Chapter 1

Introduction

Chapter 2

Methodology

2.1 The setup

A complex scalar field of mass m and charge e obeys the Klein-Gordon equation

$$\left(D_{\mu}D^{\mu}+m^{2}\right)\phi=0. \tag{2.1}$$

Confined to an interval of size normalized to 1, we consider Robin boundary conditions i.e. boundary conditions of the form of

$$n_{\mu}D^{\mu}\phi+\xi(\sigma)\phi=0 \tag{2.2}$$

with σ the boundary of the region considered. We are however, not interested in (anti) periodic boundary conditions, as these would translate into a 'particle' that gets accelerated every time it loops around the interval.

Under the Coulomb gauge, the electromagnetic potential corresponding to a constant electric field is

$$A_{0}=-\lambda\left(z-\frac{1}{2}\right)\quad A_{1}=0. \tag{2.3}$$

Assuming the separation ansatz

$$\phi(t,z)=\phi_{n}(z)e^{-i\omega_{n}t} \tag{2.4}$$

equation (2.1) can be rewritten as the time independent Klein-Gordon equation

$$\partial_1^2 \phi_n = \left(-(\omega_n - eA_0)^2 + m^2\right)\phi_n \tag{2.5}$$

$$= \left(-\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)^2 + m^2\right)\phi_n. \tag{2.6}$$

One may notice that this equation is of the form of the Weber differential equation

$$\partial_1^2 \phi_n = -\left(\nu + \frac{1}{2} - \frac{1}{4}\tilde{z}^2\right)\phi_n, \tag{2.7}$$

and can thus be solved by the parabolyc cylinder functions $D_\nu(\tilde{z})$. NOTE: Add this to the appendix? We then write the general solution to equation 2.6 as follows

$$\phi_n(z) = a_n D_{i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{1+i}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right) + b_n D_{-i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{i-1}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right). \tag{2.8}$$

NOTE: Also talk about the 0 field solution that indeed lead to vanishing vacuum polarization. We have thus far considered what in [1] is referred to as the 'external field approximation'. This implies that one does not consider the fact that the vacuum polarization induces itself another electric field. This approximation fails [1] in the calculation of the energy of the modes, as it yields complex values for some external field values.

NOTE: [2]: This seems to be the reason why in [1], where the mode sum formula is employed, it is claimed that the vacuum polarization vanishes for massless fermions in 1 + 1 spacetime dimension. As we will see, this is not correct.

We therefore use the "back reaction" approach, in which we solve the problem iteratively. Following the convention used in [1], we label by κ each of the iterations. At each step κ we solve numerically the following set of integro-differential equations

$$\partial_1^2 \phi_n^{\kappa+1} = \left(-(\omega_n - eA_0^\kappa(z))^2 + m^2\right)\phi_n^{\kappa+1} \tag{2.9}$$

$$\partial_1^2 \tilde{A}^\kappa = -\rho^\kappa, \tag{2.10}$$

with

$$A_0^\kappa(z) = -\lambda\left(z - \frac{1}{2}\right) + \tilde{A_0}^\kappa(z),$$

and $\rho^\kappa(z)$ the associated charge density at each step. Since we expect to see no screening at the boundaries, (2.10) has Neumann boundary conditions.

NOTE: [2]: The supplementary electromagnetic field produced by (36) via back- reaction is

$$E_{br} = \frac{1}{2\pi} e^2 E \left[z^2 - \left(\frac{L}{2} \right)^2 \right] \tag{2.11}$$

For the model to be physically viable, this should be small compared to the external electric field, or equivalently

$$eL \ll 1 \tag{2.12}$$

This method removes the appearance of complex eigenvalues, as the screening effects of the vacuum should be taken into account.

One should take care when calculating the charge density, since

$$\rho(z) = \langle 0 | \hat{\rho}(z) | 0 \rangle = ie \langle 0 | \phi^* D_0 \phi - \phi \left(D_0 \phi \right)^* | 0 \rangle \tag{2.13}$$

is not well defined as it involves the product of two fields. We take care of this in the following section.

2.2 Point-split renormalization

Due to the linearity of (2.1), we write the quantum field as

$$\phi(t, z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n(z) e^{-i\omega_n t} \tag{2.14}$$

with the operators a_n, b_n fulfilling

$$\left[a_n, a_m^\dagger \right] = \delta_{nm} \qquad \left[b_n, b_m^\dagger \right] = \delta_{nm} \tag{2.15}$$

and all other commutators vanishing.

We define the vacuum state $|0\rangle$ as that fulfilling

$$a_n |0\rangle = 0, \, b_n |0\rangle = 0. \tag{2.16}$$

To calculate the charge density $\rho(z)$ of the field, one would naïvely calculate

$$\rho(z) = ie \langle 0 | \phi^* D_0 \phi - \phi \left(D_0 \phi \right)^* | 0 \rangle \, , \tag{2.17}$$

which is known to be ill-defined as it involves the product of two distributions.

As any physically reasonable state is assumed to be Hadamard, NOTE: Need to read on this

$$\rho^{\kappa}(z) = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} (\omega_n - A_0^{\kappa}(z)) \|\phi_n(z)\|^2 e^{-i\omega_n(\tau + i\varepsilon)} + \frac{e}{\pi^2} A_0^{\kappa}(z) \tag{2.18}$$

2.3