SABR model calibration

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SABR model

$$
\begin{cases}
dF_t = \alpha_t F_t^{\beta} dW_t \\
d\alpha_t = v\alpha_t dZ_t \\
dW_t dZ_t = \rho dt \\
F_0 = F, \alpha_0 = \alpha
\end{cases}
$$

$0 < \beta < 1$, $\alpha > 0, v > 0$, $-1 < \rho < 1$
Optimized parameters are $\vec{\theta} = (\alpha, v, \beta, \rho)$

SABR model

$$F_m = \sqrt{FK}, \quad \zeta = \frac{v}{\alpha} F_m^{1-\beta} \log \frac{F}{K}$$

$$X(\zeta, \rho) = \log \frac{\sqrt{1 - 2\zeta\rho + \zeta^2} + \zeta - \rho}{1 - \rho}$$

$$q_1 = \frac{(\beta - 1)^2 \alpha^2 F_m^{2\beta - 2}}{24}, \quad q_2 = \frac{\rho\beta\alpha v F_m^{\beta - 1}}{4}$$

$$q_3 = \frac{2 - 3\rho^2}{24} v^2, \quad S = 1 + T(q_1 + q_2 + q_3)$$

$$D = F_m^{1-\beta} \left[1 + \frac{(\beta - 1)^2}{24} \log^2 \frac{F}{K} + \frac{(\beta - 1)^4}{1920} \log^4 \frac{F}{K} \right]$$

$$\sigma = \frac{\alpha S}{D} \times \frac{\zeta}{X(\zeta, \rho)}$$

Derivatives

$$\log \sigma = \log \alpha + \log S - \log D + \log \zeta - \log X(\zeta, \rho)$$

$$\frac{\partial \log \sigma}{\partial \alpha} = \frac{1}{\alpha} + \frac{1}{S}\frac{\partial S}{\partial \alpha} + \frac{1}{\zeta}\frac{\partial \zeta}{\partial \alpha} - \frac{1}{X}\frac{\partial X}{\partial \alpha}$$

$$\frac{\partial \log \sigma}{\partial \rho} = \frac{1}{S}\frac{\partial S}{\partial \rho} - \frac{1}{X}\frac{\partial X}{\partial \rho}$$

$$\frac{\partial \log \sigma}{\partial v} = \frac{1}{S}\frac{\partial S}{\partial v} + \frac{1}{\zeta}\frac{\partial \zeta}{\partial v} - \frac{1}{X}\frac{\partial X}{\partial v}$$

$$\frac{\partial \log \sigma}{\partial \beta} = \frac{1}{S}\frac{\partial S}{\partial \beta} - \frac{1}{D}\frac{\partial D}{\partial \beta} + \frac{1}{\zeta}\frac{\partial \zeta}{\partial \beta} - \frac{1}{X}\frac{\partial X}{\partial \beta}$$

Calibration

$$f(\vec{\theta}) = \frac{1}{2}\sum_{i=1}^{N}\Big(\sigma(K_i, T_i, \vec{\theta}) - \hat{\sigma}_i\Big)^2 = \frac{1}{2}\|\vec{r}\|_2^2$$

$$\vec{r}_i = \sigma(K_i, T_i, \vec{\theta}) - \hat{\sigma}_i$$

$$J_{ji} = \frac{\partial \vec{r}_i}{\partial \theta_j}$$

$$H_{jk}(r_i) = \frac{\partial^2 \vec{r}_i}{\partial \theta_j \partial \theta_k}$$

$$\nabla f = J\vec{r}$$

$$\nabla^2 f = JJ^T + \sum_{i=1}^{N} r_i H(\vec{r}_i) \approx JJ^T$$

Calibration

Newton Algorithm with damping(Levenberg–Marquardt algorithm):

$$\vec{\theta}_{k+1} = \vec{\theta}_k - \left(\lambda I + JJ^T\right)^{-1} J\vec{r}$$

Modified Marquardt algorithm:

$$\vec{\theta}_{k+1} = \vec{\theta}_k - \left(\lambda \mathrm{diag}(JJ^T) + JJ^T\right)^{-1} J\vec{r}$$

$$\vec{\theta}_{k+1} = \mathrm{proj}\left(\vec{\theta}_k - \left(\lambda \mathrm{diag}(JJ^T) + JJ^T\right)^{-1} J\vec{r}\right)$$

Calibration

We can calibrate on prices as well.

$$\tilde{f}(\vec{\theta}) = \frac{1}{2}\sum_{i=1}^{N}\left(C(K_i, T_i, \vec{\theta}) - \hat{C}_i\right)^2 = \frac{1}{2}\|\vec{r}\|_2^2$$

$$\vec{r}_i = C(K_i, T_i, \vec{\theta}) - \hat{C}_i$$

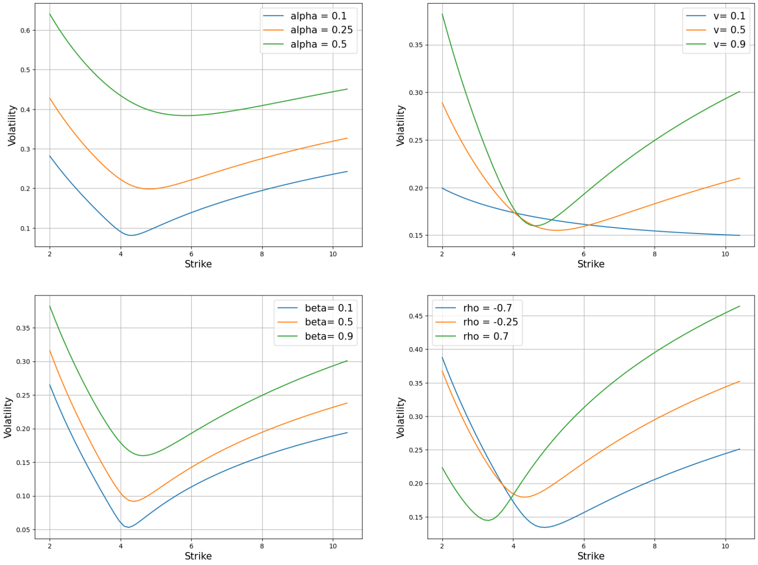
$$J_{ji} = \frac{\partial \vec{r}_i}{\partial \theta_j} = \mathsf{vega}_i \frac{\partial \sigma_i}{\partial \theta_j}$$

$$H_{jk}(r_i) = \frac{\partial^2 \vec{r}_i}{\partial \theta_j \partial \theta_k} = \mathsf{vanna}_i \frac{\partial \sigma_i}{\partial \theta_k} + \mathsf{vega}_i \frac{\partial^2 \sigma_i}{\partial \theta_k \partial \theta_j}$$

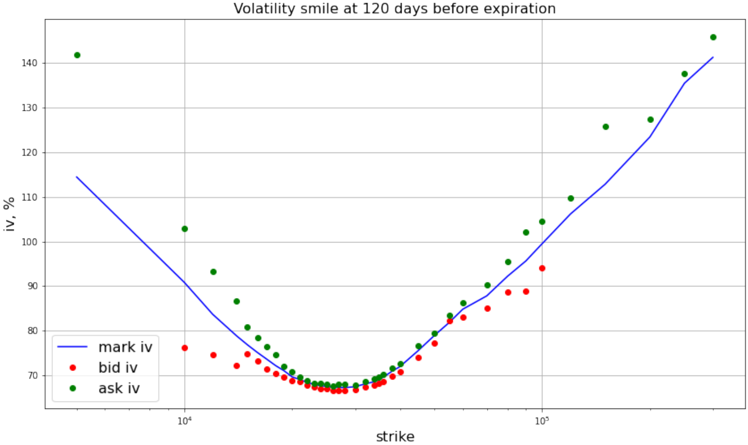
$$\nabla \tilde{f} = J\vec{r}$$

$$\nabla^2 \tilde{f} = JJ^T + \sum_{i=1}^{N} r_i H(\vec{r}_i) \approx JJ^T$$

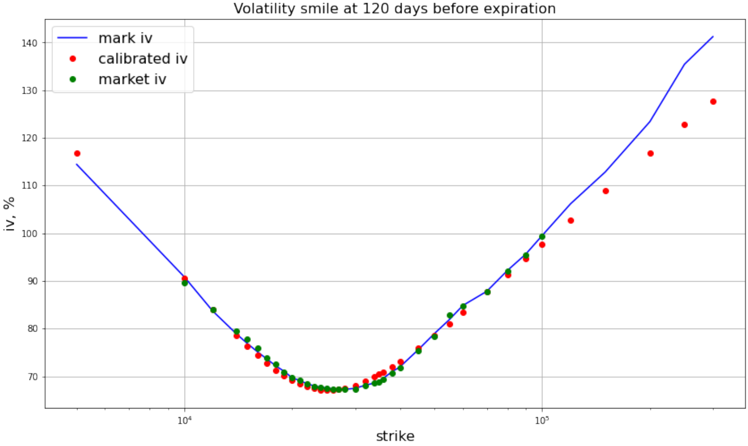
Volatility smile



Volatility smile



Calibrated smile



Calibrated price

Contour plot