Vacuum polarization in 1+1 dimensions

A thesis submitted to Universität Leipzig for the Master's degree Title

Santiago Sanz Wuhl

Institut für Theoretische Physik

January 2025

To my family

Acknowledgements

I'd like to thank the following people:

Abstract

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Contents

1 Introduction **1**

2 Preliminaries **4**

2.1 Constructing the quantum field theory 4

2.2 Hadamard two point functions and point-split renormalization . . . 7

3 Vacuum polarization **10**

3.1 The setup 10

3.1.1 Gauge fixing 11

3.2 The external field approximation 13

3.3 Quantization and vacuum polarization 15

3.3.1 Perturbation theory 17

3.3.2 Vacuum polarization at first order in perturbation theory . . 18

3.4 The iterative procedure 19

3.4.1 Numerical instabilities 20

3.5 Computational details 21

3.5.1 Noise filtering 21

4 Results **22**

4.1 Comparing the charge density from different renormalization schemes 22

4.2 Attempting to retrieve Ambjørn and Wolfram's results 22

4.3 Impact of having different mode cutoffs 22

4.4 Evolution of the induced field as a function of λ 22

4.5 Dirichlet and Neumann boundary conditions for big should behave the same 22

A Computation details **23**

A.1 Numerical methods 23

A.2 Calculating series and limits 23

B Calculating vacuum polarization at first perturbative order **24**

C Calculating vacuum polarization using Hadamard point-splitting renormalization **27**

Bibliography **30**

List of Figures

3.1 The energy of the first mode as the background electric field strength λ is increased, showing the vertical drop to 0 when $\lambda \to \lambda_c$, displaying the is the appearance of the instability. **[Show the diverging vacuum polarization as $\lambda \to \lambda_c$]** 14

3.2 The vacuum polarization at first order in λ in Dirichlet boundary conditions. In orange, ρ as calculated using the mode sum in [1], and in blue using point splitting with respect to the Hadamard parametrix as in [3]. **[Here I intend to add a third curve, with the mode sum formula for ρ for $N = 1$, which is the one used in [1].]** 19

List of Tables

Chapter 1

Introduction

Vacuum polarization is a fundamental phenomenon of quantum field theory (QFT), one of the first theoretical predictions of quantum field theory [**?**][2]. It shows how the vacuum also behaves as a dynamical medium, as a dielectric with permitivity $\epsilon > 1$.

For high enough external electromagnetic fields, the vacuumm will polarize inducing a non zero current density. The dependence of the current density on the applied electromagnetic field introduces non-linear terms in the Maxwell equation, leading to effects such as photon-photon scattering. Other interesting consequences of vacuum polarization is the screening behaviour of the vacuum, as (similarly to the behaviour of a classical dielectric) the polarized vacuum will act as a source for an electromagnetic field opposite to the one imposed. Similarly to how a dielectric breaks under strong enough external electric fields, the Schwinger mechanism predicts the generation of real electron-positron pairs. However, the particles do not have to be generated to observe the non-linear effects of the vacuum.

On another note, recent interests have arisen in the effects of considering boundary conditions for quantum field theory in the context of topological insulators.

More things to talk about

1. The lack of a clearly defined vacumm state both in quantum field theories on Minkowski spacetime coupled to external electromagnetic fields and in free quantum field theories in generic background spacetimes lead to a decent analogy between these two. Studying and understanding the semiclassical Klein-Gordon-Maxwell equation might be a crucial step in understanding the semiclassical Klein-Gordon-Einstein equation, or the influence of the matter fields in the gravitational fields. Understanding semiclassical gravity

might be the first of many steps to understand quantum gravity.

2. The calcualtion of the vacuum polarization involves some ill-defined quantities, namely the product of distributions and thus require some renormalization. [1] do these exact calculations, as one naïvely would and get the wrong results [**?**]

3. In this project, we redo the calculations done in [1], with the correct point-split renormalization w.r.t. a Hadamard parametrix.

4. We study the semiclassical approach to quantum field theory. This approach treats the background as a completely

Notation and conventions

Throughout this thesis, we use natural units $\hbar = c = 1$. We work on flat 1+1 dimensional Minkowski spacetime, with the 'mostly negative' metric (+, -). x will usually denote an event in the spacetime, with x^0, x^1 its time and spatial components, respectively. The partial derivatives with respect to these coordinates will then be

$$\partial_\mu := \frac{\partial}{\partial x^\mu}. \tag{1.1}$$

We also use a different coordinate choice, which will describe the spacetime with coordinates t, z for the time and spatial components, respectively. Similary

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_z := \frac{\partial}{\partial z}. \tag{1.2}$$

Unless noted otherwise, Einstein's summation convention is assumed, i.e.

$$a_\mu b^\mu = \sum_{n=0}^{N} a_\mu b^\mu. \tag{1.3}$$

Chapter 2

Preliminaries

2.1 Constructing the quantum field theory

The free Klein-Gordon equation for a complex field ϕ in two spacetime dimensions is

$$(D_\mu D^\mu + m^2)\phi = 0, \tag{2.1}$$

with $D_\mu = \partial_\mu + ieA_\mu$ the gauge covariant derivative, and $D^\mu = \eta^{\mu\nu} D_\nu$, with η the 2-dimensional Minkowski metric.

The general solution to equation (2.1) is

$$\phi(t,x) = \sum_{n>0} a_n \phi_n(x) e^{-i\Omega_n t} + \sum_{n<0} b_n^* \phi_n(x) e^{-i\Omega_n t}, \tag{2.2}$$

with a_n, b_n complex coefficients depending on n, and ϕ_n solutions to

$$\left[(\Omega_n - eA_0)^2 - \partial_1^2 + m^2\right] \phi_n = 0, \tag{2.3}$$

where the Ω_n are gauge dependent energy parameters that are calculated when applying the boundary conditions ϕ is subjected to.

The associated canonical momentum is calculated as

$$\pi(x) = D_0^* \phi^* = i \sum_{n>0} a_n^* (\Omega_n - eA_0)\phi^*(x) e^{i\Omega_n t} + i \sum_{n<0} b_n (\Omega_n - eA_0)\phi^*(x) e^{i\Omega_n t}, \tag{2.4}$$

and the results are equivalent to the field ϕ^*.

We quantize the field $\phi(x)$ by imposing the same time commutation relation

$$[\phi(t,x), \pi(t,y)] = [\phi^*(t,x), \pi^*(t,y)] = i\delta(x-y) \tag{2.5}$$

The quantization of the fields implies the promotion of the a_n, b_n numbers into operators, and $a_n^* \to a_n^\dagger$. Due to the canonical commutation relations,

$$[a_n, a_m^\dagger] = [b_n, b_m^\dagger] = \delta_{nm}, \tag{2.6}$$

and the annihiliation operators a_n, b_n can be written as

$$a_n = e^{-i\Omega_n t} \sum_{n>0} a_n z^n \tag{2.7}$$

When solved with compactly supported Cauchy data $\left(\phi|_\Sigma, n^\mu D_\mu \phi|_\Sigma\right)$, with Σ a space-like hypersurface (typically a constant-t surface) and n a unit vector orthogonal to Σ, the global hyperbolicity of Minkowski spacetime ensures that this problem is well posed. The collection of all possible smooth ϕ with compactly supported Cauchy data forms the configuration space $\mathcal{C}$. From $\mathcal{C}$ one constructs the phase space $\mathcal{P} = T^*\mathcal{C}$, as its cotangent space. The elements of $\mathcal{P}$ are the points (ϕ, π), with $\pi = D_0^* \phi^*$ the canonical conjugate of the field ϕ. Points in the phase space correspond to initial conditions for the Klein-Gordon equation on the Cauchy surface Σ. $\mathcal{P}$ can be endowed with symplectic structure with the symplectic* form

$$\sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = i \int_\Sigma (\pi_1^* \phi_2^* - \pi_2 \phi_1)\, d\Sigma. \tag{2.8}$$

It is not difficult to prove that σ is a time evolution invariant if ϕ_1, ϕ_2 are solutions to the Klein-Gordon equation, as long as they are compactly supprted. **[Prove this? In appendix?]**

The solution space $\mathcal{S}$ is the space of all smooth solutions to the Klein-Gordon equation. As the initial data fully determines a solution to the Klein-Gordon equation, each solution corresponds to a point in phase space and therefore $\mathcal{S}$ can be identified with $\mathcal{P}$. $\mathcal{S}$ can be given Hilbert space structure as follows:

1. Define the subspace $\mathcal{S}_+$ of positive frequency solutions. In this subspace the symplectic product can be made positive definite.

*Bilinear, antisymmetric

2. Define a Hermitian inner product in this subspace

$$(\phi_1, \phi_2) = i\sigma(\phi_1^*, \phi_2), \quad \phi_1, \phi_2 \in S_+ \tag{2.9}$$

3. Complete $\mathcal{S}_+$ with respect to this norm. **[How to go from this to the whole $\mathcal{S}$?]**

In classical mechanics, observables are the compactly supported functions $f : \mathcal{S} \to \mathbb{C}$. They can be equipped with an algebra structure via the Poisson brackets,

$$\{f, g\} := \int_\Sigma \left(\frac{\delta f}{\delta \phi} \frac{\delta g}{\delta \pi} - \frac{\delta g}{\delta \phi} \frac{\delta f}{\delta \pi} \right) d\Sigma. \tag{2.10}$$

First quantisation is done by the $\hat{}$ operator, which elevates classical observables f to operators $\hat{f}$ on the $\mathcal{S}$ obeying the algebraic rule

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}, \tag{2.11}$$

with $\widehat{\{f, g\}}$ the quantized version of the Poisson bracket.

The field observables are defined with respect to smooth compactly supported test functions f

$$\phi(f) = a(f)e^{-i\omega t} + a^\dagger(\bar{f})e^{i\omega t}, \tag{2.12}$$

where the a and $a^\dagger$ operators are defined the same way the are constructed in the quantum harmonic oscillator problem.

The measured observables of the field in a state $\Omega \in \mathcal{F}(\mathcal{S})$ are therefore the expectation value $\langle \Omega | \phi(f) | \Omega \rangle$. Through $\phi(f)$ one can define the field as the weighted average by the test function f of the spacetime distribution $\phi(x)$ through

$$\phi(f) = \int_{\mathbb{R}^n} \phi(x) f(x) d^n x, \tag{2.13}$$

with $\phi(x)$ an operator valued distribution. This also formally defines the two-point function as the distributional kernel of

$$w_\Omega^{\phi\phi^*}(f, h) = \langle \Omega | \phi(f)\phi^*(g) | \Omega \rangle = \int_{\mathcal{M}\times\mathcal{M}} f(x)g(x')w_\Omega^{\phi\phi^*}(x, x')dxdx'. \tag{2.14}$$

2.2 Hadamard two point functions and point-split renormalization

The above mentioned construction of QFT is incomplete in the sense that it fails to describe the product of fields at same spacetime points. In particular, we are interested in calculating the following expression

$$\rho(x) = \langle \hat{\rho}(x) \rangle_\Omega = ie \langle \Omega | \phi(x)^* D_0 \phi(x) - \phi(x) D_0^* \phi(x)^* | \Omega \rangle , \tag{2.15}$$

wich is a priori ill-defined and needs to be renormalized.

The direct evaluation of (2.15) is divergent, even for the vacuum, due to the terms $a_n a_n^\dagger$, $b_n b_n^\dagger$ found in the mode expansion of the product of the fields **[Mention the mode expansion here?]**. The usual procedure in Relativistic Quantum Field Theory is to define $\langle \rho(x) \rangle_\Omega$ through normal ordering,

$$\langle : \hat{\rho}(x) : \rangle_\Omega := \langle \Omega | \hat{\rho}(x) | \Omega \rangle - \langle 0 | \hat{\rho}(x) | 0 \rangle , \tag{2.16}$$

which defines the expectation values with respect to the vacuum state, or the zero-point value.

The normal ordering of an observable $\mathcal{O}(\phi) \rightarrow : \mathcal{O}(\phi) :$ is a crucial tool in Quantum Field Theory in the absence of background electromagnetic fields, when defining products of field operators at coinciding points. This prescription presents the key property $\langle : \mathcal{O}(\phi) : \rangle_0 = 0$, which is not an expected behaviour for the vacuum in the pressence of background electric fields. This prescription also presents the flaw of defining the expectation values using the vacuum as a reference point. In the pressence of more general electromagnetic fields, the definition of a vacuum state lacks uniqueness, rendering the normal ordering prescription for more general backgrounds invalid. Using this prescription also fails at correctly describing the Casimir effect, as it appears purely due to the boundaries, which are not seen in the normal ordering.

In the context of Quantum Field Theory on Curved Spacetimes, these difficulties are circumvented by the following observations

1. Even though the expectation value $\langle \Omega | \phi(x) \phi^*(x) | \Omega \rangle$ is ill-defined, the two point function (2.14) is a well defined bi-distribution, bi-solution to the field equation, divergent as $x' \rightarrow x$.

2. In the normal ordering prescription, the divergences that appear when evaluating expectation values are defined through difference of expectation values between different states. There exists a class of states for which their

difference in the expectation values is also well-defined.

These results, appearing in the study of QFTCS, will still be useful to our case. There is some parallelism to be drawn between electromagnetism and gravitation. In General Relativity, the Levi-Civita connection gives rise to the Christoffel symbols that make the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma^\nu_{\mu\rho}$ covariant. The dynamical equation is an equation on the curvature of the connection, $R_{\mu\nu}$. Similarly, in the case of electrodynamics, the connection over the $U(1)$ fiber bundle gives rise to the covariant derivative $D_\mu + ieA_\mu$, and the dynamical question of interest is Maxwell's equations, $\partial_\mu F^{\mu\nu} = j^\nu$, with $F^{\mu\nu}$ the curvature of the connection. For our specific case, we are mostly interested in the parallelism in between ∇_μ and D_μ.

We are particularly interested in the class of Hadamard states, states which have the same divergence **[Not actually the same divergence structure, since the parallel transport is what gives the very relevant gauge invariance]** structure as the vacuum state of the Klein-Gordon field in the absence of background electromagnetic fields,

$$w_\Omega^{\phi\phi^*}(x,x') = H^{\phi\phi^*}(x,x') + R_\Omega^{\phi\phi^*}(x,x'), \tag{2.17}$$

with $H^{\phi\phi^*}$ the Hadamard parametrix, a divergent bi-distribution independent of the state Ω, resembling the behaviour of the vacuum two-point function in absence of external fields, up to smooth coefficients, and $R_\Omega^{\phi\phi^*}$ a smooth function which does depend on the state.

The Hadamard parametrix in general takes the form

$$H^{\phi\phi^*} = \sum_{k=0}^{N} V_k(x,x')T_k(x,x') \tag{2.18}$$

with $V_k(x,x')$ smooth coefficients, and in $n = 2$ dimensions,

$$T_k(x,x') = -\frac{1}{4\pi k!}(-\sigma)^{-k}\log\frac{\sigma_\varepsilon}{\Lambda^2} \tag{2.19}$$

with

$$\sigma(x,x') = \frac{1}{2}(x-x')^2,\ \sigma_\varepsilon = \sigma + i\varepsilon\text{sign}(x'_0 - x_0). \tag{2.20}$$

It is important to note that the Hadamard parametrix is constructed entirely by geometrical means and is independent of the state.

To find the $V_k(x,x')$ and $W(x,x')$ functions, one imposes that $w(x,x')$ is a bi-

solution to the Klein-Gordon equation, i.e.

$$(D_\mu D^\mu + m^2) w_\Omega^{\phi\phi^*}(x, x') = ({D^*}'_\mu {D^*}'^\mu + m^2) w(x, x') = 0, \tag{2.21}$$

with $D'^*_\mu = \partial'_\mu - ieA_\mu(x')$ acting on x'. With this imposition, one finds a sequence of differential equations, which together with the initial conditions $\lim_{x' \to x} V_0(x, x') = 1$ leads to the family V_k of solutions.

Once $H^{\phi\phi^*}$ is known, one can define the expectation value of fields at coinciding points through point-splitting with respect to the Hadamard parametrix,

$$\left\langle D_\alpha \phi(x) D^*_\beta \phi^*(x) \right\rangle_\Omega = \lim_{x' \to x} D_\alpha D^{*\prime}_\beta \left(w_\Omega^{\phi\phi^*}(x, x') - H^{\phi\phi^*}(x, x') \right). \tag{2.22}$$

Due to the Hadamard form of the two-point function, in the limit of coinciding points, the divergences cancel out, resulting in a well-defined expectation value.

Chapter 3

Vacuum polarization

3.1 The setup

[This first order backreaction calculation should only be valid up to $\varepsilon\lambda \ll 1$ (?). In the problem of a charged particle of charge e colliding with a nucleus of charge Ze, vacuum polarization becomes relevant when the coupling constant becomes comparable to the source. Where to discuss this? What when this approximation isn't true?]

We study the coupling of a massive charged scalar complex field ϕ to a classical external electromagnetic field A_μ in a 1+1 dimensional flat spacetime. The simplest description this admits is the minimal coupling of the scalar field to the background classical gauge field A_μ

$$S[\phi, A_\mu] = \int_{\mathbb{R}^2} dx^0 dx^1 \left(\frac{1}{2} D_\mu \phi D^{*\mu} \phi^* + \mu^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu_{\text{external}} \right) \tag{3.1}$$

[This shows $\mathcal{L}$. Relevant to write the corresponding Hamiltonian density?]

with $D_\mu = \partial_\mu + ieA_\mu$ the gauge covariant derivative given by the minimal coupling to A_μ and e, μ the charge and mass parameter of the Klein-Gordon field, respectively. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic stress-energy tensor and j^μ_{external} is the charge current density of the external source.

From (3.1) one derives through the Euler-Lagrange equations the Klein-Gordon-Maxwell equations, namely the coupled system of differential equations given by the Klein-Gordon equation and the Maxwell equations

$$\left(D_\mu D^\mu + \mu^2 \right) \phi = 0 \tag{3.2a}$$

$$\partial_\mu F^{\mu\nu} = j^\nu_{\text{external}} + j^\nu_{\text{br}}, \tag{3.2b}$$

with

$$j^{\nu}_{\text{br}} = ie\left(\phi^* D_0\phi - \phi D_0^*\phi^*\right) \tag{3.3}$$

the charge current density of the Klein-Gordon field.

This set up attempts to act as a toy model for studying Quantum Field Theory on Curved Spacetimes (QFTCS), which is concerned in the study of quantum fields on fixed background spacetimes. For these fields one can calculate its stress-energy tensor $T_{\mu\nu}$, which acts as a gravitational source. $T_{\mu\nu}$ is not necessary 0 and will therefore have a non-zero contribution to the gravitational field. The semi-classical ansatz is assuming that the contribution of the quantum field to the Einstein field equations is through its expectation value

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = 8\pi\left(T_{\mu\nu} + \left\langle T^{\text{br}}_{\mu\nu}\right\rangle_{\omega}\right), \tag{3.4}$$

in which the field acts as a source for the gravitational field through its expectation value in a state ω. The 'quantum' contributions to the metric are assumed to be irrelevant when the spacetime curvature is less than the Planck scale.

In a similar fashion, we study the semi-classical Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^{\nu}_{\text{external}} + \langle j^{\nu}_{\text{br}}\rangle, \tag{3.5}$$

to see how the back-reaction of the Klein-Gordon field affects the background electromagnetic field.

The background electric field in the region we are studying $[0, a]$ is generated by two charges $q, -q$ placed at the boundaries. Placing the positive charge at the left boundary and the negative charge at the right boundary generates a constant, time-independent electric field $E = q$ pointing towards positive x^1. Equation (3.2a) can be solved with attractive or repulsive boundary conditions, resp. Dirichlet $\phi|_{x^1=0,a} = 0$, Neumann $\partial_1\phi|_{x^1=0,a} = 0$, or Robin boundary conditions $(\partial_1 + h)\,\phi|_{x^1=0,a} = 0$ with the $h \in \mathbb{R}$ appropiately chosen at each boundary. Equation (3.5) need however be solved with Dirichlet boundary conditions.

3.1.1 Gauge fixing

The charge current density in the laboratory frame of reference is $j^\nu = (\rho, 0)$. We therefore choose to solve this problem using Coulomb gauge, $\partial_1 A_1 = 0$. In this gauge, the space component A_1 of the vector potential is 0 up to an addition constant, and we turn our attentions to the time component A_0 of the vector potential.

In the Coulomb gauge, for $x^1 \in (0, a)$, A_0 solves the Poisson equation

$$\partial_1^2 A_0 = -\rho \tag{3.6}$$

Equation (3.6) is to be solved with the boundary conditions

$$\partial_1 A_0|_{x_1=0,a} = -E, \tag{3.7}$$

so that no screening appears in the boundaries. This fixes the solution up to a constant factor, which is to be chosen so that the problem is antisymmetric. It suffices to demand $A_0(\frac{a}{2}) = 0$.

For clarity, we artificially split the full potential A_0 into two: A_0^{external} due to the constant background electric field and A_0^{br} due to the backreaction of the Klein-Gordon field. Hence, (3.6) splits into

$$\begin{aligned} \partial_1^2 A_0^{\text{external}} &= 0 \\ \partial_1 A_0^{\text{external}}\Big|_{x_1=0,a} &= -E, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \partial_1^2 A_0^{\text{br}} &= -\rho \\ \partial_1 A_0^{\text{br}}\Big|_{x_1=0,a} &= 0. \end{aligned} \tag{3.9}$$

This splitting of the potential makes it clear to write the solution to equation (3.6) is the full potential

$$A_0(x^1) = A_0^{\text{external}}(x^1) + A_0^{\text{br}}(x^1) = \tag{3.10}$$

$$-E\left(x^1 - \frac{a}{2}\right) - \int_{\frac{a}{2}}^{x^1} \int_0^{x^{1\prime}} \rho(x^{1\prime\prime}) dx^{1\prime\prime} dx^{1\prime}. \tag{3.11}$$

Knowing how to calculate A_0 (provided we know how to calculate ρ_ϕ) we can now write (3.2a) as

$$\left(-(\partial_0 + ieA_0(x^1))^2 + \partial_1^2 + m^2\right)\phi = 0. \tag{3.12}$$

Finally, define the dimensionless parameters

$$\begin{aligned} (t,z) &:= \left(\frac{x^0}{a}, \frac{x^1}{a}\right) \\ \lambda = ea^2E \qquad & \varepsilon = ae \\ \mathrm{m} = \mathrm{a}\mu \qquad & \omega_n = a\Omega_{n,\alpha} + \lambda\alpha, \end{aligned} \tag{3.13}$$

and use the ansatz $\phi(t,z) = \phi_n(z)e^{-i\omega_n t}$, to split (3.12) into the dimensionless mode equations

$$\left((\omega_n - \varepsilon A_0)^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0. \tag{3.14}$$

Each of the modes is normalised with respect to the inner product

$$(\phi_n, \phi_m) = i\int_0^1 dz\,(\pi_n^*\phi_m^* - \pi_m\phi_n)\,, \tag{3.15}$$

with $\pi_n^* = D_0\phi_n$ the canonical conjugate of ϕ. Positive (negative) energy solutions are normalized to $+1$ (-1).

3.2 The external field approximation

To study the relevance of the backreaction of the scalar field we first study the external field approximation, ignoring the backreaction of the scalar field. In this approximation, $A_0(z) = -\lambda\left(z - \frac{1}{2}\right)$, and therefore equation (3.14) takes the form

$$\left(\left[(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right]^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0. \tag{3.16}$$

With the appropriate parameter definitions,

$$x = \frac{1+i}{\sqrt{\lambda}}\left[\omega_n + \lambda\left(z - \frac{1}{2}\right)\right], \quad n = -\frac{m^2}{2\lambda} - 1/2, \tag{3.17}$$

equation (3.16) can be identified with the Weber equation

$$\frac{d^2D_n(x)}{dx^2} + \left(n + \frac{1}{2} - \frac{1}{4}x^2\right)D_n(x) = 0, \tag{3.18}$$

which is solved by the parabolyc cylinder functions $D_n(x)$ [citation required].

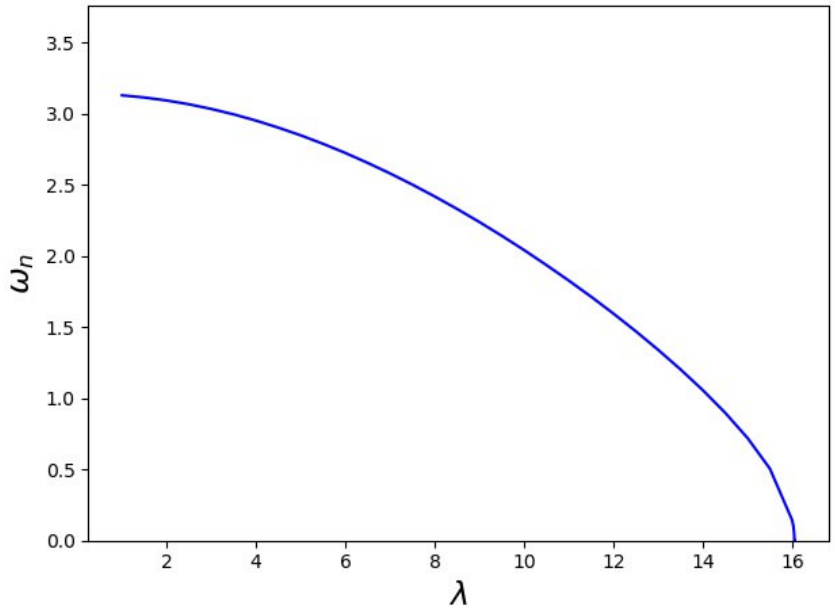

Figure 3.1: The energy of the first mode as the background electric field strength λ is increased, showing the vertical drop to 0 when $\lambda \to \lambda_c$, displaying the is the appearance of the instability. **[Show the diverging vacuum polarization as $\lambda \to \lambda_c$]**

Equation (3.2a) is then solved by

$$\begin{aligned}\phi_n(z) = a_n D_{i\frac{m_a^2}{2\lambda}-\frac{1}{2}}\left(\frac{1+i}{\sqrt{\lambda}}\left(\omega_n+\lambda\left(z-\frac{1}{2}\right)\right)\right) \\ + b_n D_{-i\frac{m_a^2}{2\lambda}-\frac{1}{2}}\left(\frac{i-1}{\sqrt{\lambda}}\left(\omega_n+\lambda\left(z-\frac{1}{2}\right)\right)\right),\end{aligned} \tag{3.19}$$

where the parameters ω_n, a_n, b_n are to be calculated by applying the corresponding boundary and normalization conditions for the field ϕ_n.

This approximation presents an instability in the sense that when increasing the external electric field strength λ past a certain critical value λ_c, the vacuum polarization diverges and the energy of the modes goes into the complex realm. This leads to runaway solutions, as the complex exponential in the single frequency ansatz $\phi_n(z)e^{i\omega_n t}$ turns into a real exponential [1]. This instability is shown in figure 3.1, where the energy of the first mode is represented, as λ goes to λ_c. **[Add more modes to the plot?]**

[1] claims that the screening nature of the Klein-Gordon vacuum is enough to raise the energy of the modes enough in order to avoid these instabilities. However, the vacuum polarization in [1] is not calculated properly [3] and therefore the question still stands. As we will see below, in the correct renormalization pro-

cedure, the Klein-Gordon vacuum is still screening enough, and the instabilities are therefore avoided.

3.3 Quantization and vacuum polarization

We write the solution to equation (3.12) after changing the variables

$$\phi(t,z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n^*(z) e^{-i\omega_n t}, \tag{3.20}$$

with a_n, b_n the annihilation operators for the positive and negative energy solutions, respectively. These operators obey the commutation relations

$$[a_n, a_m^\dagger] = [b_n, b_m^\dagger] = \delta_{nm}, \tag{3.21}$$

with all other possible commutators vanishing. These operators define the vacuum state $|0\rangle$

$$a_n |0\rangle = b_n |0\rangle = 0. \tag{3.22}$$

The vacuum polarization is calculated as

$$\rho(z) = i\varepsilon \langle 0| \phi^* D_0 \phi - \phi D_0^* \phi^* |0\rangle . \tag{3.23}$$

ϕ is an operator valued distribution. Evaluating products of distributions at the same points is a priori ill-defined, and the expectation value in (3.23) should be calculated as explained in section 2.2, i.e. by defining the expectation value of products of (derivatives of) fields at the same point via point-splitting with respect to the Hadamard parametrix as in (2.22). The Hadamard parametrix $H(x,x')$ of the Klein-Gordon field in 1+1 dimensions up to relevant order* is given by

$$H^{\phi^*\phi}(x,x') = -\frac{1}{4\pi} U(x,x') \ln \sigma_\varepsilon, \tag{3.24}$$

with $\sigma = \frac{1}{2}(x-x')^2$ the geodesic distance between x and x', and $\sigma_\varepsilon = \sigma + i\varepsilon(t-t')$, where the limit $\varepsilon \to 0^+$ is to be taken at the end of the calculation.

The smooth coefficient $U(x,x')$ is calculated by parallely transporting it with

*The charge density operator only has first order derivatives, and therefore higher orders in the Hadamard parametrix will vanish after taking the limit $x' \to x$

respect to D_μ along the geodesic (straight line) joining x and x'. This results in

$$U(x,x') = \exp\left(-i\varepsilon \int_0^1 A_\mu(x' + s(x-x'))\,(x-x')^\mu\, ds\right). \tag{3.25}$$

In the chosen gauge $A_\mu(t,z) = (A_0(z),0)$, and performing the point-splitting in the time direction, $x' = (t+\tau, z)$,

$$U(x,x') = \exp\left(-ie\tau A_0(z)\right). \tag{3.26}$$

We calculate the vacuum expectation value of the first term in equation (3.23). In this point-splitting, the two-point function $w_0^{\phi^*\phi}(x,x')$ takes the form

$$w^{\phi^*\phi}(x,x') = \langle 0|\,\phi^*(x)\phi(x')\,|0\rangle = \sum_{n<0} |\phi_n(z)|^2 e^{-i\omega_n(\tau+i\epsilon)}, \tag{3.27}$$

where an $i\epsilon$-prescription is used in order to ensure convergence. The derivative of $w^{\phi^*\phi}(x,x')$ with respect to the time coordinate of x' reads

$$D_0' w^{\phi^*\phi}(x,x') = -i\sum_{n<0}(\omega_n - eA_0(z))|\phi_n|^2 e^{-i\omega_n(\tau+i\epsilon)}, \tag{3.28}$$

and of the parametrix $H^{\phi^*\phi}$,

$$D_0' H^{\phi^*\phi}(x,x') = -\frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.29}$$

This implies

$$\langle 0|\,\phi(x)^* D_0\phi(x)\,|0\rangle = \lim_{\tau\to 0} -i\sum_{n<0}(\omega_n - eA_0)|\phi_n(z)|^2 e^{-i\omega_n(\tau+i\epsilon)} + \frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.30}$$

Similarly, for the second term in (3.23),

$$D_0^{*\prime} w^{\phi\phi^*}(x,x') = i\sum_{n>0}(\omega_n - eA_0(z))|\phi_n|^2 e^{i\omega_n(\tau+i\epsilon)}, \tag{3.31}$$

and

$$D_0^{*\prime} H^{\phi\phi^*}(x,x') = -\frac{1}{2\pi}\frac{1}{\tau+i\varepsilon} + \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau), \tag{3.32}$$

since $U^*(x,x') = U(x',x)$. This implies

$$\langle 0|\,\phi(x)D_0\phi(x)^*\,|0\rangle = \lim_{\tau\to 0} i\sum_{n>0}(\omega_n - eA_0)|\phi_n(z)|^2 e^{i\omega_n(\tau+i\epsilon)} + \frac{1}{2\pi}\frac{1}{\tau+i\epsilon} - \frac{ieA_0(z)}{2\pi} + \mathcal{O}(\tau). \tag{3.33}$$

When subtracting these two quantities as per (3.23), the divergent terms cancel each other out, resulting in the following expression for the vacuum polarization

$$\rho(z) = \lim_{\tau\to 0}\left(\sum_{n<0}(\omega_n - A_0)\|\phi_n\|^2 e^{-i\omega_n(\tau+i\epsilon)} + \sum_{n>0}(\omega_n - A_0)\|\phi_n\|^2 e^{i\omega_n(\tau+i\epsilon)}\right) + \frac{e^2}{\pi}A_0(z). \tag{3.34}$$

This expression is gauge independent, as long as the summation and the limit are taken in the correct order.

In [1], the vacuum polarization is calculated as

$$\rho(z) = \sum_{n<0}(\omega_n - A_0)\|\phi_n\|^2 + \sum_{n>0}(\omega_n - A_0)\|\phi_n\|^2, \tag{3.35}$$

which even though it is gauge invariant, it cannot be derived from a gauge invariant theory. The crucial difference is the extra $\frac{e^2}{\pi}A_0(z)$.

3.3.1 Perturbation theory

[Q: I find that this calculations are necessary for some reasons:

1. **To have an intuition on how the vacuum polarizes on this setup**
2. **To show the difference between the mode sum and the Hadamard point-split formula**
3. **Knowing this difference, that the answer to the question "does backreaction avoid instabilities?" is not obvious.**

However, I am basically copying the results from the bibliography, and I do not want to do that. Should I keep this section? Or just cite it?]

It might be enlightening to see the mode solutions and vacuum polarization at first perturbational order. These calculations were already done in [1, 3], and we therefore just state the perturbative corrections to the free field solutions, for Dirichlet boundary conditions (marked with a superscript D) and for Neumann

boundary conditions (marked with a superscript N).

$$\begin{aligned}\phi_n^D = &\left(m^2+\pi^2n^2\right)^{-\frac{1}{2}}\Bigg[\sin\pi nz \\ &+\lambda\frac{\sqrt{m^2+\pi^2n^2}}{2\pi|n|}\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg]\end{aligned} \tag{3.36}$$

$$\begin{aligned}\phi_n^N = &\left(m^2+\pi^2n^2\right)^{-\frac{1}{2}}\Bigg[\cos\pi nz \\ &+\lambda\frac{\sqrt{m^2+\pi^2n^2}}{2\pi|n|}\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\cos\pi nz + (z(1-z)+(\pi n)^{-2)})\sin\pi nz\right)\Bigg]\end{aligned} \tag{3.37}$$

$$\phi_{\pm 0}^N = (2m)^{-\frac{1}{2}} \mp \lambda\sqrt{2m}\left(\frac{1}{24}-\frac{1}{4}z^2+\frac{1}{6}z^3\right). \tag{3.38}$$

3.3.2 Vacuum polarization at first order in perturbation theory

Using (3.34) we calculate the vacuum polarization at first order in λ for the massless case using Dirichlet boundary conditions. We rewrite (3.34) as

$$\rho(z) = \lim_{\tau\to 0}\sum_{n=1}^{\infty}\left(|\phi_n|^2(\pi n-\varepsilon A_0) - |\phi_{-n}|^2(\pi n+\varepsilon A_0)\right)e^{i\pi n(\tau+i\epsilon)} + \frac{e^2}{\pi}A_0. \tag{3.39}$$

From (3.36), up to first order in λ,

$$|\phi_n|^2 = \frac{1}{\pi\|n\|}\left(\sin^2\pi nz - \lambda\sin\pi nz\left[\frac{1}{\pi n}\left(z-\frac{1}{2}\right)\sin\pi nz + z(1-z)cos\pi nz\right]\right). \tag{3.40}$$

Substituting in (3.39), and with some rearranging,

$$\begin{aligned}\rho(z) &= -2\varepsilon\lambda\lim_{\tau\to 0}\sum_{n=1}^{\infty} z(1-z)\sin\pi nz\cos\pi nz - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right) \\ &= -\varepsilon\lambda z(1-z)\lim_{\tau\to 0}\frac{1}{2i}\left(\frac{e^{i\pi(2z+\tau+i\epsilon)}}{1-e^{i\pi(2z+\tau+i\epsilon)}} - \frac{e^{i\pi(-2z+\tau+i\epsilon)}}{1-e^{i\pi(-2z+\tau+i\epsilon)}}\right) - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right) \\ &= -\varepsilon\lambda z(1-z)\cot\pi z - \frac{\lambda}{\pi}\left(z-\frac{1}{2}\right).\end{aligned} \tag{3.41}$$

In [1], the vacuum polarization at first order in λ is (wrongly) calculated as

$$\rho = -2\varepsilon\lambda z(1-z)\cot(\pi z). \tag{3.42}$$

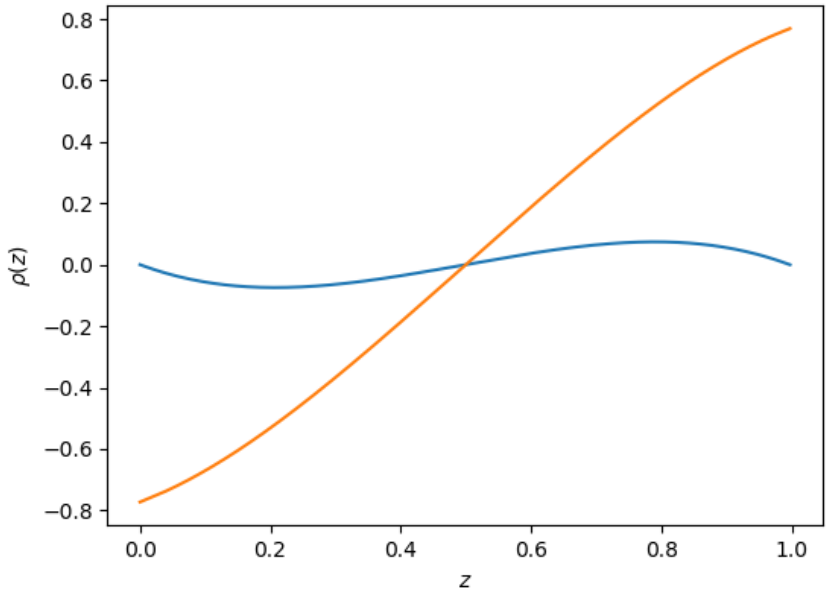

Figure 3.2: The vacuum polarization at first order in λ in Dirichlet boundary conditions. In orange, ρ as calculated using the mode sum in [1], and in blue using point splitting with respect to the Hadamard parametrix as in [3]. **[Here I intend to add a third curve, with the mode sum formula for ρ for $N = 1$, which is the one used in [1].]**

We show in the figure 3.2 the comparison between the two ρ. Notice how the addition of the potential term in (3.34) forces ρ to be 0 at the boundaries, as one would expect for Dirichlet boundary conditions. We also observe that the overall vacuum polarization is much stronger when calculating it using the mode sum formula. The main claim in [1], is that when considerring the back reaction of the scalar field, the instabilities that appear in the external field approximation disappear. However, since [1] uses a wrong expression to calculate the back-reaction of the scalar field, and the correct expression leads to a much 'weaker' ρ, it is not obvious to say whether the back-reaction of the field is enough to raise the energy of the modes enough so as to avoid the original instabilities.

3.4 The iterative procedure

To include backreaction into the calculation, we proceed iteratively. Following the convention used in [1], we label by κ each of the iterations. For a given A_0^κ, at each

step κ we solve numerically the following equations,

$$\partial_1^2 \phi_n^{\kappa+1} = \left(-(\omega_n^{\kappa+1} - \varepsilon A_0^\kappa(z))^2 + m^2\right) \phi_n^{\kappa+1} \tag{3.43}$$

$$\begin{aligned} \varepsilon A_0^\kappa(z) &= -\lambda\left(z - \frac{1}{2}\right) - \varepsilon \int_{\frac{1}{2}}^{z} \int_0^{z'} \rho^\kappa(z'')dz''dz' \\ \varepsilon A_0^0(z) &= -\lambda\left(z - \frac{1}{2}\right) \end{aligned} \quad . \tag{3.44}$$

with ρ calculated using (3.34), up to a finite mode cutoff in the sum. This cutoff at mode N causes the calculated ρ to be oscillatory, of period $\Delta z = \frac{1}{N+1}$, which should be averaged out.

Finally, we take the limit $\kappa \to \infty$, i.e. solve equation (3.43) for a given potential A_0^κ, calculate the corresponding induced potential at κ, and solve equation (3.43) using this new potential. We look for convergence following this procedure. The computation is halted when convergence with respect to some norm is found. Once a self-consistent solution is found, we use this solution as a guess for some new λ value, greater than the old one. In this way, we sample the whole λ-parameter space to study the interaction between the scalar field and the external electric field.

This procedure can be also be stated in terms of infinite dimensional fixed point problems,

$$A_0 = f(A_0), \tag{3.45}$$

with f the update rule given by equation (3.44). The function f has no closed form, as calculating ρ in (3.44) involves taking all the steps that were explained throughout this section.

3.4.1 Numerical instabilities

Even though we are considering the back-reaction of the field, this problem still presents some instabilities when crossing $\lambda \approx \lambda_c$. When sampling the λ-parameter space, if the step $\Delta\lambda$ is too big, then one of two things might happen:

1. Since we are using the previous solutions as the initial guess to find the self-consistent solution, the screening from the previous solution might not be strong enough causing the energies of some modes to not have solutions in $\mathbb{R}$, or

2. convergence is too slow.

Even though this can in theory be overcome by choosing smaller $\Delta\lambda$, in practice $\Delta\lambda$ can get to the order of 10^{-7}. At $\sim 30s$ per value of λ, this amounts to about 10 years to achieve a step of unit size in λ.

To overcome this, we slightly modify the iterative procedure with a *relaxing* procedure,

$$A_0^{\kappa+1}(z) = cA_0^{\kappa}(z) + (1-c)\left[-\lambda\left(z-\frac{1}{2}\right) - \int_{\frac{1}{2}}^{z}\int_0^{z'}\rho^{\kappa}(z'')dz''dz'\right], \;\; 0 < c \lesssim 1. \tag{3.46}$$

This scheme avoids these instabilities by allowing self consistent solutions to "relax" into one another, i.e. if for a certain λ value a self-consistent solution is found but $\Delta\lambda$ was too big, the change in the potential will not be as strong as it was in the previous scheme. The closer the parameter c is to 1, the slower the convergence, but the more relaxed.

3.5 Computational details

3.5.1 Noise filtering

To solve this problem numerically, we split the interval $(0,1)$ into M equal intervals, with the mesh points $0 < \ldots < z_i < \ldots < 1$ and $i \in [0, M-1]$.

In the (numerical) computation of (3.34) one needs to cut the mode sum at some finite mode N. This induces oscillations of period $\Delta z = \frac{1}{N+1}$ in the resulting vacuum polarization, which should be averaged out. To do this, we convolute the resulting ρ with a very specific array, designed to cancel these oscillations out. Recall the definition of the convolution of two arrays ρ_n, b_n not necessarily of the same length

$$(\rho * b)_n = \sum_{n=0}^{M} \rho_m b_{n-m}. \tag{3.47}$$

A convoluting array of the form*

$$b_n = [1,0,4,0,6,0,4,0,1] \tag{3.48}$$

approximately cancels the oscillations. For this to be effective, this array should "fit" exactly one period of the noise oscillations, and therefore the number of mesh points should be exactly $M = 8(N+1)$.

*It should be renormalized so that the sum of its elements is 1

Chapter 4

Results

4.1 Comparing the charge density from different renormalization schemes

4.2 Attempting to retrieve Ambjørn and Wolfram's results

4.3 Impact of having different mode cutoffs

4.4 Evolution of the induced field as a function of λ

4.5 Dirichlet and Neumann boundary conditions for big should behave the same

Appendix A

Computation details

A.1 Numerical methods

We solve equation (3.14) computationally, using fourth order Runge-Kutta method, and the ω_n are sought after by using the bisection method,

A.2 Calculating series and limits

The vacuum polarization (3.34) should be calculated by summing over all modes of the Klein-Gordon field, and then taking the limits $\tau \to 0$, and $\epsilon \to 0^+$. To do this compuptationally, first we note that due to the chosen gauge we can state $\omega_n = -\omega_n$. This allows us to write equation (3.34) as a single sum,

$$\rho(z) = \lim_{\tau \to 0} \left(\sum_{n>0} \left[(\omega_n - A_0)\|\phi_n\|^2 - (\omega_n + A_0)\|\phi_{-n}\|^2 \right] e^{i\omega_n(\tau + i\varepsilon)} \right) + \frac{e^2}{\pi} A_0(z). \tag{A.1}$$

Note that this expression is no longer gauge independent.

If the coefficients of the exponential functions decay fast enough as n grows, we can commute the limit and the sum so that

$$\rho(z) = \sum_{n>0} \left[(\omega_n - A_0)\|\phi_n\|^2 - (\omega_n + A_0)\|\phi_{-n}\|^2 \right] + \frac{e^2}{\pi} A_0(z). \tag{A.2}$$

However, we can only calculate a finite amount of mode solutions, so the sum is to be taken only until a certain mode cutoff N. This finite cutoff induces oscillations on the calculated vacuum polarization, of period $\Delta z = \frac{1}{N+1}$. These should be averaged out by convoluting the resulting vacuum polarization with a suitable array.

Appendix B

Calculating vacuum polarization at first perturbative order

In the perturbative approximation, for the massless case

$$\begin{aligned}
&\omega_n = n\pi \\
&\phi_n^D = \left(m^2+\pi^2 n^2\right)^{-\frac{1}{4}} \Bigg[\sin \pi n z \\
&\quad +\lambda \frac{\sqrt{m^2+\pi^2 n^2}}{2\pi|n|}\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin \pi n z - z(1-z)\cos \pi n z\right)\Bigg] \\
&\phi_n^D = \frac{1}{\sqrt{\pi|n|}}\left[\sin \pi n z + \frac{\lambda}{2}\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin \pi n z - z(1-z)\cos \pi n z\right)\right]
\end{aligned} \tag{B.1}$$

At first order in λ,

$$|\phi_n^D|^2 = \frac{1}{\pi|n|}\left[\sin^2 \pi n z - \lambda \sin \pi n z\left(\frac{1}{\pi n}\left(z-\frac{1}{2}\right)\sin \pi n z + z(1-z)\cos \pi n z\right)\right] \tag{B.2}$$

The charge density for the nth mode

$$\begin{aligned}(\pi n - \varepsilon A_0)|\phi_n^D|^2 &= (\pi n + \lambda(z-\frac{1}{2}))\frac{1}{\pi|n|}\Bigg[\sin^2\pi nz \\ &\quad +\lambda\sin\pi nz\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg] \\ &= \mathrm{sign}(n)\Bigg[\sin^2\pi nz \\ &\quad +\lambda\sin\pi nz\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg] \\ &\quad +\frac{\lambda\left(z-\frac{1}{2}\right)}{\pi|n|}\Bigg[\sin^2\pi nz \\ &\quad +\lambda\sin\pi nz\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg] \\ &= \mathrm{sign}(n)\Bigg[\sin^2\pi nz \\ &\quad +\lambda\sin\pi nz\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg] \\ &\quad +\frac{\lambda\left(z-\frac{1}{2}\right)}{\pi|n|}\sin^2\pi nz\end{aligned}$$

And for the -nth mode

$$\begin{aligned}-(\pi n + \varepsilon A_0)|\phi_{-n}^D|^2 &= -\mathrm{sign}(n)\Bigg[\sin^2\pi nz \\ &\quad -\lambda\sin\pi nz\left(\frac{1}{\pi n}\left(\frac{1}{2}-z\right)\sin\pi nz - z(1-z)\cos\pi nz\right)\Bigg] \\ &\quad +\frac{\lambda\left(z-\frac{1}{2}\right)}{\pi|n|}\sin^2\pi nz\end{aligned}$$

So the contribution to the charge density of the nth mode is given by

$$\begin{aligned}(\pi n - \varepsilon A_0)|\phi_n^D|^2 - (\pi n + \varepsilon A_0)|\phi_{-n}^D|^2 = \\ \frac{2\lambda\sin^2\pi nz}{\pi n}\left(\frac{1}{2}-z\right) - 2\lambda z(1-z)\sin\pi nz\cos\pi nz \\ +2\frac{\lambda\left(z-\frac{1}{2}\right)}{\pi|n|}\sin^2\pi nz\end{aligned}$$

$$\rho(z) = \varepsilon \lim_{\tau \to 0} \sum_{n=1}^{\infty} \left(|\phi_n|^2(\pi n - \varepsilon A_0) - |\phi_{-n}|^2(\pi n + \varepsilon A_0) \right) e^{i\pi n(\tau + i\epsilon)} + \frac{e^2}{\pi} A_0. \quad \text{(B.3)}$$

Appendix C

Calculating vacuum polarization using Hadamard point-splitting renormalization

The expression for the vacuum polarization comes from the zeroth component of the charge current density

$$\rho(z) = \langle 0| \, \phi^* D_0 \phi - \phi D_0^* \phi^* \, |0\rangle \,, \tag{C.1}$$

with the field given by its energy mode expansion

$$\phi(t,z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega t} i + \sum_{n<0} b_n^\dagger \phi_n(z) e^{-i\omega t}. \tag{C.2}$$

Using the Hadamard point-splitting renormalization of the products of (derivaatives of fields) (2.22), from equation (3.23) we need to calculate the two two-point functions $w^{\phi\phi^*}$, $w^{\phi^*\phi}$ and their corresponding Hadamard paramatrices $H^{\phi\phi^*}$, $H^{\phi^*\phi}$.

We are performing the point-splitting in the time direction, $x' = (t+\tau, z)$. We start by calculating the second term in (3.23). The relevant two-point function is

$$w^{\phi\phi^*}(x,x') = \langle 0| \, \phi(x)\phi^*(x') \, |0\rangle = \sum_{n>0} |\phi_n(z)|^2 e^{i\omega_\tau}, \tag{C.3}$$

and therefore

$$D_0^{*\prime} w^{\phi\phi^*}(x,x') = i \sum_{n>0} (\omega_n - eA_0) |\phi_n(z)|^2 e^{i\omega_\tau}. \tag{C.4}$$

The Hadamard parametrix of the two-point function $w^{\phi\phi^*}$ is (up to order rele-

vang for the charge density operator)

$$H^{\phi\phi^*} = -\frac{1}{4\pi}U(x,x')\log(\tau^2 + i\varepsilon\tau), \tag{C.5}$$

again, with $U(x,x') = \exp(ieA_0\tau)$. We calculate

$$\begin{aligned} D_0^{*\prime}H^{\phi\phi^*}(x,x') &= -\frac{1}{4\pi}(\partial_0 - ieA_0)e^{ieA0\tau}\log\left(\tau^2 + i\varepsilon\tau\right) \\ &= -\frac{1}{4\pi}ieA_0e^{ieA_0\tau}\log(\tau^2 + i\varepsilon\tau) - \frac{1}{4\pi}e^{ieA_0\tau}\frac{2\tau + i\varepsilon}{\tau^2 + i\varepsilon\tau} + \frac{1}{4\pi}ieA_0e^{ieA_0\tau}\log(\tau^2 + i\varepsilon\tau). \\ &= -\frac{1}{2\pi}(1 + ieA_0\tau)\frac{1}{\tau + i\varepsilon} \end{aligned} \tag{C.6}$$

Thus,

$$\langle 0|\,\phi D_0^{*\prime}\phi^*\,|0\rangle = \lim_{\tau\to 0} i\sum_{n>0}(\omega_n - eA_0)|\phi_n(z)|^2e^{i\omega_n(\tau+i\varepsilon)} + \frac{1}{2\pi}(1 + ieA_0\tau)\frac{1}{\tau + i\varepsilon} \tag{C.7}$$

In a similar fashion,

$$D_0'w^{\phi^*\phi}(x,x') = -i\sum_{n<0}(_n - eA_0)\,|\phi_n|^2e^{i\omega_n\tau} - \frac{1}{2\pi}(1 + ieA_0\tau)\frac{1}{\tau + i\varepsilon}, \tag{C.8}$$

and

$$\begin{aligned} D_0'H^{\phi^*\phi}(x,x') &= -\frac{1}{4\pi}(\partial_0 + ieA_0)e^{-ieA0\tau}\log\left(\tau^2 + i\varepsilon\tau\right) \\ &= \frac{1}{4\pi}ieA_0e^{-ieA_0\tau}\log(\tau^2 + i\varepsilon\tau) - \frac{1}{4\pi}e^{-ieA_0\tau}\frac{2\tau + i\varepsilon}{\tau^2 + i\varepsilon\tau} - \frac{1}{4\pi}ieA_0e^{-ieA_0\tau}\log(\tau^2 + i\varepsilon\tau). \\ &= -\frac{1}{2\pi}(1 - ieA_0\tau)\frac{1}{\tau + i\varepsilon} \end{aligned} \tag{C.9}$$

The vacuum expectation value of the first term in (3.23) is

$$\langle 0|\,\phi^*D_0'\phi\,|0\rangle = \lim_{\tau\to 0} -i\sum_{n>0}(\omega_n - eA_0)|\phi_n(z)|^2e^{i\omega_n(\tau+i\varepsilon)} + \frac{1}{2\pi}(1 - ieA_0\tau)\frac{1}{\tau + i\varepsilon} \tag{C.10}$$

Following equation (3.23), we subtract these two and multiply by *ie*. The diverging parts cancel out resulting in the following expression for the vacuum po-

larization

$$\rho(z) = e \lim_{\varepsilon \to 0^+} \lim_{\tau \to 0} \left[\sum_{n>0} (\omega_n - eA_0) |\phi_n(z)|^2 e^{i\omega_n(\tau + i\varepsilon)} + \sum_{n<0} (\omega_n - eA_0) |\phi_n(z)|^2 e^{-i\omega_n(\tau + i\varepsilon)} \right] + \frac{e^2}{\pi} A_0(z) \tag{C.11}$$

Bibliography

[1] J. Ambjørn and S. Wolfram. Properties of the vacuum. 2. electrodynamic. *Annals of Physics*, 147(1):33–56, 1983.

[2] W.K. Heisenberg and H. Euler. Folgerungen aus der Diracschen Theorie des Positrons. *Z. Phys.*, 98:714–732, 1936.

[3] J. Wernersson and J. Zahn. Vacuum polarization near boundaries.