Chapter 2

Lessons from Quantum Field Theory in Curved Spacetimes

Why this chapter? This chapter appears because in the following chapter I calculate the expression of the expectation value of the charge density through Hadamard point-split renormalization

What is this chapter's end goal? Derive an expression for Hadamard point split renormalization.

Why is Hadamard point split renormalization needed? Because we want to solve

$$\partial_\nu F^{\mu\nu} = \langle j^\mu \rangle_\omega \tag{2.1}$$

w.r.t. some state ω, and the RHS of this equation is ill-defined due to the appearance of products of fields. Hadamard point splitting renormalization defines the expectation value of these terms in mathematically precise way, yielding finite values and assuring gauge invariance of the calculated expectation values.

What do we need for this?

1. The phase space
    (a) Symplectic space and time evolution in the space.
    (b) Isomorphism between phase space $\mathcal{M}$ and solution space $\mathcal{S}$.
    (c) Classical observables $f\mathcal{C}_0^\infty(S)$ form a vector space. Can be given algebraic structure via $\{f,g\} = \Omega^{ab}\nabla_a f \nabla_b g$

(d) Quantization of the phase space. Quantum observables are self-adjoint operators on $\mathcal{S}$. Quantization map $\hat{}: \mathcal{O} \to \hat{\mathcal{O}}$, such that for any pair of classical observables we have

$$[\hat{f}, \hat{g}] = i\widehat{\{f, g\}}, \tag{2.2}$$

with $[\, ,]$ the commutator and $\{\, ,\}$ the Poisson bracket.

(e) Define observables through points in the manifold: $\forall y \in \mathcal{M}, \Omega(y, \cdot)$ defines a (classical) observable on $\mathcal{M}$. Quantize by requiring

$$[\hat{\Omega}(y_1, \cdot), \hat{\Omega}(y_2, \cdot)] = -i\Omega(y_1, y_2)I, \tag{2.3}$$

which is the usual quantization.

(f) Quantum theory of harmonic oscillators. Annihilation operator

$$a = \sqrt{\frac{\omega}{2}}q + \sqrt{\frac{1}{2\omega}}p \tag{2.4}$$

Its Heisenberg representation

$$a_H(t) = U_t^{-1} a U_t = e^{-i\omega t}a, \tag{2.5}$$

leads to the Heisenberg representation of the position operator

$$q_H = \sqrt{\frac{1}{2\omega}}(a_H + a_H^\dagger) = \sqrt{\frac{1}{2\omega}}\left(e^{-i\omega t}a + e^{i\omega t}a^\dagger\right). \tag{2.6}$$

The annihilation operator is the positive frequency part of the position operator.

2. Define One-particle Hilbert spaces $\mathcal{H} \subset \mathcal{S}$ as the space of positive frequency solutions

$$q_i = \alpha_i e^{-i\omega_i t}. \tag{2.7}$$

$\mathcal{H}$ is a Hilbert space (can define an inner product given by the symplectic form $(y_1, y_2) = -i\Omega(\overline{y_1}, y_2)$. Construct its (symmetric) Fock space $\mathcal{F}_s(\mathcal{H})$. Choose the (normalized w.r.t. (,)) solution to the classical EOM $\xi_i \in \mathcal{H}$ s.t. only the i-th oscillator is excited. Therefore define the q, p observables to be

$$q_{iH}(t) = \xi_i(t)a_i + \overline{\xi_i}(t)a^\dagger, p_{iH} = \dot{q}_{iH} \tag{2.8}$$

3. Construct the negative frequency solution space as $\overline{\mathcal{H}}$. Define K as the projector $K : \mathcal{S} \to \mathcal{H}$ (and consequently $\overline{K} : \mathcal{S} \to \overline{\mathcal{H}}$).

$$\forall y \in \mathcal{S}, \hat{\Omega}_H(y, \cdot) := ia(\overline{Ky}) - ia^{\dagger}(Ky) \tag{2.9}$$

4. Klein-Gordon in flat spacetime. Linearity of KG equation allows to study the system as infinite harmonic oscillators, decoupled. Define momentum through the lagrangian $\pi = \frac{\delta S}{\delta \dot{\phi}} = \dot{\phi}$. KG is initial value problem, define $\mathcal{M}$ as the (compact) Cauchy data defined over a Cauchy surface Σ_0

$$\mathcal{M} := \{[\phi, \pi] \in C_0^{\infty}(\Sigma) \times C_0^{\infty}(\Sigma)\} \tag{2.10}$$

Every point in $\mathcal{M}$ defines a single solution to the KG equation. There is therefore a direct identification with the solution space $\mathcal{S}$. The symplectic structure on $\mathcal{S}$

$$\Omega([\phi_1, \pi_1], [\phi_2, \pi_2]) = \int_{\Sigma} d^3x(\pi_1\phi_2 - \pi_2\phi_1). \tag{2.11}$$

5. Test functions. Given any element $f \in C_0^{\infty}(M) = \mathcal{T}$ (in flat spacetime case, $M = \mathbb{R}^4$,) define its advanced and retarded propagators as

$$(\partial_a\partial^a - m^2)(Af) = f, \; (\partial_a\partial^a - m^2)(Rf) = f, \tag{2.12}$$

such that $\text{supp}(Af) = J^-(\text{supp}(f))$, $\text{supp}(Rf) = J^+(\text{supp}(f))$. Define the causal propagator $Ef = Af - Rf$. $E : \mathcal{T} \to \mathcal{S}$ defines an onto, non degenerate map. Also verifies that

$$\int \psi f d^4x = \Omega(Ef, \psi), \forall \psi \in \mathcal{S}, f \in \mathcal{T}. \tag{2.13}$$

This allows one to define, for each $f \in \mathcal{T}$ the field as an averaging of the solution to KG weighted by f.

$$\hat{\phi}(f) = \hat{\Omega}(Ef, \cdot) = ia(\overline{K(Ef)} - ia^{\dagger}(K(Ef)) \tag{2.14}$$

is then to be understood as the weighted spacetime average of the field operator representing the field at spacetime point x. $\overline{\phi}$ is an operator-valued distribution, and solves the KG equation in the distributional sense ,

$$KG\hat{\phi}(f) = \hat{\phi}(KG(f)) = \hat{\Omega}(E(KG(f)), \cdot) = 0 \tag{2.15}$$

Choosing the vacuum state as that which verifies $a_{\mathbf{k}} |0\rangle = 0$ for all $\mathbf{k}$, then the vacuum two point function is defined as

$$w(f,g) = \langle 0| \hat{\phi}(f)\hat{\phi}(g) |0\rangle . \tag{2.16}$$

With the definition of the field being averaged weighted by f, then

$$w(f,g) = \langle 0| \int_M \phi(x)f(x)d^4x \int_M \phi(y)g(y)d^4y |0\rangle \tag{2.17}$$

$$= \int_{M\times M} \langle 0| \phi(x)\phi(y) |0\rangle f(x)f(y)d^4xd^4y. \tag{2.18}$$

This expression defines the distributional notation

$$w(x,y) = \langle 0| \phi(x)\phi(y) |0\rangle , \tag{2.19}$$

so that two point functions can be understood as a bi-solution to the Klein-Gordon equation.

6. We are interested in calculating the charge current density of the Klein-Gordon field

$$j^\mu = ie\left((D^\mu\phi)^*\phi - \phi^* D^\mu\phi\right). \tag{2.20}$$

This observable contains products of fields, whose distributional nature automatically makes its expectation value ill-defined. To calculate this quantity, it needs to be redefined.

In attempting to directly calculate this expectation value, one finds that the terms containing $aa^\dagger$ in the mode expansion of the 'field' ϕ^2 yield a diverging expectation value, with all other terms vanishing (in a real field. In a complex field the $bb^\dagger$ will also contribute to an infinite expectation value.) In QFT in flat spacetime one proceeds by 'normal ordering' i.e. by moving the creation operators to the right and the annihilation operators to the left, or more precisely by defining $\left\langle : aa^\dagger : \right\rangle_\omega = \left\langle a^\dagger a \right\rangle_\omega$. Notice how this prescription implies (in the abscence of external electric fields)

$$\langle 0| : j^\mu : |0\rangle = 0, \tag{2.21}$$

as one would expect.

Evaluating $\left\langle \phi(x)^2 \right\rangle_\omega$ for a state ω is problematic, but the two point function is well defined as a bi-solution to the Klein-Gordon equation. This distribution will not be smooth in the limit $x' \to x$, but with suitably chosen states the difference

$$F(x,y) = \langle \phi(x)\phi(y) \rangle_\omega - \langle 0 | \phi(x)\phi(y) | 0 \rangle \tag{2.22}$$

is. We therefore define the expectation value of terms quadratic in the field through a point-splitting procedure

$$\langle \phi^*(x)\phi(x) \rangle = \lim_{x' \to x} F(x,x'). \tag{2.23}$$

This definition can be extended through linearity to our quantity of interest

$$\langle j^\mu \rangle_\omega := \frac{ie}{2} \lim_{x' \to x} \left(D^*_\mu F(x,x') - D'_\mu F(x,x') \right), \tag{2.24}$$

with $D^*_\mu = \partial_\mu - ieA_\mu$ and D' the derivative over primed variables. j^μ should be parallely transported with respect to the gauge covariant derivative.

This prescription coincides with the normal ordering prescription on flat spacetimes with no background electromagnetic field, but considering curved spacetimes or background electromagnetic fields (i.e. connections over $U(1)$ with curvature) takes away having a preferred choice of a vacuum state, and therefore defining these expectation values through the vacuum state as above becomes problematic. Even when the vacuum state can be defined, we do not expect

$$\langle 0 | j_\mu | 0 \rangle = 0 \tag{2.25}$$

with a non zero background electromagnetic background.

The prescription above, however, defines the difference of these expectation values between states. In this sense, the difference

$$\langle j_\mu \rangle_1 - \langle j_\mu \rangle_2 \tag{2.26}$$

is smooth for two states ω_1, ω_2.

We can develop this idea to come up with a formal definition of the difference between expectation values on different states. To this end, we list some the properties that we expect from $\langle j_\mu \rangle$. The full list can be found in [5].

(a) The difference between states $\langle j_\mu \rangle_1 - \langle j_\mu \rangle_2$ to be smooth whenever the difference $\langle \phi(x)\phi(x') \rangle_1 - \langle \phi(x)\phi(x') \rangle_2$ is a smooth function.

(b) $D_\mu \langle j^\mu \rangle_\omega = 0$ for any state ω,

(c) $\langle 0 | j_\mu | 0 \rangle = 0$ in flat spacetime with no background electromagnetic field.

From property (a), two different definitions of the expectation value $\langle j_\mu \rangle_\omega$, $\left\langle \bar{j}_\mu \right\rangle_\omega$ must obey

$$\langle j_\mu \rangle_1 - \langle j_\mu \rangle_2 = \left\langle \bar{j}_\mu \right\rangle_1 - \left\langle \bar{j}_\mu \right\rangle_2, \tag{2.27}$$

and therefore the quantity $\mathcal{J}_\mu = \langle j_\mu \rangle_1 - \left\langle \bar{j}_\mu \right\rangle_1$ is independent of the state considered. This, with a certain additional property of the expectation value $\langle j_\mu \rangle_\omega$ implies that $\mathcal{J}_\mu(x)$ is only dependent locally. Finally, (b) imply that $\mathcal{J}_\mu$ is conserved and (c) that for no background electromagnetic field, $\mathcal{J}_\mu = 0$.

The prescription for $\langle j_\mu \rangle_\omega$ can be shown to be unique up to local background field terms, but not necessarily existent. The prescription can be built from "point-splitting" so that for any $\langle \phi(x)\phi(x') \rangle$ a bi-solution to the Klein-Gordon equation $H(x, x')$ is subtracted. Through this prescription, $\langle j_\mu \rangle_\omega$ obeys the required properties, as long as $H(x, x')$ exists. A proof of the uniqueness of the definition of non linear observables can be found in [4].

We choose an ansatz for $H(x, x')$ such that it shares a singularity structurure with vacuum two point function of the scalar field on flat (even) n dimensional spacetime with no background electromagnetic field, up to smooth coefficients V_k

$$H(x, x') = \sum_k^\infty V_k T_k + W(x, x'), \tag{2.28}$$

with

$$T_k = \begin{cases} \frac{(\frac{n}{2}-2-k)!}{2(2\pi)^{n/2}} \sigma_\varepsilon^{-(\frac{n}{2}-1-k)} & k \leq \frac{n}{2} - 2 \\ -\frac{1}{2(2\pi)^{n/2}(k-\frac{n}{2}+1)} (-\sigma)^{-k-\frac{n}{2}+1} \log(\sigma_\varepsilon) & k > \frac{n}{2} - 2, \end{cases} \tag{2.29}$$

where $\sigma(x, x')$ is the geodesic distance from x to x' (which in Minkowski spacetime corresponds to $\sigma(x, x') = (x - x')^2$, and

$$\sigma_\varepsilon = \sigma + 2i\varepsilon(t - t') + \varepsilon^2. \tag{2.30}$$

t, t' are the time coordinates of x, x' given by an appropiate choice of a time coordinate function. It is also understood that the limit $\varepsilon \to 0^+$ should be taken after applying this to test functions. The coefficients V_k and the function W are to be obtained by solving $\mathcal{K}H(x,x) = 0$. This equation gives an ordinary differential equations for each of the V_k that should be solved with the initial condition $V_0(x,x') = 1$, and integrated along the geodesics (straight lines) joining x and x'.

With a working prescription of the expectation value of non-linear terms on the field, we can finally formally define

$$\left\langle D_\alpha D^*_\beta \phi(x)\phi^*(x)\right\rangle := \lim_{x'\to x} D_\alpha D'^*_\beta \left(\left\langle \phi(x)\phi^*(x')\right\rangle - H(x,x')\right), \tag{2.31}$$

with α, β symmetrized multiindices.