STAA 574: Homework 3

Spring 2020

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1. Using the skulls dataset on Canvas (skull.csv), complete a two-sample test of difference in means between the "Era 1" and "Era 2", i.e., test H_0: $\boldsymbol{\mu}_{era1} = \boldsymbol{\mu}_{era2}$ at $\alpha = 0.05$. Does it seem reasonable to pool variances here? Please use the two-sample test appraoch rather than using MANOVA.

```
#read in the skulls data
skull_data <- read.csv("skull.csv")

#take a look at the data to see what was read in
head(skull_data)
```

```
##    MB  BH  BL NH Era
## 1 131 138  89 49   1
## 2 125 131  92 48   1
## 3 131 132  99 50   1
## 4 119 132  96 44   1
## 5 136 143 100 54   1
## 6 138 137  89 56   1
```

```
#check the dimensions of skulls data set
dim(filter(skull_data, Era ==1))[1]
```

```
## [1] 30
```

```
#define n1 and n2
n1 = dim(filter(skull_data, Era ==1))[1]
n2 = dim(filter(skull_data, Era ==2))[1]
```

Check the assumption of equal sample variances for Era's 1 and 2,

```
#Create a matrix with Era 1 and 2 data
era1_data = filter(skull_data, Era == 1)
era2_data = filter(skull_data, Era == 2)
data_M = rbind(era1_data,era2_data)

#Use the boxM function in R to complete the Box's M-test
#The boxM function can be found in the heplots package
eq_var_check <- boxM(data_M[,1:4], data_M[, "Era"])
eq_var_check
```

```
##
##  Box's M-test for Homogeneity of Covariance Matrices
##
## data:  data_M[, 1:4]
## Chi-Sq (approx.) = 10.143, df = 10, p-value = 0.428
```

State for formal hypothesis test for equal variances.

1) Test the hypothesis that $H_0 : \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ vs. $H_a : \mathbf{\Sigma}_1 \neq \mathbf{\Sigma}_2$
2) The level of significance for this test will be $\alpha = 0.05$.
3) The test statistic is 10.143.
4) The p-value is 0.428
5) Since p = 0.428 is > than 0.05 we fail to reject $H_0 : \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$.
6) Since we failed to reject H_0 we will conclude that the covariance matrices for Era 1 and 2 are equal and that pooling the variances is reasonable.

Calculate the mean vectors for Era 1 and 2,

```
#Calculate the mean vector for Era 1
x_bar_era1 <- apply(skull_data[skull_data[, "Era"] == 1, c(1:4)], 2, mean)
x_bar_era1
```

```
##        MB        BH        BL        NH
## 131.36667 133.60000  99.16667  50.53333
```

$$\overline{\mathbf{x}}_{era1} = \begin{bmatrix} 131.36667 \\ 133.60000 \\ 99.16667 \\ 50.53333 \end{bmatrix}$$

```
#Calculate the mean vector for Era 2
x_bar_era2 <- apply(skull_data[skull_data[, "Era"] == 2, c(1:4)], 2, mean)
x_bar_era2
```

```
##        MB        BH        BL        NH
## 132.36667 132.70000  99.06667  50.23333
```

$$\overline{\mathbf{x}}_{era2} = \begin{bmatrix} 132.36667 \\ 132.70000 \\ 99.06667 \\ 50.23333 \end{bmatrix}$$

Calculate the sample mean difference vector,

```
x_bar_diff <- x_bar_era1 - x_bar_era2
x_bar_diff
```

```
##   MB   BH   BL   NH
## -1.0  0.9  0.1  0.3
```

$$\overline{\mathbf{x}}_{diff} = \begin{bmatrix} -1.0 \\ 0.9 \\ 0.1 \\ 0.3 \end{bmatrix}$$

Calculate the sample covariance matrices for Era's 1 and 2,

```r
#Calculate the covariance matrix for Era 1
S_era1 <- var(skull_data[skull_data[, "Era"] == 1, 1:4])
S_era1
```

```
##           MB         BH         BL         NH
## MB 26.309195  4.1517241  0.4540230  7.2459770
## BH  4.151724 19.9724138 -0.7931034  0.3931034
## BL  0.454023 -0.7931034 34.6264368 -1.9195402
## NH  7.245977  0.3931034 -1.9195402  7.6367816
```

$$\mathbf{S}_{era1} = \begin{bmatrix} 26.309195 & 4.1517241 & 0.4540230 & 7.2459770 \\ 4.151724 & 19.9724138 & -0.7931034 & 0.3931034 \\ 0.454023 & -0.7931034 & 34.6264368 & -1.9195402 \\ 7.245977 & 0.3931034 & -1.9195402 & 7.6367816 \end{bmatrix}$$

```r
#Calculate the covariance matrix for Era 2
S_era2 <- var(skull_data[skull_data[, "Era"] == 2, 1:4])
S_era2
```

```
##           MB        BH         BL        NH
## MB 23.136782  1.010345  4.7678161 1.8425287
## BH  1.010345 21.596552  3.3655172 5.6241379
## BL  4.767816  3.365517 18.8919540 0.1908046
## NH  1.842529  5.624138  0.1908046 8.7367816
```

$$\mathbf{S}_{era2} = \begin{bmatrix} 23.136782 & 1.010345 & 4.7678161 & 1.8425287 \\ 1.010345 & 21.596552 & 3.3655172 & 5.6241379 \\ 4.767816 & 3.365517 & 18.8919540 & 0.1908046 \\ 1.842529 & 5.624138 & 0.1908046 & 8.7367816 \end{bmatrix}$$

Calculate S_{pooled},

```r
#calculate s_pooled
S_pooled <- (1/(n1 + n2 - 2))*(((n1-1)*S_era1) + ((n2-1)*S_era2))
S_pooled
```

```
##           MB        BH         BL         NH
## MB 24.722989  2.581034  2.6109195  4.5442529
## BH  2.581034 20.784483  1.2862069  3.0086207
## BL  2.610920  1.286207 26.7591954 -0.8643678
## NH  4.544253  3.008621 -0.8643678  8.1867816
```

$$\mathbf{S}_{era2} = \begin{bmatrix} 24.722989 & 2.581034 & 2.6109195 & 4.5442529 \\ 2.581034 & 20.784483 & 1.2862069 & 3.0086207 \\ 2.610920 & 1.286207 & 26.7591954 & -0.8643678 \\ 4.544253 & 3.008621 & -0.8643678 & 8.1867816 \end{bmatrix}$$

Calculate the T^2 test statistic,

```r
#calculate the T^2 test statistic
Tsquared <- t(x_bar_diff) %*% solve(((n1 + n2)/(n1*n2)) * S_pooled) %*% (x_bar_diff)
Tsquared
```

```
##           [,1]
## [1,] 1.650787
```

$T^2 = 1.650787$

Calculate the critical value at the $\alpha = 0.05$ level of significance,

```
p = 4
critical_val = (((n1 + n2 - 2)*p)/(n1 + n2 - p - 1))*qf(0.95,p,(n1+n2-p-1))
critical_val
```

```
## [1] 10.71287
```

Critical Value = 10.71287

```
#Calculate the p-value
1-pf((Tsquared*(n1 + n2 - p - 1)/((n1 + n2 - 2)*p)),p,(n1+n2-p-1))
```

```
##           [,1]
## [1,] 0.8139433
```

State the formal hypothesis test:

1) We will test the hypothesis that H_0: $\boldsymbol{\mu}_{era1} = \boldsymbol{\mu}_{era2}$ vs. H_a: $\boldsymbol{\mu}_{era1} \neq \boldsymbol{\mu}_{era2}$.
2) The level of significance for this test will be $\alpha = 0.05$.
3) The test statistic $T^2 = 1.650787$.
4) The p-value = 0.8139433.
5) Since the p-value = 0.8139433 > than the significance level of 0.05 we fail to reject the null hypothesis that H_0: $\boldsymbol{\mu}_{era1} = \boldsymbol{\mu}_{era2}$.
6) Since we fail to reject the null hypothesis with this test we conclude that we do not have enough evidence that the population mean vectors between Era's 1 and 2 differ. Therefore, $\boldsymbol{\mu}_{era1} = \boldsymbol{\mu}_{era2}$.

2. Observations on two responses are collected for three treatments. The observation vectors $[x_1 \;\; x_2]^T$ are:

$$Treatment\;\;1: \begin{bmatrix}6\\7\end{bmatrix}, \begin{bmatrix}5\\9\end{bmatrix}, \begin{bmatrix}8\\6\end{bmatrix}, \begin{bmatrix}4\\9\end{bmatrix}, \begin{bmatrix}7\\9\end{bmatrix}$$

$$Treatment\;\;2: \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$$

$$Treatment\;\;3: \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$$

(a) Construct a one-way MANOVA table by hand. Verify your calculations in R.
(b) Evaluate Wilks' lambda, Λ^*, and use table 6.3 (provided in the notes) to test (at $\alpha = 0.05$) for treatment effects, i.e., are the mean vectors equal across the three treatments.
(c) Repeat the test in (b) using the chi-square approximation and comment on any similarities or differences with the test completed in (b).

a)

Calculate mean vectors, $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$, and $\overline{\mathbf{x}}_3$ for Treatment's 1, 2, and 3 respectively,

$$\overline{\mathbf{x}}_1 = \begin{bmatrix}\frac{(6+5+8+4+7)}{5}\\ \frac{(7+9+6+9+9)}{5}\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$$

$$\overline{\mathbf{x}}_2 = \begin{bmatrix}\frac{(3+1+2)}{3}\\ \frac{(3+6+3)}{3}\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$\overline{\mathbf{x}}_3 = \begin{bmatrix}\frac{(2+5+3+2)}{4}\\ \frac{(3+1+1+3)}{4}\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

Calculate the overall mean, $\overline{\mathbf{x}}$,

$$\overline{\mathbf{x}} = \begin{bmatrix}\frac{(6+5+8+4+7+3+1+2+2+5+3+2)}{12}\\ \frac{(7+9+6+9+9+3+6+3+3+1+1+3)}{12}\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

Verify the calculations for $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$, and $\overline{\mathbf{x}}_3$, and $\overline{\mathbf{x}}$ using R,

```r
#Create data matrices, calculate mean vectors:
T1c1 = c(6,5,8,4,7)
T1c2 = c(7,9,6,9,9)
T1 = cbind(T1c1,T1c2)
x1_bar = rbind(mean(T1c1),mean(T1c2))
x1_bar
```

```
##      [,1]
## [1,]    6
## [2,]    8
```

```r
T2c1 = c(3,1,2)
T2c2 = c(3,6,3)
T2 = cbind(T2c1,T2c2)
x2_bar = rbind(mean(T2c1),mean(T2c2))
x2_bar
```

```
##      [,1]
## [1,]    2
## [2,]    4
```

```r
T3c1 = c(2,5,3,2)
T3c2 = c(3,1,1,3)
T3 = cbind(T3c1,T3c2)
x3_bar = rbind(mean(T3c1),mean(T3c2))
x3_bar
```

```
##      [,1]
## [1,]    3
## [2,]    2
```

```r
TOvc1 = c(6,5,8,4,7,3,1,2,2,5,3,2)
TOvc2 = c(7,9,6,9,9,3,6,3,3,1,1,3)
TOv = cbind(TOvc1,TOvc2)
xOv_bar = rbind(mean(TOvc1),mean(TOvc2))
xOv_bar
```

```
##      [,1]
## [1,]    4
## [2,]    5
```

Create matrices for calcuation of MANOVA sums of squares (SS) for the first variable,

$$\underset{\text{Observations}}{\begin{bmatrix} 6 & 5 & 8 & 4 & 7 \\ 3 & 1 & 2 & & \\ 2 & 5 & 3 & 2 & \end{bmatrix}} = \underset{\text{Var 1 Overall Mean}}{\begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & & \\ 4 & 4 & 4 & 4 & \end{bmatrix}} = \underset{\text{Treatment Effect}}{\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & & \\ -1 & -1 & -1 & -1 & \end{bmatrix}} = \underset{\text{Residuals}}{\begin{bmatrix} 0 & -1 & 2 & -2 & 1 \\ 1 & -1 & 0 & & \\ -1 & 2 & 0 & -1 & \end{bmatrix}}$$

For the first variable, calculate the observations SS (SS_{obs1}), mean SS (SS_{mean1}), corrected SS (SS_{corr1}), treatment SS (SS_{treat1}), and the residual SS (SS_{res1}),

$SS_{obs1} = 6^2 + 5^2 + 8^2 + 4^2 + 7^2 + 3^2 + 1^2 + 2^2 + 2^2 + 5^2 + 3^2 + 2^2 = 246$

$SS_{mean1} = 12 \cdot (4^2) = 192$

$SS_{corr1} = SS_{obs1}$ - $SS_{mean1} = 246$ - $192 = 54$

$SS_{treat1} = 5 \cdot (2^2) + 3 \cdot (-2^2) + 4 \cdot (-1^2) = 36$

$SS_{res1} = 0^2 + (-1^2) + 2^2 + (-2^2) + 1^2 + 1^2 + (-1^2) + 0^2 + (-1^2) + 2^2 + 0^2 + (-1^2) = 18$

Verify the SS calculations using R

```r
#Create the matrices of observations, overall mean for variable 1, treatment effect, and residual,
OBS1 = rbind(c(6,5,8,4,7),c(3,1,2,0,0),c(2,5,3,2,0))
OV_MEAN1 = rbind(c(4,4,4,4,4),c(4,4,4,0,0),c(4,4,4,4,0))
```

```r
TREAT1 = rbind(c(2,2,2,2,2),c(-2,-2,-2,0,0),c(-1,-1,-1,-1,0))
RES1 = rbind(c(0,-1,2,-2,1),c(1,-1,0,0,0),c(-1,2,0,-1,0))
#Calculate SSobs, SSmean, SScorr, SStreat, and SSres,
SSobs1 = sum(OBS1^2)
SSobs1
```

```
## [1] 246
```

```r
SSmean1 = sum(OV_MEAN1^2)
SSmean1
```

```
## [1] 192
```

```r
SScorr1 = SSobs1 - SSmean1
SScorr1
```

```
## [1] 54
```

```r
SStreat1 = sum(TREAT1^2)
SStreat1
```

```
## [1] 36
```

```r
SSres1 = sum(RES1^2)
SSres1
```

```
## [1] 18
```

Create matrices for calcuation of MANOVA sums of squares (SS) for the second variable,

$$\begin{bmatrix} 7 & 9 & 6 & 9 & 9 \\ 3 & 6 & 3 & & \\ 3 & 1 & 1 & 3 & \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & & \\ 5 & 5 & 5 & 5 & \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & & \\ -3 & -3 & -3 & -3 & \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ -1 & 2 & -1 & & \\ 1 & -1 & -1 & 1 & \end{bmatrix}$$

Observations Var 2 Overall Mean Treatment Effect Residuals

For the second variable, calculate the observations SS (SS_{obs2}), mean SS (SS_{mean2}), corrected SS (SS_{corr2}), treatment SS (SS_{treat2}), and the residual SS (SS_{res1}),

$SS_{obs2} = 7^2 + 9^2 + 6^2 + 9^2 + 9^2 + 3^2 + 6^2 + 3^2 + 3^2 + 1^2 + 1^2 + 3^2 = 402$

$SS_{mean2} = 12 \cdot (5^2) = 300$

$SS_{corr2} = SS_{obs2}$ - $SS_{mean2} = 402$ - $300 = 102$

$SS_{treat2} = 5 \cdot (3^2) + 3 \cdot (-1^2) + 4 \cdot (-3^2) = 84$

$SS_{res2} = (-1)^2 + (1^2) + (-2^2) + 1^2 + 1^2 + (-1^2) + 2^2 + (-1^2) + 1^2 + (-1^2) + (-1^2) + 1^2 = 18$

Verify the SS calculations using R

```r
#Create the matrices of observations, overall mean for variable 2, treatment effect, and residual,
OBS2 = rbind(c(7,9,6,9,9),c(3,6,3,0,0),c(3,1,1,3,0))
OV_MEAN2 = rbind(c(5,5,5,5,5),c(5,5,5,0,0),c(5,5,5,5,0))
TREAT2 = rbind(c(3,3,3,3,3),c(-1,-1,-1,0,0),c(-3,-3,-3,-3,0))
RES2 = rbind(c(-1,1,-2,1,1),c(-1,2,-1,0,0),c(1,-1,-1,1,0))
#Calculate SSobs, SSmean, SScorr, SStreat, and SSres,
SSobs2 = sum(OBS2^2)
SSobs2
```

```
## [1] 402
```

```r
SSmean2 = sum(OV_MEAN2^2)
SSmean2
```

```
## [1] 300
```

```r
SScorr2 = SSobs2 - SSmean2
SScorr2
```

```
## [1] 102
```

```r
SStreat2 = sum(TREAT2^2)
SStreat2
```

```
## [1] 84
```

```r
SSres2 = sum(RES2^2)
SSres2
```

```
## [1] 18
```

Calculate the cross products between variables 1 and 2,

$CP_{mean} = 12 \cdot (4 \cdot 5) = 240$

$CP_{treatment} = 5 \cdot (2 \cdot 3) + 3 \cdot (-2 \cdot -1) + 4 \cdot (-1 \cdot -3) = 48$

$CP_{residual} = (0 \cdot -1) + (-1 \cdot 1) + (2 \cdot -2) + (-2 \cdot 1) + (1 \cdot 1) + (1 \cdot -1) + (-1 \cdot 2) + (0 \cdot -1) + (-1 \cdot 1) + (2 \cdot -1) + (0 \cdot -1) + (-1 \cdot 1) = -13$

$CP_{total} = (6 \cdot 7) + (5 \cdot 9) + (8 \cdot 6) + (4 \cdot 9) + (7 \cdot 9) + (3 \cdot 3) + (1 \cdot 6) + (2 \cdot 3) + (2 \cdot 3) + (5 \cdot 1) + (3 \cdot 1) + (2 \cdot 3) = 275$

$CP_{corr} = CP_{total}$ - $CP_{mean} = 275$ - $240 = 35$

```r
#Calculate cross products
CP_mean = sum(OV_MEAN1 * OV_MEAN2)
CP_mean
```

```
## [1] 240
```

```r
CP_treat = sum(TREAT1 * TREAT2)
CP_treat
```

```
## [1] 48
```

```r
CP_res = sum(RES1 * RES2)
CP_res
```

```
## [1] -13
```

```r
CP_tot = sum(OBS1*OBS2)
CP_tot
```

```
## [1] 275
```

```r
CP_corr = CP_tot - CP_mean
CP_corr
```

```
## [1] 35
```

Create the MANOVA Table,

groups (g) = 3

parameters (p) = 2

number of observations (n) = 12

$$\underline{MANOVA\ \ Table\ \ for\ \ Treatments\ \ 1, 2, and\ \ 3}$$

SOURCE	SS and CP Matrices	df
$Treatment$	$\mathbf{B} = \begin{bmatrix} 36 & 48 \\ 48 & 84 \end{bmatrix}$	$g - 1 = 2$
$Residual$	$\mathbf{W} = \begin{bmatrix} 18 & -13 \\ -13 & 18 \end{bmatrix}$	$n - g = 9$
$Total\ Corr\ SS$	$\mathbf{T} = \begin{bmatrix} 54 & 35 \\ 35 & 102 \end{bmatrix}$	

b)

Calculate Wilkes' lambda, Λ^*,

$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B} + \mathbf{W}|} = \frac{18^2 - (-13^2)}{(54 \cdot 102) - 35^2} = 0.03619$

Check the Wilkes' lambda calculation with R

```r
#calculate Wilkes' lambda
B = cbind(c(36, 48), c(48,84))
W = cbind(c(18,-13), c(-13,18))
wilkes_lambda = det(W)/(det(B+W))
wilkes_lambda
```

```
## [1] 0.03618959
```

Calculate the test statistic from table 6.3.

$Test_{stat} = \frac{\Sigma n_l - p - 2}{p} \cdot \frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} = \frac{12 - 2 - 2}{2} \cdot \frac{1 - \sqrt{0.03619}}{\sqrt{0.03619}} = 17.026$

The test statistic is distributed $F_{2p, 2(\Sigma n_l - p - 2)}$

Using R calculate the critical value and p-value,

```r
#calculate the critical value
crit_val = qf(0.95,4,16)
crit_val
```

```
## [1] 3.006917
```

```r
p_val = 1-pf(17.026,4,16)
p_val
```

```
## [1] 1.28297e-05
```

Critical Value = 3.006917

p-value = 1.28297e-05

State the formal hypothesis test:

1) We will test the hypothesis $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \boldsymbol{\tau}_3 = 0$ vs. H_a: that at least one of the $\boldsymbol{\tau}_l$'s is $\neq 0$.
2) The significance level for this test is $\alpha = 0.05$.
3) The test statistic was calculated and is 17.026.
4) The p-value for this test is 1.28297e-05.
5) The p-value, 1.28297e-05, is less than the significance level of 0.05. Therefore, we will reject the null hypothesis, $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \boldsymbol{\tau}_3 = 0$.
6) Since the null hypothesis was rejected we will conclude that at least one of the $\boldsymbol{\tau}_l$'s is not equal to zero.

c)

Calculate the test statistic $-(n-1-\frac{p+g}{2})\ln(\Lambda)$,

$T_{statistic}$ = -$(12-1-\frac{(2+3)}{2})\ln(0.03619) = 28.21127$

Using R, calculate the critical value and p-value.

The test statistic has the approximate distribution $\chi^2_{p(g-1)}$.

```r
critical_value = qchisq(0.95,4)
critical_value
```

```
## [1] 9.487729
```

```r
pval = 1-pchisq(28.21127,4)
pval
```

```
## [1] 1.130159e-05
```

Critical Value = 9.487729

p-value = 1.130159e-05

State the formal hypothesis test:

1) We will test the hypothesis $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \boldsymbol{\tau}_3 = 0$ vs. H_a: that at least one of the $\boldsymbol{\tau}_l$'s is $\neq 0$.
2) The significance level for this test is $\alpha = 0.05$.
3) The test statistic was calculated and is 28.21127.
4) The p-value for this test is 1.130159e-05.
5) The p-value, 1.130159e-05, is less than the significance level of 0.05. Therefore, we will reject the null hypothesis, $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \boldsymbol{\tau}_3 = 0$.
6) Since the null hypothesis was rejected we will conclude that at least one of the $\boldsymbol{\tau}_l$'s is not equal to zero.

Both tests from parts b) and c) result in the same conclusion. The p-values for each test were very similar to each other and of the same order of magnitude.

3. Find the spectral decomposition of the following matrix, A, by hand. Verify your solution in R. (You can type up or scan your hand calculations)

$$\mathbf{A} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

I completed this work by hand. Please see the attached handwritten calculation and answer for this problem. I checked my work using R.

```
#Check work using R
#Create the matrix A
A = cbind(c(25,50,5),c(50,100,10),c(5,10,145))
A
```

```
##      [,1] [,2] [,3]
## [1,]   25   50    5
## [2,]   50  100   10
## [3,]    5   10  145
```

```
#find the eigenvalues and eigenvectors for this matrix
eigen_v = eigen(A)
values = eigen_v$values
values
```

```
## [1] 1.500000e+02 1.200000e+02 2.842171e-14
```

```
vectors = eigen_v$vectors
vectors
```

```
##           [,1]       [,2]          [,3]
## [1,] 0.1825742 -0.4082483  8.944272e-01
## [2,] 0.3651484 -0.8164966 -4.472136e-01
## [3,] 0.9128709  0.4082483 -7.632783e-17
```

```
#check the spectral decomposition
lambda1 = eigen_v$values[3]
lambda1
```

```
## [1] 2.842171e-14
```

```
lambda2 = eigen_v$values[2]
lambda2
```

```
## [1] 120
```

```
lambda3 = eigen_v$values[1]
lambda3
```

```
## [1] 150
```

```
eigen_vec1 = eigen_v$vectors[,3]
eigen_vec1
```

```
## [1]  8.944272e-01 -4.472136e-01 -7.632783e-17
```

```
eigen_vec2 = eigen_v$vectors[,2]
eigen_vec2
```

```
## [1] -0.4082483 -0.8164966  0.4082483
```

```r
eigen_vec3 = eigen_v$vectors[,1]
eigen_vec3
```

```
## [1] 0.1825742 0.3651484 0.9128709
```

```r
A_check = (lambda1*(eigen_vec1)%*%t(eigen_vec1)) + (lambda2*(eigen_vec2)%*%t(eigen_vec2)) + (lambda3*(ei
A_check
```

```
##      [,1] [,2] [,3]
## [1,]   25   50    5
## [2,]   50  100   10
## [3,]    5   10  145
```

STAA 574: HW 3
PROBLEM 3
JON DOLLARD

3. FIND THE SPECTRAL DECOMPOSITION OF THE FOLLOWING MATRIX, A, BY HAND. VERIFY YOUR SOLUTION IN R. (YOU CAN TYPE UP OR SCAN YOUR HAND CALCULATIONS.)

$$A = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{bmatrix} \right| = 0$$

$$(25-\lambda)\Big((100-\lambda)(145-\lambda) - (10)(10)\Big) - 50\Big(50(145-\lambda) - (5)(10)\Big) +$$
$$5\Big((50)(10) - 5(100-\lambda)\Big)$$

AFTER SIMPLIFICATION,

$$-\lambda^3 + 270\lambda^2 - 18{,}000\lambda = 0$$

PROBLEM 3 (CONT.)

SOLVE THE ROOTS OF $-\lambda^3 + 270\lambda^2 - 18{,}000\lambda = 0$

$$\lambda(\lambda^2 - 270\lambda + 18{,}000) = 0$$

ONE ROOT IS $\lambda = 0$

$\lambda^2 - 270\lambda + 18{,}000 = 0$ FACTORS TO

$(\lambda - 120)(\lambda - 150) = 0$

THEREFORE, $\lambda = 120$ + $\lambda = 150$

THE EIGENVALUES ARE $\lambda_1 = 0$, $\lambda_2 = 120$, $\lambda_3 = 150$

NEXT, FIND THE EIGENVECTORS ASSOCIATED WITH THESE EIGENVALUES.

FIRST, $\lambda_1 = 0$

$Ax = \lambda_i x$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$25x_1 + 50x_2 + 5x_3 = 0$$
$$50x_1 + 100x_2 + 10x_3 = 0$$
$$5x_1 + 10x_2 + 145x_3 = 0$$

USE ROW REDUCTION TO SOLVE FOR X_1, X_2 + X_3,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \underset{\sim}{\xleftrightarrow{L_1 \; L_3}} \begin{bmatrix} 5 & 10 & 145 \\ 50 & 100 & 10 \\ 25 & 50 & 5 \end{bmatrix} \underset{\sim}{L_1 \leftarrow \frac{L_1}{5}} \begin{bmatrix} 1 & 2 & 29 \\ 50 & 100 & 10 \\ 25 & 50 & 5 \end{bmatrix}$$

$$\underset{\sim}{L_3 \leftarrow L_2 - 2L_3} \begin{bmatrix} 1 & 2 & 29 \\ 50 & 100 & 10 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{L_2 \leftarrow 50L_1 - L_2} \begin{bmatrix} 1 & 2 & 29 \\ 0 & 0 & 1440 \\ 0 & 0 & 0 \end{bmatrix}$$

AT THIS POINT WE CAN WRITE THE EQUATIONS AND SOLVE

$$X_1 + 2X_2 + 29X_3 = 0$$

$$1440X_3 = 0 \Rightarrow X_3 = 0$$

$X_1 + 2X_2 = 0 \Rightarrow X_1 = -2X_2$ LET $X_2 = 1$, THEN $X_1 = -2$

THE EIGENVECTOR IS $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. NORMALIZING; $\sqrt{\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{5}$

THE NORMALIZED EIGENVECTOR FOR EIGENVALUE $\lambda = 0$ IS $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

NEXT FIND THE EIGENVECTOR FOR $\lambda = 120$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 120 \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$25X_1 + 50X_2 + 5X_3 = 120X_1 \Rightarrow -95X_1 + 50X_2 + 5X_3 = 0$$
$$50X_1 + 100X_2 + 10X_3 = 120X_2 \Rightarrow 50X_1 - 20X_2 + 10X_3 = 0$$
$$5X_1 + 10X_2 + 145X_3 = 120X_3 \Rightarrow 5X_1 + 10X_2 + 25X_3 = 0$$

$$\begin{bmatrix} -95 & 50 & 5 \\ 50 & -20 & 10 \\ 5 & 10 & 25 \end{bmatrix} \underset{\sim}{L_1 \leftrightarrow L_3} \begin{bmatrix} 5 & 10 & 25 \\ 50 & -20 & 10 \\ -95 & 50 & 5 \end{bmatrix} \underset{\sim}{L_1 \leftarrow L_1/5} \begin{bmatrix} 1 & 2 & 5 \\ 50 & -20 & 10 \\ -95 & 50 & 5 \end{bmatrix}$$

$$\underset{\sim}{L_2 \leftarrow -50L_1 + L_2} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -120 & -240 \\ -95 & 50 & 5 \end{bmatrix} \underset{\sim}{L_3 \leftarrow 95L_1 + L_3} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -120 & -240 \\ 0 & 240 & 480 \end{bmatrix} \underset{\sim}{L_3 \leftarrow \frac{L_3}{240}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -120 & -240 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\underset{\sim}{L_2 \leftarrow \frac{L_2}{-120}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \underset{\sim}{L_3 \leftarrow L_2 - L_3} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

WE NOW HAVE 2 INDEPENDENT EQUATIONS AND 3 UNKNOWNS.

WRITE THE EQUATIONS AND SOLVE FOR X_1, X_2, AND X_3

$$x_1 + 2x_2 + 5x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$X_2 = -2X_3$, LET $X_3 = 1$, THEN $X_2 = -2$ AND $X_1 = -1$

THE EIGENVECTOR IS $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, NORMALIZING $\sqrt{\begin{bmatrix} -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}} = \sqrt{6}$

THE NORMALIZED EIGENVECTOR FOR $\lambda = 2$ IS $\begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

FINALLY, FIND THE EIGENVECTOR FOR $\lambda = 150$,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 150 \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$25x_1 + 50x_2 + 5x_3 = 150x_1 \Rightarrow -125x_1 + 50x_2 + 5x_3 = 0$$
$$50x_1 + 100x_2 + 10x_3 = 150x_2 \Rightarrow 50x_1 - 50x_2 + 10x_3 = 0$$
$$5x_1 + 10x_2 + 145x_3 = 150x_3 \Rightarrow 5x_1 + 10x_2 - 5x_3 = 0$$

USING ROW REDUCTION SOLVE FOR X_1, X_2, + X_3:

$$\begin{bmatrix} -125 & 50 & 5 \\ 50 & -50 & 10 \\ 5 & 10 & -5 \end{bmatrix} \underset{\sim}{L_1 \leftrightarrow \tfrac{L_3}{5}} \begin{bmatrix} 1 & 2 & -1 \\ 50 & -50 & 10 \\ -125 & 50 & 5 \end{bmatrix} \underset{\sim}{L_2 \leftarrow 50L_1 - L_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 150 & -60 \\ -125 & 50 & 5 \end{bmatrix}$$

$$\underset{\sim}{L_3 \leftarrow 125L_1 + L_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 150 & -60 \\ 0 & 300 & -120 \end{bmatrix} \underset{\sim}{L_3 \leftarrow 2L_2 - L_3} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 150 & -60 \\ 0 & 0 & 0 \end{bmatrix}$$

NOW WE HAVE 2 INDEPENDENT EQUATIONS AND 3 UNKNOWNS.

WRITE THE EQUATIONS AND SOLVE

$$X_1 + 2X_2 - X_3 = 0$$
$$150X_2 - 60X_3 = 0$$
$$X_3 = 2.5X_2$$

SUBSTITUTING $X_1 + 2X_2 - 2.5X_2 = 0 \Rightarrow X_1 = 0.5X_2$

LET $X_1 = 1$ THEN $X_2 = 2$ AND $X_3 = 5$.

THE EIGENVECTOR IS $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. NORMALIZING, $\sqrt{\begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}} = \sqrt{30}$

THE NORMALIZED EIGENVECTOR FOR $\lambda = 150$ IS $\begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$

LAST WRITE THE SPECTRAL DECOMPOSITION OF A,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} = 0 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} +$$

$$120 \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} +$$

$$150 \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 1/\sqrt{30} & 2/\sqrt{30} & 5/\sqrt{30} \end{bmatrix}$$

$$= 0 \begin{bmatrix} 4/5 & 2/5 & 0 \\ -2/5 & -1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 120 \begin{bmatrix} 1/6 & 2/6 & -1/6 \\ 2/6 & 4/6 & -2/6 \\ -1/6 & -2/6 & 1/6 \end{bmatrix}$$

$$+ 150 \begin{bmatrix} 1/30 & 2/30 & 5/30 \\ 2/30 & 4/30 & 10/30 \\ 5/30 & 10/30 & 25/30 \end{bmatrix}$$

CHECK WITH R. PLEASE SEE ATTACHED PAGE.