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Fixed point and periodic point theorems

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Abstract. We introduce a weaker form of continuity which is a necessary and sufficient condition for the existence of fixed points. The obtained theorems exhibit interesting fixed point – eventual fixed point patterns. If we slightly weaken the conditions then the mappings admit periodic points besides fixed points and such mappings possess interesting combinations of fixed and periodic points. Our results are applicable to contractive type as well as non-expansive type mappings. Our theorems are independent of almost all the existing results for contractive type mappings. The last theorem of Sect. 2 is applicable to mappings having various geometric patterns as their domain and is perhaps the first result of its type that also opens up scope for the study of periodic points and periodic point structures. We also give an application of our theorem to obtain the solutions of a nonlinear Diophantine equation; and also show that various well-known fixed point theorems are not applicable in solving this equation.

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1. Introduction

Meir and Keeler [18] proved that if a selfmapping f of a complete metric space (X, d) satisfies the condition:

(a) Given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Longrightarrow d(fx, fy) < \varepsilon$$

then f has a unique fixed point.

Rhoades, Park and Moon [35] established the existence of a unique common fixed point for mappings $S, T, \{A_i\} : X \to X$ of a complete metric (X, d) satisfying:

(b) Given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{d(Sx, Ty), d(Sx, A_i x), d(Ty, A_j y), [d(Sx, A_j y) + d(Ty, A_i x)]/2\} < \varepsilon + \delta$$

$\Longrightarrow d(A_i x, A_j y) < \varepsilon.$

Jachymski [15] used a slightly different (ε, δ) condition to obtain a unique common fixed point of two selfmappings A, B of a complete metric space (X, d) satisfying the conditions:

(c) $\varepsilon < \max\{d(x, y), [d(x, Ax) + d(y, By)]/2, [d(Ax, y) + d(By, x)]/2\} < \varepsilon + \delta$
$\Longrightarrow d(Ax, By) \le \varepsilon,$

(d) $d(Ax, By) < \max\{d(x, y), [d(x, Ax)+d(y, By)]/2, [d(Ax, y)+d(By, x)]/2\},$

whenever $\max\{d(x, y), [d(x, Ax) + d(y, By)]/2, [d(Ax, y) + d(By, x)]/2\} > 0$.
Kannan [16,17] proved that if a self-mapping f of a complete metric space (X, d) satisfies:

$$d(fx, fy) \le k[d(x, fx) + d(y, fy)], 0 \le k < 1/2,$$

for each x, y in X then f has a unique fixed point. Chatterjea [9] proved that a self-mapping f of a complete metric space (X, d) satisfying the condition

$$d(fx, fy) \le k[d(x, fy) + d(y, fx)], 0 \le k < 1/2,$$

for each x, y in X has a unique fixed point.
Caristi [7] proved the following:

Theorem 1.1. ([7]). *Let* (X, d) *be a complete metric space and* $f : X \to X$. *If there exists a lower semicontinuous function* $\phi : X \to [0, \infty)$ *such that* $d(x, fx) \le \phi(x) - \phi(fx), x \in X$*, then* f *has a fixed point.*

Caristi's theorem contains the fixed point theorems due to Banach [1], Kannan [16,17], Chatterjee [9] and Ciric [11] as particular cases (see e.g. [28]). The results by Kannan [16,17] and Chatterjea [9] motivated the introduction of a large number of contractive definitions and many of these contractive mappings admit discontinuity in their domain. In 1988 Rhoades [34] examined continuity of a large number of contractive mappings at their fixed points and found that all the contractive definitions studied in [34] force the mapping to be continuous at the fixed point. The question whether there exists a contractive definition which admits discontinuity at the fixed point was listed by Rhoades in [34, p. 242] as an open problem. Pant [23–27] obtained fixed point theorems for contractive mappings which admit discontinuity at the fixed point and resolved the Rhoades problem on continuity at fixed point. Recently some new solutions to the problem of continuity at fixed point have been reported (e. g. Bisht [2], Bisht and Pant [3], Bisht and Rakocevic [4,5], Celik and Ozgur [8], Ozgur and Tas [19,21,22], Pant and Pant [28], Pant et al [31], Pant et al [32], Rashid et al [33], Tas and Ozgur [38], Tas et al [39], Zheng and Wang [42]). In some recent papers applications of such results in the analysis of discontinuous activations of neural networks have been given (e. g. Celik and Ozgur [8], Ozgur and Tas [22], Rashid et al [33], Tas and Ozgur [38], Tas et al [39]). In 1999, Pant [25] initiated the study of the (ε, δ) condition

$$\varepsilon < \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \Longrightarrow d(fx, fy) \le \varepsilon$$

for investigating existence of fixed points and obtained the following theorem in which a contractive mapping admits discontinuity at the fixed point:

Theorem 1.2. (Theorem 1 of [25]) *Let f be a self-mapping of a complete metric space* (X, d) *such that for any* x, y *in* X

(e) $d(fx, fy) \leq \phi(\max\{d(x, fx), d(y, fy)\})$, *where* $\phi : R_+ \to R_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$,

(f) *given* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$

$$\varepsilon < \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.
Pant and Pant [29] employed a slightly modified form of the above (ε, δ) condition together with weaker forms of continuity and obtained the following solution of the Rhoades problem.

Theorem 1.3. (Theorem 2.1 of [29]). *Let f be a self-mapping of a complete metric space* (X, d) *such that*

(g) $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$, *whenever* $\max\{d(x, fx), d(y, fy)\} > 0$,

(h) *Given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon < \max\{d(x, fx), d(y, fy)\} \leq \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

If f is k-continuous or f^k is continuous for some $k \geq 1$ or f is orbitally continuous then f possesses a unique fixed point.

In a recent work Pant et al [31] obtained the first Meir-Keeler type solution of the Rhoades problem on continuity of contractive mappings at the fixed point:

Theorem 1.4. (Theorem 2.1 of [31]) *Let f be a self-mapping of a complete metric space* (X, d) *such that* $d(fx, fy) \leq \max\{d(x, fx), d(y, fy)\}$ *and*

(i) *given* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \leq \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \Longrightarrow d(fx, fy) < \varepsilon.$$

Then f possesses a fixed point if and only if f is weakly orbitally continuous. Moreover, the fixed point is unique and f is continuous at the fixed point, say z, if and only if $\lim_{x\to z} m(x, z) = 0$ or, equivalently, $\lim_{x\to z} \sup d(fz, fx) = 0$.

Theorem 1.4 admits only one fixed point since it assumes $d(fx, fy) \leq \max\{d(x, fx), d(y, fy)\}$. However, the mappings satisfying Theorems 1.2 and 1.3 admit unique fixed point as well as multiple fixed points although Theorems 1.2 and 1.3 focussed on unique fixed point only. In the present paper we study the (ε, δ) conditions (f) and (h) of Theorems 1.2 and 1.3 respectively in detail and also introduce a weaker form of continuity which is a necessary and sufficient condition for the existence of a fixed point of mappings satisfying these conditions. The obtained theorems are independent of the fixed point theorems for contractive mappings except the Kannan fixed point theorem [16,17] and exhibit interesting fixed point – eventual fixed point patterns. We also show that if we further weaken the (ε, δ)-conditions then the mappings admit periodic points besides fixed points. Such mappings possess interesting combinations of fixed and periodic points and if there is no fixed point then each point is a periodic point. Our results are applicable to contractive type mappings as well as mappings that do not satisfy a contractive condition. We now give some relevant definitions.

Definition 1.5. A function f: $X \to Y$ is called continuous if $\lim_{n\to\infty} fx_n = ft$ whenever $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} x_n = t$.

Definition 1.6. ([29]) A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $\lim_{n\to\infty} f^k x_n = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f^{k-1} x_n = t$.

Continuity of f^k and k-continuity of f are independent conditions when $k > 1$ and continuity $\Longrightarrow$ 2-continuity $\Longrightarrow$ 3-continuity $\Longrightarrow \ldots$, but not conversely (see [29]). Moreover, 1- continuity is continuity. We now generalize the notions of continuity and k-continuity and introduce:

Definition 1.7. A function f: $X \to Y$ will be called asymptotically k-continuous (or equivalently, asymptotically continuous) if $\lim_{k,n\to\infty} f(f^k x_n\} = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{k,n\to\infty}(f^k x_n) = t$.

In this paper, we will use the terms asymptotic k-continuity and asymptotic continuity interchangeably.

Example 1.8. Let $X = [0, \infty)$ equipped with Euclidean metric. Define $f : X \to X$ by

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = x/3 \text{ if } x > 1.$$

Then f is neither continuous nor k-continuous for any k. To see this let us consider the sequence $\{x_n : n = 1, 2, 3, \ldots\}$ in X defined by $x_n = 3^{k-1} + 1/(3n)$, k a natural number > 1. Then $\lim_n f^{k-1} x_n = \lim_n(1 + (1/(n3^k))) = 1$ and $\lim_n f^k x_n = \lim_n((1/3) + (1/(n3^{k+1}))) = 1/3 \neq f1$. This shows that f is not k-continuous for any k. This also shows that f^k is not continuous since $\lim_n x_n = 3^{k-1}, \lim_n f^k x_n = 1/3$ while $f^k 3^{k-1} = 1$. Now, for each x in X we have $\lim_{k\to\infty} f^k x = 1$. This implies that for any sequence $\{x_n\}$ in X we have $\lim_{k,n\to\infty} f^k x_n = 1$ and $\lim_{k,n\to\infty} f(f^k x_n) = 1 = f1$, that is, f is an asymptotically continuous (equivalently, asymptotically k-continuous) mapping.

Definition 1.9. ([10,11]) If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i ff^{m_i} x$.

Continuity implies orbital continuity but not conversely [10,11].

Definition 1.10. ([31]) A self-mapping f of a metric space (X; d) is called weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i} y = u \Longrightarrow \lim_i ff^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.

Definition 1.11. ([12,13]). If f is a self-mapping of a set X then a point x in X is called a periodic point of period n if $f^n x = x$. The least positive integer n for which $f^n x = x$ is called the prime period of x. The set of all iterates of a periodic point forms a periodic orbit which equals $\{x, fx, f^2x, \ldots, f^{n-1}x\}$ if x is a periodic point of prime period n. A point x is called eventually periodic of period k if there exists $N > 0$ such that $f^{n+k}(x) = f^n(x)$ for $n \geq N$.

Definition 1.12. ([12,13]). If f is a self-mapping of a set X then a point x in X is called an eventually fixed point of f if there exists N such that $f^{n+1}(x) = f^n(x)$ for $n \geq N$. If $fx = x$ then x is called a fixed point of f.

Definition 1.13. ([12,13]). Let f be a selfmapping of the set of real numbers and p be a periodic point of prime period n. The point p is called hyperbolic if, $|(f^n)'(p)| \neq 1$. The point p is called non-hyperbolic if, $|(f^n)'(p)| = 1$. Here $(f^n)'(p)$ denotes the derivative of $f^n(x)$ at p.

Definition 1.14. The function $f : (-\infty, \infty) \to (-\infty, \infty)$ such that f(x) is the least integer not less than x is called the least integer function or the ceiling function and is denoted by $f(x) = \lceil x \rceil$.

2. Main results

In the following theorems, we shall denote $m(x, y) = \max\{d(x, fx), d(y, fy)\}$.

Theorem 2.1. *Let* f *be a self-mapping of a complete metric space* (X, d). *Suppose that given* $\varepsilon > 0$ *there exists* $0 < \delta(\varepsilon) < \varepsilon$ *such that for each* x, y *in* X

(i) $\varepsilon < m(x, y) \leq \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon$, *and*
(ii) $\varepsilon - \delta < m(x, y) \leq \varepsilon \Longrightarrow d(fx, fy) \leq \varepsilon - \delta$.

Then either f has a unique fixed point and for each x in X the sequence of iterates $\{f^n x\}$ converges to the fixed point, or each x in X is a fixed point or an eventually fixed point. The mapping f possesses a unique fixed point if and only if $x \neq y \Longrightarrow \max\{d(x, fx), d(y, fy)\} > 0$.

Proof. If each x in X is a fixed point of f then the conditions (i) and (ii) are satisfied since there exists no pair (x, y) in X that violates (i) and (ii). Thus, either each point of X is a fixed point of f or there exists a point x_0 in X such that $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Condition (ii) implies that f cannot have periodic points of prime period ≥ 2. For example, suppose y is a periodic point of prime period $n \geq 2$, that is, $f^n y = y$ but $f^i y \neq f^n y$ for $1 \leq i < n$. Then $\max\{d(y, fy), d(fy, f^2y), \ldots, d(f^{n-1}y, f^n y\} > 0$. Let $\max\{d(y, fy), d(fy, f^2y), \ldots, d(f^{n-1}y, f^n y\} = d(f^{k-1}y, f^k y) = \varepsilon > 0, 1 \leq k \leq n$. If $k = 1$ then $d(y, fy) = \varepsilon$ and

$$\max\{d(y, fy), d(f^{n-1}y, f^n y)\} = \varepsilon > 0.$$

This can also be written as

$$m(y, f^{n-1}y) = m(f^n y, f^{n-1}y) = \max\{d(f^{n-1}y, f^n y), d(f^n y, f^{n+1}y)\} = \varepsilon > 0.$$

Using condition (ii) this yields $d(f^n y, f^{n+1} y) = d(y, fy) < \varepsilon$, a contradiction. Similarly, if $k > 1$ then $d(f^{k-1} y, f^k y\} = \varepsilon$ and

$$\begin{aligned} m(f^{k-2} y, f^{k-1} y) &= \max\{d(f^{k-2} y, f^{k-1} y), d(f^{k-1} y, f^k y)\} = d(f^{k-1} y, f^k y) \\ &= \varepsilon > 0. \end{aligned}$$

By virtue of (ii) this gives $d(f^{k-1} y, f^k y) < \varepsilon$, a contradiction. Hence, f cannot have periodic points of prime period ≥ 2 if f satisfies (ii). It may be observed that this conclusion remains true if we replace condition (ii) by any of the following two conditions:

$$\begin{aligned} \max\{d(x, fx), d(y, fy)\} = \varepsilon \quad &\Rightarrow d(fx, fy) < \varepsilon, \\ \max\{d(x, fx), d(y, fy)\} = \varepsilon \quad &\Rightarrow d(fx, fy) > \varepsilon. \end{aligned} \tag{2.1}$$

The above computation shows that f can possess periodic points only if the set $\{y \in X : \max\{d(f^{k-2} y, f^{k-1} y), d(f^{k-1} y, f^k y)\} = d(f^{k-1} y, f^k y) > 0, k \geq 2\}$ is nonempty, that is, condition (ii) will not hold in that case and some weaker condition, say,

$$d(fx, fy) \leq m(x, y), x, y \in X, \tag{2.2}$$

will hold. Also, if

$$\begin{aligned} \{y \in X : \max\{d(f^{k-2} y, f^{k-1} y), d(f^{k-1} y, f^k y)\} &= d(f^{k-1} y, f^k y) > 0, k \geq 2\} \\ &= X \end{aligned} \tag{2.3}$$

then each point of X will be a periodic point of period ≥ 2.

Therefore, condition (ii) implies that either $x_n = x_{n+1}$ for some $n \geq 1$ or $x_n \neq x_{n+1}$ for each $n > 0$. If $x_n = x_{n+1}$ for some $n \geq 1$ then $f^n x_0 = f^{n+1} x_0$, that is, x_n is a fixed point of f and x_0 is an eventually fixed point. On the other hand, if $x_n \neq x_{n+1}$ for each n, then x_0 is neither a fixed point nor an eventually fixed point. This implies that either each point in X is a fixed point or an eventually fixed point, or there exists a point x_0 in X such that the sequence of iterates $\{f^n x_0\}$ consists of distinct points. Suppose the sequence of iterates $\{f^n x_0\}$ of x_0 consists of distinct points. Then $d(x_n, fx_n) > 0, d(x_{n+1}, fx_{n+1}) > 0$ and there exists $\varepsilon > 0$ such that

$$\max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} = \varepsilon.$$

By virtue of (ii) the last equation implies $d(fx_n, fx_{n+1}) \leq \varepsilon - \delta(\varepsilon)$, that is, $d(x_{n+1}, x_{n+2}) \leq \varepsilon - \delta$. Thus, $d(x_n, x_{n+1}) = \varepsilon$ and $d(x_{n+1}, x_{n+2}) \leq \varepsilon - \delta$. Therefore, $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence and, hence, tends to a limit $r \geq 0$. Suppose $r > 0$. Then there exists positive integer N such that

$$n \geq N \Longrightarrow r < d(x_n, x_{n+1}) < r + \delta(r), 0 < \delta(r) < r. \tag{2.4}$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$, which by virtue of (i) yields $d(fx_n, fx_{n+1}) \leq r$, that is, $d(x_{n+1}, x_{n+2}) \leq r$. This contradicts (2.4). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Since $\{d(x_n, x_{n+1})\}$ is a strictly decreasing, for each $p \geq 1$ we get:

$$\max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} = d(x_{n-1}, x_n) > 0.$$

Let $\max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\} = \varepsilon > 0$, that is, $d(x_{n-1}, x_n) = \varepsilon > 0$. Using (ii), we get $d(fx_{n-1}, fx_{n+p-1}) \leq \varepsilon - \delta(\varepsilon)$, that is, $d(x_n, x_{n+p}) < d(x_{n-1}, x_n)$. Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. Hence $\{x_n\} = \{f^n x_0)$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n = f^n x_0 \to z$. Also, $f^p x_n \to z$ for each $p > 0$. We claim that $z = fz$. If not, suppose $z \neq fz$ and $d(z, fz) = \varepsilon$. Then for sufficiently large values of n we get

$$m(x_n, z) = \max\{d(x_n, fx_n), d(z, fz)\}) = d(z, fz) = \varepsilon. \tag{2.5}$$

Using condition (ii) the above inequality yields $d(fx_n, fz) \leq \varepsilon - \delta(\varepsilon)$. Making $n \to \infty$ we get $d(z, fz) \leq \varepsilon - \delta$, which contradicts (2.5). Hence $z = fz$ and z is a fixed point of f. Now, suppose that z and u are fixed points of f, that is, $z = fz$ and $u = fu$. Then

$$\max\{d(x_n, fx_n), d(u, fu)\} = d(x_n, fx_n) > 0.$$

By virtue of (ii), this implies $d(fx_n, fu) < d(x_n, fx_n)$. Taking limit as $n \to \infty$ we get $d(z, fu) = 0$, that is, $d(z, u) = 0$. Hence $z = u$ and z is the unique fixed point of f. Moreover, if $y(\neq z)$ is any point in X then, by using (i) and (ii) it follows that $f^n y \to z$ as $n \to \infty$.

The above analysis implies that if for some x_0 in X the sequence of iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point, say z, and $\lim_{n\to\infty} f^n y = z$ for each y in X. It further implies that if there does not exist an x_0 in X such that $\{f^n x_0\}$ consists of distinct points, then each point of X is either a fixed point or an eventually fixed point. Examples 2.7 and 2.8 given below show that existence of a point x_0 such that $\{f^n x_0\}$ consists of distinct points is not necessary for the existence of a unique fixed point. Under the assumptions of this theorem, it is easy to verify that the condition:

$$x \neq y \Longrightarrow \max\{d(x, fx), d(y, fy)\} > 0$$

is a necessary and sufficient condition for the existence of a unique fixed point. This proves the theorem. $\square$

We now generalise Theorem 2.1 to include mappings that satisfy weaker conditions than (ii).

Theorem 2.2. *Let f be a self-mapping of a complete metric space* (X, d) *such that*

(iii) $d(fx, fy) < m(x, y)$, *whenever* $m(x, y) > 0$,
(iv) *Given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon < m(x, y) \leq \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

Then f has a unique fixed point or each x in X is a fixed point or an eventually fixed point if and only if f is asymptotically k-continuous. If f has a unique fixed point, then for each x in X the sequence of iterates $\{f^n x\}$ converges to the fixed point.

Proof. If each x in X is a fixed point of f then the conditions (iii) and (iv) are satisfied trivially since there exists no pair (x, y) in X that violates (iii) and (iv). Thus, either each X is a fixed point of f or there exists some point which is not a fixed point. Suppose x_0 is not a fixed point of f, that is, $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. By virtue of (2.1), condition (iii) ensures that f does not possess periodic points of prime period ≥ 2. Therefore, either $x_n = x_{n+1}$ for some $n \geq 1$ or $\{x_n = f^n x_0\}$ consists of distinct points. If $x_n = x_{n+1}$ for some $n \geq 1$ then $x_n = f^n x_0$ is a fixed point and x_0 is an eventually fixed point. If $x_n \neq x_{n+1}$ for each n, then $d(x_n, fx_n) > 0$ for each n, and

$$\max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} > 0.$$

Using (iii) this gives $d(fx_n, fx_{n+1}) < d(x_n, x_{n+1})$. Therefore, $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence. Now, proceeding on the lines of the proof of Theorem 2.1 it follows that $\{x_n\} = \{f^n x_0)$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n = f^n x_0 \to z$. Also, $\lim_{n\to\infty} f^k x_n = z$ for each integer $k \geq 1$ and $\lim_{k,n\to\infty}(f^k x_n) = z$. Moreover, using (iii), for each x in X we have $\lim_{n\to\infty} d(f^n x_0, f^n x) = 0$, that is, $\lim_{n\to\infty} f^n x = z$. Suppose that f is asymptotically continuous. Then, since $\lim_{k,n\to\infty} f^k x_n = z$, asymptotic continuity implies $\lim_{k,n\to\infty} f(f^k x_n) = fz$. This implies $z = fz$ since $\lim_{k,n\to\infty} f(f^k x_n) = \lim_{k,n\to\infty} f^{k+1} x_n = z$. Therefore, z is a fixed point of f. If z and u are fixed points of f, then using (iii) it follows that $\lim_{n\to\infty} d(f^n x_0, f^n u) = 0$, that is, $d(z, u) = 0$. Hence, z is the unique fixed point of f.

If there does not exist an x_0 in X such that $x_n \neq x_{n+1}$ for each n, then each point of X is either a fixed point or an eventually fixed point. In this case the mapping f is obviously asymptotically continuous.

Conversely suppose that a mapping f satisfying (iii) and (iv) either has a unique fixed point or each x in X is a fixed point or an eventually fixed point. First suppose that f has a unique fixed point, say z. Then, by virtue of (iii) and (iv), for each x in X the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} f^n x = z = fz$. If $\{x_n\}$ is any sequence in X then for each n we have $\lim_{k\to\infty} f^k x_n = z$ and $\lim_{k,n\to\infty} f(f^k x_n) = z = fz$. This implies that f is asymptotically continuous. On the other hand, if each point of X is either a fixed point or an eventually fixed point then f is obviously asymptotically continuous. This completes the proof of the theorem. □

Example 2.3. Let (X, d) be a metric space and $f : X \to X$ be defined by $fx = x$ for each x in X. Then for any x, y in X we have

$$\max\{d(x, fx), d(y, fy)\} = \max\{d(x, x), d(y, y)\} = 0.$$

This implies that there exists no pair of points (x, y) in X which violates conditions (i) and (ii). Hence conditions (i) and (ii) of Theorem 2.1 are trivially satisfied and each x in X is a fixed point of f.

Example 2.4. Let $X = (-\infty, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ by $fx = 0$, for each x in X.

Then f satisfies the conditions of Theorems 2.1 and 2.2, possesses a unique fixed point $x = 0$ and every nonzero x is an eventually fixed point since $fx = f^2x$ if $x \neq 0$.

Example 2.5. Let $X = \{\pm 4^n : n = 0, 1, 2, 3, \ldots\}$ and d be the usual metric. Let $f : X \to X$ be the signum function $fx = \text{sgn } x$ defined as

$$fx = 1 \text{ if } x > 0, \qquad f0 = 0, \qquad fx = -1 \text{ if } x < 0.$$

Then f is a continuous mapping which satisfies all the conditions of Theorems 2.1 and 2.2, has two fixed points $x = \pm 1$ and all other points are eventually fixed.
Examples 2.3 and 2.5 show that a mapping satisfying the conditions of Theorems 2.1 and 2.2 may not satisfy any contractive condition.

Example 2.6. Let $X = \{re^{i\theta} : 0 \leq \theta \leq 2\pi, r = 1, 4, 4^2, \ldots\}$ be the self-similar family of circles, each lying within larger cercles having radii in geometric progression, in the XY-plane and let d be the usual metric. Define $f : X \to X$ by

$$f(re^{i\theta}) = \lceil r/4 \rceil e^{i\theta}$$

where $\lceil x \rceil$ denotes the least integer not less than x. Then f is a continuous mapping which satisfies all the conditions of Theorem 2.1 and has the points on the unit circle as fixed points, that is, the unit circle is a fixed circle of f. The study of fixed circles is presently an active area of research (e. g. see [14, 19–22, 36, 39]).

Example 2.7. Let $X = [0, 2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by

$$fx = 1 \text{ if } x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Then f satisfies the conditions of the Theorem 2.1, has a unique fixed point $x = 1$, every other point is an eventually fixed point since $f^2x = f^3x$ for each $x \neq 1$, and for each x in X the sequence of iterates $\{f^nx\}$ obviously converges to the fixed point as $n \to \infty$. The mapping f satisfies conditions (i) and (ii) with $\delta(\varepsilon) = (\varepsilon - 1)/2$ if $\varepsilon > 1, \delta(1) = 1/2, \delta(\varepsilon) = (1 - \varepsilon)/2$ if $1/2 \leq \varepsilon < 1$ and $\delta(\varepsilon) = \varepsilon/2$ if $\varepsilon < 1/2$; and satisfies the contractive condition $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ whenever $\max\{d(x, fx), d(y, fy)\} > 0$. The mapping f is discontinuous at the fixed point and, therefore, the above theorem contains solutions to Rhoades problem [34] on continuity of contractive mappings at the fixed point. Also, f is a 2-continuous mapping and, hence, an asymptotically continuous mapping that satisfies the conditions of Theorem 2.2.

Example 2.8. Let $X = [0, \infty)$ and let d be the Euclidean metric. Define $f : X \to X$ by

$$fx = 1 \text{ if } x \leq 1, \qquad fx = x/3 \text{ if } x > 1.$$

This example is Example 1.8 of Sect. 1. Hence, as shown in Sect. 1, f is an asymptotically continuous mapping. Also, f satisfies the conditions of Theorem 2.2 and has a unique fixed point $x = 1$ at which f is discontinuous. This example provides a new solution of the Rhoades problem [34] in the form of an asymptotically continuous mapping. It is easy to see that each $y \neq 1$ is an eventually fixed point and $f^N y = f^{N+1} y = 1$ for some integer $N = N(y) \geq 1$. The mapping f satisfies the (ε, δ) condition with $\delta(\varepsilon) = 2/3 - \varepsilon$ if $\varepsilon < 2/3, \delta(\varepsilon) = 2 - \varepsilon$ if $2/3 \leq \varepsilon < 2$ and $\delta(\varepsilon) = \varepsilon$ if $\varepsilon \geq 2$.

Example 2.9. Let X denote the set of nonnegative real numbers equipped with the Euclidean metric. Each real number y in X can be written as $y = 10^3 a_3 + 10^2 a_2 + 10 a_1 + a_0$ where a_1, a_2, a_3 are nonnegative integers and $0 \leq a_0 < 10$. Let $p \geq 1$ and $0 \leq q \leq 9$ be given integers and let $x = 10^3 p + q$. Corresponding to $x = 10^3 p + q$ let us define a self-mapping f_x on X such that

$$\begin{aligned} f_x(y) &= f_x(10^3 a_3 + 10^2 a_2 + 10 a_1 + a_0) \\ &= |(10^2 a_2 + 10 a_1 + a_0)p - a_3 q|, \end{aligned} \tag{2.6}$$

where a_1, a_2, a_3 are nonnegative integers and $0 \leq a_0 < 10$. These mappings are called Multiple Lowering Mappings of Order 3 (see [30]). Using Remark 2.4 in [30] and Theorem 3 of Pant [28], it follows that: (i) f_x is a chaotic mapping when $x \geq 2000$ and (ii) for parameter values x satisfying $1000 \leq x < 2000$ the mappings f_x possess non-hyperbolic fixed points and each y in [0, 2000] is either a fixed point or an eventually fixed point.

Let us now consider the multiple lowering mapping $f_x, x = 1000$ corresponding to $p = 1$ and $q = 0$. Then $f_x = f_{1000}$ possesses non-hyperbolic fixed points and it is easy to show that f_{1000} is a 2-continuous mapping. f_{1000} satisfies conditions (iii) and (iv) of Theorem 2.2 with $\delta(\varepsilon) = \varepsilon$ for $\varepsilon \geq 1000$ and $\delta(\varepsilon) = (1000 - \varepsilon)/2$ for $\varepsilon < 1000$. Thus, f_{1000} satisfies all the conditions of Theorem 2.2 and each point in X is either a fixed point or an eventually fixed point. However, f_{1000} does not satisfy condition (ii) of Theorem 2.1.

We now give an example of a mapping that satisfies conditions (iii) and (iv) of Theorem 2.2 but does not have a fixed point; this shows that asymptotic continuity is necessary for the existence of a fixed point.

Example 2.10. Let $X = [0, 2]$ and d be the Euclidean metric. Define $f : X \to X$ by

$$fx = (2 + x)/3 \text{ if } x < 1, \qquad fx = 0 \text{ if } x \geq 1.$$

Then f is not asymptotically continuous, satisfies the conditions (iii) and (iv) of Theorem (2.2) and does not have a fixed point. f satisfies condition (iv) with $\delta(\varepsilon) = \min(\varepsilon, 1 - \varepsilon)$ if $\varepsilon < 1$ and $\delta(\varepsilon) = \varepsilon$ when $\varepsilon \geq 1$.

Example 2.11. Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by

$$fx = x - [x], [x] \text{ being the greatest integer } \leq x.$$

Then f is a 2-continuous mapping that satisfy es the conditions of Theorem 2.2. Each x in the interval [0, 1) is a fixed point while each y in [1, 2] is an eventually fixed point since $f^2 y = fy$.

Example 2.12. Let $\{n_i, i \geq 1\}$ be the Fibonacci type sequence given by

$$n_1 = 1, n_2 = 5, n_i = 6n_{i-1} - n_{i-2} \text{ for } i \geq 3.$$

This gives $n_3 = 29$, $n_4 = 169$, $n_5 = 985$, $n_6 = 5741, \ldots$. Let $X = [1, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ by

$$fn_i = n_{i-1} \text{ if } i \geq 2, \qquad fx = 1 \text{ otherwise.}$$

Then f is a nonlinear mapping that satisfies all the conditions of Theorem 2.2 with $\delta(\varepsilon) = \varepsilon/3$ and has a unique fixed point at 1. If we take $x = 29$ and $y = 140$ then $f(x + y) = 29$ while $f(x) + f(y) = 5 + 0 = 5$ showing that f is nonlinear.

Example 2.13. Let $\{m_i, i \geq 1\}$ be the Fibonacci type sequence given by

$$m_1 = 0, m_2 = 3, m_i = 6m_{i-1} - (m_{i-2} - 2) \text{ for } i \geq 3.$$

This gives $m_3 = 20$, $m_4 = 119$, $m_5 = 696$, $m_6 = 4059, \ldots$. Let $X = [0, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ such that $fm_i = m_{i-1}$ if $i \geq 2$, $fx = 0$ otherwise. Then, as in Example 2.12, f is a nonlinear mapping that has a unique fixed point at 0.

In the proof of Theorem 2.1 we have seen that if we replace condition (ii) by a weaker condition like (2.2), that is, $d(fx, fy) \leq m(x, y)$, then f may possess periodic points also. We now modify conditions (ii) and (iii) of Theorem 2.1 and Theorem 2.2 respectively to include mappings which admit periodic points besides fixed points. In the following, by a periodic point we shall mean a periodic point of prime period > 1.

Theorem 2.14. (The (ε, δ) Fixed and Periodic Points Theorem). *Let* f *be a k-continuous or asymptotically continuous self-mapping of a complete metric space* (X, d) *such that*

(v) $\max\{d(x, fx), d(y, fy)\} > 0 \Longrightarrow d(fx, fy) \leq \max\{d(x, fx), d(y, fy)\}$, *strict inequality holding if* x, y *are non-periodic,*

(vi) *Given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon < \max\{d(x, fx), d(y, fy)\} \leq \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

Then either (a) each x in X is a fixed point or an eventually fixed point, or (b) if there exists an x_0 in X such that the sequence iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point, say z, and $\lim_{n\to\infty} f^n y = z$ provided $y (\neq z)$ is not a periodic point, or (c) if for each x in X, the sequence of iterates consists of only finitely many distinct points then f may possess periodic points besides fixed points or (d) each point is a periodic point of prime period > 1.

Proof. We first note that if each x in X is a fixed point of f then the conditions (v) and (vi) are satisfied trivially since there exists no pair (x, y) that violates (v) and (vi). Thus, either each x in X is a fixed point of f or there exists some point which is not a fixed point. Suppose x_0 is not a fixed point, that is, $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Then two cases arise: either there exists $n > 1$ such that $x_n = x_i$ for some $i < n$ or for each n we have $x_n \neq x_i$ for $i < n$. We first consider the case

$x_n \neq x_i$ for $i < n, n > 1$. In this case, x_0 is neither an eventually fixed point nor a periodic or eventually periodic point, that is, the sequence $\{x_n = f^n x_0\}$ consists of distinct points. Since $\{f^n x_0\}$ consists of distinct points, using (v) we get $d(f^n x_0, f^{n+1} x_0) < \max\{d(f^{n-1} x_0, f^n x_0), d(f^n x_0, f^{n+1} x_0)\}$. This implies that $\{d(f^n x_0, f^{n+1} x_0)\} = \{d(x_n, x_{n+1}\}$ is a strictly decreasing sequence and, hence, converges to a limit $r \geq 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \geq N \Longrightarrow r < d(x_n, x_{n+1}) < r + \delta(r).$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$. By virtue of (vi) this gives $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq r$, a contradiction. Hence $r = 0$ and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. From this and (v) it follows easily that $\{x_n\} = \{f^n x_0)$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n = f^n x_0 \to z$. Also, $\lim_{n\to\infty} f^k x_n = z$ for each integer $k \geq 1$ and $\lim_{k,n\to\infty} f^k x_n = z$. Suppose f is asymptotically continuous. Since $\lim_{k,n\to\infty} f^k x_n = z$, asymptotic continuity of f implies $\lim_{k,n\to\infty} f(f^k x_n) = fz$. This implies $z = fz$ since $\lim_{k,n\to\infty} f(f^k x_n) = \lim_{k,n\to\infty} f^{k+1} x_n = z$. Therefore, z is a fixed point of f. Now, suppose that z and u are fixed points of f, that is, $z = fz$ and $u = fu$. Then $\max\{d(x_n, fx_n), d(u, fu)\} = d(x_n, fx_n) > 0$. By virtue of (v), this implies $d(fx_n, fu) < d(x_n, fx_n)$. Taking limit as $n \to \infty$ we get $d(z, fu) = 0$, that is, $z = u$ and z is the unique fixed point of f. This implies that if $y(\neq z)$ is not a periodic point then $f^m y = z$ for some $m > 1$ or $\{f^n y\}$ consists of distinct points. If $\{f^n y\}$ consists of distinct points then using (v) and (vi) it follows that $\{f^n y\}$ is a Cauchy sequence and $f^n y \to z$ as $n \to \infty$. Thus, if there exists an x_0 in X such that the sequence iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point z and $\lim_{n\to\infty} f^n y = z$ for each nonperiodic point $y(\neq z)$. This further implies that if $y(\neq z)$ does not satisfy $\lim_{n\to\infty} f^n y = z$ then y is either a periodic point of prime period > 1 or $f^m y = z$ for some integer $m \geq 1$.

Now let us consider the case when, for each x in X, the sequence of iterates consists of only finitely many distinct points. In this case if each point in X is not a fixed point or an eventually fixed point then f possesses periodic points besides fixed points and eventually fixed points. This implies that if f does not possess a fixed point then each point is a periodic point (e.g. Example 2.15 below). If f is k-continuous then the theorem holds since k-continuity implies asymptotic continuity. It is clear from the proof of this theorem that a mapping f satisfying the conditions of Theorem 2.12 is not a chaotic mapping since the sequence of iterates for each x in X either converges to a fixed point or forms a periodic orbit. □

Example 2.15. (Example 2.10 of [29]). Let $X = [-1, 1]$ and d be the usual metric. Define $f : X \to X$ by

$$fx = -|x|\,x, \text{ that is}, fx = -x^2 \text{ if } x \geq 0 \text{ and } fx = x^2 \text{ if } x < 0.$$

Then f satisfies condition (vi) of Theorem 2.13 with $\delta(\varepsilon) = (\sqrt{(\varepsilon/2)}) - (\varepsilon/2)$ if $\varepsilon < 2$ and $\delta(\varepsilon) = \varepsilon$ if $\varepsilon \geq 2$, and also satisfies the condition (v). If we take

$x = 1$ and $y = -1$ then $fx = y, f^2x = x, fy = x, f^2y = y$ and

$$\max\{d(x, fx), d(y, fy)\} = d(fx, fy) = 2.$$

(1, -1) is the only pair of elements in X satisfying such a condition, and all other pairs of points (x, y) satisfy the condition $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$. Thus, as observed in the proof of Theorem 2.14, f has a unique fixed point $x = 0$, two periodic points $y = \pm 1$, and for each $y \neq \pm 1$ we have $\lim_{n\to\infty} f^n y = 0$. This also shows that f does not have eventually fixed points.

Example 2.16. Let $X = [-1, 1]$ and d be the usual metric. Define $f : X \to X$ by $fx = -[|x|]x$, where [y] denotes the greatest integer not greater than x. Then $fx = 0$ if $-1 < x < 1$, $f1 = -1$, $f(-1) = 1$. As in Example 2.15 it is easy to show that f satisfies the conditions of Theorem 2.14, has a unique fixed point $x = 0$ and two periodic points $y = \pm 1$ while every other point is an eventually fixed point.

Example 2.17. Let $X = \{\pm n : n = 1, 2, 3, \ldots\}$ and d be the usual metric. Define $f : X \to X$ by $fx = -x$ for each x in X. Then each x in X is a periodic point of period 2. It can be verified that condition (v) of Theorem 2.14 is satisfied for all pairs (x, y) in X with $\delta(\varepsilon) = (m + 2 - \varepsilon)/2$ if $m \leq \varepsilon < m + 2$ where $m \geq 2$ is any even integer and $\delta(\varepsilon) = (2 - \varepsilon)/2$ for $0 < \varepsilon < 2$. Further, f satisfies the condition $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ for each $y \neq -x$, while for $y = -x$ we have $m(x, y) = 2x = d(fx, fy)$.

Example 2.18. Let X be set of integers equipped with the Euclidean metric. Define $f : X \to X$ by $fx = -x$ for each x in X. Then f possesses a unique fixed point $x = 0$ and every other point is a periodic point of period 2. Also, $\delta(\varepsilon) = (m + 2 - \varepsilon)/2$ if $m \leq \varepsilon < m + 2$ where $m \geq 2$ is any even integer, $\delta(\varepsilon) = (2-\varepsilon)/2$ for $\varepsilon < 2$, $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ for each $y \neq -x$, and for $y = -x$ we have $m(x, y) = 2x = d(fx, fy)$.

Example 2.19. Let A, B, C be the vertices of an equilateral triangle with each side of length 1. Let D be the circumcentre of the triangle ABC and E, F be points on the perpendicular line to the plane of ABC through D such that $DE = DF = 1/2$. Let $X = \{A, B, C, D, F, E]$ and d be the Euclidean metric on X. Define $f : X \to X$ by $fA = B, fB = C, fC = A, fD = D, fE = F, fF = E$. Then f satisfies all the conditions of Theorem 2.14 and has a unique fixed point D, two periodic points E, F of prime period 2 and three periodic points A, B, C of prime period 3.

Remark 2.20. The points in Example 2.19 form a symmetrical geometric pattern showing that Theorem 2.14 is applicable to functions having various geometric patterns as domain. If we extend Example 2.19 on the lines of Example 2.6 by taking the points A, B, C on the unit circle in the XY-plane, take E, F along the Z-axis, add the vertices of larger equilateral triangles in the XY-plane and similarly add points along the Z-axis then we will get a more intricate and interesting geometrical pattern that satisfies the conditions of Theorem 2.14 and has fixed points and periodic points.

Remark 2.21. Theorems 2.1, 2.2 and 2.14 contain the Kannan fixed point theorem as a particular case. These theorems are independent of the theorems due to Banach [1], Boyd and Wong [6], Meir-Keeler [18], Chatterjea [9], Ciric [11], Suzuki [37], Wardowski [40,41] since $fx = 4x/5$ defined on $(-\infty, \infty)$ satisfies the theorems due to Banach [1], Boyd and Wong [6], Meir-Keeler [18], Chatterjea [9], Ciric [11], Suzuki [37] and Wardowski [40,41] but not Theorems 2.1, 2.2, 2.14 while $fx = x$ defined on $(-\infty, \infty)$ satisfies Theorems 2.1, 2.2, 2.14 but not the theorems due to Banach [1], Meir-Keeler [18], Chatterjea [9], Ciric [11], Suzuki [37] and Wardowski [40,41]. Also, $fx = 4x/5$ defined on $(-\infty, \infty)$ satisfies Theorem 1 of Rhoades et al [35] with $S = T =$ identity map, $A_i = f$ for each i, and satisfies Theorem 4.3 of Jachymski [15] but not the Theorems 2.1, 2.2, 2.14. On the other hand, $fx = x$ defined on $(-\infty, \infty)$ satisfies Theorems 2.1, 2.2, 2.14 but not the theorems of Rhoades et al [35] and Jachymski [15]. Theorems 2.1, 2.2, 2.14 are, therefore, independent of Theorem 1 of Rhoades et al and Theorem 4.3 of Jachymski. Thus Theorem 2.14 is independent of almost all the existing fixed point theorems including the Caristi fixed point theorem [7].

Remark 2.22. Theorem 2.1 is also independent of the corresponding Meir-Keeler type Theorem 1.4 of Sect. 1 (Theorem 2.1 of [31]). The function f in Example 2.7 satisfies conditions (i) and (ii) of Theorem 2.1 above with $\delta(\varepsilon) = (\varepsilon - 1)/2$ if $\varepsilon > 1$, $\delta(1) = 1/2$, $\delta(\varepsilon) = (1 - \varepsilon)/2$ if $1/2 \leq \varepsilon < 1$ and $\delta(\varepsilon) = \varepsilon/2$ if $\varepsilon < 1/2$ but does not satisfy condition (i) of Theorem 1.4 (condition (iv) of Theorem 2.1 of [31]). For example, if we take $x = 1$ and $y > 1$ then $d(fx, fy) = 1, 1 < m(x, y) \leq 2$ and, hence, given $\varepsilon = 1$ we cannot find a $\delta(\varepsilon)$ such that

$$1 \leq m(x, y) < 1 + \delta(1) \Longrightarrow d(fx, fy) < 1.$$

Therefore, the function f in Example 2.7 does not satisfy the Meir-Keeler type condition (iv) of Theorem 2.1 of [31]. The mapping f in the next example satisfies the Meir-Keeler condition (iv) of Theorem 2.1 of [31] but does not satisfy condition (ii) of Theorem 2.1 above.

Example 2.23. Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by

$$fx = 1 \text{ if } 0 \leq x \leq 1, \qquad fx = x - 1 \text{ if } x > 1.$$

Then f satisfies the Meir-Keeler contractive condition (iv) of [31] with $\delta(\varepsilon) = 1 - \varepsilon$ if $\varepsilon < 1$ and $\delta(\varepsilon) = \varepsilon$ for $\varepsilon \geq 1$ and has a unique fixed point $z = 1$ at which f is discontinuous. However, f does not satisfy condition (ii) of Theorem 2.1 above. For example if we take $y > x > 1$ then $d(fx, fy) = y - x < 1$ and $m(x, y) = 1$. Therefore, f does not satisfy condition (ii) when $\varepsilon = 1$. Hence f satisfies the Meir-Keeler condition (iv) of [31] but not the condition (ii) of Theorem 2.1 above. These examples prove that the (ε, δ)-Theorem 2.1 above is independent of the corresponding Meir-Keeler type Theorem 2.1 of [31].

3. Applications

Fibonacci numbers are very interesting numbers and these are of particular interest to biologists and physicists because they are observed in various objects and phenomena. Because of the importance and applications of Fibonacci sequences the journal Fibonacci Quarterly is devoted to the studies on Fibonacci sequences. The Pythagorean triples are interesting triplets of nonzero numbers that find a number of applications. In this section we give an application Theorem 2.2 and examples 2.12 and 2.13 to: (a) find the relationship of the Fibonacci type sequences $\{m_i, i \geq 1\}$ and $\{n_i, i \geq 1\}$ introduced by us with Pythagorean triples and (b) find the integral solutions of the nonlinear Diophantine equation

$$X^2 + (X+1)^2 = Y^2.$$

The integral solutions of the equation $X^2+(X+1)^2 = Y^2$ are Pythagorean triples.

Theorem 3.1. *The Fibonacci type sequences* $\{n_i, i \geq 1\}$ *and* $\{m_i, i \geq 1\}$ *of Examples 2.12 and 2.13 yield all the integral solutions of the Diophantine equation* $X^2 + (X+1)^2 = Y^2$.

Proof. Let us select the values of X from the Fibonacci type sequence $\{m_i, i \geq 1\}$ defined in Example 2.13 and the values of Y from the Fibonacci type sequence $\{n_i, i \geq 1\}$ defined in Example 2.12, that is, $\{m_i, i \geq 1\} = \{0, 3, 20, 119, 696, 4059, 23660, 137903, 803760, \ldots\}$ and $\{n_i, i \geq 1\} = \{1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, \ldots\}$. Then we obtain $0^2 + 1^2 = 1^2, 3^2 + 4^2 = 5^2, 20^2 + 21^2 = 29^2, 119^2 + 120^2 = 169^2, 696^2 + 697^2 = 985^2, 4059^2 + 4060^2 = 5741^2, 23660^2 + 23661^2 = 33461^2, 137903^2 + 137904^2 = 195025^2, 803760^2 + 803761^2 = 1136689^2$, that is , $m_i^2 + (m_i + 1)^2 = n_i^2$, $i = 1, 2, 3, \ldots, 9$.

This shows that the first nine elements of the sequences $\{m_i\}$ and $\{n_i\}$ satisfy $m_i^2 + (m_i+1)^2 = n_i^2$ and yield solution of the equation $X^2 + (X+1)^2 = Y^2$. However, the remaining elements of the sequences $\{m_i\}$ and $\{n_i\}$ are defined in the same manner as the first nine elements, it can be verified that we get $m_i^2 + (m_i + 1)^2 = n_i^2$ for $i > 9$ also. This proves the theorem.

We observe that the nonlinear mappings defined in Examples 2.12 and 2.13 satisfy Theorems 2.2 and 2.14 but do not satisfy the well-known theorems due to Banach [1], Boyd and Wong [6], Meir and Keeler [18], Suzuki [37], Wardowski [40, 41] and the extensions and generalizations of these and various other theorems. □

Remark 3.2. Results like Theorem 3.1 can be very helpful in making mathematics teaching interesting at school and pre-university levels and in motivating the students to explore similar properties of natural numbers.

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