2.1

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

We will mark the Fourier transform of $sinc(x)$ as $F(u) = \mathcal{F}[sinc(x)]$.

From the convolution theorem we get that

$$\mathcal{F}[sinc(x) * sinc(x)] = \mathcal{F}[sinc(x)] \times \mathcal{F}[sinc(x)]$$

So, our process will be like so-

1. Calculate $\mathcal{F}[sinc(x)]$
2. Calculate $\mathcal{F}[sinc(x) * sinc(x)]$
3. Using the inverse Fourier transform, calculate $sinc(x) * sinc(x)$

<u>Step 1</u>

We'll define the box function $h(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & otherwise \end{cases}$.

$$\mathcal{F}[h(t)] = \int_{-0.5}^{0.5} e^{-j2\pi f t} dt = \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-0.5}^{0.5} = -\frac{1}{2j\pi f}\left[e^{-j\pi f} - e^{j\pi f}\right]$$
$$= \frac{1}{\pi f} \times \frac{e^{j\pi f} - e^{-j\pi f}}{2j} = \frac{\sin(\pi f)}{\pi f} = sinc(f)$$

So $\mathcal{F}[h(t)] = sinc(f)$

$sinc(x)$ is even, meaning $sinc(-x) = sinc(x)$. So, we know that

$$(1)\ \mathcal{F}[sinc(f)] = \mathcal{F}[sinc(-f)]$$

From the duality property of Fourier transformations, we know that

$$(2)\ \mathcal{F}[sinc(-f)] = h(t)$$

Combining (1) and (2), we get that $\mathcal{F}[sinc(f)] = h(t)$.

<u>Step 2</u>

We know that the Fourier transformation of a convolution is the product of the Fourier transformations.

So, $\mathcal{F}[sinc(x) * sinc(x)] = \mathcal{F}[sinc(x)] \times \mathcal{F}[sinc(x)] = h^2(t) = h(t)$

<u>Step 3</u>

Since we know that $h(t) = \mathcal{F}[sinc(f)]$, by applying $\mathcal{F}^{-1}$, we get that $\mathcal{F}^{-1}[h(t)] = sinc(f)$.

And so, we got that

$$sinc(x) * sinc(x) = \mathcal{F}^{-1}\big[\mathcal{F}[sinc(x) * sinc(x)]\big] = \mathcal{F}^{-1}\big[\mathcal{F}[sinc(x)] \times \mathcal{F}[sinc(x)]\big]$$
$$= \mathcal{F}^{-1}[h^2(t)] = \mathcal{F}^{-1}[h(t)] = sinc(x)$$

So $sinc(x) * sinc(x) = sinc(x)$.

2.2

Zooming means inserting new pixels to the image with some method to define the values of the new pixels.

For example, zooming a 100×100 image to a 200×200 image involves adding 3 new pixels for every existing pixel.

We do it in three steps:

1. Take the original image and apply a Fourier Transform to it.
2. Zero-pad the Fourier Transformation.
3. Apply the Inverse-Fourier Transformation.

The resolution of the image increased when we zero-padded the Fourier Transformation, increasing the sampling rate.

This way, we didn't introduce new higher frequencies, and so maintained the frequencies within the Nyquist limit.

We did however add 'artificial' pixels that are an interpolation of the initial data, thanks to the zero-padding and the effects it had on the Inverse-Fourier Transformation.

2.3

$$f(t) = \sin(2\pi n t)$$

Need to prove-$\forall n \in \Re,\ \mathcal{F}\big(f(t)\big) = \frac{i}{2}\big(\delta(f+n) - \delta(f-n)\big)$

$$\begin{aligned}
\mathcal{F}\big(f(t)\big) &= \int_{-\infty}^{\infty} f(t) \times e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} \sin(2\pi n t) \times e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{j2\pi n t} - e^{-j2\pi n t}}{2j}\right) \times e^{-j2\pi f t}\, dt \\
&= \frac{1}{2j}\int_{-\infty}^{\infty} e^{j2\pi(n-f)t} - e^{-j2\pi(n+f)t}\, dt \\
&= \frac{1}{2j}\left(\int_{-\infty}^{\infty} e^{-j2\pi(f-n)t}\, dt - \int_{-\infty}^{\infty} e^{-j2\pi(f+n)t}\, dt\right) \\
&= \frac{1}{2j}(I_1 - I_2) = -\frac{j}{2}(I_1 - I_2) = \frac{j}{2}(I_2 - I_1)
\end{aligned}$$

From the definition of impulse, we get that

$$\delta'(f) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) \times e^{-j2\pi f t}\, dt = e^{-j2\pi \times 0 \times t} = 1$$

We also know that $\delta'(f) = 1(t) = \mathcal{F}[\delta(-t)] = \delta'(-f)$

And from the $(*)$ duality property we get that

$$\mathcal{F}[1(t)] = \mathcal{F}[\delta'(f)] = \mathcal{F}[\delta'(-f)] \overset{(*)}{\triangleq} \delta(f)$$

Meaning that $\mathcal{F}[1(t)] = \delta(f)$.

From the definition of Fourier transformations, we know that

$$\delta(f) = F[1(t)] = \int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt$$

And so,

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} e^{-j2\pi(f-n)t}\, dt = \delta(f-n) \\
I_2 &= \int_{-\infty}^{\infty} e^{-j2\pi(f+n)t}\, dt = \delta(f+n)
\end{aligned}$$

Substituting this result back, we get that

$$\mathcal{F}\big(f(t)\big) = \frac{j}{2}(I_2 - I_1) = \frac{j}{2}\big(\delta(f+n) - \delta(f-n)\big)$$

As required.