \quad (7.5\text{-}2)$$

or

$$E\left\{y_{Trgt}(t, x_i)\, y^*_{Trgt}(t, x_j)\right\} = a_i a_j E\left\{\left|s(t)\right|^2\right\}, \qquad i, j = -N', \ldots, 0, \ldots, N', \quad (7.5\text{-}3)$$

since $s(t)$ is WSS. The second moment (mean-squared value) $E\left\{\left|s(t)\right|^2\right\}$ is the *constant* average power of $s(t)$, where $s(t)$ is given by (7.3-53). Substituting (7.5-3) into (7.1-11) yields

$$E\left\{\left|y_{Trgt}(t)\right|^2\right\} = E\left\{\left|s(t)\right|^2\right\} \sum_{i=-N'}^{N'} a_i \sum_{j=-N'}^{N'} a_j = E\left\{\left|s(t)\right|^2\right\} \left[\sum_{i=-N'}^{N'} a_i\right]^2, \quad (7.5\text{-}4)$$

which is the *constant* average power of the signal $y_{Trgt}(t)$ from the array due to the target. Equation (7.5-4) is the numerator of SNR_A.

The denominator of SNR_A is given by (7.1-16). In order to compute the average power of the ambient noise from the array, first rewrite (7.1-13) as follows:

$$E\left\{\left|y_{n_a}(t)\right|^2\right\} = \sum_{i=-N'}^{N'} E\left\{\left|y_{n_a}(t, x_i)\right|^2\right\} + \sum_{\substack{i=-N' \\ i \neq j}}^{N'} \sum_{j=-N'}^{N'} E\left\{y_{n_a}(t, x_i)\, y^*_{n_a}(t, x_j)\right\}. \quad (7.5\text{-}5)$$

Since the output electrical signal from element i in the array due to ambient noise after complex weighting is given by [see (7.4-9)]

$$y_{n_a}(t, x_i) = a_i n_a(t - \tau'_i, x_i), \qquad i = -N', \ldots, 0, \ldots, N', \quad (7.5\text{-}6)$$

where $n_a(t, x_i)$ is given by (7.4-7), the average power of $y_{n_a}(t, x_i)$ is

$$E\left\{\left|y_{n_a}(t, x_i)\right|^2\right\} = a_i^2 E\left\{\left|n_a(t - \tau'_i, x_i)\right|^2\right\} = a_i^2 E\left\{\left|n_a(t, x_0)\right|^2\right\}, \quad (7.5\text{-}7)$$

or

$$E\left\{\left|y_{n_a}(t, x_i)\right|^2\right\} = a_i^2 E\left\{\left|n_a(t, x_0)\right|^2\right\}, \qquad i = -N', \ldots, 0, \ldots, N', \quad (7.5\text{-}8)$$

since $n_a(t, x_i)$ is WSS in time and space $\forall \; i$. The second moment (mean-squared value) $E\left\{\left|n_a(t, x_0)\right|^2\right\}$ is the *constant* average power of the ambient noise at the output of element $i = 0$ in the array after complex weighting. Since $n_a(t, x_i)$ is zero-mean $\forall \; i$, and $n_a(t, x_i)$ and $n_a(t, x_j)$ are uncorrelated for $i \neq j$, the cross-correlation

$$\begin{aligned} E\left\{y_{n_a}(t, x_i)\, y_{n_a}^*(t, x_j)\right\} &= a_i a_j E\left\{n_a(t - \tau_i', x_i)\, n_a^*(t - \tau_j', x_j)\right\} \\ &= a_i a_j E\left\{n_a(t - \tau_i', x_i)\right\} E\left\{n_a^*(t - \tau_j', x_j)\right\} = 0, \qquad i \neq j. \end{aligned} \tag{7.5-9}$$

Recall that a cross-correlation function is equal to a cross-covariance function whenever one or both random processes are zero-mean. Since $y_{n_a}(t, x_i)$ is zero-mean $\forall \; i$, (7.5-9) agrees with the statement made in Section 7.4 that $y_{n_a}(t, x_i)$ and $y_{n_a}(t, x_j)$ are uncorrelated for $i \neq j$. Therefore, by substituting (7.5-8) and (7.5-9) into (7.5-5), we obtain

$$E\left\{\left|y_{n_a}(t)\right|^2\right\} = E\left\{\left|n_a(t, x_0)\right|^2\right\} \sum_{i=-N'}^{N'} a_i^2 \,, \tag{7.5-10}$$

which is the *constant* average power of the ambient noise $y_{n_a}(t)$ from the array. Also recall that a mean-squared value is equal to a variance for a zero-mean random process.

In order to compute the average power of the receiver noise from the array, first rewrite (7.1-14) as follows:

$$E\left\{\left|n_r(t)\right|^2\right\} = \sum_{i=-N'}^{N'} E\left\{\left|n_r(t, x_i)\right|^2\right\} + \sum_{\substack{i=-N' \\ i \neq j}}^{N'} \sum_{j=-N'}^{N'} E\left\{n_r(t, x_i)\, n_r^*(t, x_j)\right\}. \tag{7.5-11}$$

Since the output electrical signal from element i in the array due to receiver noise after complex weighting is given by [see (7.4-12)]

$$n_r(t, x_i) = a_i n(t - \tau_i', x_i), \qquad i = -N', \ldots, 0, \ldots, N' \,, \tag{7.5-12}$$

the average power of $n_r(t, x_i)$ is

$$E\left\{\left|n_r(t, x_i)\right|^2\right\} = a_i^2 E\left\{\left|n(t - \tau_i', x_i)\right|^2\right\} = a_i^2 E\left\{\left|n(t, x_0)\right|^2\right\}, \tag{7.5-13}$$

or

$$E\left\{\left|n_r(t,x_i)\right|^2\right\}=a_i^2E\left\{\left|n(t,x_0)\right|^2\right\}, \qquad i=-N',\ldots,0,\ldots,N', \tag{7.5-14}$$

since $n(t,x_i)$ is WSS in time and space $\forall\ i$. The second moment (mean-squared value) $E\left\{\left|n(t,x_0)\right|^2\right\}$ is the *constant* average power of the receiver noise at the output of element $i=0$ in the array after complex weighting. Since $n(t,x_i)$ is zero-mean $\forall\ i$, and $n(t,x_i)$ and $n(t,x_j)$ are uncorrelated for $i\neq j$, the cross-correlation

$$\begin{aligned}E\left\{n_r(t,x_i)n_r^*(t,x_j)\right\}&=a_ia_jE\left\{n(t-\tau_i',x_i)n^*(t-\tau_j',x_j)\right\}\\&=a_ia_jE\left\{n(t-\tau_i',x_i)\right\}E\left\{n^*(t-\tau_j',x_j)\right\}=0, \qquad i\neq j.\end{aligned} \tag{7.5-15}$$

Since a cross-correlation function is equal to a cross-covariance function whenever one or both random processes are zero-mean, and since $n_r(t,x_i)$ is zero-mean $\forall\ i$, (7.5-15) agrees with the statement made in Section 7.4 that $n_r(t,x_i)$ and $n_r(t,x_j)$ are uncorrelated for $i\neq j$. Therefore, by substituting (7.5-14) and (7.5-15) into (7.5-11), we obtain

$$E\left\{\left|n_r(t)\right|^2\right\}=E\left\{\left|n(t,x_0)\right|^2\right\}\sum_{i=-N'}^{N'}a_i^2, \tag{7.5-16}$$

which is the *constant* average power of the receiver noise $n_r(t)$ from the array. Substituting (7.5-10) and (7.5-16) into (7.1-16) yields

$$E\left\{\left|z(t)\right|^2\right\}=\left[E\left\{\left|n_a(t,x_0)\right|^2\right\}+E\left\{\left|n(t,x_0)\right|^2\right\}\right]\sum_{i=-N'}^{N'}a_i^2. \tag{7.5-17}$$

Equation (7.5-17) is the denominator of SNR_A. The output signal-to-noise ratio of the array can now be obtained by substituting (7.5-4) and (7.5-17) into (7.1-10). Doing so yields

$$\mathrm{SNR}_\mathrm{A}=\frac{E\left\{\left|s(t)\right|^2\right\}}{E\left\{\left|n_a(t,x_0)\right|^2\right\}+E\left\{\left|n(t,x_0)\right|^2\right\}}\frac{\left[\sum_{i=-N'}^{N'}a_i\right]^2}{\sum_{i=-N'}^{N'}a_i^2}. \tag{7.5-18}$$

Next, compute the output signal-to-noise ratio at element $i=0$ in the array, SNR_0, given by (7.1-17). Evaluating (7.5-3) at $i=0$ and $j=0$, (7.5-8) at $i=0$, and (7.5-14) at $i=0$ yields

$$E\left\{\left|y_{Trgt}(t,x_0)\right|^2\right\} = a_0^2 E\left\{\left|s(t)\right|^2\right\}, \tag{7.5-19}$$

$$E\left\{\left|y_{n_a}(t,x_0)\right|^2\right\} = a_0^2 E\left\{\left|n_a(t,x_0)\right|^2\right\}, \tag{7.5-20}$$

and

$$E\left\{\left|n_r(t,x_0)\right|^2\right\} = a_0^2 E\left\{\left|n(t,x_0)\right|^2\right\}, \tag{7.5-21}$$

respectively. Substituting (7.5-20) and (7.5-21) into (7.1-18) yields

$$E\left\{\left|z(t,x_0)\right|^2\right\} = a_0^2\left[E\left\{\left|n_a(t,x_0)\right|^2\right\} + E\left\{\left|n(t,x_0)\right|^2\right\}\right], \tag{7.5-22}$$

and by substituting (7.5-19) and (7.5-22) into (7.1-17), we obtain

$$\text{SNR}_0 = \frac{E\left\{\left|s(t)\right|^2\right\}}{E\left\{\left|n_a(t,x_0)\right|^2\right\} + E\left\{\left|n(t,x_0)\right|^2\right\}}. \tag{7.5-23}$$

And finally, substituting (7.5-18) and (7.5-23) into (7.1-19) yields the following expression for array gain:

$$\boxed{\text{AG} = 10\log_{10}\frac{\left[\sum\limits_{i=-N'}^{N'} a_i\right]^2}{\sum\limits_{i=-N'}^{N'} a_i^2}\ \text{dB}} \tag{7.5-24}$$

If rectangular amplitude weights are used, then $a_i = 1 \;\; \forall \;\; i$, and (7.5-24) reduces to

$$\boxed{\text{AG} = 10\log_{10} N \text{ dB}} \tag{7.5-25}$$

where N is the total number of elements in the array. For example, if the number of elements is doubled and if (7.5-25) is applicable, then the AG is increased by $3\,\text{dB}$. Later, in Chapter 12, we will show how AG can increase the probability of detecting a very weak signal with a low SNR at the output of a single element in an array.

In summary, the AG formula given by (7.5-25) is based on the following major assumptions:

1) The target is modeled as an *omnidirectional point-source with arbitrary time dependence* $s_0(t)$, where $s_0(t)$ is the source strength of the target.

2) The source strength of the target $s_0(t)$ is a *wide-sense stationary* (WSS) *random process*.

3) The fluid medium is *unbounded*, *viscous*, and *homogeneous* (constant speed of sound).

4) The ambient noise is *isotropic* and the velocity potential of the output acoustic signal from the fluid medium due to ambient noise and incident upon element i in the array, $y_{M,n_a}(t,x_i)$, is a *zero-mean*, *time-domain* and *spatial-domain*, *WSS random process* $\forall \; i$. Also, $y_{M,n_a}(t,x_i)$ and $y_{M,n_a}(t,x_j)$ are *uncorrelated* for $i \neq j$.

5) The receiver noise $n(t,x_i)$ is a *zero-mean*, *time-domain* and *spatial-domain*, *WSS random process* $\forall \; i$, and $n(t,x_i)$ and $n(t,x_j)$ are *uncorrelated* for $i \neq j$. Also, the receiver noise and ambient noise are *statistically independent*.

6) The target is in either the *Fresnel* or *far-field* region of the array.

7) All the individual output electrical signals due to the target after complex weighting, $y_{Trgt}(t,x_i)$, $i = -N',\ldots,0,\ldots,N'$, are *in-phase*.

8) *Rectangular amplitude weights* are used.

Problems

Section 7.2

7-1 If the attenuation coefficient of seawater is 4.425 dB/km at a frequency of 25 kHz, then what is its value in nepers per meter?

Section 7.3

7-2 Compute the autocorrelation function

$$R_{y_{Trgt}}(t_1,t_2)=E\left\{y_{Trgt}(t_1)\,y^*_{Trgt}(t_2)\right\}$$

of $y_{Trgt}(t)$ given by (7.3-51). At the end of the derivation, let $t_1=t$ and $t_2=t-\tau$.

Section 7.4

7-3 Compute the autocorrelation function

$$R_{y_{n_a}}(t_1,t_2)=E\left\{y_{n_a}(t_1)\,y^*_{n_a}(t_2)\right\}$$

of $y_{n_a}(t)$ given by (7.4-10). At the end of the derivation, let $t_1=t$ and $t_2=t-\tau$.

7-4 Compute the autocorrelation function

$$R_{n_r}(t_1,t_2)=E\left\{n_r(t_1)\,n^*_r(t_2)\right\}$$

of $n_r(t)$ given by (7.4-14). At the end of the derivation, let $t_1=t$ and $t_2=t-\tau$.

Section 7.5

7-5 For $N=7$,

(a) use (7.5-25) to compute the array gain for the set of rectangular amplitude weights.

(b) use (7.5-24) to compute the array gain for the set of Hamming amplitude weights and the normalized, Dolph-Chebyshev amplitude weights computed in Example 6.3-1.

Appendix 7A Attenuation Coefficient of Seawater

An accurate equation for the frequency-dependent attenuation coefficient of sound in seawater, valid for all oceanic conditions and frequencies from 200 Hz to 1 MHz , is given by[1]

[1] R. E. Francois and G. R. Garrison, "Sound Absorption Based on Ocean Measurements. Part II: Boric Acid Contribution and Equation for Total Absorption," *J. Acoust. Soc. Am.*, vol. 72, pp. 1879-1890, 1982.

$$\alpha'(f) = \frac{A_1 P_1 f_1}{f^2 + f_1^2} f^2 + \frac{A_2 P_2 f_2}{f^2 + f_2^2} f^2 + A_3 P_3 f^2 , \tag{7A-1}$$

where $\alpha'(f)$ has units of decibels per kilometer (dB/km) and frequency f is in kilohertz (kHz). The various parameters appearing in (7A-1) are defined next.

Boric Acid Contribution

$$A_1 = \frac{8.86}{c} \times 10^{(0.78\,p\mathrm{H} - 5)} \tag{7A-2}$$

has units of dB/(km-kHz), where pH is the acidity (dimensionless) of seawater,

$$c = 1412 + 3.21T + 1.19S + 0.0167y \tag{7A-3}$$

is the speed of sound in meters per second (m/sec), where T is the temperature in degrees Celsius (°C), S is the salinity in parts per thousand (ppt), and y is the depth in meters (m),

$$P_1 = 1 , \tag{7A-4}$$

where P_1 is dimensionless,

$$f_1 = 2.8\sqrt{\frac{S}{35}} \times 10^{[4-(1245/\theta)]} \tag{7A-5}$$

is the relaxation frequency of boric acid in kHz , and

$$\theta = 273 + T \tag{7A-6}$$

is the absolute temperature in degrees Kelvin (°K).

Magnesium Sulfate Contribution

$$A_2 = 21.44 \frac{S}{c} (1 + 0.025T) \tag{7A-7}$$

has units of dB/(km-kHz),

$$P_2 = 1 - (1.37 \times 10^{-4})y + (6.2 \times 10^{-9})y^2 , \tag{7A-8}$$

where P_2 is dimensionless, and

$$f_2 = \frac{8.17 \times 10^{[8-(1990/\theta)]}}{1 + 0.0018(S-35)} \tag{7A-9}$$

is the relaxation frequency of magnesium sulfate in kHz .

Pure Water Contribution

$$P_3 = 1 - (3.83 \times 10^{-5})y + (4.9 \times 10^{-10})y^2 , \tag{7A-10}$$

where P_3 is dimensionless.

For $T \leq 20°\,\mathrm{C}$:

$$A_3 = 4.937 \times 10^{-4} - (2.59 \times 10^{-5})T + (9.11 \times 10^{-7})T^2 - (1.50 \times 10^{-8})T^3 \tag{7A-11}$$

has units of $\mathrm{dB}/(\text{km-kHz}^2)$.

For $T > 20°\,\mathrm{C}$:

$$A_3 = 3.964 \times 10^{-4} - (1.146 \times 10^{-5})T + (1.45 \times 10^{-7})T^2 - (6.5 \times 10^{-10})T^3 \tag{7A-12}$$

has units of $\mathrm{dB}/(\text{km-kHz}^2)$.

Appendix 7B Fourier Transform, Fourier Series Coefficients, Time-Average Power, and Power Spectrum via the DFT

Fourier Transform

An estimate $\hat{X}(f)$ of the Fourier transform $X(f)$ of $x(t)$ can be computed numerically as follows:

$$\boxed{\hat{X}(q) = T_S\, \mathcal{X}(q), \qquad q = 0, 1, \ldots, N + Z - 1} \tag{7B-1}$$

where

$$\hat{X}(q) \equiv \hat{X}(qf_0) = \hat{X}(f)\Big|_{f=qf_0} \tag{7B-2}$$

and

$$\mathcal{X}(q) \equiv \mathcal{X}(qf_0) = \mathcal{X}(f)\big|_{f=qf_0}, \tag{7B-3}$$

where

$$\mathcal{X}(q) = DFT\{x(i)\} = \sum_{i=0}^{N-1} x(i) W_{N+Z}^{-qi}, \qquad q = 0,1,\ldots,N+Z-1, \tag{7B-4}$$

is the *discrete Fourier transform* (DFT) of $x(t)$, $x(i) \equiv x(t_i) = x(iT_S)$, and

$$W_{N+Z} = \exp\left(+j\frac{2\pi}{N+Z}\right), \tag{7B-5}$$

where T_S is the sampling period in seconds, N is the total number of samples taken of $x(t)$, and Z is a *positive integer* ($Z \geq 0$). The waveform $x(t)$ is defined for $0 \leq t \leq T$, where T is the duration of $x(t)$ in seconds. If $x(t)$ has units of volts, then $\hat{X}(q)$ has units of volts per hertz, and if $x(t)$ has units of amperes, then $\hat{X}(q)$ has units of amperes per hertz.

The frequency components f in hertz are given by

$$f = qf_0, \qquad q = 0,1,\ldots,N+Z-1, \tag{7B-6}$$

where

$$\boxed{f_0 = \frac{1}{T_0} = \frac{1}{(N+Z)T_S} = \frac{f_S}{N+Z}} \tag{7B-7}$$

is the DFT *bin spacing* [the fundamental frequency of $x(t)$] in hertz,

$$\boxed{T_0 = (N+Z)T_S} \tag{7B-8}$$

is the fundamental period of $x(t)$ in seconds, and $f_S = 1/T_S$ is the sampling frequency in hertz (a.k.a. the sampling rate in samples per second). As can be seen from (7B-7), when integer $Z = 0$, the DFT bin spacing $f_0 = 1/(NT_S)$, which is the *maximum* allowed value in hertz in order to avoid aliasing when performing a time-domain DFT. Choosing $Z > 0$ decreases f_0 and increases $N+Z$. Increasing $N+Z$ means that $\hat{X}(q)$ is evaluated at more bins yielding a more smooth curve for plotting purposes. For example, if we want to evaluate $\hat{X}(q)$ at twice as many bins by reducing the bin spacing f_0 by a factor of two, then set $Z = N$ so that $N+Z = 2N$. This is equivalent to adding $Z = N$ zeros to the original data sequence so that $N+Z = 2N$. Choosing $Z > 0$ is referred to as "padding-with-zeros".

In order to determine the value for N to be used in the DFT, set $Z=0$ and $T_0 = T$, where T is the duration of $x(t)$ in seconds. Doing so yields

$$T_0 = T = NT_S, \tag{7B-9}$$

or

$$\boxed{N = \frac{T}{T_S} = f_S T} \tag{7B-10}$$

The corresponding DFT bin spacing for $Z=0$ is given by

$$f_0 = \frac{1}{T_0} = \frac{1}{T} = \frac{1}{NT_S} = \frac{f_S}{N}. \tag{7B-11}$$

However, if we want $f_0 < 1/T$, then the required value for Z is given by

$$\boxed{Z = \frac{f_S}{f_0} - N} \tag{7B-12}$$

where $f_0 < 1/T$ is the desired DFT bin spacing. Note that $T \le T_0$.

Fourier Series Coefficients

An estimate of the set of complex Fourier series coefficients of $x(t)$ can be computed numerically as follows:

$$\boxed{\hat{c}_q = \frac{1}{N+Z}\mathcal{X}(q), \qquad q = 0,1,\ldots,N+Z-1} \tag{7B-13}$$

where $\hat{c}_q$ is an estimate of the complex Fourier series coefficient c_q of $x(t)$ at harmonic q (frequency $f = qf_0$ Hz), and $\mathcal{X}(q)$ is the DFT of $x(t)$. If $x(t)$ has units of volts, then $\hat{c}_q$ has units of volts, and if $x(t)$ has units of amperes, then $\hat{c}_q$ has units of amperes.

Time-Average Power

The time-average power of $x(t)$ in the time interval $[0,T]$ can be computed numerically as follows: