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SML Assignment-1

1) i) Given $P(\omega_1) = \frac{1}{4}$ $P(\omega_2) = \frac{3}{4}$

$P(x|\omega_1) = N(2,1)$ $P(x|\omega_2) = N(5,1)$

Decision boundary for normal distribution given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Therefore

$$P(x|\omega_1) = N(2,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}$$

$$P(x|\omega_2) = N(5,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-5)^2}$$

For zero-one loss

$$P(x|\omega_1)P(\omega_1) = P(x|\omega_2)P(\omega_2)$$

$$\Rightarrow \frac{1}{\cancel{4}} \cdot \frac{\cancel{1}}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}(x-2)^2} = \frac{\cancel{1}}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}(x-5)^2} \cdot \frac{3}{\cancel{4}}$$

$$\Rightarrow e^{-\frac{1}{2}(x-2)^2} = 3 \cdot e^{-\frac{1}{2}(x-5)^2}$$ (taking log on both the side)

$$\Rightarrow -\frac{1}{2}(x-2)^2 = \ln 3 - \frac{1}{2}(x-5)^2$$

$$\Rightarrow -(x-2)^2 + (x-5)^2 = 2\ln 3$$

$$\Rightarrow (x-5)^2 - (x-2)^2 = 2\ln 3$$

$$\Rightarrow x^2 + 25 - 10x - (x^2 + 4 - 4x) = 2\ln 3$$

$$\Rightarrow \cancel{x^2} + 25 - 10x - \cancel{x^2} - 4 + 4x = 2\ln 3$$

$$\Rightarrow 21 - 6x = 2\ln 3$$

$$\Rightarrow x = \frac{21 - 2\ln 3}{6}$$

$$\boxed{x = 3.1338}$$

Teacher's Signature ....

Given

(ii) $\lambda_{12} = 2$, $\lambda_{21} = 3$, $\lambda_{11} = 0$, $\lambda_{22} = 0$

$\Rightarrow$ $(\lambda_{21} - \lambda_{11})\, P(x|\omega_1)\, P(\omega_1) = (\lambda_{12} - \lambda_{22})\, P(x|\omega_2)\, P(\omega_2)$

$\Rightarrow$ $(3-0) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2} \cdot \frac{1}{4} = (2-0) \cdot \frac{1}{\sqrt{2\pi}} e^{-1/2(x-5)^2} \cdot \frac{3}{4}$

$\Rightarrow$ $\cancel{3} \cdot e^{-\frac{1}{2}(x-2)^2} = 2 \cdot \cancel{3} \cdot e^{-\frac{1}{2}(x-5)^2}$

$\Rightarrow$ $\ln\left(\frac{-1}{2}(x-2)^2\right) = \ln\left(2 \cdot \frac{-1}{2}(x-5)^2\right)$

$\Rightarrow$ $\frac{-1}{2}(x-2)^2 = \ln 2 - \frac{1}{2}(x-5)^2$

$\Rightarrow$ $(x-5)^2 - (x-2)^2 = 2\ln 2$

$\Rightarrow$ $\cancel{x^2} - 10x + 25 - \cancel{x^2} + 4x - 4 = 2\ln 2$

$\Rightarrow$ $21 - 6x = 2\ln 2$

$\Rightarrow$ $x = \frac{21 - 2\ln 2}{6}$

$\Rightarrow$ $x = \frac{21 - 2 \times 0.693}{6} = 3.269$

$$\boxed{x = 3.269}$$

Explanation :- NO, we'll not use the zero-one loss in a real-world dataset for a task like cancer prediction. The real world data is unbalanced, if someone has cancer and this predicts false, so with a false negative rate greater than zero, it could be quite concerning for the patient, whose health is at risk.

2) Given $X = [x_1, x_2, x_3]$ random vector

mean vector $\mu = [5, -5, 6]$

co-variance matrix $\begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}$

$Y = A^T X + B$ $\quad A = [2, -1, 2]^T \quad B = 5$

we have to calculate mean of Y which is given by

$\Rightarrow E(Y) = E(A^T X) + E(B)$

$\Rightarrow E(Y) = E(A^T X) + 5$

$\Rightarrow E(Y) = A^T E(X) + 5$

$\Rightarrow E(Y) = [2, -1, 2] \cdot \begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} + 5$

$\Rightarrow E(Y) = (10 + 12 + 5) + 5$

$\Rightarrow \boxed{E(Y) = 32}$

given

3) a) $P(x|w_i) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x - a_i}{b}\right)^2}$, $\forall i = 1, 2,$

here $P(w_1) = P(w_2) = \frac{1}{2}$ —— (1)

Decision Boundary for zero one loss

$P(w_1) P(w_1|x) = P(w_2|x) P(\cancel{w_2})$

as $P(w_1) = P(w_2)$

$P(w_1|x) = P(w_2|x)$

$$\frac{\cancel{1}}{\cancel{\pi b}} \cdot \frac{1}{1 + \left(\frac{x - a_1}{\cancel{b}}\right)^2} = \frac{\cancel{1}}{\cancel{\pi b}} \cdot \frac{1}{1 + \left(\frac{x - a_2}{\cancel{b}}\right)^2}$$

$$\Rightarrow (x-a_1)^2 = (x-a_2)^2$$

$$\Rightarrow \cancel{x^2} - 2a_1x + a_1^2 = \cancel{x^2} - 2a_2x + a_2^2$$

$$\Rightarrow x = \frac{a_2^2 - a_1^2}{2(a_2-a_1)}$$

$$\Rightarrow x = \frac{\cancel{(a_2-a_1)}(a_2+a_1)}{2\cancel{(a_2-a_1)}}$$

$$\Rightarrow \boxed{x = \frac{1}{2}(a_2+a_1)} \text{ here } a_1 \neq a_2$$

c) Overall error rate given by

$$P(\text{error}) = \int_{-\infty}^{\infty} \min[P(\omega_1|x), P(\omega_2|x)]\, P(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\frac{1}{2}(a_1+a_2)} P(\omega_2|x)\, P(x)\, dx + \int_{\frac{1}{2}(a_1+a_2)}^{\infty} P(\omega_1|x)\, P(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\frac{1}{2}(a_1+a_2)} \frac{1}{2\pi b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}\, dx + \int_{\frac{1}{2}(a_1+a_2)}^{\infty} \frac{1}{2\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}\, dx$$

as we know $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$

$$\Rightarrow \frac{1}{2\pi b}\left[\tan^{-1}\left(\frac{x-a_2}{b}\right)\right]_{-\infty}^{\frac{(a_1+a_2)}{2}} + \frac{1}{2\pi}\left[\tan^{-1}\left(\frac{x-a_1}{b}\right)\right]_{\frac{(a_1+a_2)}{2}}^{\infty}$$

$$\Rightarrow \frac{1}{2\pi b}\left\{\tan^{-1}\left(\frac{a_1-a_2}{2b}\right) + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}\left(\frac{a_2-a_1}{2b}\right)\right\}$$

$$\Rightarrow \frac{1}{2\pi b}\left\{2\tan^{-1}\left(\frac{a_1-a_2}{2b}\right) + \pi\right\}$$

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$P(error) = \frac{1}{\pi}\left(\tan^{-1}\frac{(3-5)}{2} + \frac{\pi}{2}\right)$ | $\boxed{P(error) = \frac{1}{4}}$

$= \frac{1}{\pi}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) \Rightarrow \frac{1}{4}$

Here, $a_1 = 3$ $a_2 = 5$ $b = 1$

$\Rightarrow \boxed{\frac{1}{\pi b}\left(\tan^{-1}\left(\frac{a_1 - a_2}{2b}\right) + \frac{\pi}{2}\right)}$

~~$P(error) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{5-3}{2}\right) + \frac{\pi}{2}\right) \Rightarrow \frac{1}{\pi}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{3}{4}$~~

4) a) Given

$Pdf(x) = [a\ b]$

$covariance = \begin{bmatrix} \theta(1-\theta) & 0 \\ 0 & \sigma^2 \end{bmatrix}$

$P(a=1) = \theta$

as b follow Gaussian distribution

$P(b) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{b-m}{\sigma}\right)^2}$

$P(a) = \theta^a (1-\theta)^{1-a}$

Therefore, $P(x) = P(a) \cdot P(b)$

$$\boxed{P(x) = \theta^a (1-\theta)^{1-a}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{b-m}{\sigma}\right)^2}\right)}$$

b) Given $pdf = P(x)$

No. of iid samples $= N$

$\Rightarrow q(x) = P(x_1) \cdot P(x_2) \cdot P(x_3) \dots P(x_N)$

$\Rightarrow q(x) = \prod_{i=1}^{x} P(x_i)$

as $P(x)$ is drawn from Pdf of part (a)

$q(x)$ given by

$$\Rightarrow q(x) = \prod_{i=1}^{N} \theta^a (1-\theta)^{1-a}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{b_i - m}{\sigma}\right)^2}\right)$$

$$\Rightarrow \ln(q(x)) = \sum_{i=1}^{N} \Big[a_i \ln(\theta) + (1-a_i)\ln(1-\theta) + \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2}\left(\frac{b_i - m}{\sigma}\right)^2 \Big]$$

as we have to maximize $q(x)$, so differentiating w.r.t $d\theta$ and equating to 0.

$$\Rightarrow \frac{d \ln(q(x))}{d\theta} = 0$$

$$\Rightarrow \frac{d}{d\theta}\Big(a_i \ln(\theta) + (1-a_i)\ln(1-\theta)\Big) + \underbrace{\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2}\left(\frac{b_i - m}{\sigma}\right)^2}_{\text{constant}} \Big) = 0$$

$$\Rightarrow \frac{d}{d\theta}\Big(a_i \ln(\theta) + (1-a_i)\ln(1-\theta)\Big) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \left(\frac{a_i}{\theta} - \left(\frac{1-a_i}{1-\theta}\right)\right) = 0$$

$$\Rightarrow \frac{1}{\theta(1-\theta)} \sum_{i=1}^{N} \big(a_i(1-\theta) - \theta(1-a_i)\big) = 0$$

$$\Rightarrow \frac{1}{\theta(1-\theta)} \sum_{i=1}^{N} \big(a_i - \cancel{\theta a_i} - \theta + \cancel{\theta a_i}\big) = 0$$

$$\Rightarrow \sum_{i=1}^{N} (a_i - \theta) = 0$$

$$\Rightarrow \sum_{i=1}^{N} a_i - N\theta = 0$$

$$\Rightarrow \boxed{\theta = \frac{1}{N}\sum_{i=1}^{N} a_i}$$