Fixed point problems — an introduction

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***Abstract* – This paper surveys a number of fundamental results on the existence and uniqueness of fixed points for certain classes of possibly nonlinear operators. I do not try to be exhaustive, but merely to present the results that are more useful in the context of signal and image reconstruction. Some specific aspects pertaining to linear operators, and linear operators in finite dimensional spaces, are also discussed. It is shown that the set of fixed points of a nonexpansive operator is either empty or convex. Under rather general conditions this shows that the minimum norm solution of an operator equation of the form $x = Ax$ exists and is unique, provided that A is nonexpansive. This is a new result, which has interesting practical consequences in signal and image reconstruction problems and in other engineering applications.**

I. INTRODUCTION

Briefly speaking, a fixed point of an operator T is a solution of the equation $x = Tx$. This paper surveys the basic theorems in fixed point theory, that is, Banach's theorem, for contractive mappings, Brouwer's theorem, which holds for any continuous mapping in a finite-dimensional space, and Schauder's theorem, which generalizes Brouwer's result to infinite-dimensional Banach spaces.

The interest of these results, and their practical consequences and applications to problems such as signal and image reconstruction, tomography, telecommunications, interpolation, extrapolation, quantizer design, signal enhancement, signal synthesis, filter synthesis, among many others, are quite well-known and need no further comments. A quick glance through [1], for example, should convince any reader of the practical interest of the subject: many interesting practical problems can be recast as fixed point problems.

For example, let x be a signal of interest, and let y be a distorted version of x. Assume further that y and x are related through the operator equation $y = Dx$ (y might be the signal measured at the receiving end of a transmission system D, and x the transmitted signal). The problem is how to estimate x given y and the model D of the distortion that x underwent.

If x satisfies a constraint equation $x = Cx$, then the identity

$$x = Cx + \mu(y - DCx)$$

holds. Under rather general conditions, the solution to this equation will be the unknown signal x. This observation is the key to many iterative constrained restoration algorithms

$$x_{i+1} = Cx_i + \mu(y - DCx_i). \qquad (1)$$

The fixed point results discussed in this paper are the fundamental theoretical components of such techniques, and lead to conditions under which the sequence defined by (1) converges to a solution of the initial problem.

The notation used is standard. Symbols such as $\mathbb{R}$ and $\mathbb{C}^n$ have the usual meaning, and $\mathbb{M}_n$ denotes the set of $n \times n$ matrices over the complex field. The end of a proof is marked with the symbol $\blacksquare$.

A. Structure of the paper

The structure of the paper is as follows. Section II introduces the main concepts and definitions, and section III deals with Banach's fixed point theorem. Brouwer's theorem is discussed in section IV. Its proof is based on a combinatorial lemma known as Sperner's lemma. Schauder's theorem is the subject of section V.

To keep the paper reasonably self-contained, the basic elements of graph theory, which are required for the proof of Sperner's lemma, are presented. This limits the mathematical prerequisites to the rudiments of functional analysis, such as presented in [2].

In section VI, it is shown that the set of fixed points of a nonexpansive operator is either empty or closed and convex. This is, as far as I know, a new result. It has important consequences, namely, it shows that the minimum norm solution of an operator equation of the form $x = Ax$ exists and is unique provided that A is nonexpansive.

Section VII addresses some specific problems related to the convergence of sequences of successive approximations and matrix theory.

The paper closes with section VIII, which contains comments and refers to a number of additional works that might be useful to readers interested in comprehensive treatments of this subject.

II. CONCEPTS AND DEFINITIONS

I begin with some standard definitions. The metric function of any of the mentioned metric spaces is denoted by d.

Definition 1 (contraction mapping) Let X be a metric space. A mapping $T : X \to X$ is a contraction mapping if there exists a positive real $\lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda\, d(x, y), \qquad (2)$$

for every two $x, y \in X$.

If X is a normed space, then T is a contraction if

$$\|Tx - Ty\| \leq \lambda\, \|x - y\|.$$

If T is linear, this reduces to

$$\|Tx\| \leq \lambda\, \|x\|$$

for all $x \in X$. Thus, a linear operator $T : X \to X$ is a contraction if its norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$$

is bounded by a number $0 < \lambda < 1$.

Contraction mappings are well behaved and easy to study. Unfortunately, condition (2) is often too strong for practical applications. The following definition introduces a less stringent concept which is quite useful.

Definition 2 (nonexpansive mapping) Let X be a metric space. A mapping $T : X \to X$ is nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \tag{3}$$

for every two $x, y \in X$.

If X is a normed space, then T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|.$$

If T is linear, this reduces to

$$\|Tx\| \leq \|x\|$$

for all $x \in X$. Thus, a linear operator $T : X \to X$ is nonexpansive if its norm satisfies $\|T\| \leq 1$.

The following class of mappings is less general but nevertheless important.

Definition 3 (strictly nonexpansive mapping) Let X be a metric space. A nonexpansive mapping $T : X \to X$ is strictly nonexpansive if

$$d(Tx, Ty) = d(x, y) \quad \Rightarrow \quad x = y, \tag{4}$$

for every two $x, y \in X$.

If X is a normed space, the condition for strict nonexpansiveness reduces to

$$\|Tx - Ty\| < \|x - y\|$$

if and only if $x \neq y$. A linear operator T is strictly nonexpansive if

$$\|Tx\| < \|x\|$$

for all nonzero $x \in X$.

Note that any of these properties imply the continuity of T. It is time to define fixed point of a mapping.

Definition 4 (fixed point) Let T be a mapping of a metric space X into itself. A point $x \in X$ is called a fixed point of T in X if $x = Tx$.

Definition 5: A topological space is said to possess the fixed point property if every continuous mapping of the space into itself has a fixed point.

Of course, if a topological space has the fixed point property, any other topological space homeomorphic to the first will also possess the fixed point property. In other words, the fixed point property is a topological property.

A. Finite dimensional spaces

The spectral norm [3] of an arbitrary matrix T is denoted by $\|T\|$ and defined by

$$\|T\| = \sup_{\|x\|=1} \|Tx\|,$$

where the symbol $\|\cdot\|$ in the right-hand side denotes the Euclidean norm in $\mathbb{C}^n$.

The spectral radius $\rho(T)$ of a matrix $T \in \mathbb{M}_n$, with eigenvalues $\{\lambda_i\}_{1 \leq i \leq n}$, is defined by

$$\rho(T) = \max_{1 \leq i \leq n} |\lambda_i|.$$

In words, the spectral radius of a matrix is the greatest of its eigenvalues, in absolute value. The following theorem relates the spectral norm and the spectral radius of a matrix.

Theorem 1: The spectral norm of $T \in \mathbb{M}_n$ is given by

$$\|T\| = \sqrt{\rho(T^H T)}, \tag{5}$$

where $\rho(T)$ denotes the spectral radius of T.

Proof: Seeking the stationary points of the continuous function $\phi(x) = \|Tx\|^2$, with the constraint $\|x\| = 1$, leads to the equation

$$T^H T x = \lambda x, \tag{6}$$

where λ appears as a Lagrange multiplier. This shows that the unitary vectors x that render $\|T\|$ stationary are the eigenvectors of $T^H T$. If x is an eigenvector of $T^H T$ then

$$\|Tx\|^2 = x^H T^H T x = |\lambda| \, \|x\|^2 = |\lambda|.$$

The function ϕ is continuous and the set defined by $\|x\| = 1$ is compact. Therefore, the absolute maximum (minimum) of $\phi(x)$ is attained by the eigenvector of $T^H T$ that corresponds to its largest (smallest) eigenvalue in absolute value. These eigenvalues will be denoted by λ_{max} and λ_{min}, respectively, and the associated eigenvectors by v_{max} and v_{min}.

Relation (5) follows from this and the definition of norm. Note that $\|Tx\| \leq \|T\| \, \|x\|$. Also, since $T^H T$ is Hermitian, $\|T\|^2 = \rho(T^H T) = \|T^H T\|$. ■

Note that the spectral norm of a Hermitian matrix equals its spectral radius.

Corollary 1: If $T \in \mathbb{M}_n$ is nonexpansive, then $\rho(T) \leq 1$. If $T \in \mathbb{M}_n$ is strictly nonexpansive, then $\rho(T) < 1$.

Proof: A direct consequence of the inequality $\rho(T) \leq \|T\|$. ■

The converse of the corollary is not true. It is easy to exhibit examples of expanding linear operators, that is, matrices with norm greater than one, which have spectral radius less than unity. This can never occur for Hermitian or normal matrices, whose norms and spectral radii are necessarily equal. A simple example of a matrix whose norm and spectral radius differ is given by Varga [4].

III. BANACH'S THEOREM

The simplest result about the existence of a fixed point is Banach's theorem, sometimes called the contraction mapping theorem. It was first stated and proved by Banach around 1922 [5], as part of his doctoral thesis.

One word regarding the notation: the symbol T^k is a shorthand for the composition of T, k times, that is

$$T^k \equiv \underbrace{T \cdot T \cdot T \cdots T}_{k \text{ times}}.$$

Theorem 2 (Banach) Let M be a nonempty, complete metric space, and let $T : M \to M$ be a contraction mapping. Then, the equation $x = Tx$ has one and only one solution in M, given by

$$x = \lim_{n\to\infty} T^n y,$$

for every $y \in M$.

Proof: I will first show that, for every positive ϵ, there is a positive integer N such that, for every integers $n \geq N$ and $k > 0$,

$$d(T^n y, T^{n+k} y) < \epsilon.$$

The triangle inequality implies

$$\begin{aligned} d(T^n y, T^{n+k} y) &\leq d(T^n y, T^{n+1} y) + \\ &+ d(T^{n+1} y, T^{n+2} y) + \cdots + d(T^{n+k-1} y, T^{n+k} y). \end{aligned} \tag{7}$$

But, since T is a contraction mapping,

$$d(T^m y, T^{m+1} y) \leq \lambda\, d(T^{m-1} y, T^m y),$$

and therefore

$$d(T^m y, T^{m+1} y) \leq \lambda^m d(y, Ty).$$

Inserting this in (7) yields

$$\begin{aligned} d(T^n y, T^{n+k} y) &\leq (\lambda^n + \cdots + \lambda^{n+k-1})\, d(y, Ty) \\ &\leq \frac{\lambda^n}{1-\lambda}\, d(y, Ty). \end{aligned} \tag{8}$$

The second member of (8) can be made arbitrarily small by a suitable choice of n, because λ is positive and less that 1. Since the space M was assumed nonempty and complete, the sequence $T^n y$ has a limit in M.

The limit of the sequence $T^n y$ is a fixed point of T. This follows from the continuity of T, since

$$T \lim_{n\to\infty} T^n y = \lim_{n\to\infty} T^{n+1} y = \lim_{n\to\infty} T^n y.$$

■

In spite of its importance, the contraction mapping theorem is not strong enough for certain purposes. In fact, many operators frequently found do not satisfy condition (2).

The question of whether or not a result similar to theorem 2 holds for the more general class of nonexpansive operators has a negative answer. In fact, as the following examples show, a nonexpansive mapping may have any number of fixed points.

Example 1: The translation $f \to f + g$ $(g \neq 0)$ in a Banach space is nonexpansive and does not have any fixed points.

If $g = 0$ this changes drastically.

Example 2: The identity mapping is nonexpansive, and each point in its domain is a fixed point.

This raises the problem of defining and characterizing classes of nonexpansive mappings having fixed points. There is also the related and important uniqueness problem. I will restrict myself to a few results that are often useful in applications.

IV. BROUWER'S THEOREM

This section discusses Brouwer's theorem, which asserts that every continuous mapping of a closed n-ball to itself has a fixed point. Unlike Banach's theorem, which applies to contractions only, Brouwer's theorem applies to any continuous mapping. It is restricted, however, to finite-dimensional spaces.

Theorem 3 (Brouwer) Every continuous mapping f of a closed n-ball to itself has a fixed point.

The proof depends on a very simple result from graph theory.

A. Graphs

In simple terms, a *graph* G is a triple consisting of a set V of vertices, a set E of edges, and an *incidence* function which associates with each edge a pair of vertices. The sets V and E are assumed to be nonempty and finite.

As the name suggests, graphs have a simple graphical representation. The set of vertices can be represented as a set of points in the plane, and each edge e can be represented as a line joining the pair of vertices which the incidence function associates with e. For an example, see figure 1.

For simplicity I will consider simple graphs only. By definition, a graph is *simple* if it has no loops (that is, edges with identical ends) and if no two of its edges join the same pair of vertices.

The number of edges incident with a vertex v of a simple graph G is called the *degree* $(\deg v)$ of that vertex. Graphically, the degree of v is the number of lines which connect v to other vertices of the same graph.

Proposition 1: In any graph G, the number of vertices with odd degree is even.

Proof: Denote by V_e and V_o the sets of vertices of the graph with even and odd degrees, respectively. The sets V_e and V_o are disjoint and their union is V. Adding the degrees of each vertex of a graph gives an even number, since every edge is counted twice in the process. Therefore

$$\sum_{v\in V_e} \deg v + \sum_{v\in V_o} \deg v$$

must be an even number. But the sum of the degrees of the vertices with even degree must be even, and, consequently,

$$\sum_{v\in V_o} \deg v$$

must be an even number. ■

The fixed point theorem due to Brouwer, which counts so many applications in analysis, can be established on the basis of as simple a result as proposition 1. I will show, following Bondy [6], that Brouwer's theorem is a consequence of a combinatorial result due to Sperner, which is itself a consequence of proposition 1.

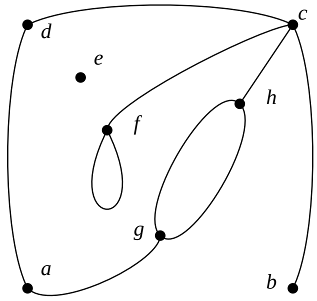

Figure 1 - An example of a graph with $V = \{a, b, c, d, e, f, g, h\}$.

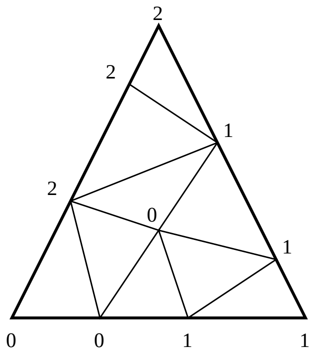

Figure 2 - An example of simplicial subdivision and proper labeling. Note the existence of one distinguished triangle.

B. Sperner's lemma

A few definitions are needed before stating Sperner's lemma.

Let T be a closed triangle in the plane. A subdivision of T into a finite number of smaller triangles is said to be *simplicial* if any two intersecting triangles have either a vertex or a whole side in common.

Definition 6: Given a simplicial subdivision of T, a labeling of the vertices of triangles in the subdivision in three symbols 0, 1 and 2 is said to be *proper* if the following conditions are satisfied.

1. The three vertices of T are labeled 0, 1 and 2 in any order.
2. For $0 \leq i < j \leq 2$, each vertex on the side of T joining vertices labeled i and j is labeled either i or j.

A triangle in the subdivision may or may not receive all three possible labels. If it does, it is called a *distinguished triangle*.

An example of two-dimensional simplicial subdivision and proper labeling is shown in figure 2. Note that, although we have been talking about triangles, there is nothing that prevents us from extending the definitions above to other simplices (tetrahedrons, for example).

Sperner's lemma, which was proved by Sperner in 1928, states the following.

Lemma 1 (Sperner) Every properly labeled simplicial subdivision of a triangle has an odd number of distinguished triangles.

The following proof is essentially the one given in [6].

Denote by $T_1, T_2, \ldots, T_n$ the regions (triangles) of the subdivision, and let T_0 be the region of the plane outside the triangle T. Consider a graph G with as many vertices $v_0, v_1, v_2, \ldots, v_n$ as regions T_i, and edges defined by the following rule: v_i is connected to v_j if and only if the regions T_i and T_j have a common boundary whose vertices have labels 0 and 1.

The following results are required.

Proposition 2: None of the vertices of G can have degree three.

Proof: Assume that there is a vertex v_i of G with degree three, and let T_i be the triangle of the subdivision associated with v_i. Let the vertices of the triangle T_i be a, b and c, and ab, bc and ac its sides.

Since, by hypothesis, v_i has degree three, the vertices a and b of the side ab of T_i have labels 0 and 1, in some order. The same is true of the vertices b and c of side bc, which implies that a and c have the same label. Thus, v_i cannot possibly have degree three. ■

Proposition 3: A triangle T_i is distinguished if and only if the corresponding vertex v_i of G has degree one.

Proof: For if T_i is a distinguished triangle one of its vertices will have label 0 and another will have label 1. Consider the set S of all triangles of the subdivision which have a common boundary with T_i. The previous observation and the fact that the subdivision is simplicial imply that exactly one of the triangles of S will be adjacent to the edge of T_i with labels 0 and 1. This implies that the vertex v_i of G will have degree 1. ■

Proposition 4: The vertex v_0 of G, which corresponds to the region T_0 outside T, has odd degree.

Proof: The two edges of T incident with the vertex with label 2 may be ignored in the discussion. For, if the labeling is proper, any two labeled points lying in one of these edges will not have labels 0 and 1 simultaneously.

Consider the edge e of T incident with the vertices labeled 0 and 1, and imagine that the simplicial division took place sequentially in time. At step $k = 0$, the labels 0 and 1 were added to the two vertices of e. Had the subdivision stopped at this point the degree of v_0 in G would be one.

Each step $k > 0$ of the subdivision may bring an additional labeled point to e. This can only leave the degree of v_0 unchanged or increase it by two (consider adding a point labeled 0 between two points labeled 1 or vice-versa). This and the previous observation imply that the degree of v_0 in G must be odd. ■

Sperner's lemma can now be demonstrated. By proposition 4 the vertex v_0 has odd degree, and therefore by proposition 1 an odd number of vertices $v_1, v_2, \ldots, v_n$ have odd degree. By proposition 2 none of these may have degree 3, and so those with odd degree must have degree one. By proposition 3 the existence of a vertex with degree one is equivalent to the existence of a distinguished triangle. ■

C. Brouwer's theorem in one dimension

Before discussing Brouwer's theorem in n dimensions, it is enlightening to seek its meaning in one dimension, which turns out to be quite clear (see figure 3).

A rigorous proof of the result is just as clear. To begin with, notice that it is possible to assume without loss of generality that $x = 0$ and $x = 1$ are not fixed points (if they are there

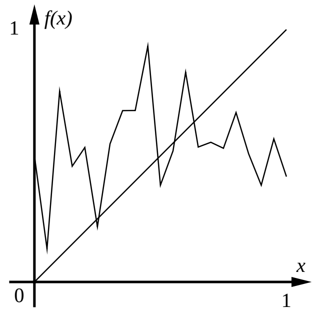

Figure 3 - One-dimensional interpretation of Brouwer's theorem: the continuous function f must cross the line $y = x$ at least once.

is nothing else to show). Let $g(x) = f(x) - x$, and consider

$$g(0) = f(0),$$

$$g(1) = f(1) - 1.$$

Since $0 \leq f(x) \leq 1$, $g(0) > 0$ and $g(1) < 0$. But g is a continuous function, and therefore there must exist a real $x \in [0, 1]$ such that $g(x) = 0$, which means $f(x) = x$. ■

D. Brouwer's theorem in two dimensions

The proof of Brouwer's theorem in n dimensions is considerably harder. It is preferable to begin by discussing the two-dimensional situation, which asserts that a continuous mapping of the closed circle into itself has a fixed point. Since a closed triangle and a closed circle are homeomorphic it is sufficient to prove the result for a closed triangle T.

Since T is convex, any of its points can be expressed as a triple (a_0, a_1, a_2), such that each a_i is nonnegative and $\sum a_i = 1$. These are called the *barycentric coordinates* of T.

Let f be a continuous function of the closed triangle T in itself. I will use the notation

$$f(a_0, a_1, a_2) = (b_0, b_1, b_2)$$

where the a_i are the barycentric coordinates of a point of T, and the b_i are the barycentric coordinates of its image under f.

Denote by S_i $(i = 0, 1, 2)$ the set of points $x = (a_0, a_1, a_2)$ such that $b_i \leq a_i$. The vertex of T with coordinate $(1, 0, 0)$ belongs to S_0, and thus S_0 is not empty. A similar argument shows that S_1 and S_2 are not empty.

I claim that any point belonging to the intersection of S_0, S_1 and S_2 is a fixed point of f. In fact, if $x = (a_0, a_1, a_2)$ is one such point and (b_0, b_1, b_2) its image under f, then $b_i \leq a_i$ for $i = 0, 1, 2$ by the definition of S_i. Since $\sum a_i = \sum b_i$ it follows that $b_i = a_i$, that is, $y = x$ and x is a fixed point of f.

To show that the intersection of the sets S_i is nonempty the following results are required.

Proposition 5: Any simplicial subdivision of T can be properly labeled, in the sense of definition 6, in such a way that each vertex labeled i belongs to S_i.

Proof: Let x_0 be the vertex of T with coordinates $(1, 0, 0)$. The coordinates (b_0, b_1, b_2) of its image under f are nonnegative and satisfy $\sum_i b_i = 1$. This implies $b_0 \leq 1$, and therefore $x_0 \in S_0$. Similarly one can show that vertices $x_1 = (0, 1, 0)$ and $x_2 = (0, 0, 1)$ belong to S_1 and S_2, respectively. Thus, the condition mentioned in the proposition is compatible with condition 1 of definition 6.

Let x be any point in the side of T with vertices x_0 and x_1, and therefore with coordinates $(\alpha, 1 - \alpha, 0)$, for some $0 \leq \alpha \leq 1$.

Consider the image $y = f(x)$ of x. It either belongs to the same side of T as x or not. If it does $y = (\beta, 1 - \beta, 0)$, with $0 \leq \beta \leq 1$, and at least one of the inequalities $\beta \leq \alpha$ or $1 - \beta \leq 1 - \alpha$ will hold, meaning that x belongs to at least one of the sets S_0 or S_1.

If y and x are not on the same side of T its third coordinate will be nonzero. Since the third coordinate of x is zero, x can not belong to S_3. However, it does belong to one of the sets S_0 or S_1.

This shows that any point x of the side of T with vertices x_0 and x_1 belongs to S_0 or S_1. Thus, any vertex of the subdivision of T belonging to the side of T with vertices x_0 and x_1 may be labeled 0 or 1.

It is possible to proceed similarly for points in one of the other sides of T. This establishes that the condition mentioned in the proposition is compatible with condition 2 of definition 6. ■

Proposition 6: The sets S_i $(i = 0, 1, 2)$ are closed.

Proof: Consider a convergent sequence $y^{(n)}$ $(n = 1, 2, \ldots)$ of points of S_i and let y be its limit. If

$$y^{(n)} = (a_0^{(n)}, a_1^{(n)}, a_2^{(n)})$$

and

$$f(y^{(n)}) = (b_0^{(n)}, b_1^{(n)}, b_2^{(n)}),$$

then $b_i^{(n)} \leq a_i^{(n)}$. Since f is continuous, the coordinates b_i and a_i of the limit points x and y will have to satisfy a similar relation, meaning that $y \in S_i$. ■

To proceed with the proof of Brouwer's theorem, consider an arbitrary simplicial subdivision of T and a proper labeling satisfying proposition 5. By Sperner's lemma there is a triangle in the subdivision whose vertices belong to S_0, S_1 and S_2. Since this may happen when the subdivisions have arbitrarily small diameter there are points of S_0, S_1 and S_2 arbitrarily close to one another. Since the sets S_i are closed and non-empty their intersection is non-empty, and f will have at least one fixed point. ■

E. The general case

The arguments used to prove Brouwer's theorem in two dimensions still hold in n-dimensional spaces. The necessary generalization of Sperner's lemma is straightforward. Instead of three sets S_i one would need $n + 1$ sets, the barycentric coordinates would consist of $n + 1$ numbers, and so on.

V. SCHAUDER'S THEOREM

Brouwer's theorem is valid in a finite-dimensional setting only. The following theorem, due to Schauder, gives a sufficient condition for a possibly infinite-dimensional Banach space to possess the fixed point property.

Theorem 4 (Schauder) Any compact convex nonempty subset of a Banach space has the fixed point property.

This is the same as stating that any continuous mapping T of a compact convex subset S of a Banach space into itself has a fixed point.

The proof is patterned after [7], [8]. The main idea is to build an infinite sequence of mappings of S into itself, such that each member of the sequence has a fixed point. From these, a subsequence is extracted which is showed to converge to a fixed point of T. It all starts with the following lemma.

Lemma 2: Let X be a Banach space and E a compact convex subset of X. Then, for any $\epsilon > 0$, there exists an $F : E \to C$ (C is a finite dimensional subset of E) such that $\|Fx - x\| < \epsilon$.

Proof: Since E is compact there are points $x_1, x_2, \ldots, x_n$ belonging to E and such that

$$\min_{1\le i\le n} \|x - x_i\| < \epsilon,$$

that is, any $x \in E$ is at a distance less that ϵ from at least one of the x_i.

Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $g(t) = 0$ if and only if $t \ge 1$. The functions

$$h_i(x) = \frac{g\left(\dfrac{\|x - x_i\|}{\epsilon}\right)}{\displaystyle\sum_{i=1}^{n} g\left(\dfrac{\|x - x_i\|}{\epsilon}\right)}, \qquad (1 \le i \le n)$$

are continuous on E, nonnegative and satisfy

$$\sum_{i=1}^{n} h_i(x) = 1,$$

$$h_i(x) = 0, \quad \text{if } \|x - x_i\| \ge \epsilon.$$

Let C be the convex hull of $x_1, x_2, \ldots, x_n$, which is finite-dimensional and contained in E, since E is convex. Define $F : E \to C$ by

$$Fx = \sum_{i=1}^{n} h_i(x) x_i.$$

Note that Fx is a convex combination of the x_i, and consequently $Fx \in C$. Also,

$$\begin{aligned} \|Fx - x\| &= \left\| \sum_{i=1}^{n} h_i(x)\, x_i - \sum_{i=1}^{n} h_i(x)\, x \right\| \\ &\le \sum_{\|x-x_i\|<\epsilon} h_i(x)\, \|x_i - x\|, \end{aligned}$$

which is less that ϵ. ■

It is now possible to proceed with the proof of Schauder's theorem. The lemma implies the existence of finite-dimensional subsets C_k of X and continuous mappings F_k such that

$$F_k : E \to C_k \cap E, \qquad (1 \le k \le n)$$

and

$$\|F_k x - x\| \le \frac{1}{k}.$$

The mappings T_k defined by

$$T_k \equiv F_k T$$

from E into F_k are continuous. Their restriction to $C_k \cap E$,

$$T_k : C_k \cap E \to C_k \cap E,$$

by Brouwer's theorem, has a fixed point x_k. Now,

$$\|Tx - T_k x\| = \|Tx - F_k Tx\| \le \frac{1}{k},$$

and consequently $\|Tx - T_k x\| \to 0$ as $k \to \infty$. Replacing x by x_k, and using the fact that x_k is a fixed point of T_k, leads to

$$\|Tx_k - T_k x_k\| = \|Tx_k - x_k\| \to 0$$

as $k \to \infty$.

Since E is compact the sequence x_k has a convergent subsequence x_k^*. Let its limit be x^*. Since $\|Tx_k^* - x_k^*\| \to 0$, $\|Tx^* - x^*\| = 0$, and $x^* = Tx^*$. ■

The following corollaries are immediate consequences of Schauder's theorem.

Corollary 2: Any nonexpansive mapping of a compact and convex subset of a Banach space into itself has a fixed point.

Proof: A nonexpansive mapping is continuous. ■

Corollary 3: Any strictly nonexpansive mapping of a compact and convex subset of a Banach space into itself has one and only one fixed point.

Proof: The existence of fixed points being guaranteed by Schauder's theorem, it remains to be shown that there can be no more than one fixed point. But this is a trivial consequence of definition 4. For, if a and b are distinct fixed points, then

$$\|Ta - Tb\| = \|a - b\| < \|a - b\|,$$

which is impossible. The contradiction implies $a = b$. ■

VI. CONVEXITY OF THE SET OF FIXED POINTS

The examples previously given show that the set of fixed points of a given nonexpansive mapping may contain any number of elements. In spite of this, there is an useful property that the set of fixed points associated with the mapping has, as the following lemma shows.

Lemma 3: Let X be a Hilbert space. The set of fixed points of a nonexpansive mapping $T : X \to X$ is either empty or closed and convex.

Proof: Example 1 shows that the set of fixed points can be empty. If there is only one fixed point there is nothing to show. Consequently, let x and y be two fixed points, and put

$$z = \alpha x + (1 - \alpha) y.$$

Consider the inequalities

$$\|Tz - x\| = \|Tz - Tx\| \le \|z - x\| = (1 - \alpha)\|x - y\|,$$

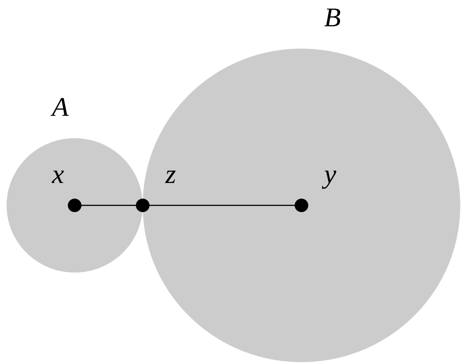

Figure 4 - Geometrical interpretation of theorem 3.

$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = \alpha\|x - y\|.$$

Adding them together leads to

$$\|Tz - x\| + \|Tz - y\| \leq \|x - y\|.$$

But

$$\|Tz - x\| + \|Tz - y\| \geq \|x - y\|,$$

by the triangle inequality, which shows that

$$\|Tz - x\| + \|Tz - y\| = \|x - y\|.$$

This means that

$$\|Tz - x\| = \|z - x\| = (1 - \alpha)\|x - y\|,$$

$$\|Tz - y\| = \|z - y\| = \alpha\|x - y\|.$$

Applying the parallelogram law [2]

$$\|a - b\|^2 + \|a + b\|^2 = 2\|a\|^2 + 2\|b\|^2 \qquad (9)$$

with $a = Tz - x$ and $b = z - x$ leads to

$$\|Tz-z\|^2+\|Tz-x+z-x\|^2 = 2\|Tz-x\|^2+2\|z-x\|^2.$$

Therefore

$$\begin{aligned} \|Tz - z\|^2 &\leq 2\|z - x\|^2 + 2\|z - x\|^2 - \\ &\quad - \|Tz - Tx + z - x\|^2 \\ &\leq 4\|z - x\|^2 - (\|Tz - Tx\| + \|z - x\|)^2 \\ &\leq 4\|z - x\|^2 - (\|z - x\| + \|z - x\|)^2 \\ &\leq 4\|z - x\|^2 - 4\|z - x\|^2, \end{aligned}$$

which means that $\|Tz - z\| = 0$ and consequently $z = Tz$. This shows that the set of fixed points of T is convex.

It remains to show that it is closed. Let z_n be a sequence of fixed points, and let z be its limit. To see that z is also a fixed point of T note that

$$\|z_n - Tz\| = \|Tz_n - Tz\| \leq \|z_n - z\|.$$

This implies that

$$\lim_{n\to\infty} \|z_n - Tz\| = 0,$$

and consequently z_n converges to Tz. Hence $Tz = z$, which shows that z is also a fixed point of T. ■

A geometrical interpretation of the lemma is given in figure 4. The sets A and B are defined by

$$A = \{x : \|Tz - x\| \leq (1 - \alpha)\|x - y\|\},$$

$$B = \{x : \|Tz - y\| \leq \alpha\|x - y\|\}.$$

The image of z by T must lie in the intersection of A and B, which reduces to the point z. Thus, Tz must be equal to z.

Note that it is the strict convexity of the balls A and B that implies $z = Tz$. The lemma ceases to be true for Banach spaces that are not strictly convex, such as L_1.

Lemma 4: Let S be a closed and convex subset in a Hilbert space. Then S contains a unique element of smallest norm.

Proof: This is a well-known result. Choose a sequence $\{x_n\}$ of elements of S whose norms converge to

$$\alpha = \inf_{x\in S} \|x\|.$$

The convexity of S implies that $\|x_m + x_n\| \geq 2\alpha$, since $(x_m + x_n)/2 \in S$. On the other hand, again using the parallelogram identity (9),

$$\|x_m + x_n\|^2 + \|x_m - x_n\|^2 = 2(\|x_m\|^2 + \|x_n\|^2)$$

and therefore

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0,$$

since $\lim_{m,n}(\|x_m\|^2 + \|x_n\|^2) = 2\alpha^2$ and $\|x_m + x_n\|^2 \geq 4\alpha^2$. But S is closed, and therefore the sequence has a limit in S which can only be a point a with norm α.

There is only one such point a, since the existence of another point b with that property would imply the convergence of the sequence

$$a, b, a, b, \ldots$$

by the above argument. ■

Theorem 5: Let S be a compact and convex subset of a Hilbert space, and let A be a nonexpansive mapping which carries S into itself. Then, the minimum-norm solution of $x = Ax$ exists and is unique.

Proof: By lemma 3, A has a nonempty and convex set of fixed points. By lemma 4, this set contains a unique element with minimum norm. ■

VII. FIXED POINTS, SUCCESSIVE APPROXIMATIONS AND MATRIX THEORY

The following simple result is often useful.

Theorem 6: Let X be a Banach space. If the operator $T : X \to X$ satisfies $\|T\| < 1$, then

$$(I - T)^{-1} = \sum_{i=-\infty}^{+\infty} T^i,$$

the convergence being in norm. The equation $x = Tx + b$ has exactly one solution $(I - T)^{-1}b$ which can be found by successive approximations

$$x_{i+1} = Tx_i + b,$$

independently of $x_0 \in X$.

Proof:

$$(I-T)\sum_{i=0}^{n-1} T^i = I - T^n,$$

and $\|T^n\| \leq \|T\|^n$, which tends to zero as $n \to \infty$. ■
The theorem can not be obtained from Banach's theorem, since T is not required to be a contraction.

Consider the sequence of vectors $x_i \in \mathbb{C}^n$ defined by $x_{i+1} = Mx_i + b$, M and b being, respectively, a $n \times n$ matrix and a $n \times 1$ column vector. This is a model of the general linear stationary iterative algorithm of first order with *iteration matrix* M. This is a special case of the problem of theorem 6, in which the Banach space X is $\mathbb{C}^n$.

Defining the operator $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$Tx \equiv Mx + b,$$

one sees that $x_{i+1} = Tx_i$. If T is a contraction there exists a real λ, satisfying $0 \leq \lambda < 1$, and such that for any $u, v \in \mathbb{C}^n$,

$$\|Tu - Tv\| = \|M(u-v)\| \leq \lambda \|u - v\|.$$

This means that the norm of M is bounded by λ, and consequently its spectral radius $\rho(M)$ is also bounded by λ. By Banach's theorem, the sequence x_i converges to the unique fixed point x of T, that is, to a solution of $x = Mx + b$. The contraction condition implies that $\rho(M) \leq \lambda < 1$, that is, M has no eigenvalues equal to one. Consequently $I - M$ has no zero eigenvalues and is nonsingular.

Assume now that T is strictly nonexpansive. Then,

$$\|Tu - Tv\| = \|M(u-v)\| \leq \|u - v\|,$$

which implies that $\rho(M) < 1$. Although Banach's theorem does not hold in this case, one sees that $I - M$ must also be nonsingular, and that there exists a unique fixed point of T.

In fact, if $\rho(M) < 1$, the sequence x_i converges to a limit vector x which satisfies $x = Mx + b$, since

$$x = \sum_{k=0}^{+\infty} M^k b = (I - M)^{-1} b.$$

The error or residual at iteration k, $e_k = x_k - x$, is given by

$$e_k = Me_{k-1},$$

being related to the initial error e_0 by $e_k = M^k e_0$. Thus

$$\|e_k\| \leq \|M^k\| \, \|e_0\|.$$

Without imposing further restrictions on M one can only say that $\|M^k\|$ tends to zero with k, monotonically for sufficiently high k. For Hermitian, or, more generally, for normal matrices,

$$\|M^k\| = \rho(M)^k,$$

and the norm of the error decreases with each iteration, that is, the iterative method has the error-reduction property, independently of k. In this case, knowledge of $\|M\|$ allows a simple upper-bound for the error to be established. In general this is not the case, and it is necessary to study $\|M^k\|$ instead.

VIII. NOTES

There are many fixed point theorems beyond those presented here. See [7], [9], [10] for many other results. The background in functional analysis required for understanding these results is presented in [2]. A more advanced treatment can be found in [11].

Section II-A follows [12]. For an alternative proof of theorem 1 see [4]. A matrix T such that all its eigenvalues are less than unity in absolute value is sometimes called convergent to zero [4].

The combinatorial proof of Brouwer's theorem given in section IV is a detailed version of the proof given on [6]. A version of the proof written in Portuguese is available [13]. It is interesting to compare this proof with others, based on analytic arguments, such as those given in [8], [14], [15].

The proof of Schauder's theorem given is essentially the one found in [7], [8]. A proof based on topological arguments can be found in [9]. It has two main steps: first, it is showed that any compact and convex subset of a Banach space is homeomorphic to a compact convex subset of the Hilbert cube. Then, it is proved that the Hilbert cube has the fixed point property.

For a discussion and comparison of many more (125!) definitions that lead to fixed point theorems see [16].

Theorem 5 and lemma 3 are probably new.

Applications of the theorems discussed in this paper can be found throughout the literature. Some applications in signal enhancement, restoration and more are addressed in [1], [12], [17–23]. The books [24], [25] might also be of interest. A review of several important fixed point theorems can be found in [26].

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