Vacuum polarization in 1+1 spacetime dimensions

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Abstract

In my M.Sc. project, I study the effects of the backreaction of a charged Klein-Gordon field coupled to an external electric field, in 1+1 dimensional spacetime. In this talk, I summarize the statement of the problem I intend to solve, and present preliminary results and observations from my project.

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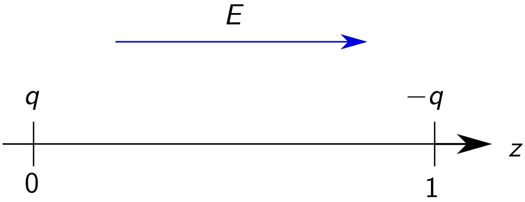
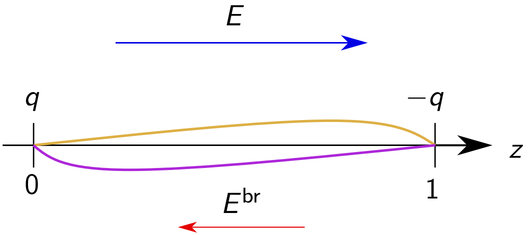
The set-up



The set-up

1 Preliminaries

$$E \neq 0 \implies \rho(z) \neq 0$$



Semi classical Klein-Gordon-Maxwell (KGM) equations

The goal is to solve the system of coupled differential equations

$$\begin{cases} \left[D_\mu D^\mu + m^2\right] \phi(t, z) = 0 \\ \partial_\mu F^{\mu\nu} = j^\nu_{\text{source}} + \langle j^\nu \rangle_\phi \end{cases}$$

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With the gauge covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu,$$

and the boundary conditions

$$\begin{cases} \phi\,|_{z=0,1} &= 0 \\ F^{\mu\nu}|_{z=0,1} &= \lambda \end{cases}$$

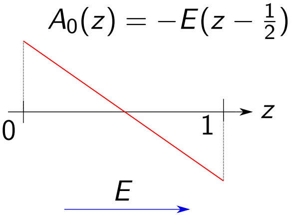


Figure: The background electric potential

1. Constant electric field of strength λ pointing towards positive z.

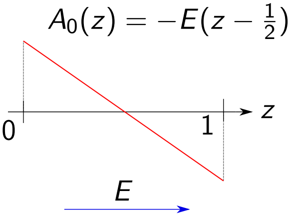


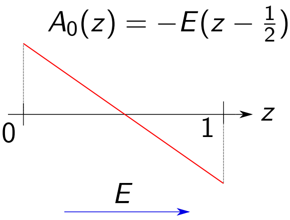
Figure: The background electric potential

1. Constant electric field of strength λ pointing towards positive z.
2. Under the Coulomb gauge

$$A_0(t, z) = -Ez,\ A_1(t, z) = 0$$

up to additive constant

Gauge fixing



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$$A_0(t, z) = -E(z - \frac{1}{2})$$

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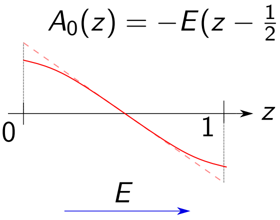

Figure: The full electric potential (background+backreaction)

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3. To ensure anti-symmetric solutions
$$A_0(t,z) = -E(z - \frac{1}{2}) + A_0^{\mathrm{br}}(z)$$

Klein-Gordon equation

With the chosen gauge the KGM equations turn into

$$
\begin{aligned}
((\partial_t + ieA_0)^2 - \partial_z^2 + m^2)\phi(t,z) = 0, \\
\partial_z^2 A_0^{br} &= - \langle \rho(z) \rangle_\phi \,,
\end{aligned}
$$

with

$$
A_0(z) = -E\left(z - \frac{1}{2}\right) + A_0^{\mathrm{br}}(z),
$$

$$
\rho(z) = ie\left((D_0\phi)^*\phi - \phi^* D_0\phi\right)
$$

Mode equation

$$\phi(t,z) = \phi_n(z)e^{-i\omega_n t} \implies$$

Time independent Klein-Gordon equation

Mode equation

$$\phi(t,z) = \phi_n(z) e^{-i\omega_n t} \implies \left([\omega_n - eA_0(z)]^2 + \frac{d^2}{dz^2} - m^2 \right) \phi_n = 0$$

Mode equation

$$\phi(t,z)=\phi_n(z)e^{-i\omega_n t} \implies \left([\omega_n - eA_0(z)]^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0$$

Ignoring the backreaction of the scalar field (External field approximation)

$$\left(\left[\omega_n + \lambda\left(z-\frac{1}{2}\right)\right]^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n = 0, \quad \lambda = eE$$

Analytic solutions

When $A_0(z) = -\lambda(z - \frac{1}{2})$, the KG equation can be solved analytically

$$\phi_n(z) = a_n D_{i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{1+i}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right)$$
$$+ b_n D_{-i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{i-1}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z - \frac{1}{2}\right)\right)\right)$$

with $D_\nu(z)$ the parabolic cylinder functions.

Instabilities of the external field approximation

The external field approximation yields instabilities for critical λ values [AW83]:

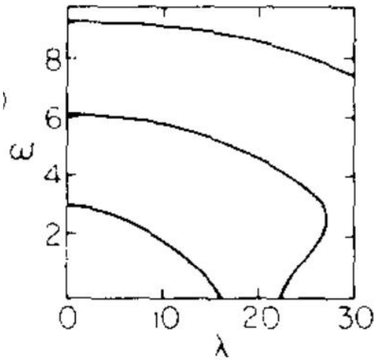

Figure: Energy of the first three modes as the background field strength λ increases, without considering the backreaction of the field. Courtesy of [AW83].

Backreaction avoids those instabilities?

The main claim in [AW83] is that considering the backreaction of the scalar field raises the energy levels, avoiding instabilities

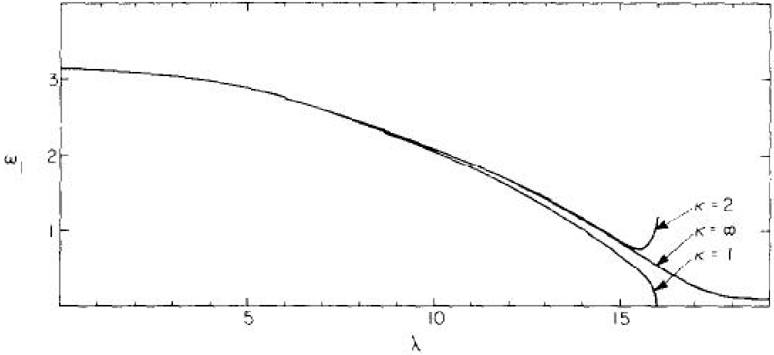

Figure: Positive energy levels for increasing λ **with** backreaction considered. [AW83]

Vacuum polarization is calculated as the zeroth component of the charge density current

$$\rho(z) = ie\left((D_0\phi)^*\,\phi - \phi^*\,D_0\phi\right).$$

This operator is non-linear on the fields $\implies$ ill-defined expectation value.

Mode expansion of the vacuum polarization

In [AW83] the vacuum polarization is directly calculated as

$$\langle 0|\,\rho(z)\,|0\rangle = ie\,\langle 0|\,\phi^* D_0\phi - \phi\,(D_0\phi)^*\,|0\rangle =$$
$$e\left(\sum_{n>0}(\omega_n - eA_0)|\phi_n|^2 + \sum_{n<0}(\omega_n - eA_0)|\phi_n|^2\right)$$

Assume the two-point function $w(x, x')$ is of Hadamard form

$$\langle 0| \phi(x)\phi^*(x') |0\rangle = w(x, x') = \underbrace{H(x, x')}_{\text{Divergent}} + \underbrace{R(x, x')}_{\text{Smooth}}$$

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$w(x,x')$ is a divergent distribution as $x' \to x$, but so is $H(x,x')$. In a way similar to the normal ordering prescription of RQFT, define

$$\langle 0|\, D_\alpha\phi(x)\,(D_\beta\phi(x))^*\,|0\rangle := \lim_{x'\to x}\left[D_\alpha D'^*_{\beta'}\left(w(x,x') - H(x,x')\right)\right].$$

Renormalized vacuum polarization

Hadamard Point-splitting[WZ20]

$$\langle 0|\,\rho(z)\,|0\rangle = ie\,\langle 0|\,\phi^* D_0\phi - \phi\,(D_0\phi)^*\,|0\rangle =$$

$$e\lim_{\tau\to 0}\left(\sum_{n>0}(\omega_n - eA_0)|\phi_n|^2 e^{i\omega_n(\tau+i\epsilon)} + \sum_{n<0}(\omega_n - eA_0)|\phi_n|^2 e^{-i\omega_n(\tau+i\epsilon)}\right) + \frac{e^2}{\pi}A_0$$

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Renormalized vacuum polarization

Hadamard Point-splitting[WZ20]

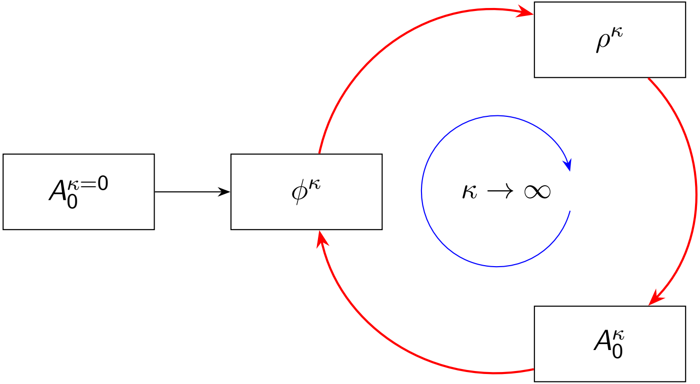
$$\langle 0|\, \rho(z)\, |0\rangle = e \sum_{n=-N, n\neq=0}^{N} (\omega_n - eA_0)|\phi_n|^2 + \frac{e^2}{\pi} A_0$$

Mode sum formula [AW83]

$$\langle 0|\, \rho(z)\, |0\rangle = e \sum_{n=-1, n\neq=0}^{1} (\omega_n - eA_0)|\phi_n|^2$$

Closing the loop

3 Quantisation and renormalisation



Closing the loop

$$\left(\left[\omega_n^{\kappa+1} - eA_0^{\kappa}(z)\right]^2 + \frac{d^2}{dz^2} - m^2\right)\phi_n^{\kappa+1} = 0$$

$$A_0^{\kappa}(z) = -\lambda\left(z - \frac{1}{2}\right) - \int_{\frac{1}{2}}^{z}\int_{0}^{z'} \rho^{\kappa}(z'')dz''dz'.$$

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As a fixed point problem

Reminiscent of fixed point problems

$$A_0^{\kappa+1} = f(A_0^{\kappa})$$

Fixed point problems

Definition

For X a metric space, a fixed point $x \in X$ of a function $f : X \to X$ is defined as

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$$A_0 \longrightarrow \boxed{(\omega_n, \phi_n) \longrightarrow \rho \longrightarrow -\int\int \rho} \longrightarrow A_{\text{background}} + A_{\text{induced}}(A_0)$$

Fixed point

Find, if it exists

$$\lim_{n\to\infty} f^n(x), f^n := \underbrace{f \circ \ldots \circ f}_{\text{n times}}$$

One dimensional fixed point problems

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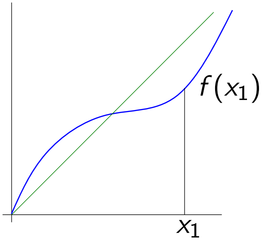

Figure: Iterations in fixed point problems

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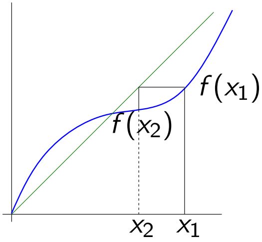

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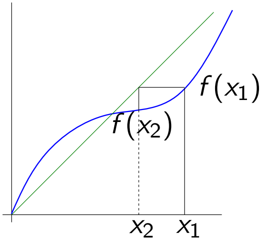

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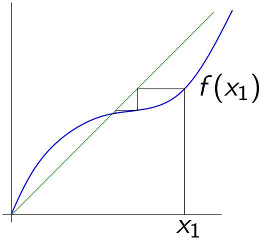

Figure: Iterations in fixed point problems

The relaxing update rule

To avoid numerical instabilities due to $\Delta\lambda$ being too big,

$$A_0^{\kappa+1} = cA_0^{\kappa} + (1-c)(A_{\text{background}} + A_{\text{induced}}^{\kappa}), 0 < c \lesssim 1$$

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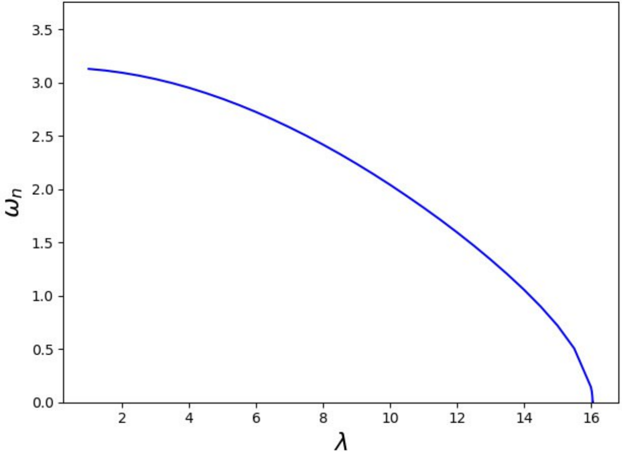
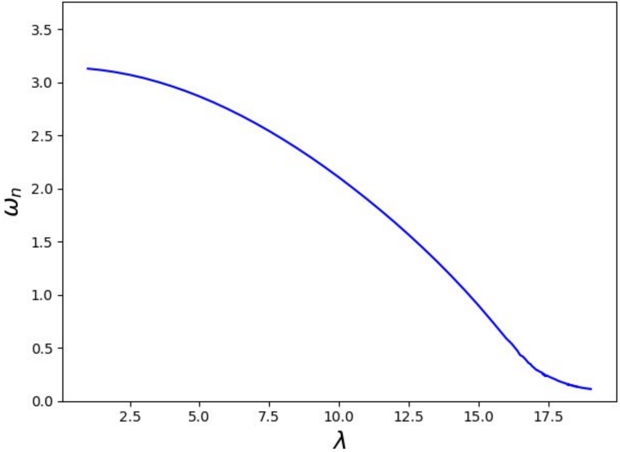
The energy of the modes for each λ

Figure: ω_1 as a function of λ for the massless case in the **external field approximation**.



The energy of the modes for each λ

Figure: ω_1 as a function of λ in the mode sum prescription with $N = 1$.



The energy of the modes for each λ

Figure: The energy of the first three modes of the scalar field in the Hadamard point-splitting prescription with cutoff $N = 12$ as λ increases.

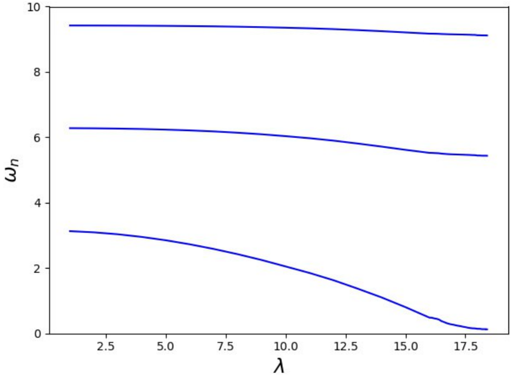
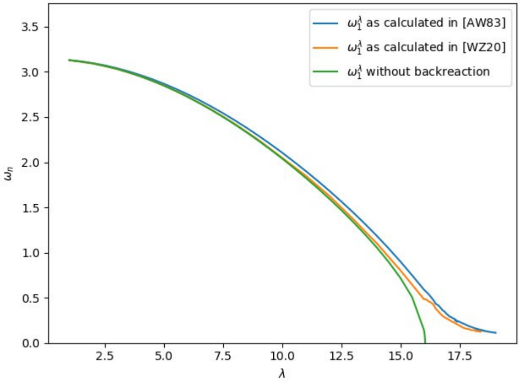


Figure: ω_1 as a function of λ for the mode sum formula prescription, Hadamard point-splitting prescription and external field approximation.



Self consistent vacuum polarizations for low λ

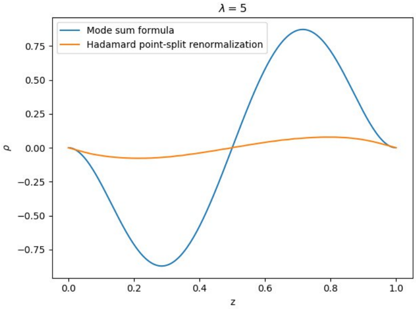


Figure: $\lambda = 5$ comparison of the self consistent vacuum polarization.

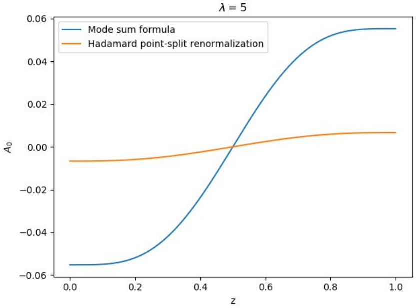


Figure: $\lambda = 5$ comparison of the self consistent induced A_0

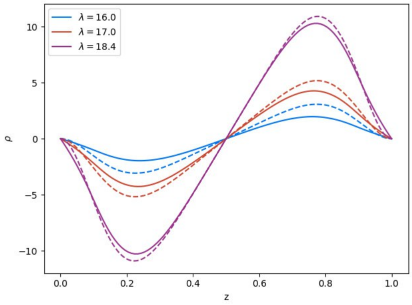


Figure: Higher values of λ comparison of the self consistent vacuum polarization.

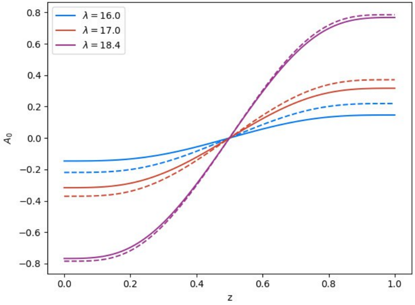


Figure: Higher values of λ comparison of the self consistent vacuum polarization.

Dashed lines: mode sum formula.

Solid lines: Hadamard point-splitting renormalization.

Vacuum polarization for different λ

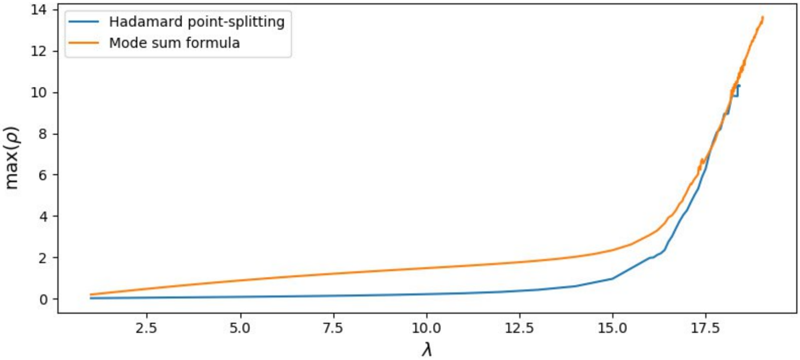


Figure: The maximum of the vacuum polarization for different values of the background electric field strength.

Asymptotic behaviour?

It is not yet safe to say that backreaction completely avoids instabilities of the solutions for each background electric field strength λ.

Asymptotic behaviour?

It is not yet safe to say that backreaction completely avoids instabilities of the solutions for each background electric field strength λ.
There are two main problems that can arise for a given configuration

1. The root finding algorithm for the boundary conditions finds no ω_1
2. There is no convergence

These two issues can be solved by a smaller $\Delta\lambda$, or a relaxing parameter c closer to 1.
It can get however to the order of $\Delta\lambda \sim 10^{-7}$.

Good convergence

6 Standing questions

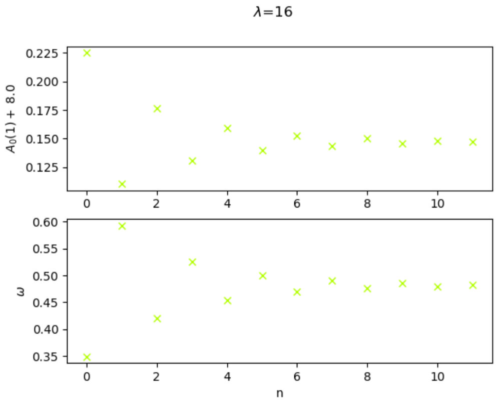


Figure: The iterations in the potential and the energy of the first mode in a case of good convergence.

No convergence

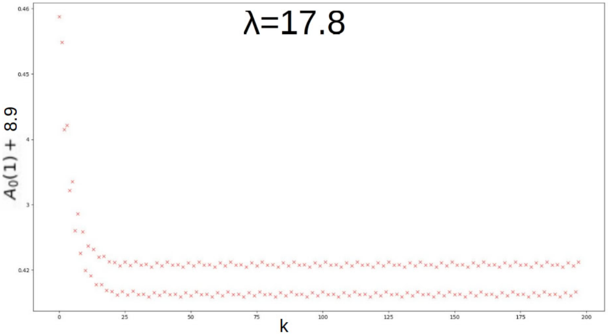


Figure: The values of the potential for constant $\lambda = 17.8$, as $\kappa \to \infty$.

Periodic points in the context of fixed points

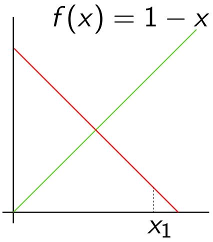
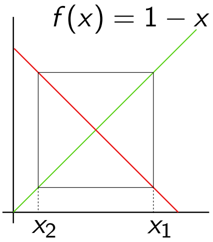


Figure: The periodic points of the function $f(x) = 1 - x$

Eventhough $x = \frac{1}{2}$ is a fixed point of $f(x)$, it cannot be found by taking the limit $\lim_{n\to\infty} f^n(x_1)$. Indeed,

$$x_{2n} := f^{2n}(x_1) = x_1$$
$$x_{2n+1} := f^{2n+1}(x_1) = 1 - x_1$$



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Recall the relaxing update law

$$A_0^{\kappa+1} = cA_0^{\kappa} + (1-c)(A_{\text{background}} + A_{\text{induced}}^{\kappa}), 0 < c \lesssim 1$$

Dynamic relaxation

In the 1-D case, no convergence appeared when $f'(x) = -1$.
Recall the relaxing update law

$$A_0^{\kappa+1} = cA_0^{\kappa} + (1-c)(A_{\text{background}} + A_{\text{induced}}^{\kappa}), 0 < c \lesssim 1$$

Choose c at every mesh point z_n, so that convergence is fastest, e.g. The convergence for the function $f(x) = \frac{1}{2}$ is immediate.

Extraplation

Since the problem arises from starting too far away from the next self consistent solution, try to predict by extrapolation.

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Point-wise extrapolation

$A_0^{\lambda_n}(z), A_0^{\lambda_{n-1}}(z)$ the self consistent potentials for λ_n, λ_{n-1}, respectively. Guess the next self consistent potential (and use it as a starting A_0) by

$$A_0^{\lambda_{n+1}}(z) = \frac{A_0^{\lambda_n}(z) - A_0^{\lambda_{n-1}}(z)}{\lambda_n - \lambda_{n-1}}(\lambda_{n+1} - \lambda_n) + A_0^{\lambda_n}(z) \tag{1}$$

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- We used the proper prescription for the vacuum polarization $\langle \rho(z) \rangle$ calculated in [WZ20], to study the effect the backreaction of the Klein-Gordon field has on a background constant electric field.
- We find that the backreaction raises the energy of the modes enough so as to avoid instabilities found when ignoring backreaction.
- However, convergence is still not fast enough to be able to correctly study the whole of the λ parameter space.

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1. Redoing the calculations for Neumann boundary conditions,
2. Studying the effect of the size of the interval has for the different boundary conditions,
3. Redoing the calculations for mixed (Robin) boundary conditions.

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[AW83] Jan Ambjørn and Stephen Wolfram. "Properties of the vacuum. 2. Electrodynamic". In: *Annals of Physics* 147.1 (1983), pp. 33–56. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(83)90066-0. URL: https://www.sciencedirect.com/science/article/pii/0003491683900660.

[WZ20] Jonathan Wernersson and Jochen Zahn. "Vacuum polarization near boundaries". In: *Phys. Rev. D 103, 016012 (2021)* (Oct. 12, 2020). DOI: 10.1103/PhysRevD.103.016012. arXiv: 2010.05499v2 [hep-th].

Vacuum polarization in 1+1 spacetime dimensions

Thank you for listening