Electrodynamic properties of the vacuum near boundaries

Santiago Sanz Wuhl
Institut für Theoretische Physik
Universität Leipzig

Abstract

In my M.Sc. project, I study the effects of the backreaction of a charged Klein-Gordon field coupled to an external electric field, in 1+1 dimensional spacetime.
In this talk, I motivate and state the problem I intend to solve, discuss existing results from the literature, and present preliminary results from my project.

Table of Contents

Motivation

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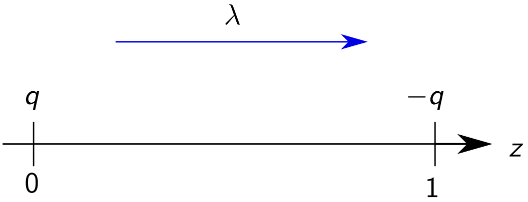
  1. Use of a mode sum formula for the charge density
  2. Neglecting broken down modes for Neumann boundary conditions

The problem



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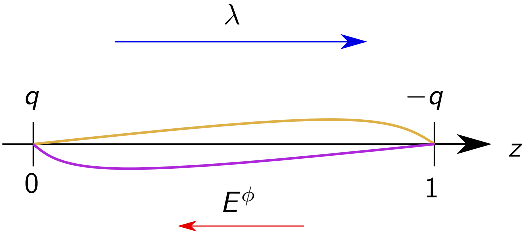

Semi classical Klein-Gordon-Maxwell equations

The goal is to solve the coupled system of equations[a]

$$\begin{cases} \left[D_\mu D^\mu + m^2\right] \phi(x) = 0 \\ \partial_\mu F^{\mu\nu} = \langle : j^\nu : \rangle_\phi \end{cases} \tag{1}$$

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and the boundary conditions

$$\begin{cases} \phi\,\big|_{z=0,1} & = 0 \\ E^\phi\big|_{z=0,1} & = 0 \end{cases} \tag{2}$$

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The problem

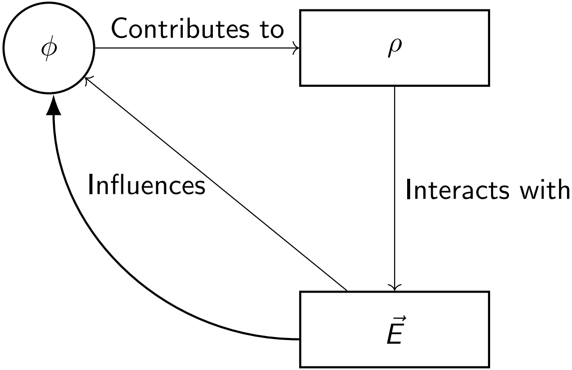

Table of Contents

Fixing the gauge

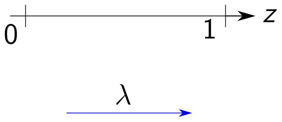

Figure: The background electric field

1. Constant electric field of strength λ pointing towards positive z.

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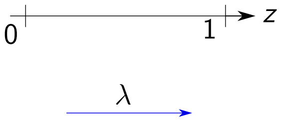

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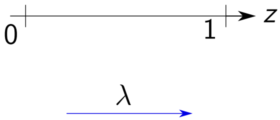

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$$A_0(t,z) = -\lambda(z - \frac{1}{2})$$

Electromagnetic potential

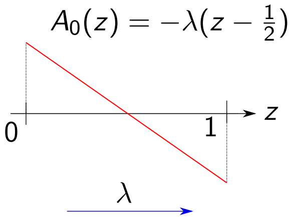

Figure: The background classical electric field and potential

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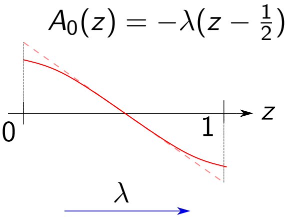

Figure: The background electric field and potential, with an additional (antisymmetric) potential.

1. Constant electric field of strength λ pointing towards positive z.
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3. To ensure antisymmetric solutions

$$A_0(t,z) = -\lambda(z - \frac{1}{2}) + A_0^{\phi}(z)$$

Klein-Gordon equation

With the chosen gauge the KGM equations turn into

$$((\partial_t + ieA_0)^2 - \partial_1^2 + m^2)\phi(x) = 0, \tag{3}$$

$$\partial_1^2 A_0^\phi = -\langle\rho(z)\rangle_\phi\,, \tag{4}$$

with

$$A_0(z) = -\lambda\left(z - \frac{1}{2}\right) + A_0^\phi(z), \tag{5}$$

$$x = (t, z) \in \mathbb{R} \times [0, 1] \tag{6}$$

Time independent Klein-Gordon equation

With the variable separation ansatz $\phi(x) = \phi_n(z)e^{-i\omega_n t}$,

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Mode equation

For some potential $A_0(z)$

$$\left([\omega_n - eA_0(z)]^2 + \frac{d^2}{dz^2} + m^2 \right) \phi_n = 0 \tag{7}$$

Without backreaction

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$$\left(\left[\omega_n + e\lambda \left(z - \frac{1}{2} \right) \right]^2 + \frac{d^2}{dz^2} + m^2 \right) \phi_n = 0 \tag{8}$$

Without taking backreaction into account, the KG equation can be solved by

$$\begin{aligned}\phi_n(z) = {} & a_n D_{i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{1+i}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z-\frac{1}{2}\right)\right)\right) \\ & + b_n D_{-i\frac{m^2}{2\lambda}-\frac{1}{2}}\left(\frac{i-1}{\sqrt{\lambda}}\left(\omega_n + \lambda\left(z-\frac{1}{2}\right)\right)\right)\end{aligned}$$

with $D_\nu(z)$ the parabolic cylinder functions.

The external field approximation limitations

Results by [AW83]:

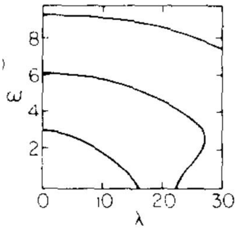

Figure: Positive energy levels for increasing λ **without** considering the backreaction of the massless Klein-Gordon field.

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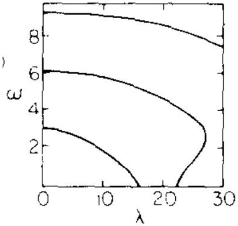

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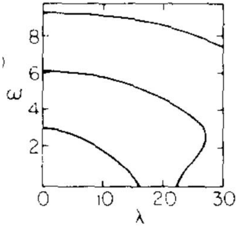

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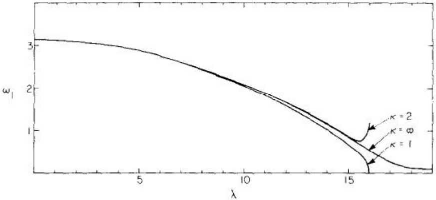

Figure: Positive energy levels for increasing λ **with** backreaction.

We quantize the field based on the mode solutions

$$\phi(t,z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n(z) e^{-i\omega_n t}, \tag{9}$$

with a_n, b_n the operators fulfilling

$$\left[a_n, a_m^\dagger\right] = \delta_{nm}, \qquad \left[b_n, b_m^\dagger\right] = \delta_{nm}, \tag{10}$$

$$[a_n, a_m] = [a_n, b_m] = \left[a_n, b_m^\dagger\right] = [b_n, b_m] = 0. \tag{11}$$

This also defines the vacuum by

$$a_n \left|0\right\rangle = b_n \left|0\right\rangle = 0 \tag{12}$$

Charge density

We calculate the **charge density** as the 0th component of the charge current

$$\rho(z) = ie \langle 0| \phi^*(z) D_0 \phi(z) - \phi(z) D_0 \phi^*(z) |0\rangle \tag{13}$$

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Ambjørn and Wolfram calculate it as one naïvely would,

$$\rho(z) = e \left(\sum_{n>0} (\omega_n - eA_0(z)) |\phi_n(z)|^2 + \sum_{n<0} (\omega_n - eA_0(z)) |\phi_n(z)|^2 \right) \tag{14}$$

Hadamard states and point-splitting

Assume that the state is of Hadamard form,

$$w_{\Omega}^{\phi\phi^*}(x,x') = \langle\Omega|\,\phi(x)\phi^*(x')\,|\Omega\rangle = H^{\phi\phi^*}(x,x') + R_{\Omega}^{\phi\phi^*}(x,x'), \tag{15}$$

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with

$$H^{\phi\phi^*}(x,x') = -\frac{1}{4\pi}U(x,x')\log\left(-(x-x')^2 + i\epsilon(x-x')^0\right) \tag{16}$$

and

$$U(x,x') = \exp\left(-i\int_0^1 A_\mu(x'+s(x-x'))(x-x')^\mu ds\right).$$

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Define the product of derivatives of fields as

$$\langle\Omega|\,D_\alpha\phi(x)(D_\beta\phi)^*(x)\,|\Omega\rangle = \lim_{x'\to x}\left[D_\alpha D_\beta'^{\,*}\left(w_{\Omega}^{\phi\phi^*}(x,x') - H^{\phi\phi^*}(x,x')\right)\right], \tag{17}$$

Renormalizing charge densities

Recall the mode decomposition of the field

$$\phi(t,z) = \sum_{n>0} a_n \phi_n(z) e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n(z) e^{-i\omega_n t}, \tag{18}$$

the expression for charge density

$$\rho(x) = ie \langle 0| \phi^*(x) D_0 \phi(x) - \phi(x) D_0 \phi^*(x) |0\rangle \,, \; D_0 = \partial_0 + ieA_0, \tag{19}$$

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Realizing the point splitting in the time direction $x'=(t+\tau,z)$, we calculate

$$\langle 0|\,\phi^*(x)D_0\phi(x')\,|0\rangle=-i\sum_{n<0}(\omega_n-eA_0(z))|\phi_n(z)|^2e^{-i\omega_n(\tau+i\epsilon)} \tag{20}$$

$$\langle 0| \, \phi^*(x) D_0 \phi(x') \, |0\rangle = -i \sum_{n<0} (\omega_n - eA_0(z)) |\phi_n(z)|^2 e^{-i\omega_n(\tau + i\epsilon)} \tag{21}$$

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$$D_0' H^{\phi\phi^*}(x, x') = -\frac{1}{2\pi}\frac{1}{\tau + i\epsilon} U(x', x) + \mathcal{O}(\tau) = -\frac{1}{2\pi}\left(\frac{1}{\tau + i\epsilon} - ieA_0\right) + \mathcal{O}(\tau) \tag{23}$$

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The renormalized charge density

$$\rho(z) = e \lim_{\tau \to 0} \left(\sum_{n>0} (\omega_n - eA_0(z)) |\phi_n(z)|^2 e^{i\omega_n(\tau + i\epsilon)} + \sum_{n<0} (\omega_n - eA_0(z)) |\phi_n(z)|^2 e^{-i\omega_n(\tau + i\epsilon)} \right) + \frac{e^2}{\pi} A_0(z) \tag{25}$$

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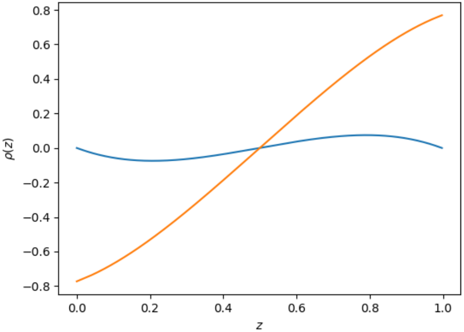
The renormalized charge density

2 Methods



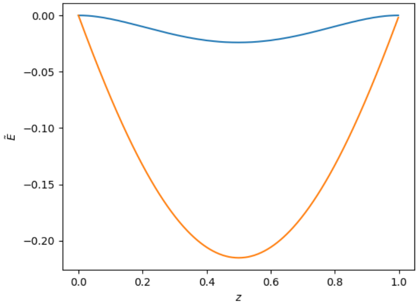
The induced charge density ρ resulting from the two different renormalization techniques with Dirichlet boundary conditions. $m = 0$, $\lambda = 5$.

- In orange, the charge density ρ calculated through the mode sum formula.
- In blue, the charge density ρ calculated through Hadamard point-splitting.

The induced fields

The induced electric field:

$$E^{\phi}(z) = \int_0^z \rho(z')dz' \qquad (27)$$



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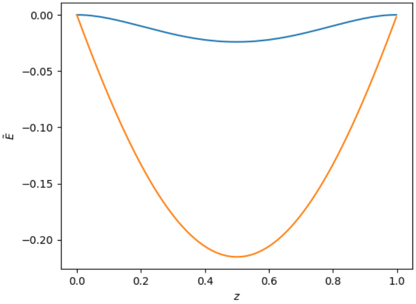
The induced fields

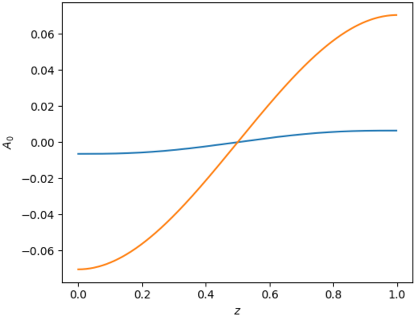
The induced electric field:

$$E^{\phi}(z) = \int_0^z \rho(z')dz' \qquad (27)$$

The induced electric potential

$$A_0^{\phi}(z) = -\int_0^z E^{\phi}(z')dz' \qquad (28)$$





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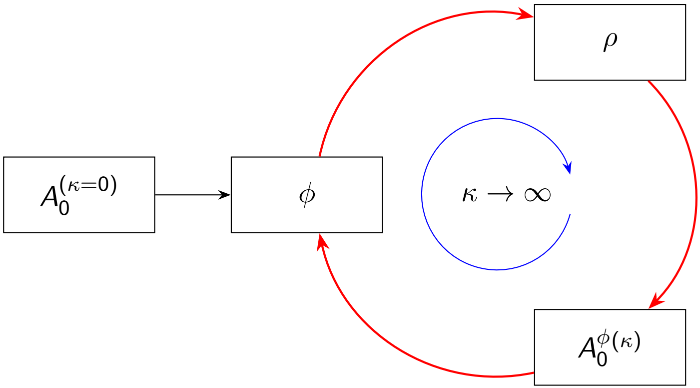
Closing the loop



$$\left(\left[\omega_n^{(\kappa+1)} + \lambda\left(z - \frac{1}{2}\right) - \left(A_0^{\phi}\right)^{(\kappa)}(z)\right]^2 + \frac{d^2}{dz^2} + m^2\right)\phi_n^{(\kappa+1)} = 0 \tag{29}$$

$$\left(A_0^{\phi}\right)^{(\kappa)}(z) = -\int_0^z \int_0^{z'} \rho^{(\kappa)}(z'')dz''dz', \tag{30}$$

Visualizing the impact of backreaction

2 Methods

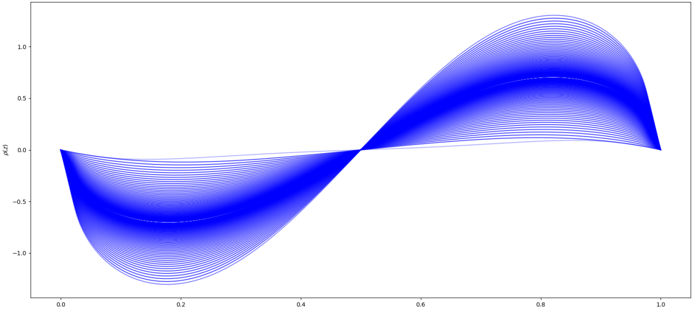


Figure: The convergence of $\rho^{\kappa}(z)$ as $\kappa \to \infty$ for $m = 0$, $\lambda = 15.5$.

Table of Contents

ω_n dependence on λ considering backreaction

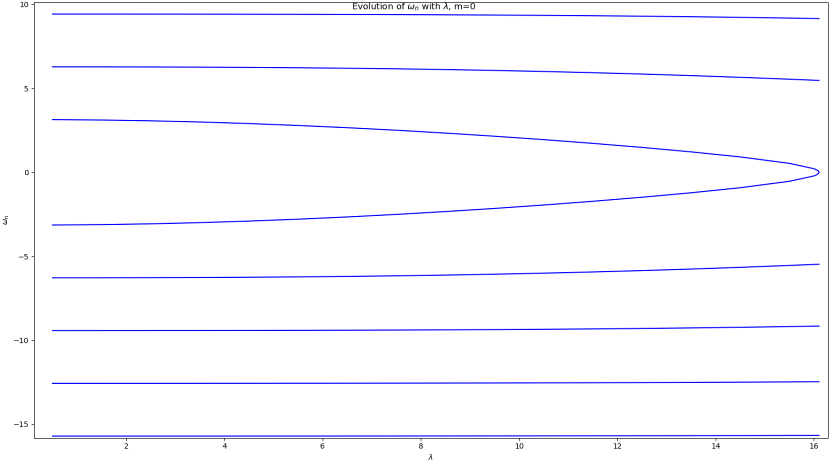


Figure: ω_n as a function of λ for the massless case.

Charge densities and E^{ϕ}

3 Preliminary results

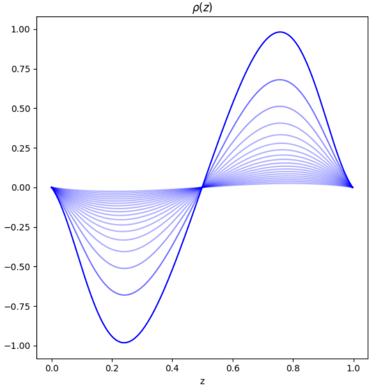


Figure: Charge densities ρ for increasing background electric field λ

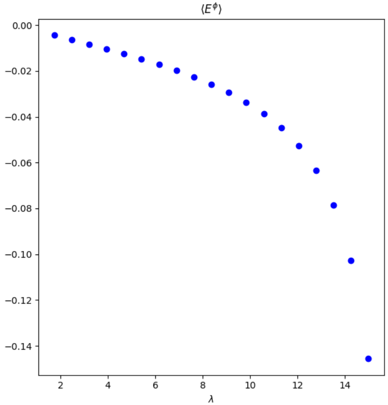


Figure: Average polarization for different background electric fields.

There is still work to be done:

1. For Dirichlet boundary conditions, study deeper the behavior of the critical λ value.
2. Repeat these calculations for Neumann boundary conditions, and finally for more general Robin boundary conditions

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = h_0 \phi(0) \tag{31}$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=1} = -h_1 \phi(1) \tag{32}$$

Table of Contents

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Electrodynamic properties of the vacuum near boundaries

Thank you for listening