Chapter 2

Preliminaries

2.1 Constructing the quantum field theory

The free Klein-Gordon equation for a complex field ϕ is

$$(D_\mu D^\mu + m^2)\phi = 0. \tag{2.1}$$

When solved with compactly supported Cauchy data $\left(\phi|_\Sigma, n^\mu D_\mu \phi|_\Sigma\right)$, with Σ a space-like hypersurface (typically a constant-t surface) and n a unit vector orthogonal to Σ, the global hyperbolicity of Minkowski spacetime ensures that this problem is well posed. The collection of all possible smooth ϕ with compactly supported Cauchy data forms the configuration space $\mathcal{C}$. From $\mathcal{C}$ one constructs the phase space $\mathcal{P} = T^*\mathcal{C}$, as its cotangent space. The elements of $\mathcal{P}$ are the points (ϕ, π), with $\pi = D_0^* \phi^*$ the canonical conjugate of the field ϕ. Points in the phase space correspond to initial conditions for the Klein-Gordon equation on the Cauchy surface Σ. $\mathcal{P}$ can be endowed with symplectic structure with the symplectic* form

$$\sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = i \int_\Sigma (\pi_1^* \phi_2^* - \pi_2 \phi_1)\, d\Sigma. \tag{2.2}$$

It is not difficult to prove that σ is a time evolution invariant if ϕ_1, ϕ_2 are solutions to the Klein-Gordon equation, as long as they are compactly supprted. **[Prove this? In appendix?]**

The solution space $\mathcal{S}$ is the space of all smooth solutions to the Klein-Gordon equation. As the initial data fully determines a solution to the Klein-Gordon equation, each solution corresponds to a point in phase space and therefore $\mathcal{S}$ can be identified with $\mathcal{P}$. $\mathcal{S}$ can be given Hilbert space structure as follows:

*Bilinear, antisymmetric

1. Define the subspace $\mathcal{S}_+$ of positive frequency solutions. In this subspace the symplectic product can be made positive definite.

2. Define a Hermitian inner product in this subspace

$$(\phi_1, \phi_2) = i\sigma(\phi_1^*, \phi_2), \quad \phi_1, \phi_2 \in S_+ \tag{2.3}$$

3. Complete $\mathcal{S}_+$ with respect to this norm. **[How to go from this to the whole $\mathcal{S}$?]**

In classical mechanics, observables are the compactly supported functions $f : \mathcal{S} \to \mathbb{C}$. They can be equipped with an algebra structure via the Poisson brackets,

$$\{f, g\} := \int_\Sigma \left(\frac{\delta f}{\delta \phi} \frac{\delta g}{\delta \pi} - \frac{\delta g}{\delta \phi} \frac{\delta f}{\delta \pi} \right) d\Sigma. \tag{2.4}$$

First quantisation is done by the $\hat{}$ operator, which elevates classical observables f to operators $\hat{f}$ on the $\mathcal{S}$ obeying the algebraic rule

$$[\hat{f}, \hat{g}] = i\{\hat{f,g}\}, \tag{2.5}$$

with $\{\hat{f,g}\}$ the quantized version of the Poisson bracket.

The field observables are defined with respect to smooth compactly supported test functions f

$$\phi(f) = a(f)e^{-i\omega t} + a^\dagger(\overline{f})e^{i\omega t}, \tag{2.6}$$

where the a and $a^\dagger$ operators are defined the same way the are constructed in the quantum harmonic oscillator problem.

The measured observables of the field in a state $\Omega \in \mathcal{F}(\mathcal{S})$ are therefore the expectation value $\langle \Omega | \phi(f) | \Omega \rangle$. Through $\phi(f)$ one can define the field as the weighted average by the test function f of the spacetime distribution $\phi(x)$ through

$$\phi(f) = \int_{\mathbb{R}^n} \phi(x) f(x) d^n x, \tag{2.7}$$

with $\phi(x)$ an operator valued distribution. This also formally defines the two-point function as the distributional kernel of

$$w_\Omega^{\phi\phi^*}(f, h) = \langle \Omega | \phi(f)\phi^*(g) | \Omega \rangle = \int_{\mathcal{M}\times\mathcal{M}} f(x)g(x')w_\Omega^{\phi\phi^*}(x, x')dxdx'. \tag{2.8}$$

2.2 Hadamard two point functions and point-split renormalization

The above mentioned construction of QFT is incomplete in the sense that it fails to describe the product of fields at same spacetime points. In particular, we are interested in calculating the following expression

$$\rho(x) = \langle \hat{\rho}(x) \rangle_\Omega = ie \langle \Omega | \phi(x)^* D_0 \phi(x) - \phi(x) D_0^* \phi(x)^* | \Omega \rangle , \tag{2.9}$$

wich is a priori ill-defined and needs to be renormalized.

The direct evaluation of (2.9) is divergent, even for the vacuum, due to the terms $a_n a_n^\dagger$, $b_n b_n^\dagger$ found in the mode expansion of the product of the fields **[Mention the mode expansion here?]**. The usual procedure in Relativistic Quantum Field Theory is to define $\langle \rho(x) \rangle_\Omega$ through normal ordering,

$$\langle : \hat{\rho}(x) : \rangle_\Omega := \langle \Omega | \hat{\rho}(x) | \Omega \rangle - \langle 0 | \hat{\rho}(x) | 0 \rangle , \tag{2.10}$$

which defines the expectation values with respect to the vacuum state, or the zero-point value.

The normal ordering of an observable $\mathcal{O}(\phi) \rightarrow : \mathcal{O}(\phi) :$ is a crucial tool in Relativistic Quantum Field Theory when defining products of field operators at coincident points. This prescription has the key property $\langle : \mathcal{O}(\phi) : \rangle_0 = 0$, which is not an expected behaviour for the vacuum in the pressence of background electric fields. This prescription also presents the flaw of defining the expectation values using the vacuum as a reference point. In the pressence of more general electromagnetic fields, the definition of a vacuum state lacks uniqueness, rendering the normal ordering prescription for more general backgrounds invalid.

In the context of Quantum Field Theory on Curved Spacetimes, these difficulties are circumvented by the following observations

1. Even though the expectation value $\langle \Omega | \phi(x) \phi^*(x) | \Omega \rangle$ is ill-defined, the two point function (2.8) is a well defined bi-distribution, bi-solution to the field equation, divergent as $x' \rightarrow x$.

2. In the normal ordering prescription, the divergences that appear when evaluating expectation values are defined through difference of expectation values between different states. There exists a class of states for which their difference in the expectation values is also well-defined.

We are particularly interested in the class of Hadamard states, states which have the same divergence structure as the vacuum state of the Klein-Gordon field in the

absence of background electromagnetic fields,

$$w_\Omega^{\phi\phi^*}(x,x') = H^{\phi\phi^*}(x,x') + R_\Omega^{\phi\phi^*}(x,x'), \tag{2.11}$$

with $H^{\phi\phi^*}$ the Hadamard parametrix, a divergent bi-distribution independent of the state Ω, resembling the behaviour of the vacuum two-point function in absence of external fields, up to smooth coefficients, and $R_\Omega^{\phi\phi^*}$ a smooth function which does depend on the state.

The Hadamard parametrix in general takes the form

$$H^{\phi\phi^*} = \sum_{k=0}^{N} V_k(x,x') T_k(x,x') \tag{2.12}$$

with $V_k(x,x')$

$$T_k(x,x') := \begin{cases} \frac{\left(\frac{n}{2}-2-k\right)!}{2(2\pi)^{\frac{n}{2}}} \left(-\sigma_\varepsilon\right)^{-\left(\frac{n}{2}-1-k\right)} & k \leq \frac{n}{2}-2, \\ -\frac{1}{2(2\pi)^{\frac{n}{2}}\left(k-\frac{n}{2}+1\right)!} \left(-\sigma\right)^{\frac{n}{2}-1-k} \log \frac{\sigma_\varepsilon}{\Lambda^2} & k > \frac{n}{2}-2, \end{cases} \tag{2.13}$$

with

$$\sigma(x,x') = \frac{1}{2}(x-x')^2,\ \sigma_\varepsilon = \sigma + i\varepsilon \mathrm{sign}(x_0'-x_0). \tag{2.14}$$

It is important to note that the Hadamard parametrix is constructed entirely by geometrical means and is independent of the state.

To find the $V_k(x,x')$ and $W(x,x')$ functions, one imposes that $w(x,x')$ is a bi-solution to the Klein-Gordon equation, i.e.

$$(D_\mu D^\mu + m^2) w_\Omega^{\phi\phi^*}(x,x') = ({D^*}'_\mu {D^*}'^\mu + m^2) w(x,x') = 0, \tag{2.15}$$

with $D_\mu'^* = \partial_\mu' - ieA_\mu(x')$ acting on x'. With this imposition, one finds a sequence of differential equations, which together with the initial conditions $\lim_{x'\to x} V_0(x,x') = 1$ leads to the family V_k of solutions.

Once $H^{\phi\phi^*}$ is known, one can define the expectation value of fields at coinciding points through point-splitting with respect to the Hadamard parametrix,

$$\left\langle D_\alpha \phi(x) D_\beta^* \phi^*(x) \right\rangle_\Omega = \lim_{x'\to x} D_\alpha D_\beta^{*\prime} \left(w_\Omega^{\phi\phi^*}(x,x') - H^{\phi\phi^*}(x,x') \right). \tag{2.16}$$

Due to the Hadamard form of the two-point function, in the limit of coinciding points, the divergences cancel out, resulting in a well-defined expectation value.

Chapter 3

Vacuum polarization

3.1 The setup

[This first order backreaction calculation should only be valid up to $\varepsilon\lambda \ll 1$ (?). In the problem of a charged particle of charge e colliding with a nucleus of charge Ze, vacuum polarization becomes relevant when the coupling constant becomes comparable to the source. Where to discuss this? What when this approximation isn't true?]

We study the coupling of a massive charged scalar complex field ϕ to a classical external electromagnetic field A_μ in a 1+1 dimensional flat spacetime. The simplest description this admits is the minimal coupling of the scalar field to the background classical gauge field A_μ

$$S[\phi, A_\mu] = \int_{\mathbb{R}^2} dx^0 dx^1 \left(\frac{1}{2} D_\mu \phi D^{*\mu} \phi^* + m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu_{\text{external}} \right) \tag{3.1}$$

[This shows $\mathcal{L}$. Relevant to write the corresponding Hamiltonian density?]

with $D_\mu = \partial_\mu + ieA_\mu$ the gauge covariant derivative given by the minimal coupling to A_μ and e, μ the charge and mass parameter of the Klein-Gordon field, respectively. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic stress-energy tensor and j^μ_{external} is the charge current density of the external source.

From (3.1) one derives through the Euler-Lagrange equations the Klein-Gordon-Maxwell equations, namely the coupled system of differential equations given by the Klein-Gordon equation and the Maxwell equations

$$\left(D_\mu D^\mu + \mu^2 \right) \phi = 0 \tag{3.2a}$$

$$\partial_\mu F^{\mu\nu} = j^\nu_{\text{external}} + \langle j^\nu_{\text{br}} \rangle_\omega \,, \tag{3.2b}$$

with

$$j^{\nu}_{\text{br}} = ie\left(\phi^* D_0 \phi - \phi D_0^* \phi^*\right) \tag{3.3}$$

the charge current density of the Klein-Gordon field. We consider the semiclassical approximation **[Relevant to check [3] for breakdowns of the semiclassical approximation?]**, in which ϕ interacts with the classical external electromagnetic field through the expectation value of its charge current density with respect to some state ω of the field.

Equation (3.2a) can be solved with attractive or repulsive boundary conditions, resp. Dirichlet $\phi|_{x^1=0,a}$, Neumann $\partial_1\phi|_{x^1=0,a}$ boundary conditions, or a linear combination of the two. Equation (3.2b) need however be solved with Dirichlet boundary conditions.

In this setup, the external electric field is generated by placing two charges q, $-q$ at $x_1 = a$ and $x_1 = a > 0$, respectively. In one spatial dimension, these charges generate a uniform electric field between them, of strength $E = 2q$. Thus, in the laboratory frame of reference,

$$j^{\mu}_{\text{external}} = (\rho, 0). \tag{3.4}$$

The pressence of these charges breaks the translational symmetry of the problem which keeps us from using tools from Relativistic Quantum Field Theory.

3.1.1 Gauge fixing

We choose to solve this problem using Coulomb's gauge, $\partial_1 A_1 = 0$. In the chosen gauge, for a time-independent electric field, $A_1 = 0$ up to an addition constant. We therefore turn our attention to the zeroth component of the potential, or the electrostatic potential, which in Coulomb gauge is a solution to the Poisson equation

$$\partial_1^2 A_0 = -\rho = -\left(\rho_{\text{external}} + \rho_\phi\right), \tag{3.5}$$

This differential equation is to be solved with the boundary conditions

$$\partial_1 A_0|_{x_1=0,a} = -E, \tag{3.6}$$

so that no screening appears in the boundaries. This fixes the solution up to a constant factor, which is to be chosen so that the problem is antisymmetric. It suffices to demand $A_0(\frac{a}{2}) = 0$.

Due to the linearity of (3.5) and (3.6), we decompose the electrostatic potential into a background A_0^{external} and an induced A_0^{br} potential. This way (3.5) splits into

$$\begin{aligned} \partial_1^2 A_0^{\text{external}} &= 0 \\ \partial_1 A_0^{\text{external}}\Big|_{x_1=0,a} &= -E, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \partial_1^2 A_0^{\text{br}} &= 0 \\ \partial_1 A_0^{\text{br}}\Big|_{x_1=0,a} &= 0. \end{aligned} \tag{3.8}$$

The full potential reads

$$A_0(x^1) = A_0^{\text{external}}(x^1) + A_0^{\text{br}}(x^1) = \tag{3.9}$$

$$-E\left(x^1 - \frac{a}{2}\right) - \int_{\frac{a}{2}}^{x^1}\int_0^{x^{1\prime}} \rho_\phi(x^{1\prime\prime})dx^{1\prime\prime}dx^{1\prime}. \tag{3.10}$$

Knowing how to calculate A_0 (provided we know how to calculate ρ_ϕ) we can now write (3.2a) as

$$\left(-(\partial_0 + ieA_0(x^1))^2 + \partial_1^2 + m^2\right)\phi = 0. \tag{3.11}$$

Finally, define the dimensionless parameters

$$\begin{aligned} (t,z) &:= \left(\frac{x^0}{a}, \frac{x^1}{a}\right) \\ \lambda &= ea^2E \\ \varepsilon &= ae \\ m &= a\mu \\ \omega_n &= a\Omega_n, \end{aligned} \tag{3.12}$$

and use the ansatz $\phi(t,z) = \phi_n(z)e^{-i\omega_n t}$, to split (3.11) into the dimensionless mode equations

$$\left((\omega_n - \varepsilon A_0)^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0. \tag{3.13}$$

Each of the modes should be normalised with respect to the inner product

$$(\phi_n, \phi_m) = i\int_0^1 dz\left(\pi_n^*\phi_m^* - \pi_m\phi_n\right), \tag{3.14}$$

with $\pi_n^* = D_0\phi_n$ the canonical conjugate of ϕ. Positive (negative) energy solutions are normalized to $+1$ (-1).

3.2 The external field approximation

To study the relevance of the backreaction of the scalar field we first study the external field approximation, ignoring the backreaction of the scalar field. In this approximation, $A_0(z) = -\lambda\left(z - \frac{1}{2}\right)$, and therefore equation (3.13) takes the form

$$\left(\left[(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right]^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0. \tag{3.15}$$

With the appropriate parameter definitions,

$$x = \frac{1+i}{\sqrt{\lambda}}\left[\omega_n + \lambda\left(z - \frac{1}{2}\right)\right], \quad n = -\frac{m^2}{2\lambda} - 1/2, \tag{3.16}$$

equation (3.15) can be identified with the Weber equation

$$\frac{d^2 D_n(x)}{dx^2} + \left(n + \frac{1}{2} - \frac{1}{4}x^2\right) D_n(x) = 0, \tag{3.17}$$

which is solved by the parabolyc cylinder functions $D_n(x)$ [citation required]. Equation (3.2a) is then solved by

$$\begin{aligned}\phi_n(z) = a_n D_{i\frac{m_a^2}{2\lambda} - \frac{1}{2}}\left(\frac{1+i}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right) \\ + b_n D_{-i\frac{m_a^2}{2\lambda} - \frac{1}{2}}\left(\frac{i-1}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right),\end{aligned} \tag{3.18}$$

where the parameters ω_n, a_n, b_n are to be calculated by applying the corresponding boundary and normalization conditions for the field ϕ_n.

This approximation is unstable in the sense that increasing the external electric field strength λ past a certain critical value λ_c the induced potential diverges and the energy of the lower modes go to the complex realm. This leads to runaway solutions, as the complex exponential in the single frequency ansatz $\phi_n(z)e^{i\omega_n t}$ turns into a real exponential [1]. This instability is shown in figure 3.1, where the energy of the first mode is represented, as λ goes to 0. **[Add more modes to the plot?]**

[1] claims that the screening nature of the Klein-Gordon vacuum is enough to raise the energy of the modes enough in order to avoid these instabilities. However, the vacuum polarization in [1] is not calculated properly [4] and therefore

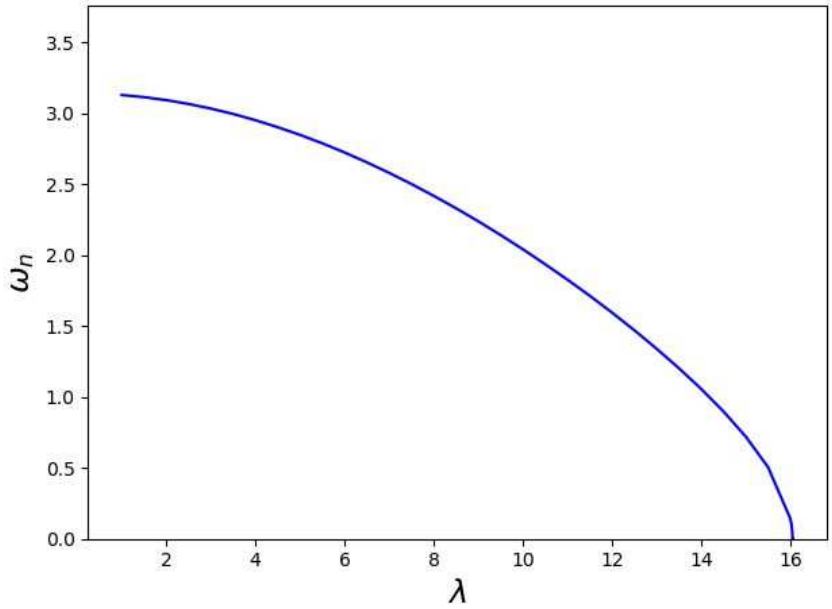

Figure 3.1: The energy of the first mode as the background electric field strength λ is increased, showing the vertical drop to 0 when $\lambda \to \lambda_c$, displating the is the appearance of the instability. **[Show the diverging vacuum polarization as $\lambda \to \lambda_c$]**

the question still stands. As we will see below, in the correct renormalization procedure, the Klein-Gordon vacuum is still screening enough, and the instabilities are therefore avoided.

3.3 Quantization and vacuum polarization

We write the solution to equation (3.11) after changing the variables

$$\phi(t,z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n^*(z) e^{-i\omega_n t}, \tag{3.19}$$

with a_n, b_n the annihilation operators for the positive and negative energy solutions, respectively. These operators obey the commutation relations

$$[a_n, a_m^\dagger] = [b_n, b_m^\dagger] = \delta_{nm}, \tag{3.20}$$

with all other possible commutators vanishing. These operators define the vacuum state $|0\rangle$

$$a_n |0\rangle = b_n |0\rangle = 0. \tag{3.21}$$

The vacuum polarization is calculated as

$$\rho(z) = i\varepsilon \langle 0| \phi^* D_0 \phi - \phi D_0^* \phi^* |0\rangle . \tag{3.22}$$

ϕ is an operator valued distribution. Evaluating products of distributions at the same points is a priori ill-defined, and the expectation value in (3.22) should be calculated as explained in section 2.2, i.e. by defining the expectation value of products of (derivatives of) fields at the same point via point-splitting with respect to the Hadamard parametrix as in (2.16). The Hadamard parametrix $H(x, x')$ of the Klein-Gordon field in 1+1 dimensions up to relevant order* is given by

$$H^{\phi^*\phi}(x, x') = -\frac{1}{4\pi} U(x, x') \ln \sigma_\varepsilon, \tag{3.23}$$

with $\sigma = \frac{1}{2}(x - x')^2$ the geodesic distance between x and x', and $\sigma_\varepsilon = \sigma + i\varepsilon(t - t')$, where the limit $\varepsilon \to 0^+$ is to be taken at the end of the calculation.

The smooth coefficient $U(x, x')$ is calculated by parallely transporting it with respect to D_μ along the geodesic (straight line) joining x and x'. This results in

$$U(x, x') = \exp\left(-i\varepsilon \int_0^1 A_\mu(x' + s(x - x')) \left(x - x'\right)^\mu ds \right). \tag{3.24}$$

In the chosen gauge $A_\mu(t, z) = (A_0(z), 0)$, and performing the point-splitting in the time direction, $x' = (t + \tau, z)$,

$$U(x, x') = \exp\left(-ie\tau A_0(z)\right). \tag{3.25}$$

We calculate the vacuum expectation value of the first term in equation (3.22). In this point-splitting, the two-point function $w_0^{\phi^*\phi}(x, x')$ takes the form

$$w^{\phi^*\phi}(x, x') = \langle 0| \phi^*(x)\phi(x') |0\rangle = \sum_{n<0} |\phi_n(z)|^2 e^{-i\omega_n(\tau + i\epsilon)}, \tag{3.26}$$

where an $i\epsilon$-prescription is used in order to ensure convergence. The derivative of $w^{\phi^*\phi}(x, x')$ with respect to the time coordinate of x' reads

$$D_0' w^{\phi^*\phi}(x, x') = -i \sum_{n<0} (\omega_n - eA_0(z)) |\phi_n|^2 e^{-i\omega_n(\tau + i\epsilon)}, \tag{3.27}$$

*The charge density operator only has first order derivatives, and therefore higher orders in the Hadamard parametrix will vanish after taking the limit $x' \to x$

and of the parametrix $H^{\phi^*\phi}$,

$$D_0' H^{\phi^*\phi}(x,x') = -\frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.28}$$

This implies

$$\langle 0|\,\phi(x)^* D_0\phi(x)\,|0\rangle = \lim_{\tau\to 0} -i\sum_{n<0}(\omega_n - eA_0)|\phi_n(z)|^2 e^{-i\omega_n(\tau+i\epsilon)} + \frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.29}$$

Similarly, for the second term in (3.22),

$$D_0^{*\prime} w^{\phi\phi^*}(x,x') = i\sum_{n>0}(\omega_n - eA_0(z))|\phi_n|^2 e^{i\omega_n(\tau+i\epsilon)}, \tag{3.30}$$

and

$$D_0^{*\prime} H^{\phi\phi^*}(x,x') = -\frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} + \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau), \tag{3.31}$$

since $U^*(x,x') = U(x',x)$. This implies

$$\langle 0|\,\phi(x) D_0\phi(x)^*\,|0\rangle = \lim_{\tau\to 0} i\sum_{n>0}(\omega_n - eA_0)|\phi_n(z)|^2 e^{i\omega_n(\tau+i\epsilon)} + \frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.32}$$

When subtracting these two quantities as per (3.22), the divergent terms cancel each other out, resulting in the following expression for the vacuum polarization

$$\rho(z) = \lim_{\tau\to 0}\left(\sum_{n<0}(\omega_n - A_0)\|\phi_n\|^2 e^{-i\omega_n(\tau+i\varepsilon)} + \sum_{n>0}(\omega_n - A_0)\|\phi_n\|^2 e^{i\omega_n(\tau+i\varepsilon)}\right) + \frac{e^2}{\pi}A_0(z). \tag{3.33}$$

This expression is gauge independent, as long as the summation and the limit are taken in the correct order.

In [1], the vacuum polarization is calculated as

$$\rho(z) = \sum_{n<0}(\omega_n - A_0)\|\phi_n\|^2 + \sum_{n>0}(\omega_n - A_0)\|\phi_n\|^2, \tag{3.34}$$

which even though it is gauge invariant, it cannot be derived from a gauge invariant theory. The crucial difference is the extra $\frac{\varepsilon^2}{\pi}A_0(z)$.

3.3.1 Perturbation theory

[Q: I find that this calculations are necessary for some reasons:

1. **To have an intuition on how the vacuum polarizes on this setup**
2. **To show the difference between the mode sum and the Hadamard point-split formula**
3. **Knowing this difference, that the answer to the question "does backreaction avoid instabilities?" is not obvious.**

However, I am basically copying the results from the bibliography, and I do not want to do that. Should I keep this section? Or just cite it?] Following the steps of [1, 4], it might be instructive to study the problem perturbatively. We state the problem as a Schrödinger-like equation

$$i\partial_t |\Psi\rangle = H |\Psi\rangle \tag{3.35}$$

with

$$|\Psi\rangle = \begin{pmatrix} \phi \\ \pi^* \end{pmatrix}, \text{and } \pi^* = D_0\phi. \tag{3.36}$$

The space of all $|\Psi\rangle$ equipped with the inner product (3.14) forms the phase space of our system.

We can construct the corresponding Hamiltonian operator as

$$H = i\begin{pmatrix} 0 & 1 \\ D_1^2 - m^2 & 0 \end{pmatrix} + \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix} = H_0 + H_1. \tag{3.37}$$

The Hamiltonian H is split into a free field hamiltonian H_0 and consider H_1 to be a perturbation. H is hermitean with respect to the inner product (3.14). The eigenstates Ψ_n of H_0 with eigenvalues ω_n are the solutions to a free particle in a box f size 1. For Dirichlet boundary conditions ($\Psi(0) = \Psi(1) = 0$),

$$\omega_n^D = \text{sign}(n)\sqrt{m^2 + \pi^2 n^2}, \qquad \phi_n^D(z) = |\omega_n|^{-\frac{1}{2}} \sin \pi n z, \quad n \in \mathbb{Z} \setminus \{0\}, \tag{3.38}$$

For Neumann boundary conditions ($\Psi'(0) = \Psi'(1) = 0$) one has

$$\omega_{n\neq 0}^N = \text{sign}(n)\sqrt{m^2 + \pi^2 n^2}, \quad \phi_{n\neq 0}^N(z) = |\omega_n|^{-\frac{1}{2}} \cos \pi n z \tag{3.39}$$

$$\omega_{\pm 0}^N = \pm m, \quad \phi_{\pm 0}^N(z) = (2m)^{-\frac{1}{2}}. \tag{3.40}$$

These eigenstates are the zeroth order correction $\left|\Psi_n^{(0)}\right\rangle$ to the H_0 eigenstates. The first order correction is given by perturbation theory

$$\omega_n^{(1)} = \frac{\left\langle \Psi_n^{(1)} \middle| H_1 \Psi_n^{(1)} \right\rangle}{\left\langle \Psi_n^{(0)} \middle| \Psi_n^{(0)} \right\rangle} \tag{3.41}$$

$$\left|\Psi_n^{(1)}\right\rangle = \sum_{k\neq n} \frac{1}{\left\langle \Psi_n^{(0)} \middle| \Psi_n^{(0)} \right\rangle} \frac{\left\langle \Psi_k^{(0)} \middle| H_1 \Psi_n^{(0)} \right\rangle}{\omega_n^0 - \omega_k^{(0)}} \left|\Psi_k^{(0)}\right\rangle \tag{3.42}$$

After performing the corresponding calculations, one finds that there are no corrections to the eigenvalues ω_n, and that the solutions both for Dirichlet and Neumann boundary conditions (indicated by a D or N superscript, respectively) obtain the form

$$\begin{aligned}\phi_n^D = \left(m^2+\pi^2n^2\right)^{-\frac{1}{2}} \Bigg[& \sin \pi n z \\ & + \lambda \frac{\sqrt{m^2+\pi^2n^2}}{2\pi|n|} \left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin \pi n z - z(1-z)\cos \pi n z \right) \Bigg]\end{aligned} \tag{3.43}$$

$$\begin{aligned}\phi_n^N = \left(m^2+\pi^2n^2\right)^{-\frac{1}{2}} \Bigg[& \cos \pi n z \\ & + \lambda \frac{\sqrt{m^2+\pi^2n^2}}{2\pi|n|} \left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\cos \pi n z + (z(1-z) + (\pi n)^{-2)})\sin \pi n z \right) \Bigg]\end{aligned} \tag{3.44}$$

$$\phi_{\pm 0}^N = (2m)^{-\frac{1}{2}} \mp \lambda\sqrt{2m}\left(\frac{1}{24} - \frac{1}{4}z^2 + \frac{1}{6}z^3\right). \tag{3.45}$$

3.3.2 Vacuum polarization at first order in perturbation theory

Using (3.33) we calculate the vacuum polarization at first order in λ for the massless case using Dirichlet boundary conditions. We rewrite (3.33) as

$$\rho(z) = \lim_{\tau\to 0} \sum_{n=1}^{\infty} \left(|\phi_n|^2(\pi n - \varepsilon A_0) - |\phi_{-n}|^2(\pi n + \varepsilon A_0)\right) e^{i\pi n(\tau+i\epsilon)} + \frac{e^2}{\pi}A_0. \tag{3.46}$$

From (3.43), up to first order in λ,

$$|\phi_n|^2 = \frac{1}{\pi\|n\|}\left(\sin^2 \pi n z - \lambda \sin \pi n z \left[\frac{1}{\pi n}\left(z-\frac{1}{2}\right)\sin \pi n z + z(1-z) cos \pi n z\right]\right). \tag{3.47}$$

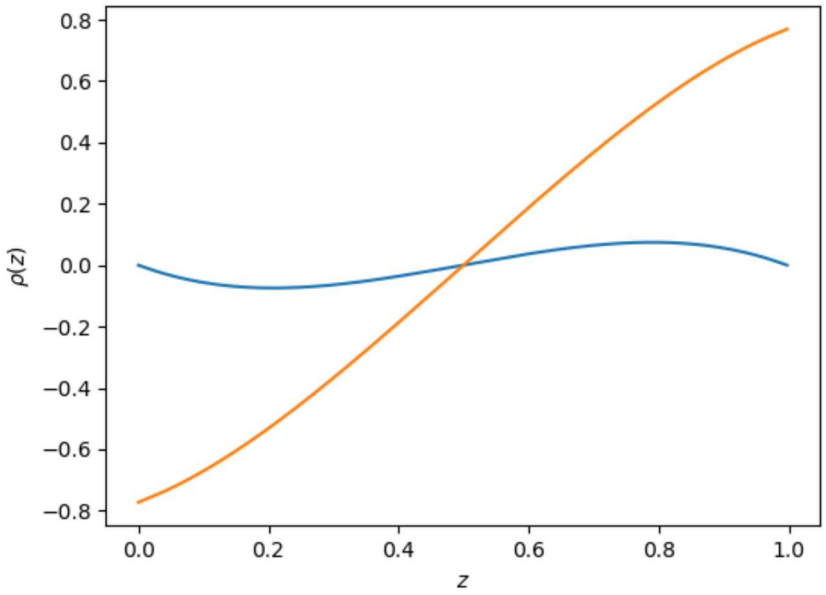

Figure 3.2: The vacuum polarization at first order in λ in Dirichlet boundary conditions. In orange, ρ as calculated using the mode sum in [1], and in blue using point splitting with respect to the Hadamard parametrix as in [4]. **[Here I intend to add a third curve, with the mode sum formula for ρ for $N = 1$, which is the one used in [1].]**

Substituting in (3.46), and with some rearranging,

$$\begin{aligned}\rho(z) &= -2\varepsilon\lambda \lim_{\tau\to 0} \sum_{n=1}^{\infty} z(1-z)\sin \pi n z \cos \pi n z - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right) \\ &= -\varepsilon\lambda z(1-z) \lim_{\tau\to 0} \frac{1}{2i}\left(\frac{e^{i\pi(2z+\tau+i\epsilon)}}{1-e^{i\pi(2z+\tau+i\epsilon)}} - \frac{e^{i\pi(-2z+\tau+i\epsilon)}}{1-e^{i\pi(-2z+\tau+i\epsilon)}}\right) - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right) \\ &= -\varepsilon\lambda z(1-z)\cot \pi z - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right).\end{aligned} \tag{3.48}$$

In [1], the vacuum polarization at first order in λ is (wrongly) calculated as

$$\rho = -2\varepsilon\lambda z(1-z)\cot(\pi z). \tag{3.49}$$

We show in the figure 3.2 the comparison between the two ρ. Notice how the addition of the potential term in (3.33) forces ρ to be 0 at the boundaries, as one would expect for Dirichlet boundary conditions. We also observe that the overall vacuum polarization is much stronger when calculating it using the mode sum formula. The main claim in [1], is that when considerring the back reaction of the

scalar field, the instabilities that appear in the external field approximation disappear. However, since [1] uses a wrong expression to calculate the back-reaction of the scalar field, and the correct expression leads to a much 'weaker' ρ, it is not obvious to say whether the back-reaction of the field is enough to raise the energy of the modes enough so as to avoid the original instabilities.

3.4 The iterative procedure

To include backreaction into the calculation, we proceed iteratively. Following the convention used in [1], we label by κ each of the iterations. For a given A_0^κ, at each step κ we solve numerically the following equations,

$$\partial_1^2 \phi_n^{\kappa+1} = \left(-(\omega_n^{\kappa+1} - \varepsilon A_0^\kappa(z))^2 + m^2\right) \phi_n^{\kappa+1} \tag{3.50}$$

$$\begin{aligned} \varepsilon A_0^\kappa(z) &= -\lambda\left(z - \frac{1}{2}\right) - \varepsilon \int_{\frac{1}{2}}^{z} \int_0^{z'} \rho^\kappa(z'') dz'' dz' \\ \varepsilon A_0^0(z) &= -\lambda\left(z - \frac{1}{2}\right) \end{aligned} \quad . \tag{3.51}$$

with ρ calculated using (3.33), up to a finite mode cutoff in the sum. This cutoff at mode N causes the calculated ρ to be oscillatory, of period $\Delta z = \frac{1}{N+1}$, which should be averaged out.

Finally, we take the limit $\kappa \to \infty$, i.e. solve equation (3.50) for a given potential A_0^κ, calculate the corresponding induced potential at κ, and solve equation (3.50) using this new potential. We look for convergence following this procedure. The computation is halted when convergence with respect to some norm is found. Once a self-consistent solution is found, we use this solution as a guess for some new λ value, greater than the old one. In this way, we sample the whole λ-parameter space to study the interaction between the scalar field and the external electric field.

This procedure can be also be stated in terms of infinite dimensional fixed point problems,

$$A_0 = f(A_0), \tag{3.52}$$

with f the update rule given by equation (3.51). The function f has no closed form, as calculating ρ in (3.51) involves taking all the steps that were explained throughout this section.

3.4.1 Numerical instabilities

Even though we are considering the back-reaction of the field, this problem still presents some instabilities when crossing $\lambda \approx \lambda_c$. When sampling the λ-parameter space, if the step $\Delta\lambda$ is too big, then one of two things might happen:

1. Since we are using the previous solutions as the initial guess to find the self-consistent solution, the screening from the previous solution might not be strong enough causing the energies of some modes to not have solutions in $\mathbb{R}$, or

2. convergence is too slow.

Even though this can in theory be overcome by choosing smaller $\Delta\lambda$, in practice $\Delta\lambda$ can get to the order of 10^{-7}. At $\sim 30s$ per value of λ, this amounts to about 10 years to achieve a step of unit size in λ.

To overcome this, we slightly modify the iterative procedure with a *relaxing* procedure,

$$A_0^{\kappa+1}(z) = cA_0^{\kappa}(z) + (1-c)\left[-\lambda\left(z-\frac{1}{2}\right) - \int_{\frac{1}{2}}^{z}\int_{0}^{z'}\rho^{\kappa}(z'')dz''dz'\right],\ 0 < c \lesssim 1. \tag{3.53}$$

This scheme avoids these instabilities by allowing self consistent solutions to "relax" into one another, i.e. if for a certain λ value a self-consistent solution is found but $\Delta\lambda$ was too big, the change in the potential will not be as strong as it was in the previous scheme. The closer the parameter c is to 1, the slower the convergence, but the more relaxed.