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Design of Nearly Constant Velocity Track Filters for Brief Maneuvers

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***Abstract*—When tracking maneuvering targets with conventional algorithms, the process noise standard deviation used in the nearly constant velocity Kalman filter is selected vaguely in relation to the maximum acceleration of the target. The deterministic tracking index has been introduced and used to develop a relationship between the maximum acceleration and the process noise variance that minimizes the maximum mean squared error (MMSE) in position. A lower bound on the process noise variance has also been expressed in terms of the maximum acceleration and deterministic tracking index. Both of these results were developed for sustained maneuvers in that all of the transient effects of the maneuver have diminished. In this paper, the results for sustained maneuvers are extended to brief maneuvers. For each case, the process noise variance is expressed in terms of the maximum acceleration, duration of the maneuver in number of measurements, and deterministic tracking index for both the discrete white noise acceleration and continuous white noise acceleration models for nearly constant velocity motion. With the use of Monte Carlo simulations, the method for choosing process noise variance for tracking maneuvering targets is demonstrated to be effective for both sustained and brief maneuvers.**

Keywords: Target tracking, Kalman filtering, filter design, estimation.

I. Introduction

Due to the random modeling of deterministic-type maneuvers that are inherent in the problem of tracking maneuvering targets, the selection of the process noise variance is not straightforward and the error covariance of the Kalman filter is not always an accurate indicator of performance. When tracking maneuvering targets with conventional algorithms, the process noise variance used in the nearly constant velocity (NCV) Kalman filter is selected vaguely in relation with the maximum acceleration of the target [1]. Therefore, it is desirable to have a better expression that would enable one to find the process noise variance that minimizes the maximum mean squared error (MMSE) given the target's maximum acceleration. Toward this goal, the deterministic tracking index is introduced in [5] and used to develop a relationship between the maximum acceleration and the process noise variance that minimizes the MMSE in either position or velocity.

The development of the filter design techniques for NCV track filters in [5] utilized the alpha-beta filter. An alpha-beta filter [2], which is a steady-state form of the NCV Kalman filter, can be used to characterize the performance in terms of the sensor-noise only (SNO) covariance matrix and the position and velocity biases [3] [4]. Decoupling the measurement noise effects from the maneuver effects on performance allows the performance degradation due to deterministic-type maneuvers to characterized. Hence, knowledge of the SNO covariance matrix and the steady-state position and velocity biases give more insight into the choice of filter gains for deterministic-type maneuvers [6]. By introducing the deterministic tracking index, which unlike the tracking index for random maneuvers is dependent on the maximum acceleration of the target, a better means of characterizing deterministic-type maneuvers is possible.

In [5], the optimal process noise variance for sustained maneuvers was expressed in terms of the maximum acceleration and deterministic tracking index for both discrete white noise acceleration and continuous white noise acceleration models for the NCV filter. The design criteria included minimizing the MMSE for sustained maneuvers. A sustained maneuver is defined as a maneuver that persists sufficiently long that the transient effects from the start of the maneuver have diminished. A lower bound on the process noise variance for sustained maneuvers was also expressed in terms of the maximum acceleration and deterministic tracking index. The use of a process noise variance larger than the lower bound ensures that the MMSE in position is less than the average error in the measurements. Since the design techniques in [5] are restricted to sustained maneuvers, this paper extends those design techniques to tracking targets with brief maneuvers. The process noise variance is expressed in terms of the maximum acceleration, duration of the maneuver in number of measurements, and deterministic tracking index for both the discrete white noise acceleration and continuous white noise acceleration models for NCV filters. The effectiveness of the design methods for choosing the process noise variance for tracking targets with brief maneuvers are verified with Monte Carlo simulations.

The Kalman filter is reviewed in Section II, and the alpha-beta filter is defined in Section III along with the SNO covariance and maneuver bias. Section IV provides the design techniques for sustained maneuvers developed in [5] for the NCV filter with discrete white noise acceleration and extends those techniques to brief maneuvers. Section V presents the design techniques for the NCV filter with continuous white nose acceleration [1]. Concluding remarks are given in Section VI.

II. Kalman Filter

A Kalman filter is often employed to filter the kinematic measurements for estimating the position, velocity, and acceleration of a target [1]. The kinematic model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k v_k \tag{1}$$

where $v_k \sim N(0, Q_k)$ is the process noise that models the unknown target acceleration and F_k defines the linear dynamics. The target state vector X_k contains the position, velocity, and possibly acceleration of the target at time t_k, as well as other variables used to model the time-varying acceleration. The linear measurement model is given by

$$Y_k = H_k X_k + w_k \tag{2}$$

where Y_k is typically the measurement of the position of the target and $w_k \sim N(0, R_k)$ is the observation error. Both w_k and v_k are assumed to be independent "white" noise processes. When designing the Kalman filter, Q_k is conventionally selected such that the 65% to 95% confidence region about zero contains the maximum acceleration level of the target. However, when targets maneuver, the acceleration changes in a deterministic manner. Thus, the white noise assumption associated with v_k is violated and the filter develops a bias in the state estimates. If a larger Q_k is chosen, the bias in the state estimates is less during a maneuver, but then Q_k poorly characterizes the target motion when the target is not maneuvering and the filter performance is far from optimal. Furthermore, the error in modeling the two modes of flight (*i.e.,* nonmaneuvering and maneuvering) with a single model and the error in the white noise assumption for the process noise during maneuvers result in an inaccurate state error covariance that cannot be used reliably for performance prediction. While an Interacting Multiple Model (IMM) estimator [1] can be used to address this conflict in situations of demanding requirements, the focus of this work is on the filter design (*i.e.,* selection of Q_k) of a NCV Kalman filter when an IMM estimator is not warranted by the requirements or achievable given the computational limitations. In [5], the optimal process noise variance Q_k for sustained maneuvers was expressed in terms of the maximum acceleration and deterministic tracking index for both discrete white noise acceleration and continuous white noise acceleration models for the NCV filter. The design criteria included minimizing the MMSE for sustained maneuvers. A lower bound on the process noise variance Q_k for sustained maneuvers was also expressed in terms of the maximum acceleration and deterministic tracking index. The focus of this paper is the extension of those techniques to brief maneuvers.

The Kalman filter is a predictor-corrector algorithm that is given in terms of a time update and measurement update. The $X_{k|j}$ denotes the state estimate at time t_k given measurements through time t_j and $P_{k|j}$ denotes the state error covariance at time t_k given measurements through time t_j. The K_k is referred to as the Kalman gain at time t_k.

III. Alpha-Beta Filter

The alpha-beta filter is based on the assumption that the target is moving with constant velocity plus zero-mean, white Gaussian acceleration errors. For the alpha-beta filter and NCV Kalman filter with discrete white noise acceleration, the state and measurement equations of (1) and (2) are defined by

$$X_k = \begin{bmatrix} x_k & \dot{x}_k \end{bmatrix}^T \tag{3}$$
$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{4}$$
$$G_k = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} \tag{5}$$
$$H_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6}$$

where T is the time interval between measurements. Also, $R_k = \sigma_w^2$ is the variance of the measurement errors in m^2, and $Q_k = \sigma_v^2$ is the variance of the "acceleration" errors in m^2/s^4. In order to simplify the example and permit analytical predictions of the filter performance, the motion of the target is defined in a single coordinate and the measurements are the positions of the target (*i.e.,* a linear function of the state).

The steady-state form of the NCV filter can be used for analytical predictions of filter performance. For a filter to achieve these steady-state conditions, the error processes v_k and w_k must be stationary and the data rate must be constant. While these conditions are seldom satisfied in practice, the steady-state form of the filter can be used to predict average or expect tracking performance. The alpha-beta filter is equivalent to the Kalman filter for this motion model in steady-state . For the alpha-beta filter, the steady-state gains that occur after the transients associated with filter initialization diminish are given by

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \tag{7}$$

where α and β are the optimal gains for discrete white noise acceleration model given in [1], [2] as

$$\Gamma_{DWNA}^2 = \frac{\sigma_v^2 T^4}{\sigma_w^2} = \frac{\beta^2}{1-\alpha} \tag{8}$$
$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{9}$$

where Γ_{DWNA} is the tracking index [2]. The steady-state error covariance of the alpha-beta filter covariance in [1] and [3] is given by

$$P_{k|k}^{\alpha\beta} = \sigma_w^2 \begin{bmatrix} \alpha & \dfrac{\beta}{T} \\ \dfrac{\beta}{T} & \dfrac{\beta(2\alpha-\beta)}{2(1-\alpha)T^2} \end{bmatrix} \tag{10}$$

For the kinematic model with continuous white noise acceleration, $G_k Q_k G_k^T$ takes on a different form resulting in a different relation between the optimal gains α and β. In this case,

$$G_k Q_k G_k^T = \tilde{q} \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix} \tag{11}$$

where $\tilde{q}$ is the power spectral density of the continuous white noise acceleration. The optimal α and β for continuous white noise acceleration is given in [1] as

$$\Gamma_{CWNA}^2 = \frac{\tilde{q}T^3}{\sigma_w^2} = \frac{\beta^2}{1-\alpha} \quad (12)$$

$$\beta = 3(2-\alpha) - \sqrt{3(\alpha^2 - 12\alpha + 12)} \quad (13)$$

where Γ_{CWNA} is the tracking index for this model.

The covariance of the state estimate $X_{k|k}$ is given by

$$P_{k|k} = E[(X_{k|k} - \bar{X}_{k|k})(X_{k|k} - \bar{X}_{k|k})^T] \quad (14)$$

where $E[\cdot]$ denotes the expected value operator, and $\bar{X}_{k|k} = E[X_{k|k}]$. When $E[X_{k|k}] = X_k$, the true value, the estimator is unbiased and the state error covariance computed by the Kalman filter is a good predictor of performance. However, when the estimator is biased, the covariance is a poor predictor of performance. When a target undergoes a deterministic maneuver (i.e., a constant acceleration), the estimates are biased and the covariance matrix tends to be a poor estimate of track filter performance. When a target undergoes no maneuver (i.e., a zero acceleration), the covariance matrix tends to also be a biased estimate of track filter performance, because process noise is included in the filter for maneuver response. Thus, in order to address both conditions of the performance prediction, the mean-squared error (MSE) will be written in terms of a sensor-noise only (SNO) covariance for no maneuver and a maneuver bias for the constant acceleration maneuver.

Let

$$B_{k|k} = \bar{X}_{k|k} - X_k \quad (15)$$

where $B_{k|k}$ denotes the filter bias. Thus,

$$E[(X_{k|k} - X_k)(X_{k|k} - X_k)^T] = P_{k|k} + B_{k|k}B_{k|k}^T \quad (16)$$

Considering the case of deterministic maneuvers of either zero acceleration or constant acceleration, the MSE of the filter is given by the SNO covariance when the acceleration is zero and the SNO covariance plus the bias error squared when the acceleration is a nonzero constant. Letting $S_{k|k}^{\alpha\beta}$ denote the SNO covariance of the alpha-beta filter and $B_{k|k}^{\alpha\beta}$ denote the bias due to an acceleration gives

$$E[(X_{k|k} - X_k)(X_{k|k} - X_k)^T] = S_{k|k}^{\alpha\beta} + B_{k|k}^{\alpha\beta}(B_{k|k}^{\alpha\beta})^T \quad (17)$$

Expressions for the SNO covariance and the bias are computed by representing the alpha-beta filter as a linear, time-invariant system with an input that can be expressed as a deterministic signal (*i.e.,* a constant acceleration rather than zero-mean white process noise) with white noise measurement errors. The input-output relationships between the measurements Y_k and state estimate $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = \bar{F}_k X_{k-1|k-1} + \bar{G}_k Y_k \quad (18)$$

where

$$\bar{F}_k = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix} \quad (19)$$

$$\bar{G}_k = \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \quad (20)$$

The error covariance of $X_{k|k}$ that results from the SNO is given in [3] and [4] for arbitrary α and β as

$$S_{k|k}^{\alpha\beta} = \frac{\sigma_w^2}{\alpha(4-2\alpha-\beta)} \begin{bmatrix} 2\alpha^2 + \beta(2-3\alpha) & \frac{\beta}{T}(2\alpha-\beta) \\ \frac{\beta}{T}(2\alpha-\beta) & \frac{2\beta^2}{T^2} \end{bmatrix} \quad (21)$$

where T is the time period between consecutive measurements and σ_w^2 is the variance of measurement errors. Since (21) includes only the effects of sensor measurement errors, it is referred to as the SNO covariance matrix. For a maneuvering target, the bias in the state estimate for arbitrary α and β is given by

$$B_{k|k}^{\alpha\beta} = \begin{bmatrix} (1-\alpha)\frac{T^2}{\beta} \\ (\frac{\alpha}{\beta} - 0.5)T \end{bmatrix} A_k \quad (22)$$

where A_k is the acceleration of the target at time t_k in the coordinate of interest. Note that $B_{k|k}^{\alpha\beta}$ corresponds to the bias obtained after the transient effects from the start of a sustained maneuver have diminished.

Let A_{max} denote the maximum acceleration of the target. Then the MMSE in the position estimates of the alpha-beta filter is given by

$$MMSE^p = \frac{\sigma_w^2(2\alpha^2 + \beta(2-3\alpha))}{\alpha(4-2\alpha-\beta)} + (1-\alpha)^2\frac{T^4}{\beta^2}A_{max}^2 \quad (23)$$

Let Γ_D denote the deterministic tracking index given by

$$\Gamma_D = \frac{A_{max}T^2}{\sigma_w} \quad (24)$$

Then the MMSE in the position estimates of the alpha-beta filter can be written as

$$MMSE^p = \sigma_w^2\Big[\frac{(2\alpha^2 + \beta(2-3\alpha))}{\alpha(4-2\alpha-\beta)} + \frac{(1-\alpha)^2}{\beta^2}\Gamma_D^2\Big] \quad (25)$$

Given Γ_D, (25), and the relationship between α and β of (9), the steady-state gains for the NCV filter that minimize the MMSE can be computed. Thus, specification of Γ_D defines the optimal filter gains of α and β.

IV. Design of NCV Filter with Piecewise Constant Acceleration Errors

The design techniques for sustained maneuvers developed in [5] for the NCV Kalman filter with discrete white noise acceleration are reformulated and presented in Section IV-A. In Section IV-B, those techniques are extended to design techniques for NCV Kalman filters for brief maneuvers. The effectiveness of the design methods for choosing the process noise variance for tracking targets with both sustained and brief maneuvers are verified with Monte Carlo simulations.

A. Sustained Maneuvers

Given the random tracking index Γ_{DWNA} of (8) and the relationship between α and β of (9) [2], the steady-state Kalman gains for the NCV filter with discrete white noise acceleration are specified. The optimal gains are unique function of the random tracking index [1], [2] and shown in

Figure 1. Utilizing the deterministic tracking index Γ_D of (24) and the relationship between α and β of (9), the steady-state gains for the NCV filter that minimize the MMSE of (25) are specified and plotted in Figure 2.

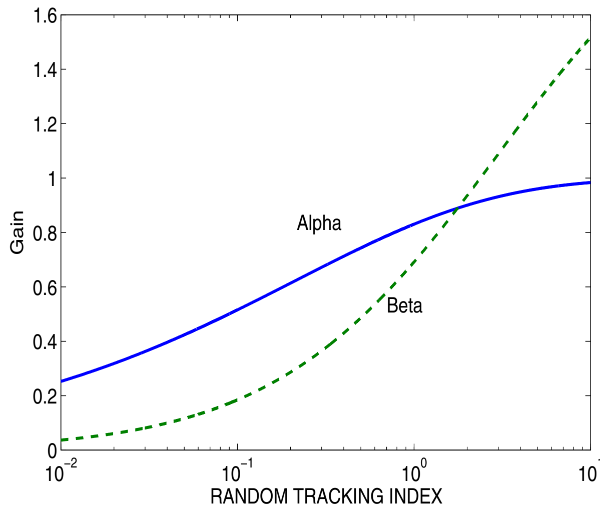

Figure 1. Steady-State Kalman Gains Versus the Random Tracking Index for the NCV Filter with Discrete White Noise Acceleration

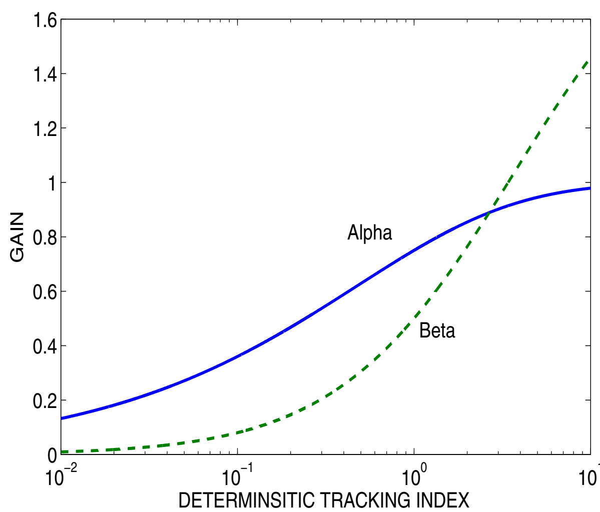

Figure 2. Steady-State Kalman Gains That Minimize the MMSE in Position for the NCV Filter with Discrete White Noise Acceleration Versus the Deterministic Tracking Index

Utilizing the results in Figures 1 and 2 gives the relationship between Γ_{DWNA} and Γ_D as shown in Figure 3. Thus, the random tracking index can be expressed in terms of the deterministic tracking index as

$$\Gamma_{DWNA} = \kappa_1(\Gamma_D)\Gamma_D \tag{26}$$

Utilizing the (8) and (24) in (26) gives

$$\sigma_v = \kappa_1(\Gamma_D)A_{max} \tag{27}$$

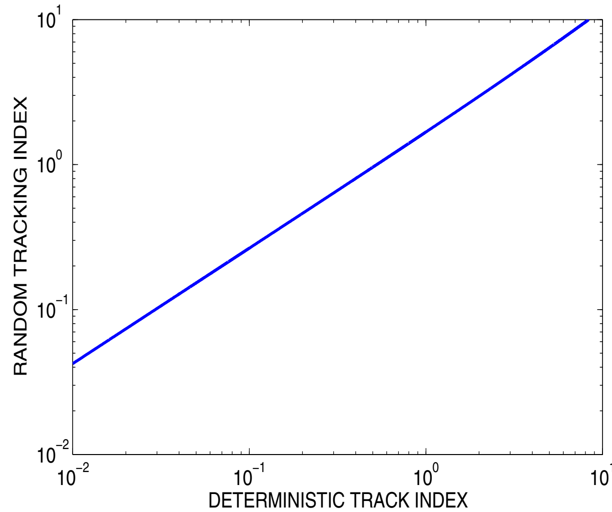

Figure 3. Random Tracking Index for Discrete White Noise Acceleration That Minimizes the MMSE Versus the Corresponding Deterministic Tracking Index

For a given value of Γ_D, there is a unique α and β that minimizes (25) in conjunction with (9). The α and β define a unique Γ_{DWNA} that is related to Γ_D by κ_1. The κ_1 that corresponds to minimizing (25) is given versus Γ_D in Figure 4 and denoted by $\kappa_{1,\infty}^{max}$ and given approximately by

$$\kappa_{1,\infty}^{max}(\Gamma_D) = 1.67 - 0.74\log(\Gamma_D) + 0.26(\log(\Gamma_D))^2 \tag{28}$$
$$0.01 \le \Gamma_D \le 10$$

Thus, (28) along with (27) and the maximum acceleration of the target defines the process noise variance for the NCV filter. Using the constraint that the MMSE in the filtered position estimate should not exceed the measurement error variance gives that $MMSE^p \le \sigma_w^2$ or

$$\frac{(2\alpha^2 + \beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)} + \frac{(1-\alpha)^2}{\beta^2}\Gamma_D^2 \le 1 \tag{29}$$

For a given value of Γ_D, there is a unique α and β that satisfies (29) with equality. These α and β are the minimum gains and define the minimum Γ_{DWNA} that is related to Γ_D by κ_1. The κ_1 that corresponds to the constraint of (29) with equality is given versus Γ_D in Figure 4 and denoted by $\kappa_{1,\infty}^{min}$ and given approximately by

$$\kappa_{1,\infty}^{min}(\Gamma_D) = 0.87 - 0.09\log(\Gamma_D) - 0.02(\log(\Gamma_D))^2 \tag{30}$$

Thus, Figure 4 shows that the constraint is always satisfied if the process noise variance is chosen to minimize the MMSE. However, Figure 4 shows that for $\Gamma_D < 1$, a wide range of values for κ_1 satisfy the constraint in (29). Thus, from Figure 4, the design rule for selecting the process noise standard deviation (*i.e.*, σ_v) as greater or equal to the maximum acceleration of the target will give acceptable filter performance for sustained maneuvers. However, picking the process noise standard deviation equal to one half of the maximum acceleration of the target will give unacceptable filter performance for sustained maneuvers.

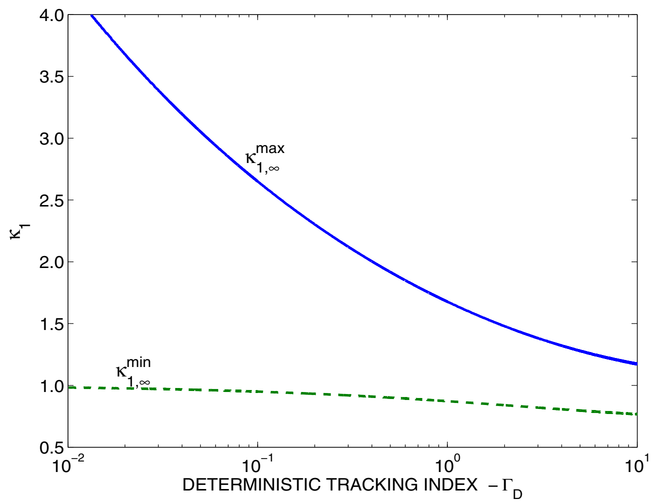

Figure 4. κ_1 for Defining the Process Noise Variance in Terms of the Maximum Acceleration for Sustained Maneuvers

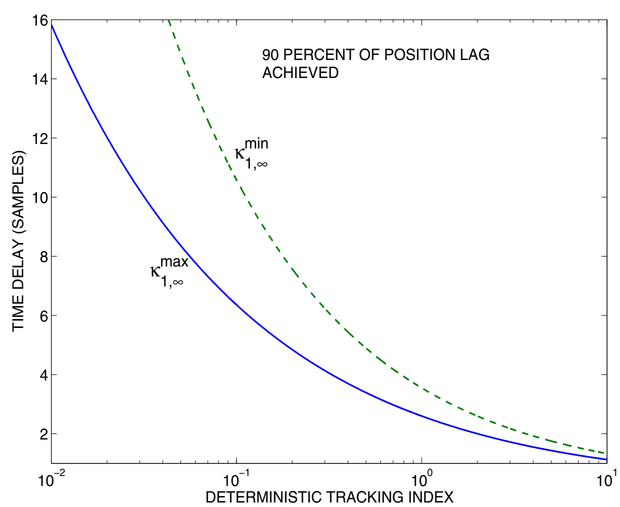

Figure 5. Approximate Time Delay in Filters Achieving 90% of Steady-State Lag in Position

When designing NCV filters for deterministic maneuvers, the question of duration of typical maneuvers arises. Figure 5 gives the approximate time period in measurement sample periods for the maneuver bias to achieve 90% of the bias in position during a sustained maneuver [7]. Thus, for $\Gamma_D \approx 1$, the design method for selecting the process noise variance for sustained maneuvers is valid for maneuvers persisting four measurement periods or longer. For $\Gamma_D \approx 0.1$, this method for selecting the process noise variance is valid for maneuvers persisting 10 measurement periods or longer. A method for targets that maneuver with maximum acceleration for a brief period (*i.e.* not sustained) is addressed in the Section IV-B.

For illustrating the filter design methodology, consider a target that maneuvers with 40 m/s^2 of acceleration from 40 to 60 s. The sensor measures the target position at a 1 Hz rate with errors defined by $\sigma_w = 120$ m. Thus, $\Gamma_D = 0.33$. Since the maneuver is sustained for 20 measurements, considering the maximum bias for design is acceptable. Then, (30) gives $\kappa_{1,\infty}^{min} = 0.92$ and $\sigma_v = 36.8$ as the minimum acceptable value for σ_v. Then, $\kappa_{1,\infty}^{max} = 2.1$ gives $\sigma_v = 84$ as the process noise standard deviation that minimizes the MMSE. Figure 6 shows the RMSE results of Monte Carlo simulations with 1000 experiments. Note that the peak RMSE in position for $\kappa_1 = 0.92$ and $\sigma_v = 36.8$ is closely matched to the standard deviation of the measurements of 120 m as anticipated. Also, note for $\kappa_1 = 2.1$ and $\sigma_v = 84$, the maximum RMSE in position is minimized and is only slightly larger than the RMSE in the absence of a maneuver. Thus, the design rule for selecting the process noise standard deviation (*i.e.*, σ_v) greater than or equal to the maximum acceleration of the target gives acceptable filter performance for sustained maneuvers. However, picking the process noise standard deviation equal to one half of the maximum acceleration of the target will give unacceptable filter performance for sustained maneuvers.

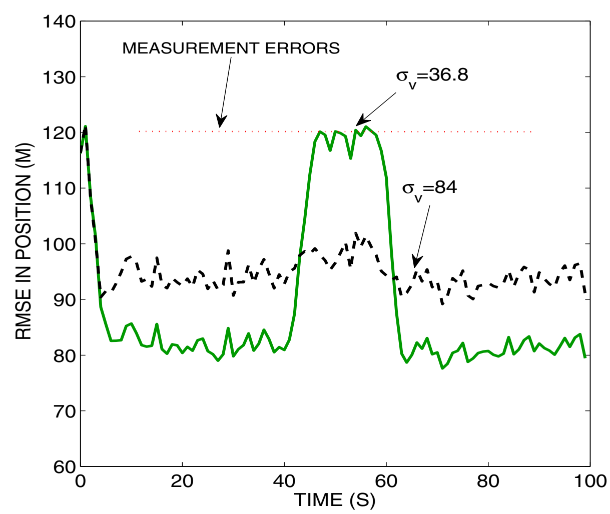

Figure 6. RMSE in Position Estimates of Two Filter Designs for Tracking a Target with a Sustained Maneuver

B. Brief Maneuvers

For a target maneuvering with a constant acceleration that begins at time k, the bias in the state estimate at time $k+n$ for arbitrary α and β is given approximately by

$$B_{k+n|k+n}^{\alpha\beta} \approx \begin{bmatrix} (1-(1-\alpha)^{\frac{5n-4}{8}})(1-\alpha)\frac{T^2}{\beta} \\ (\alpha - 0.5\beta - (1-\alpha)^{\frac{n}{2}})\frac{T}{\beta} \end{bmatrix} A_{k,k+n} \quad (31)$$

where $A_{k,k+n}$ is the acceleration of the target from time t_k to t_{k+n} in the coordinate of interest.

The MMSE in the position estimates of the alpha-beta filter at time $k+N$ for a maneuver that persists for N measurements can be approximated by

$$\begin{aligned} MMSE^p &= \sigma_w^2\Big[\frac{1}{\alpha(4-2\alpha-\beta)}(2\alpha^2+\beta(2-3\alpha) \\ &+ [(1-(1-\alpha)^{\frac{5N-4}{8}})(1-\alpha)]^2\frac{\Gamma_D^2}{\beta^2}\Big] \end{aligned} \tag{32}$$

For a given value of Γ_D and N, there is a unique α and β that minimizes (32) in conjunction with (9). The α and β define a unique Γ_{DWNA} that is related to Γ_D by κ_1. The κ_1 that corresponds to minimizing (32) for maneuvers of three and six measurements is given versus Γ_D in Figure 7 and denoted by the lines labeled with $\kappa_{1,3}^{max}$ and $\kappa_{1,6}^{max}$, respectively. Thus, Figure 7 along with (27) and the maximum acceleration of the target that lasts for N measurements defines the process noise variance for the NCV filter. Using the constraint that the MMSE in the filtered position estimate should not exceed the measurement error variance gives that $MMSE^p \leq \sigma_w^2$ or

$$\frac{(2\alpha^2+\beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)}+\frac{[(1-(1-\alpha)^{\frac{5N-4}{8}})(1-\alpha)]^2}{\beta^2}\Gamma_D^2 \leq 1 \tag{33}$$

For a given value of Γ_D and N, there is a unique α and β that satisfies (33) with equality. These α and β are the minimum gains and define the minimum Γ_{DWNA} that is related to Γ_D by κ_1. The κ_1 that defines the minimum Γ_{DWNA} for maneuvers of three and six measurements are given versus Γ_D in Figure 7 and denoted by the lines labeled with $\kappa_{1,3}^{min}$ and $\kappa_{1,6}^{min}$, respectively. Thus, Figure 7 shows the constraints for the process noise variance for sustained and brief maneuvers. Figure 7 shows that the process noise standard deviation (*i.e.*, σ_v) can be picked less than the maximum acceleration of the target for brief maneuvers.

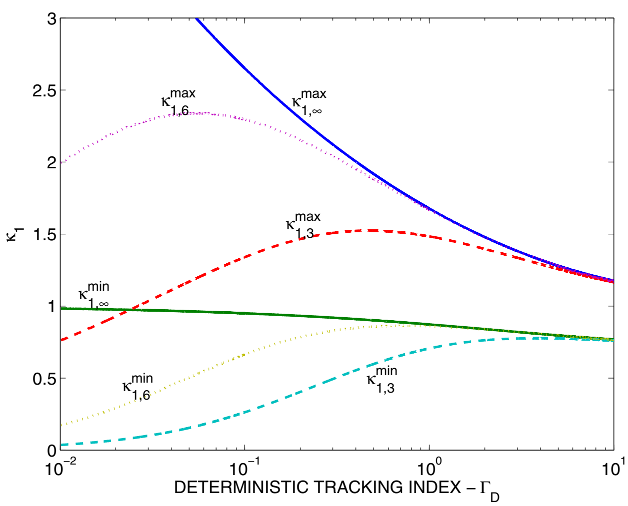

Figure 7. κ_1 for Defining the Process Noise Variance in Terms of the Maximum Acceleration for Brief Maneuvers

The graphical results of Figure 7 can be represented as a third-order polynomial by

$$\begin{aligned} \kappa_1(\Gamma_D) &= a_0 + a_1\log(\Gamma_D) + a_2(\log(\Gamma_D))^2 \\ &+ a_3(\log(\Gamma_D))^3 \end{aligned} \tag{34}$$

for $0.01 \leq \Gamma_D \leq 10$ with Table I providing the coefficients.

Table I
COEFFICIENTS FOR COMPUTING κ_1

	a_0	a_1	a_2	a_3
$\kappa_{1,\infty}^{max}$	1.68	-0.72	0.23	-0.02
$\kappa_{1,\infty}^{min}$	0.87	-0.10	-0.02	0.00
$\kappa_{1,3}^{max}$	1.49	-0.11	-0.26	0.00
$\kappa_{1,3}^{min}$	0.70	0.32	-0.20	-0.10
$\kappa_{1,6}^{max}$	1.67	-0.72	0.07	0.18
$\kappa_{1,6}^{min}$	0.87	0.03	-0.17	0.01

To illustrate the filter design methodology for brief maneuvers, consider a target that maneuvers with 40 m/s^2 of acceleration from 40 to 43 s. The sensor measures the target position at a 1 Hz rate with errors defined by $\sigma_w = 120$ m. Thus, $\Gamma_D = 0.33$. The maneuver is brief in that it lasts for three measurement periods. Using Figure 7 or Table I gives $\kappa_{1,3}^{min} = 0.51$ and $\sigma_v = 20.4$ as the minimum acceptable value for σ_v. Using Table I gives $\kappa_{1,3}^{max} = 1.48$ and $\sigma_v = 59.2$ as the process noise variance that minimizes the MMSE. Figure 8 shows the RMSE results of Monte Carlo simulations with 2000 experiments. Note that for the brief maneuver of three measurement periods that the peak RMSE in position for $\kappa_1 = 0.51$ and $\sigma_v = 20.4$ is slightly less than the standard deviation of the measurements of 120 m. Also, note for $\kappa_1 = 1.48$ and $\sigma_v = 59.2$, the maximum RMSE in position is minimized and only slightly larger than the RMSE in the absence of a maneuver. Note that the focus here is on the maneuver response after the larger errors associated with filter initialization have diminished. Thus, Figure 7 and the results in Figure 8 show that the design rule for selecting the process noise standard deviation (*i.e.*, σ_v) as greater than or equal to the maximum acceleration of the target does not hold for brief maneuvers. For this case of a brief maneuver, selecting the process noise standard deviation equal to one half of the maximum acceleration of the target will give acceptable filter performance. However, Figure 7 shows that even smaller values of the process noise can be used for brief maneuvers and smaller tracking indexes.

Now, consider a target that maneuvers with 40 m/s^2 of acceleration from 40 to 46 s. The sensor measures the target position at a 1 Hz rate with errors defined by $\sigma_w = 120$ m. Thus, $\Gamma_D = 0.33$. The maneuver is brief in that it lasts for six measurement periods. Using Table I gives $\kappa_{1,6}^{min} = 0.81$ and $\sigma_v = 32.4$ as the minimum acceptable value for σ_v. Using Table I gives $\kappa_{1,6}^{max} = 2.01$ and $\sigma_v = 80.4$ as the process noise variance that minimizes the MMSE. Figure 9 shows the RMSE results of Monte Carlo simulations with 2000 experiments. Note that for the brief maneuver of six measurement periods that the peak RMSE in position for $\kappa_1 = 0.81$ and $\sigma_v = 32.4$ is only slightly larger than the standard deviation of the measurements of 120 m. Also, note for $\kappa_1 = 2.01$ and $\sigma_v = 80.4$, the peak RMSE in position is minimized and is only slightly larger than the RMSE in the absence of a maneuver.

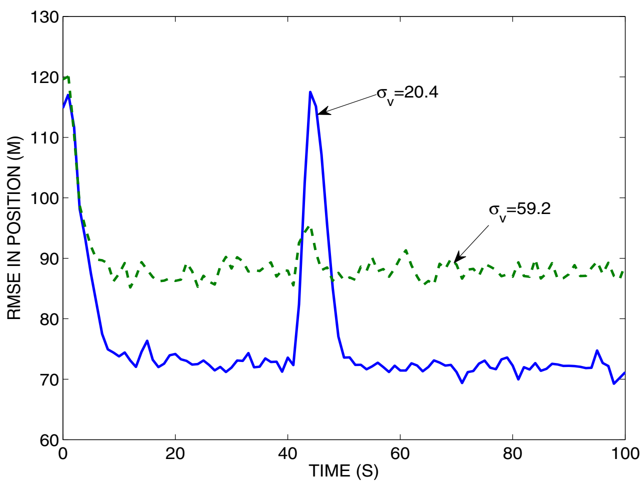

Figure 8. RMSE in Position Estimates of Two Filter Designs for Tracking a Target that Maneuvers for Three Measurements and $\Gamma_D = 0.33$

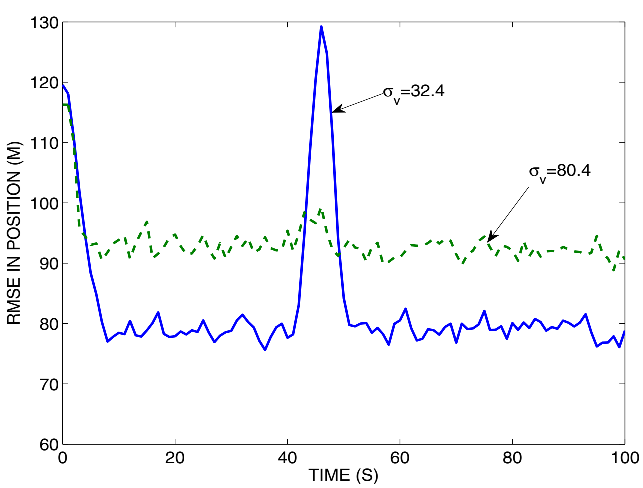

Figure 9. RMSE in Position Estimates of Two Filter Designs for Tracking a Target that Maneuvers for Six Measurements and $\Gamma_D = 0.33$

Changing to a sensor that measures the target position at a 1 Hz rate with errors defined by $\sigma_w = 600$ m gives $\Gamma_D = 0.067$ for maneuvers of 40 m/s^2 For a brief maneuver of three measurement periods, Table I gives $\kappa_{1,3}^{min} = 0.20$ and $\sigma_v = 8$ as the minimum acceptable value for σ_v. Using Table I gives $\kappa_{1,3}^{max} = 1.26$ and $\sigma_v = 50.4$ as the process noise variance that minimizes the MMSE. Figure 10 shows the RMSE results of Monte Carlo simulations. Note that for the brief maneuver of three measurement periods that the peak RMSE in position of 500 m for $\kappa_1 = 0.20$ and $\sigma_v = 8$ is about 15% less than the standard deviation of the measurements of 600 m. Also, note for $\kappa_1 = 1.26$ and $\sigma_v = 50.4$, the peak RMSE in position is minimized and is only slightly larger than the RMSE in the absence of a maneuver.

For a brief maneuver of six measurement periods with the sensor errors defined by $\sigma_w = 600$ m, Table I gives gives $\kappa_{1,6}^{min} = 0.58$ and $\sigma_v = 23.2$ as the minimum acceptable value for σ_v. Using Table I gives $\kappa_{1,6}^{max} = 2.33$ and $\sigma_v = 93.2$ as the process noise variance that minimizes the MMSE. Figure 11 shows the RMSE results. Note that for the brief maneuver of six measurement periods that the peak RMSE in position of 550 m for $\kappa_1 = 0.58$ and $\sigma_v = 23.2$ is about 10% less than the standard deviation of the measurements of 600 m. Also, note for $\kappa_1 = 2.33$ and $\sigma_v = 93.2$, the peak RMSE in position is minimized and is only slightly larger than the RMSE in the absence of a maneuver.

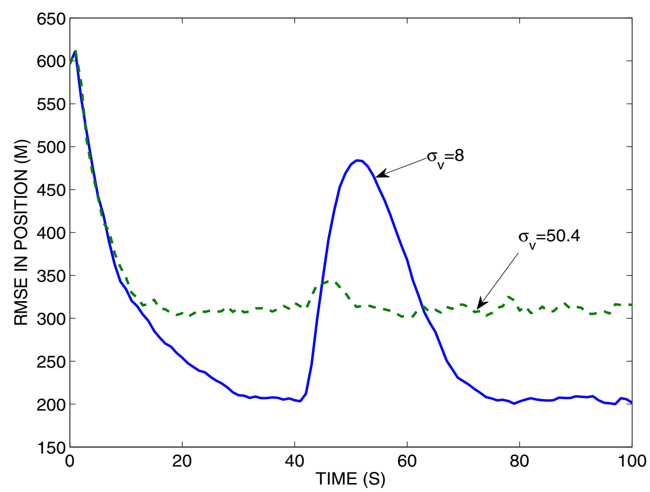

Figure 10. RMSE in Position Estimates of Two Filter Designs for Tracking a Target that Maneuvers for Three Measurements and $\Gamma_D = 0.067$

The simulation results of Figures 8 through 11 indicate that the design methods of this section can be used to design NCV filters for brief maneuvers. The process noise variance can be selected for brief maneuvers so that the RMSE in position is approximately equal to the standard deviation of the measurement errors or the MMSE is minimized during the maneuver. Of course, the filter designer can also select any process noise variance between these two extremes.

V. NCV Filter with Continuous White Noise Acceleration

Given the random tracking index Γ_{CWNA} of (12) and the relationship between α and β of (13), the steady-state Kalman gains for the NCV filter with continuous white noise acceleration are specified. Utilizing the deterministic tracking

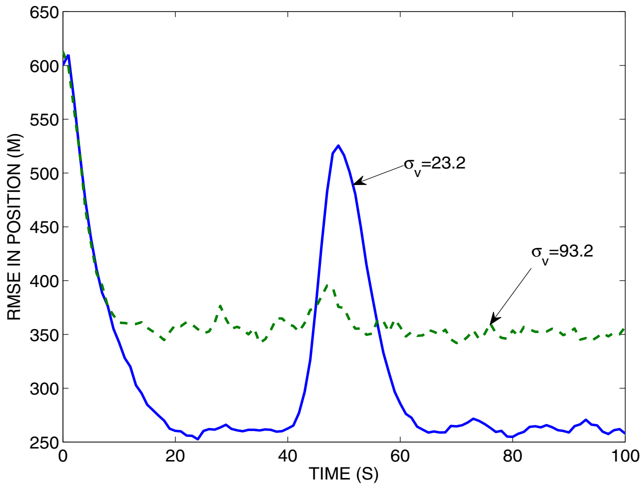

Figure 11. RMSE in Position Estimates of Two Filter Designs for Tracking a Target that Maneuvers for Six Measurements and $\Gamma_D = 0.067$

index Γ_D of (24) and the relationship between α and β of (13), the steady-state gains for the NCV filter with continuous white noise acceleration that minimize the MMSE of (25) are specified. Utilizing these results defines a relationship between Γ_{CWNA} and Γ_D. Thus, the random tracking index can be expressed in terms of the deterministic tracking index as

$$\Gamma_{CWNA} = \kappa_2(\Gamma_D)\Gamma_D \tag{35}$$

Utilizing (12) and (24) in (35) gives

$$\sqrt{\frac{\tilde{q}}{T}} = \kappa_2(\Gamma_D)A_{max} \tag{36}$$

For a given value of Γ_D and the relationship between α and β of (13), there is a unique α and β that minimizes (25). The α and β define a unique Γ_{DCWA} that is related to Γ_D by κ_2. The κ_2 that minimizes (25) is given versus Γ_D approximately by $\kappa_{2,\infty}^{max}(\Gamma_D) \approx \kappa_{1,\infty}^{max}(\Gamma_D)$ as defined in Table I. Using the constraint that the MMSE in the filtered position estimate should not exceed the measurement error variance gives that $MMSE^p \leq \sigma_w^2$. For a given value of Γ_D, there is a unique α and β that satisfies (29) with equality. The α and β are the minimum gains and define the minimum Γ_{DCWA} that is related to Γ_D by κ_2. The κ_2 that defines the minimum $\tilde{q}$ is given given approximately by $\kappa_{2,\infty}^{min}(\Gamma_D) \approx \kappa_{1,\infty}^{min}(\Gamma_D)$ as as defined in Table I. For brief maneuvers of three measurements, $\kappa_{2,3}^{max}(\Gamma_D) \approx \kappa_{1,3}^{max}(\Gamma_D)$ and $\kappa_{2,3}^{min}(\Gamma_D) \approx \kappa_{1,3}^{min}(\Gamma_D)$. For brief maneuvers of six measurements, $\kappa_{2,6}^{max}(\Gamma_D) \approx \kappa_{1,6}^{max}(\Gamma_D)$ and $\kappa_{2,6}^{min}(\Gamma_D) \approx \kappa_{1,6}^{min}(\Gamma_D)$. Thus, Figure 7 or Table I can also be used for the selection of κ_2 for sustained and brief maneuvers.

VI. Conclusions

When given the parameters of a sensor, the measurement rate, and the maximum acceleration of the target, the variance of the discrete white noise acceleration or power spectral density of the continuous white noise acceleration errors can be computed for optimal performance of track filters under steady-state conditions. However, the conditions of a fixed measurement rate and stationary measurement statistics are rarely satisfied in practice. Thus, nominal conditions can be used to select the process noise variance or power spectral density for the nominal operating point of the filter and the Kalman gains will adjust for the changes in the data rate and measurement variance. A design procedure for tracking targets with sustained maneuvers with a NCV filter was presented in [5] and it was reformulated here for clarity. This paper extends those results to brief maneuvers and simulation results confirm the design procedure. Using the techniques presented here, the process noise variance can be selected for brief maneuvers so that the RMSE in position is approximately equal to the standard deviation of the measurement errors or the MMSE is minimized during the maneuver. Of course, the filter designer can also select any process noise variance between these two extremes.

The conventional design rule for selecting the process noise standard deviation greater than or equal to the maximum acceleration of the target will give acceptable filter performance for sustained maneuvers. For the case of brief maneuvers, the process noise standard deviation can be selected less than the maximum acceleration of the target and achieve acceptable filter performance. Future work will involve the extension of the design technique to tracking with radars that measure the target position in polar or spherical measurements, while the track state is estimated in Cartesian coordinates. Also, the design techniques developed in this paper will used for the model selection and design for the the IMM estimator for tracking maneuvering targets.

VII. Acknowledgements

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References

[1] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation,* John Wiley and Sons, Inc., New York, NY, 2001.

[2] P. R. Kalata, "The tracking Index: A Generalized Parameter for Alpha-Beta-Gamma Target Trackers," *IEEE Trans. Aerospace and Electronic Systems,* March 1984, pp. 174-182.

[3] W. D. Blair, *Fixed-Gain, Two-Stage Estimators for Tracking Maneuvering Targets,* Technical report NSWCDD/TR-92/297, Naval Surface Warfare Center Dahlgren Division, Dahlgren, VA, 1992.

[4] W. D. Blair, "Fixed-Gain, Two-Stage Estimators for Tracking Maneuvering Targets," *IEEE Transactions On Aerospace and Electronic Systems*, July 1993, pp. 1004-1014.

[5] W. D Blair "Design of Nearly Constant Velocity Track Filters for Tracking Maneuvering Targets," in *Proceedings of* 11_{th} *International Conference on Information Fusion,* Cologne, Germany, June 30 - July 3, 2008.

[6] W. D. Blair, and Y. Bar-Shalom, "Tracking Maneuvering Targets With Multiple Sensors: Does More Data Always Mean Better Estimates?" *IEEE Transactions On Aerospace and Electronic Systems*, January 1996, pp. 450-456.

[7] J. E. Gray and W. Murray, "The Response of the Transfer Function of an Alpha-Beta Filter to Various Measurement Models," in *Proceedings of 23th IEEE Southeastern Symposium on System Theory,* Mar. 1991, pp. 389-393.