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Dual and Axiomatic Systems for Constructive S4, a Formally Verified Equivalence

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Abstract

We present a proof of the equivalence between two deductive systems for the constructive modal logic S4. On one side, an axiomatic characterization inspired by Hakli and Negri's Hilbert-style system of derivations from assumptions for modal logic K. On the other side, the judgmental reconstruction given by Pfenning and Davies by means of a so-called dual natural deduction approach that makes a distinction between valid, true and possible formulas. Both systems and the proof of their equivalence are formally verified using the Coq proof assistant.

Keywords: modal logic S4, constructive logic, natural deduction, dual calculus, formal verification, Coq

1 Introduction

The relevance of modal logic in philosophy and mathematics is well established. Moreover, its importance in computer science has come of age (see [14] for a brief introduction to modal logic from the computer science point of view). For instance, the monadic and computational interpretations of modal logic [6,4] for various type theories are deeply related to concurrent and distributed computations [21,32]. These applications entail the need for a verification process which nowadays cannot be thought of without the help of a proof assistant.

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To be able to carry out serious verification tasks with the help of these software programs, we need to provide them with an implementation of deductive methodologies akin to human (mathematical) thinking. This poses a problem in the case of modal reasoning, where the typical deductive systems in the literature are axiomatic or elaborated sequent calculi (see for example [26]). Moreover, some known natural deduction presentations of modal logic are sophisticated formalisms, that either involve the semantics, like labelled deduction systems ([31], see also [15] for a comprehensive overview of modal natural deduction systems); or require some clumsy conventions that make them difficult to be implemented in a proof assistant, for example, the use of special subproofs (boxes) and conventions in Fitch-style systems ([5], [9], [10, Section 1.6], [19, p. 34]) or elaborated cases like [3] for the specific case of constructive necessity.

In the case of constructive modal logic $\mathsf{S4}$ (see [29, Section 1.5] for a brief introduction to constructive modal logics), these problems are mitigated by considering the judgmental reconstruction of this logic proposed by Pfenning and Davies [24]. They use an approach that follows the constructive philosophy of Martin-Löf [22] by means of hypothetical and categorical judgments. Our specific goal here is to show the equivalence between this system, called dual natural deduction by Kavvos [17] and an axiomatic system in the style of Hakli and Negri [13]. This ensures that the dual system exactly corresponds to the more common axiomatic formalisms for modal logic. To the best of our knowledge the literature lacks of such proof (though, for the case of constructive necessity, this has been settled by Kavvos [17] and by our previous work [12]). This equivalence yields two deductive formalisms of necessity, one providing an amicable system to perform actual modal proofs of constructive necessity. As already mentioned, this is relevant to computer science applications but also for mathematical or philosophical discussions around necessity that require the development of actual derivations that could be quite tortuous to obtain in the axiomatic system.

The results exposed here extend our previous work around constructive necessity [12] by adding the $\Diamond$ possibility operator as primitive, thus the modalities present are $\Box A$ and $\Diamond A$ with their usual meanings: "A is necessary", and "A is possible", respectively. Recall that in the constructive setting there is no duality between $\Box$ and $\Diamond$ modal operators. The extension of the axiomatic system is straightforward, for we only need to add the axioms pertinent to $\Diamond$. However, in the case of natural deduction, the system needs to handle not only formulas, but modal judgments ($A\,\mathit{true}, A\,\mathit{poss}$) stating that a formula can be either true or possible. It is important to remark that these qualifiers, and also that for validity, are defined following the semantic intuition of Kripke's possible worlds, but in this paper there is no formal treatment of semantics. This feature allows us to describe modal knowledge syntactically, while still posing some challenges with respect to our mechanization.

The proof approach here taken was gained from a full synergy between pure mathematical paper-and-pencil reasoning and mechanized proof attempts (see the proof of Theorem 3.1 for an actual example). This path led us to some novel proof techniques. For instance, the admissibility of several inference rules here presented is not proved by structural induction on (the height or length of)[6] derivations. Instead, a simpler structural induction on contexts of proved sequents suffices (for example see the proofs of Theorem 3.3 and Lemma 4.11). This succeeds due to the fact that contexts are implemented as an inductive list data structure, not as a set or multiset. Moreover, most of the proofs here presented make heavy use of structural rules, whose particular shape was gained from the interactive verification process. This is an important difference with respect to other deductive system treatments, where the structural rules are either absent or its use is implicit, a feature that cannot be translated to the Coq formal development.

Our exposition begins with the syntax of formulas and contexts for modal logic S4. The axiomatic system is briefly discussed in Section 3. The natural deduction system $\mathcal{N}_{\mathsf{S4}}$ is developed in Section 4 and a discussion of its substitution properties appears in Section 5. The equivalence of these systems is proved in Section 6. To close, we give some final remarks in Section 7. The full Coq development is available at https://bitbucket.org/luglzhuesca/mlogic-formalverif/src/master/S4

2 Formulas and contexts

We adopt in this work the characterization of the notion of modal derivability by means of hypothetical judgments (sequents) $\Gamma \vdash A$ where Γ is a finite collection of hypotheses, called the context, and A is a formula. Such a sequent can be read as 'A follows from the collection of hypotheses Γ'. In this brief section we recall the definition of formulas and contexts.

Modal formulas are generated by the following grammar:

$$A, B ::= p_n \mid A \to B \mid \Box A \mid \Diamond A$$

where p_n denotes an element taken from an infinite supply of propositional variables, indexed by a natural number. Observe that implication is the only propositional connective, in particular we do not consider neither negation nor the constant $\bot$. Therefore we will be dealing with the implicational fragment of minimal logic [28] extended with modal operators of necessity and possibility. The logic here considered is called constructive due to its specific Kripke semantics not discussed here (see [1]). The constructive modal logics differ from the classical and intuitionistic versions in several aspects (see [16,27,29]). For instance, in classical logic the duality axiom $\Diamond A \leftrightarrow \neg\Box\neg A$ holds, but not in the intuitionistic systems, which include the formula $\neg\Diamond\bot$ as an axiom, same formula that is rejected in constructive modal logics.

Contexts are implemented as finite lists of formulas built from the empty list, denoted here by $\cdot$, and a constructor that generates a new list from a given one by

[6] On the height, if derivations are considered trees or on the length, if they are sequences like in this work.

adding a new element to its right-end. They are formally defined as follows:

$$\Gamma ::= \cdot \mid \Gamma, A$$

It is worth noting that the formula A in the context Γ, A does not have to be a new formula. Furthermore, the append operation of two contexts Γ and Γ' is denoted with a semicolon $\Gamma; \Gamma'$. This operation is recursively defined in the obvious way.

Our implementation choice for contexts, an inductive definition of the above grammar, is guided by the proof approach mentioned in the introduction. Several so-called low-level properties about contexts are required by the proof assistant to complete the formal verification process. Such properties, like the fact that the append of lists is associative, correspond to the kind of background results employed in paper-and-pencil proofs without further notice. Their particular form is designed to be able to automate them in an optimal way. However, they are absent in our exposition due to lack of space, but see the comment after the proof of Theorem 3.1 for an actual important example.

3 Axiomatic system for S4

In contrast to the usual axiomatic deductive systems, the one here presented, originally given by Hakli and Negri in [13] for the classical modal logic K, is specifically designed to derive consequences from hypotheses and not only from axioms or previous theorems. As deeply discussed by Hakli and Negri this is the key to be able to validate the deduction theorem, which sometimes is considered invalid for modal logic due to the fact that, in the absence of a precise definition of what a proof using assumptions is; such assumptions are considered as additional temporary axioms, which is the source of the invalidity of the deduction theorem.

The set $\mathcal{A}$ of axioms defining the modal logic S4 is

A1 $A \to (B \to A)$

A2 $(A \to (A \to B)) \to (A \to B)$

A3 $(A \to (B \to C)) \to (B \to (A \to C))$

A4 $(B \to C) \to ((A \to B) \to (A \to C))$

$\mathbb{K}$ $\Box(A \to B) \to (\Box A \to \Box B)$

$\mathbb{T}$ $\Box A \to A$

$\mathit{4}$ $\Box A \to \Box\Box A$

$\Diamond\mathbb{K}$ $\Box(A \to B) \to (\Diamond A \to \Diamond B)$

$\Diamond\mathbb{T}$ $A \to \Diamond A$

$\Diamond\mathit{4}$ $\Diamond\Diamond A \to \Diamond A$

From these axioms we generate a relation of modal derivability $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A$ by means of the following inductive definition where, as previously defined, contexts are finite lists of formulas.

$$\frac{A \in \Gamma}{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}\ (\textsc{Hyp}) \qquad\qquad \frac{A \in \mathcal{A}}{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}\ (\textsc{Ax})$$

$$\frac{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \qquad \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B}{\Gamma'; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} B}\ (\mathrm{MP}) \qquad\qquad \frac{\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} A}{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A}\ (\textsc{Nec})$$

Let us observe that the necessitation rule concedes us to introduce a $\Box$ operator only if the formula to be boxed is a theorem. This restriction solves the controversy around the validity of the deduction theorem. Moreover, the (MP) rule (MP) is stated in a multiplicative (independent contexts) style. The reason is that this is the natural way for the establishment of a correspondence of $\mathcal{H}_{\mathsf{S4}}$ with usual axiomatic systems, that is, those that do not handle hypotheses. This choice departs from the treatment in Kavvos [17] and also poses some challenges for the formal verification, as discussed in our previous work [12]. Let us also note that, since axiom $\Diamond\mathbb{T}$ is present, there is no need for an inference rule involving the $\Diamond$ operator (though see Lemma 3.6 below).

The deduction theorem and other admissible rules are discussed next.

Theorem 3.1 (Deduction) *For any context* Γ *and* A, B *propositions, if* $\Gamma, A \vdash_{\mathcal{H}_{\mathsf{S4}}} B$ *then* $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$.

Proof Structural induction[7] on $\Gamma, A \vdash_{\mathcal{H}_{\mathsf{S4}}} B$.
The case of (Hyp) follows by analysis of the hypothesis $B \in \Gamma, A$. Two cases arise: either B is A or $B \in \Gamma$. In the first case $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} B \to B$ is directly derived from axioms $\mathsf{A1}$ and $\mathsf{A2}$. For the second case, $B \in \Gamma$ implies that $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} B$. From this sequent and $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} B \to (A \to B)$, which is an instance of axiom $\mathsf{A1}$, we get the desired $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$ by (MP).
For the case of the axiom rule (Ax) the proof succeeds by (MP) and an adequate instance of axiom $\mathsf{A1}$.
For the (Nec) rule, the goal is to prove $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to \Box C$, assuming that $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} C$. This is achieved by (MP) from axiom $\mathsf{A1}$, in the form $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box C \to (A \to \Box C)$, and $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box C$, which comes out of the hypothesis by (Nec).
Finally, for the case of rule (MP), we have $\Gamma_1 \vdash_{\mathcal{H}_{\mathsf{S4}}} C$ (**1**) and $\Gamma_2 \vdash_{\mathcal{H}_{\mathsf{S4}}} C \to B$ (**2**) where $\Gamma_2; \Gamma_1 = \Gamma, A$. From this equation two cases are generated: either Γ_1 is empty or its last element is A.

(i) If $\Gamma_1 = \cdot$, then $\Gamma, A = \Gamma_2$ and hypothesis (1) becomes $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} C$. From (2), the induction hypothesis yields $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to (C \to B)$. This sequent together with an instance of axiom $\mathsf{A3}$ produces, by (MP), the sequent $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} C \to (A \to B)$. Finally, $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$ is obtained by applying (MP) again to the last sequent and to the hypothesis (1).

[7] This is the actual native inductive principle of the Coq mechanization. The proof can also be seen as employing an induction on the length of the derivation.

(ii) If A is the last element of Γ_1 then there exists another context Γ' such that $\Gamma_1 = \Gamma', A$. This implies that $\Gamma = \Gamma_2; \Gamma'$. From $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} (C \to B) \to ((A \to C) \to (A \to B))$ (axiom $\mathsf{A4}$) and hypothesis (2) we obtain $\Gamma_2 \vdash_{\mathcal{H}_{\mathsf{S4}}} (A \to C) \to (A \to B)$. Finally, by (MP) of this sequent with $\Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to C$ (coming from the induction hypothesis applied to (1)), it follows that $\Gamma_2; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$, which is the same as $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$.

□

It is worth noting that the deduction theorem is valid without further constraints, due to the fact that the essential restriction for its validity is already enforced by the specific definition of the necessitation rule.

The above proof provides an example of an alternative demonstration gained from the interaction between man and machine. One that stems from a mechanization attempt of an already well-established proof given in [13]. The two instances of case analysis occurring in the above proof are mechanized in Coq with the help of two previously formalized low-level results, namely the fact that if $B \in \Gamma, A$ then either $B \in \Gamma$ or B is exactly A; and a lemma on context decomposition saying that if $\Gamma_2; \Gamma_1 = \Gamma, A$ then either Γ_1 is empty and $\Gamma_2 = \Gamma, A$ or there is a context Γ' such that $\Gamma_1 = \Gamma', A$ and $\Gamma_2; \Gamma' = \Gamma$.

The converse implication of the deduction theorem, also known as the principle of detachment, is also easily achieved.

Theorem 3.2 (Principle of detachment) *For any context Γ and A, B propositions, if $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$ then $\Gamma, A \vdash_{\mathcal{H}_{\mathsf{S4}}} B$.*

Proof By (MP) of the hypothesis and $A \vdash_{\mathcal{H}_{\mathsf{S4}}} A$. □

From an strict point of view, the sequent $A \vdash_{\mathcal{H}_{\mathsf{S4}}} A$ in the above proof is incorrect (A is not a context). Instead we should use $\cdot, A \vdash_{\mathcal{H}_{\mathsf{S4}}} A$. Thus, the above argument yields the sequent $\Gamma; \cdot, A \vdash_{\mathcal{H}_{\mathsf{S4}}} B$ and a simple equational reasoning in contexts using the fact that $\Gamma; \cdot, A = \Gamma, A$ yields the desired sequent. This is again a low-level justification process required by Coq.

Since contexts are lists, the deduction theorem allows us to discharge only the last assumption in the context. To be able to discharge any hypothesis we must either use structural rules implicitly or prove a generalization. We choose the second option and spell out the proof which, though quite simple, serves to show the, to the best of our knowledge, novel method of proving the admissibility of an inference rule by induction on the context. Of course, what is new here is to use a routinary induction on lists (contexts) to prove the admissibility of a rule.

Theorem 3.3 (Generalized Deduction Theorem) *Let Γ, Γ' be contexts and A, B be propositions. The rule*

$$\frac{\Gamma, A; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} B}{\Gamma; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B} \text{ (GDT)}$$

is admissible.

Proof Induction on Γ'.
The base case $\Gamma' = \cdot$ is just the Deduction Theorem 3.1. For the inductive step assume that $\Gamma, A; \Gamma', C \vdash_{\mathcal{H}_{\mathsf{S4}}} B$, which implies $\Gamma, A; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} C \to B$ by Theorem 3.1. The I.H. yields now $\Gamma; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to C \to B$. From this by (MP) with an adequate instance of axiom A3 we get $\Gamma; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} C \to A \to B$. Finally the principle of detachment (Theorem 3.2) allows us to conclude $\Gamma; \Gamma', C \vdash_{\mathcal{H}_{\mathsf{S4}}} A \to B$. □

Several further properties of system $\mathcal{H}_{\mathsf{S4}}$ are gained thanks to the availability of the deduction theorem and its inverse. We present next two of them, regarding context permutation and contraction, that will be needed later to show the equivalence between the axiomatic and natural deduction approaches to S4.

Lemma 3.4 (Structural rules) *For any contexts Γ, Γ' and formula A, the rules*

$$\frac{\Gamma; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} A}{\Gamma'; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}\ (\textsc{Ctx-Perm}) \qquad \frac{\Gamma; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}\ (\textsc{Ctx-cont})$$

are admissible.

Proof Induction on Γ in each case. Theorem 3.3 and the detachment principle will be quite useful. □

It is important to remark that the above rules guarantee that the derivation process is preserved under the permutation of two contexts and by contracting a duplicated context. On the other hand, the append operation ; is neither commutative nor idempotent.

Next we prove the admissibility of two rules that allow to derive modal formulas, namely a rule for diamond introduction and a generalization of the neccesitation rule when a context is non-empty but consists only of boxed formulas.

Definition 3.5 Given a context $\Gamma = [A_0, \ldots, A_n]$ we define the boxed context $\Gamma^{\Box}$ as $\Gamma^{\Box} = [\Box A_0, \ldots, \Box A_n]$.

Lemma 3.6 (Modal introduction rules) *For any contexts Γ, Γ' and formula A, the rules*

$$\frac{\Gamma^{\Box} \vdash_{\mathcal{H}_{\mathsf{S4}}} A}{\Gamma^{\Box}; \Gamma' \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A}\ (\textsc{GenNec}) \qquad \frac{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A}{\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond A}\ (\textsc{Dia})$$

are admissible.

Proof For the first rule, by the (Ctx-Perm) rule it suffices to show that $\Gamma'; \Gamma^{\Box} \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A$, which is verified by induction on Γ. The second rule is directly derivable from axiom $\Diamond\mathbb{T}$ and (MP). □

This finishes our presentation of the axiomatic system for S4. Next we discuss the natural deduction formalism.

4 Natural deduction for S4

The here discussed formalism for S4, denoted $\mathcal{N}_{\mathsf{S4}}$, is a natural deduction system in sequent form, called S-system in [15], where propositions are analyzed judgmentally. It corresponds to the formal system for necessity and possibility presented by Pfenning and Davies in [24, Section 5], a work based in the philosophical development of Martin-Löf Type Theory [23]. This formal system makes use of intuitive semantic notions implicit in an essential distinction of three basic judgments whose purpose is to describe modal knowledge without an explicit use of worlds. The A *true* judgment denotes how to verify A under hypothetical judgments, that is, in the actual given world. A *valid* represents, by means of categorical judgments, a proposition whose truth does not depend on true hypotheses, that is, a proposition that holds at any given world. Finally, A *poss* represents a possible truth, which implies that A must be the only available true hypothesis in a hypothetical judgment, thus representing a particular world where we do not know anything but the truth of A. It is important to observe that the meaning of the labels in the definition of these judgments is mere intuitive, for there is no formal semantics involved in the following. Instead we could use labels like *global, local* and *unique*. The distinction between valid and true hypotheses is related to the notion of local and global assumptions of Fitting [8, p.177]. but we prefer to keep the terminology of [24].

Our natural deduction approach handles sequents of the form $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J$, where the antecedents Δ and Γ are contexts for valid and true hypotheses respectively, and the succedent J is a conclusion judgment, which are judgments of truth or possibility, formally defined as:

$$J ::= A\,\mathit{true} \mid A\,\mathit{poss}$$

Let us observe that contexts contain formulas and not judgments as in the original formulation where, from a strict point of view, there is only one context split in two parts, an option that would be quite complicated to manipulate in Coq. Instead, our choice simplifies the presentation and implementation of both contexts and sequents. For instance, there is no need for an explicit judgment of validity (A *valid*), since the contexts are implemented as two disjoint lists, making the use of *true* and *valid* labels redundant. That is, both metavariables Δ and Γ represent lists generated by the grammar in section 2. This implies in particular that Δ can ocurr as the right context in the antecedent of a sequent (see propositions 5.4 and 5.5), and that there is no syntactic restriction on the elements of Δ or Γ. For instance, a boxed formula $\Box A$ can ocurr both in Δ and Γ (but see section 4.2 for a way to move formulas between contexts). This kind of separated contexts is referred to as a dual context notion by Kavvos in [17]. Also, there is no need to explicitly consider neither hypotheses of possibility, as they are modelled by unique true assumptions, nor judgments of validity as conclusions, since they are represented by the judgment $\Box A$ *true*.

Another difference with the original formulation lies in the fact that, as the authors mention, their system has an explicit propositional reasoning and only an implicit judgment reasoning, saying that the derivability relation is *subject to the inclusion of $A\,true$ in $A\,poss$.* The more convenient way to implement this feature is by an explicit rule called here (TP), which converts a True judgment into a Possibility judgment. Here, the derivability relation is mechanized by an inductive definition, given by the inference rules below, which corresponds to a shallow embedding of derivability. Therefore the explicit statement and use of rule (TP) is mandatory. This rule, which can be considered as an introduction of possibility, is mentioned in [24] as an alternative formulation, less efficient than the current system, including only the definitions for modal operators and the rule (TP) as the introduction of the $A\,poss$ judgment.

System $\mathcal{N}_{\mathsf{S4}}$ is defined by the following inference rules:

$$\frac{}{\Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}\ (\textsc{Thyp}) \qquad \frac{}{\Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}\ (\textsc{Vhyp})$$

$$\frac{\Delta|\Gamma, A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\,true}\ (\to\mathrm{I}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\,true \qquad \Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} B\,true}\ (\to\mathrm{E})$$

$$\frac{\Delta|\cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true}\ (\Box\mathrm{I}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true \qquad \Delta, A|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}\ (\Box\mathrm{E})$$

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true \qquad \Delta, A|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}\ (\Box\mathrm{E}\text{-}\textsc{Poss})$$

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,poss}\ (\mathrm{TP}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,poss}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\,true}\ (\Diamond\mathrm{I})$$

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\,true \qquad \Delta|A \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}\ (\Diamond\mathrm{E})$$

We have two starting rules corresponding to both types of hypotheses, they are stated in the most general way granting us to infer any formula that belongs to a context, not only the first or the last one. The rule for introduction of necessity ($\Box$ I) exactly captures the definition of validity through a categorical judgment, whilst the rule of necessity elimination ($\Box$ E) behaves as a substitution or cut rule where the formula $\Box A$ is used as lemma A in the valid hypotheses in order to prove a conclusion judgment about C. The judgment $A\,poss$ is explained with a combination of valid and true judgments by the rules ($\Box$E-Poss), (TP), ($\Diamond$ I) and ($\Diamond$ E). The second elimination rule for the $\Box$ operator is given in order to conclude a local truth as

$C\ poss$, this rule takes advantage of a lemma of the form $\Box A\ true$. The introduction and elimination rules for the $\Diamond$ operator are indeed the definition of possibility given in [24].

Let us show next the derivations in $\mathcal{N}_{\mathsf{S4}}$ corresponding to the modal scheme axioms of $\mathsf{S4}$.

Example 4.1 [Scheme $\mathbb{T}$] The sequent $\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to A\ true$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

1.	$\cdot \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\ true$	(THYP)	
2.	$A \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ true$	(VHYP)	
3.	$\cdot \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ true$	($\Box$ E)	1, 2
4.	$\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to A\ true$	($\to$ I)	3

Example 4.2 [Scheme $\mathit{4}$] The sequent $\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Box\Box A\ true$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

1.	$A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ true$	(VHYP)	
2.	$A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\ true$	($\Box$ I)	1
3.	$A \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box\Box A\ true$	($\Box$ I)	2
4.	$\cdot \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\ true$	(THYP)	
5.	$\cdot \mid \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box\Box A\ true$	($\Box$ E)	4, 3
6.	$\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Box\Box A\ true$	($\to$ I)	5

Example 4.3 [Scheme $\mathbb{K}$] The sequent $\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to (\Box A \to \Box B)\ true$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

1.	$A, A \to B \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ true$	(VHYP)	
2.	$A, A \to B \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\ true$	(VHYP)	
3.	$A, A \to B \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ true$	($\to$ E)	2, 1
4.	$A, A \to B \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box B\ true$	($\Box$ I)	3
5.	$A \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B)\ true$	(THYP)	
6.	$A \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box B\ true$	($\Box$ E)	5, 4
7.	$\cdot \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\ true$	(THYP)	
8.	$\cdot \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box B\ true$	($\Box$ E)	7, 6
9.	$\cdot \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Box B\ true$	($\to$ I)	8
10.	$\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to (\Box A \to \Box B)\ true$	($\to$ I)	9

Example 4.4 [Scheme $\Diamond\mathbb{T}$] The sequent $\cdot \,|\, \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to \Diamond A\ \mathit{true}$ is derivable in $\mathcal{N}_{\mathsf{S4}}$

$$\begin{array}{llll}
1. & \cdot \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{true} & (\textsc{Thyp}) & \\
2. & \cdot \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{poss} & (\mathrm{TP}) & 1 \\
3. & \cdot \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\ \mathit{true} & (\Diamond\,\mathrm{I}) & 2 \\
4. & \cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to \Diamond A\ \mathit{true} & (\to \mathrm{I}) & 3
\end{array}$$

Example 4.5 [Scheme $\Diamond 4$] The sequent $\cdot \,|\, \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond\Diamond A \to \Diamond A\ \mathit{true}$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

$$\begin{array}{llll}
1. & \cdot \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{true} & (\textsc{Thyp}) & \\
2. & \cdot \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{poss} & (\mathrm{TP}) & 1 \\
3. & \cdot \mid \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\ \mathit{true} & (\textsc{Thyp}) & \\
4. & \cdot \mid \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{poss} & (\Diamond\mathrm{E}) & 3,2 \\
5. & \cdot \mid \Diamond\Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond\Diamond A\ \mathit{true} & (\textsc{Thyp}) & \\
6. & \cdot \mid \Diamond\Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{poss} & (\Diamond\mathrm{E}) & 5,4 \\
7. & \cdot \mid \Diamond\Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\ \mathit{true} & (\Diamond\mathrm{I}) & 7 \\
8. & \cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond\Diamond A \to \Diamond A\ \mathit{true} & (\to \mathrm{I}) & 8
\end{array}$$

Example 4.6 [Scheme $\Diamond\mathbb{K}$] The sequent $\cdot \,|\, \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to (\Diamond A \to \Diamond B)\ \mathit{true}$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

$$\begin{array}{llll}
1. & A \to B \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\ \mathit{true} & (\textsc{Vhyp}) & \\
2. & A \to B \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{true} & (\textsc{Thyp}) & \\
3. & A \to B \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true} & (\to \mathrm{E}) & 1,2 \\
4. & A \to B \mid A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{poss} & (\mathrm{TP}) & 3 \\
5. & A \to B \mid \Box(A \to B), \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\ \mathit{true} & (\textsc{Thyp}) & \\
6. & A \to B \mid \Box(A \to B), \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{poss} & (\Diamond\mathrm{E}) & 4,5 \\
7. & A \to B \mid \Box(A \to B), \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond B\ \mathit{true} & (\Diamond\mathrm{I}) & 6 \\
8. & \cdot \mid \Box(A \to B), \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B)\ \mathit{true} & (\textsc{Thyp}) & \\
9. & \cdot \mid \Box(A \to B), \Diamond A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond B\ \mathit{true} & (\Box\mathrm{E}) & 7,8 \\
10. & \cdot \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A \to \Diamond B\ \mathit{true} & (\to \mathrm{I}) & 9 \\
11. & \cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to (\Diamond A \to \Diamond B)\ \mathit{true} & (\to \mathrm{I}) & 10
\end{array}$$

Finally a more elaborated example involving the use of both rules for $\Box$ elimination.

Example 4.7 The sequent $\cdot \,|\, \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to \Diamond(\Box A \to \Diamond B)\ \mathit{true}$ is derivable in $\mathcal{N}_{\mathsf{S4}}$.

1.	$A \to B \mid \Box(A \to B), \Box A, A \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\ \mathit{true}$	$(\textsc{VHyp})$
2.	$A \to B \mid \Box(A \to B), \Box A, A \vdash_{\mathcal{N}_{\mathsf{S4}}} A\ \mathit{true}$	$(\textsc{THyp})$
3.	$A \to B \mid \Box(A \to B), \Box A, A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true}$	$(\to \mathrm{E})$ 1, 2
4.	$A \to B \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\ \mathit{true}$	$(\textsc{THyp})$
5.	$A \to B \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true}$	$(\Box \mathrm{E})$ 4, 3
6.	$A \to B \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{poss}$	(TP) 5
7.	$A \to B \mid \Box(A \to B), \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond B\ \mathit{true}$	$(\Diamond \mathrm{I})$ 6
8.	$A \to B \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Diamond B\ \mathit{true}$	$(\to \mathrm{I})$ 7
9.	$A \to B \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Diamond B\ \mathit{poss}$	(TP) 8
10.	$\cdot \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B)\ \mathit{true}$	$(\textsc{THyp})$
11.	$\cdot \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Diamond B\ \mathit{poss}$	$(\Box\, \mathrm{E}\text{-}\textsc{poss})$ 10, 9
12.	$\cdot \mid \Box(A \to B) \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond(\Box A \to \Diamond B)\ \mathit{true}$	$(\Diamond \mathrm{I})$ 11
13.	$\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to \Diamond(\Box A \to \Diamond B)\ \mathit{true}$	$(\to \mathrm{I})$ 12

4.1 Admissible rules

In this section we present some useful admissible rules which, though quite natural are, to the best of our knowledge and with the exception of Lemma 4.10, an original contribution gained again from the full interaction with the Coq proof assistant. The next statements are an abridge presentation of the ones included in our formalization. In there, the different rules are similar statements formulated separately for valid and true contexts combined with true and possible conclusions.

Lemma 4.8 (Weakening) *Let Δ, Γ, Γ' be contexts, A, B be formulas and J a conclusion judgment. The rules*

$$\frac{\Delta|\Gamma;\Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|\Gamma, B;\Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{Weak-THyps}) \qquad \frac{\Delta;\Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta, B;\Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{Weak-VHyps})$$

are admissible.

Proof Each rule is proved by structural induction on the given derivation. □

Next we prove that rules $(\to I)$ and $(\Diamond I)$ are invertible.

Lemma 4.9 (Inversion for $\to$ and $\Diamond$) *Let Δ, Γ be contexts and A, B formulas.*

The rules

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\,\mathit{true}}{\Delta|\Gamma, A \vdash_{\mathcal{N}_{\mathsf{S4}}} B\,\mathit{true}}\ (\textsc{Det}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\,\mathit{true}}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,\mathit{poss}}\ (\Diamond\ \text{I-}\textsc{Inv})$$

are admissible.

Proof The rule of detachment (Det) is a direct consequence of weakening and ($\to$ E). The inversion of $\Diamond$-introduction is proved using rules (TP) and ($\Diamond$ E). $\square$

Lemma 4.10 (Exchange) *Let Δ, Γ, Γ' be contexts, A, B be formulas and J be a conclusion judgment. The rules*

$$\frac{\Delta|\Gamma, A, B \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|\Gamma, B, A \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{Exch-THyps}) \qquad \frac{\Delta, A, B|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta, B, A|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{Exch-VHyps})$$

are admissible.

Proof The cases where J is $C\,\mathit{true}$ are proved directly with the help of weakening, while those for $C\,\mathit{poss}$ are proved by induction on the given derivation. $\square$

The next lemma presents some generalizations for contexts of the weakening and exchange rules of lemmas 4.8 and 4.10.

Lemma 4.11 (Structural context rules) *Let $\Delta, \Delta', \Gamma, \Gamma'$ be contexts, A be a formula and J be a true or possible judgment. The rules*

- *Context Weakening:*

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|\Gamma;\Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\text{TC}\textsc{tx-Weak}\text{R}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|\Gamma';\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\text{TC}\textsc{tx-Weak}\text{L})$$

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta;\Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\text{VC}\textsc{tx-Weak}\text{R}) \qquad \frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta';\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\text{VC}\textsc{tx-Weak}\text{L})$$

- *Context Exchange:*

$$\frac{\Delta|\Gamma, A;\Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|(\Gamma;\Gamma'), A \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{ctx-exch-conc}) \qquad \frac{\Delta|(\Gamma;\Gamma'), A \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta|\Gamma, A;\Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{ctx-exch-snoc})$$

are admissible.

Proof Each weakening rule is proved by induction on the context Γ', respectively Δ'. The proof of the exchange rules, when J is $C\,\mathit{poss}$, proceeds by structural induction on the premise, otherwise the proof succeeds by induction on the context Γ'. We show next the case for (ctx-exch-conc). The base case is straightforward, for $\Gamma, A; \cdot = (\Gamma; \cdot), A$ and therefore both sequents, premise and conclusion are the same. For the inductive step we assume that $\Delta|(\Gamma, A);(\Gamma', B) \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,\mathit{true}$ and need to prove $\Delta|(\Gamma;\Gamma', B), A \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,\mathit{true}$. From the assumption we get

$\Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} B \to C\ \mathit{true}$, by means of some low-level equational reasoning for contexts and the rule $(\to E)$. The IH yields now $\Delta|(\Gamma;\Gamma'), A \vdash_{\mathcal{N}_{\mathsf{S4}}} B \to C\ \mathit{true}$ which, by the rule (Det) (lemma 4.9) leads us to $\Delta|(\Gamma;\Gamma'), A, B \vdash_{\mathcal{N}_{\mathsf{S4}}} C\ \mathit{true}$. Finally by exchange (lemma 4.10) and some low-level equational reasoning we arrive to $\Delta|(\Gamma;\Gamma', B), A \vdash_{\mathcal{N}_{\mathsf{S4}}} C\ \mathit{true}$. □

The analogous to the above rules, for the case of contexts of valid hypotheses, can be proved in a similar way. We can now prove a generalization of the implication introduction rule that allows us to discharge any true hypotheses.

Lemma 4.12 (Generalized Implication Introduction) *Let Δ, Γ, Γ' be contexts and A, B formulas. The rule*

$$\frac{\Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true}}{\Delta|\Gamma; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\ \mathit{true}}\ (\textsc{Gen} \to \mathrm{I})$$

is admissible.

Proof Induction on Γ', using the exchange rules adequately. □

4.2 Formula transference between contexts

We discuss next a result not present in [24], namely a transfer process between contexts. This is an inference principle that captures the fact that valid formulas are necessary truths, meaning that a formula A belonging to a valid context can be considered as a necessary truth. This is achieved by deleting A from the context of valid assumptions, while adding $\Box A$ to the context of true hypothesis.

On the other direction, we can get rid of a boxed formula in the true assumptions context by moving it without the box to the valid assumptions context. Thus converting a boxed hypothesis into a pure propositional one.

Proposition 4.13 (Valid formulas are necessary truths) *Let Δ, Δ', Γ be contexts, A be a proposition and J be a conclusion judgment. The rule*

$$\frac{\Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta; \Delta'|\Gamma, \Box A \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\textsc{ValToTrue})$$

is admissible.

Proof Induction on Δ'.
If J is $C\ \mathit{true}$ then the base case is a consequence of the $(\Box E)$ elimination rule, using weakening appropriately. Otherwise the base case is settled by induction on the premise. □

We can iterate the application of the above property in such a way that the context of valid formulas becomes empty, this option is not adequate for actual proof construction in $\mathcal{N}_{\mathsf{S4}}$, but plays an important role in the desired equivalence proof of our deductive systems

Corollary 4.14 *Let Δ, Γ be contexts and J a conclusion judgment. If $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J$ then $\cdot|\Delta^{\Box};\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J$.*

As another corollary we obtain a rule that allows for the discharge of valid hypotheses.

Corollary 4.15 (Implication introduction for validity) *Let Δ, Γ be contexts and A, B be propositions. The rule*

$$\frac{\Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true}}{\Delta, \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to B\ \mathit{true}}\ (\to\mathrm{I\ V{\small AL}})$$

is admissible.

Proof Induction on Δ'. □

The above rule seems to be important for some special interpretations of the conditional, as in the case of lax logic where the lax implication can be defined as $\Box A \to B$ (see [24, Section 7]). Furthermore, the rule is invertible according to the following proposition.

Proposition 4.16 (Detachment for boxed formulas) *Let Δ, Γ, Γ' be contexts and A, B be formulas. The rule*

$$\frac{\Delta; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to B\ \mathit{true}}{\Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} B\ \mathit{true}}\ (\Box\ \mathrm{D{\small ET}})$$

is admissible.

Proof Use $(\to E)$, $(\Box I)$ and weakening. □

This proposition permits us to prove the following version of the inverse rule of Proposition 4.13.

Proposition 4.17 (Necessary truths are valid) *Let Δ, Γ, Γ' be contexts, A be proposition and J be a judgment. The rule*

$$\frac{\Delta|\Gamma, \Box A, \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}{\Delta, A|\Gamma, \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} J}\ (\mathrm{T{\small RUE}T{\small O}V{\small AL}})$$

is admissible.

Proof Induction on Γ'.
The case when J is $C\ \mathit{true}$ is proved directly while the case for $C\ \mathit{poss}$ is proved by induction on the given derivation. □

Propositions 4.13 and 4.17 together provide the formula transfer principle between contexts. This is an important tool for the actual development of derivations, one that allows us to manipulate modal formulas in the contexts by considering them as pure (non-modal) propositions and vice versa. This feature considerably simplifies the actual construction of proofs and sometimes allows us to replace a modal reasoning by a simpler propositional one. Like in the example below, where there is no need of using the modal reasoning provided by the rule $(\Box E)$.

Example 4.18 The sequent $\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to \Box(\Box A \to \Box B)\, \mathit{true}$ is derivable.

1.	$A \to B,\, A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A \to B\, \mathit{true}$	(VHYP)
2.	$A \to B,\, A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\, \mathit{true}$	(VHYP)
3.	$A \to B,\, A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} B\, \mathit{true}$	$(\to \mathrm{E})\, 1, 2$
4.	$A \to B,\, A \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box B\, \mathit{true}$	$(\Box\, \mathrm{I})$ 3
5.	$A \to B \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A \to \Box B\, \mathit{true}$	$(\to$ I VAL$)$ 4
6.	$A \to B \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(\Box A \to \Box B)\, \mathit{true}$	$(\Box\, \mathrm{I})$ 5
7.	$\cdot \mid \cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(A \to B) \to \Box(\Box A \to \Box B)\, \mathit{true}$	$(\to$ I VAL$)$ 6

Next we discuss another powerful tool for actual derivations, namely the substitution or cut rule.

5 Substitution properties in $\mathcal{N}_{\mathsf{S4}}$

In this section we show that $\mathcal{N}_{\mathsf{S4}}$ enjoys the substitution or composition property (the analogous to the cut rule in sequent calculus), which ensures that derivations are closed under composition, or from a more practical point of view, guarantees the correctness of proofs organized with help of an auxiliary lemma (the hypothesis being cut or substituted in favor of an actual proof). As the system makes distinction between two kinds of hypotheses or conclusions, there are multiple substitution properties gained by following a combination of a specific hypothesis (either valid or true) and a particular conclusion judgment (either true or possible). The total number of combinations is eight, but only five are appropriated and already given by Pfenning and Davies [24, Section 5.1] where the proofs, though not present, are mentioned to being by structural induction on a premise. However, in our development, only two of five proofs follow this methodology, while the others are obtained directly from primitive and structural rules. The enumeration of the rules is the one given in [24, Theorem 2].

The first two principles involve an auxiliary lemma A, which is proven to be true (meaning that the judgment $A\, \mathit{true}$ holds) and used as a true or valid hypothesis in order to conclude a judgment $C\, \mathit{true}$.

Lemma 5.1 (Substitution for a true conclusion) *Let* $\Delta, \Delta', \Gamma, \Gamma'$ *be contexts* *and A, C be formulas. The rules*

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true \qquad \Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}{\Delta|\Gamma; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}\ (\textsc{subst-1})$$

$$\frac{\Delta|\cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true \qquad \Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}{\Delta; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true}\ (\textsc{subst-3})$$

are admissible.

Proof The rule (SUBST-1) is directly derivable from ($\to E$) using context weakening (TCtx-WeakR) and (Gen $\to I$). The rule (SUBST-3) is derived using (VCtx-WeakR) and ($\to$ I Val). □

Let us observe that, in the above proofs, we are using the transference process indirectly by making use of rule ($\to$ I Val). This powerful rule avoids the use of induction. The next rules use and prove a lemma in the same way as the previous rules, but this time to conclude a possibility judgment $C\,poss$.

Lemma 5.2 (Substitution for a possible conclusion) *Let $\Delta, \Delta', \Gamma, \Gamma'$ be contexts and A, C be formulas. The rules*

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true \qquad \Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}{\Delta|\Gamma; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}\ (\textsc{subst-2})$$

$$\frac{\Delta|\cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true \qquad \Delta, A; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}{\Delta; \Delta'|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}\ (\textsc{subst-4})$$

are admissible.

Proof Both rules are proved by induction on the second premise, using some of the structural rules defined above. □

The above four substitution principles involve a lemma A for which the judgment $A\,true$ holds. The last option considers the weak case where the lemma A only verifies the judgment $A\,poss$. Therefore A can only be used as a local truth, which is captured by requiring that A is the unique true hypothesis in the second premise. Due to locality, we can only conclude $C\,poss$.

Lemma 5.3 (Substitution of a possible lemma) *Let Δ, Γ be contexts and A, C be formulas. The rule*

$$\frac{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,poss \qquad \Delta|A \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss}\ (\textsc{subst-5})$$

is admissible.

Proof A direct consequence of the rules ($\Diamond E$) and ($\Diamond I$). □

Interestingly enough the original proof of Lemma 5.3 points to an induction on the first premise, which was also our first option, only to discover later the above direct proof, with the help of the proof-assistant.

Note that the primitive elimination rules, ($\Box$E), ($\Box$E-Poss) and ($\Diamond$E) are instances of the above substitution principles where the lemma is an explicit modal proposition. On the other hand, as we said before, from the eight combinations, only the five just discussed are adequate. The other three combinations for substitution principles cannot be feasible as they do not respect the intuitive semantic notions. For instance, consider the case where a lemma $A\,poss$ is considered as a truth in order to derive $C\,true$, that is, $\Delta|\Gamma, A; \Gamma' \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,true$. This case is not correct since A only verifies $A\,poss$ and hence can only be used as local, meaning that the truth context must only contain A as in rule (SUBST-5). In this situation the conclusion must be again local, that is of the form $C\,poss$.

To close this section we present some examples of other known modal admissible rules about necessity (see v.gr. [17]) and proved here with the help of the substitution rules.

Proposition 5.4 (Scott's Rule with weakening) *For any contexts Δ, Γ and formula A the rule*

$$\frac{\cdot|\Delta \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true}\ (\Box\, I_K)$$

is admissible.

Proof Induction on Δ.
The base case holds by rule ($\Box$ I). The inductive case is proved by substitution (Subst-1) using an instance of axiom $\mathbb{K}$ (Example 4.3) and $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box(B \to A)\,true$, which is a consequence of the induction hypothesis. $\Box$

Proposition 5.5 (Introduction rule for 4) *For any contexts Δ, Γ and formula A the rule*

$$\frac{\Delta|\Delta \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true}{\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true}\ (\Box\, I_{K4})$$

is admissible.

Proof Induction on Δ.
The base case is proved by ($\Box$ I) and weakening. For the inductive case, substitution rule (Subst-3) using $\Box B\,true$ as lemma, requires to prove $\Delta, B, \Box B|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true$. By exchange and ($\to$ I Val) the goal is to prove $\Delta, \Box B|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box B \to \Box A\,true$ which is achieved by the I.H. and two applications ($\to$E) with an appropiate instance of the $\mathbb{K}$ scheme (example 4.3). $\Box$

6 Equivalence

We come to our main contribution, a proof of the equivalence between the natural deduction system $\mathcal{N}_{\mathsf{S4}}$ and its axiomatic counterpart $\mathcal{H}_{\mathsf{S4}}$.

6.1 From $\mathcal{H}_{\mathsf{S4}}$ to $\mathcal{N}_{\mathsf{S4}}$

The translation of Hilbert-style to dual natural deduction proofs is straightforward. The proof transformation consists mainly in substituting every occurrence of an axiom for its explicit derivation in natural deduction. Moreover, as expected, the translation of a $\mathcal{H}_{\mathsf{S4}}$-sequent will be a $\mathcal{N}_{\mathsf{S4}}$-sequent whose context of valid hypotheses is empty. In our case, we also need weakening structural rules in $\mathcal{N}_{\mathsf{S4}}$ (lemmas 4.8 and 4.11), since in $\mathcal{H}_{\mathsf{S4}}$ the (MP) rule is multiplicative whereas the ($\to$ E) rule of $\mathcal{N}_{\mathsf{S4}}$ is additive, that is, it shares contexts in both premisses.

Theorem 6.1 (From axiomatic to natural deduction proofs) *Let Γ be a context and A be a formula. The following holds:*

$$\textit{If } \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A \textit{ then } \cdot|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,\textit{true}.$$

Proof Induction on $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} A$.
The derivability of the propositional axioms is left as a simple exercise on $\mathcal{N}_{\mathsf{S4}}$ derivations and that for modal axioms, is provided in the examples starting in page 10. The case of (MP) has the hypotheses $\Gamma_1 \vdash_{\mathcal{H}_{\mathsf{S4}}} B \to A$ and $\Gamma_2 \vdash_{\mathcal{H}_{\mathsf{S4}}} B$ with $\Gamma = \Gamma_1;\Gamma_2$. By the induction hypothesis we have $\cdot|\Gamma_1 \vdash_{\mathcal{N}_{\mathsf{S4}}} B \to A\,\textit{true}$ and $\cdot|\Gamma_2 \vdash_{\mathcal{N}_{\mathsf{S4}}} B\,\textit{true}$. We apply to these sequents the Lemma 4.11, that is rules (TCTX-WEAKR) and (TCTX-WEAKL) respectively, which yield $\cdot|\Gamma_1;\Gamma_2 \vdash_{\mathcal{N}_{\mathsf{S4}}} B \to A\,\textit{true}$ and $\cdot|\Gamma_1;\Gamma_2 \vdash_{\mathcal{N}_{\mathsf{S4}}} B\,\textit{true}$. The case is now settled by the ($\to E$) rule. The remaining cases are straightforward. □

An evident corollary makes explicit the translation of a $\diamond$-formula.

Corollary 6.2 *Let Γ be a context and A be a formula. If $\Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \diamond A$ then $\cdot|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,\textit{poss}$.*

Let us observe that this translation was direct, due to the fact that the Hilbert-style system already handles sequents and not only formulas. If, instead, a derivation in $\mathcal{H}_{\mathsf{S4}}$ was a sequence of formulas, we would need a mechanism to collect the assumptions in a given deduction to be able to construct a context in $\mathcal{N}_{\mathsf{S4}}$, as it is done for example in [25].

6.2 From $\mathcal{N}_{\mathsf{S4}}$ to $\mathcal{H}_{\mathsf{S4}}$

In this section we prove the translation of natural deduction proofs to axiomatic derivations. The idea is to follow Proposition 4.13, or rather Corollary 4.14 to empty the validity context, thus getting only a context of true assumptions which can be directly managed in $\mathcal{H}_{\mathsf{S4}}$. The translation proof will be achieved by induction on $\mathcal{N}_{\mathsf{S4}}$-derivations.

Theorem 6.3 (From natural deduction to axiomatic proofs) *Let* Δ, Γ *be contexts and* J *be a conclusion judgment. Then*

$$\textit{If } \Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J \textit{ then } \Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} J^t$$

where J^t *is defined as* $(A\,true)^t = A$ *and* $(A\,poss)^t = \Diamond A$.

Proof Structural induction on $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} J$.
The case (THYP) is direct, whereas that for (VHYP) follows by (MP) of $\cdot \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A \to A$ (axiom $\mathbb{T}$) and $\Delta^{\Box}, \Box A, \Delta'^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A$.
The case of rule ($\to$ I) is proved directly from the induction hypothesis by the deduction theorem. For the case of ($\to$ E) we apply (MP) to both inductive hypotheses to get $(\Delta^{\Box}; \Gamma); (\Delta^{\Box}; \Gamma) \vdash_{\mathcal{H}_{\mathsf{S4}}} B$, which is simplified to the desired $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} B$ by the contraction rule (CTX-CONT) of Lemma 3.4. For the case of ($\Box$ I) we assume $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Box A\,true$ coming from $\Delta|\cdot \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,true$, from which the I.H. yields $\Delta^{\Box} \vdash_{\mathcal{H}_{\mathsf{S4}}} A$. The desired sequent $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A$ is obtained by the (GENNEC) rule Lemma 3.6.
We show next the case for ($\Box$E-Poss) (for ($\Box$ E) the reasoning is analogous). Here, we have to translate $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss$ and the I.H. yields $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box A$ and $\Delta^{\Box}, \Box A; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond C$. From these sequents, using the rule (GDT), (MP) and (CTX-CONT), we get the desired $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond C$.
For the rule ($\Diamond I$), assume that $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} A\,poss$. In this case the I.H yields $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond A$ which is exactly the translation of $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\,true$. For the case ($\Diamond$E), let us assume that $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss$ comes from $\Delta|\Gamma \vdash_{\mathcal{N}_{\mathsf{S4}}} \Diamond A\,true$ and $\Delta|A \vdash_{\mathcal{N}_{\mathsf{S4}}} C\,poss$. Our goal is to show $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond C$. By the axiom $\Diamond 4$ it suffices to show $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond\Diamond C$. By using the I.H $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond A$, and thanks to the rules (MP) and (CTX-CONT), it is enough to show $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond A \to \Diamond\Diamond C$. But this sequent is a consequence of the axiom $\Diamond\mathbb{K}$ and the sequent $\Delta^{\Box}; \Gamma \vdash_{\mathcal{H}_{\mathsf{S4}}} \Box(A \to \Diamond C)$, which is derivable from the I.H. $\Delta^{\Box}; A \vdash_{\mathcal{H}_{\mathsf{S4}}} \Diamond C$ by the deduction theorem and the rule (GENNEC) of general necessitation.
The case of rule (TP) is a direct consequence of the I.H. and the axiom $\Diamond\mathbb{T}$. $\Box$

7 Closing remarks

In this paper we discussed the equivalence of two deductive systems for the modal logic S4, namely the axiomatic system $\mathcal{H}_{\mathsf{S4}}$ inspired by Hakli and Negri [13] and the natural deduction system $\mathcal{N}_{\mathsf{S4}}$ from Pfenning and Davies [24]. The former is a Hilbert-style system incorporating the notion of derivability from assumptions, a device that helps to solve the controversy around the provability of the deduction theorem in modal logic. The latter is a natural deduction formalism in sequent form influenced by Martin-Löf's methodology of judgments and propositions, allowing us to give constructive explanations for necessity and possibility.

To the best of our knowledge, this equivalence has not been addresed before in a dedicated manner, although there are works whose goals indirectly force to define systems like the ones developed here and their equivalence proofs. For instance the

Coq formalization of Litak, et al. [20] or the work of Kobayashi [18] whose methods, employing the sophisticated machinery of category theory and even introducing new concepts like that of $\mathcal{L}$-strong monad, are far from elementary.

All the exposed work has been verified using the Coq proof assistant. Two main choices gave guideline to complete this paper and its formal verification. On one hand the implementation of hypotheses collections using a simple data structure, instead of sticking to sets or multisets, which require more sophisticated data structures that complicate not only the automation but the correspondence with the usual proofs. On the other hand to reconcile a high-level reasoning at the mathematical level, with low-level procedures related to interactive proof-development. These choices originated the proof approach taken here, which allowed us to present mathematical arguments involving some uncommon proof advances like the use of induction on contexts to show the admissibility of structural rules instead of the structural induction on derivations which is the preferred proof methodology in the literature. These arguments were obtained from the interaction with the reasoning facilites provided by the proof assistant. Moreover, the mechanization of proofs by using the Coq hints mechanisms allowed us to simplify some of the low-level steps, thus generating proof scripts that correspond more closely to the pure mathematical proofs.

A forthcoming line of research consists on the study of alternative rules to manipulate and ease the proofs of possibility judgments. For instance an analogous rule to ($\to$ I), but where the conclusion is a possibility judgment, would provide a way to introduce a hypothesis to such kind of conclusions. Another future research thread of our interest is related to the design of deductive systems adequate for interactive proof-search, along the lines of [25, Chapter 4]. Last but not least, we intend to investigate the benefits of our current implementation, in pursuing actual verification cases, like the formalization of modal lambda calculus in [11] or lax logic in [7], as well as the mechanization of Gödel's interpretation of intuitionistic propositional calculus in $\mathsf{S4}$ [30].

Regarding the semantical aspects of modal logic, an important line of future research consists of establishing completeness theorems for the two systems discussed in this paper. In the case of $\mathcal{H}_{\mathsf{S4}}$ we conjecture this can be achieved with respect to the semantics presented in [1]. For the dual system $\mathcal{N}_{\mathsf{S4}}$, the completeness theorem would confirm the correctness of the intuitive semantic notions employed here as the motivation for its definition. A precise definition of semantics would also answer some questions about the nature of the consequence relation captured by $\mathcal{N}_{\mathsf{S4}}$, for example if it is a local or global consequence relation. The relationship between our concept of valid and true hypotheses and the global and local assumptions of [8] should also be investigated. Finally, a starting point for this research thread could be the work of [2], which discusses modal logic encodings by using separate consequence relations (semantical and deductive) related to classical validity and truth.

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