Stochastic volatility models
Lecture 7: Heston model-1: vanilla option prices

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1 The model

1.1 Equations for the price and volatility processes

In the Heston model, the T-forward price process $F_t = S_t B_T / B_t$ is defined by the system of equations

$$dF_t = \sqrt{V_t} F_t dW_t^1, \quad F_0 > 0, \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2, \quad V_0 > 0, \tag{2}$$

where W_t^1 and W_t^2 are correlated Brownian motions:

$$dW_t^1 dW_t^2 = \rho dt.$$

The variables $\kappa > 0$, $\theta > 0$, $\sigma > 0$, $\rho \in (-1, 1)$ are the model parameters. The process V_t is the stochastic variance, and $\sigma_t = \sqrt{V_t}$ is the stochastic volatility.

Equations (1)-(2) have a unique strong solution (F_t, V_t). The process F_t is strictly positive and the process V_t is non-negative. If $2\kappa\theta \geq \sigma^2$, then the process V_t is strictly positive. The two-dimensional process (F, V) has the strong Markov property.

Proof. The existence, uniqueness and non-negativity or positivity of V_t were proved in Lecture 5. The existence, uniqueness and positivity of F_t follow from that F_t is a stochastic exponent and can be found explicitly by Ito's formula:

$$F_t = F_0 \exp\left(\int_0^t \sqrt{V_s} dW_s^1 - \frac{1}{2}\int_0^t V_s ds\right).$$

The strong Markov property follows from the existence and uniqueness of the solution. □

1.2 Martingale property of the price process (*)

In the Heston model, the forward price F_t is a martingale.

Proof. Since F_t is a stochastic exponent, in order to show that it is a martingale, it is sufficient to verify Novikov's condition

$$E\exp\left(\frac{1}{2}\int_0^T V_t dt\right) < \infty.$$

To that end, it is enough to find $\varepsilon > 0$ such that for all $t \in [0, T-\varepsilon]$ it holds that

$$E\exp\left(\frac{1}{2}\int_t^{t+\varepsilon} V_s ds\right) < \infty$$

(see Karatzas, Shreve, *Brownian motion and Stochastic Calculus*, Corollary 5.14).

By Jensen's inequality

$$E\exp\left(\frac{1}{2}\int_t^{t+\epsilon} V_s ds\right) \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} E e^{\frac{\varepsilon}{2}V_s} ds.$$

Then we have to estimate $Ee^{\frac{\varepsilon}{2}V_s}$. Let $n = 4[\kappa\theta]$, $\alpha = \kappa\theta/2$ and consider the process V_t'

$$dV_t' = (n/4 - 2\alpha V_t')dt + \sqrt{V_t'} dW_t^2, \quad V_0' = V_0.$$

Since $n/4 - 2\alpha v \geq \kappa(\theta - v)$ for all $v \geq 0$, using the comparison theorem for SDEs, we find

$$V_t' \geq V_t \text{ a.s. for all } t \geq 0.$$

We can represent

$$V_t' = \sum_{i=1}^n (Y_t^i)^2,$$

where

$$dY_t^i = -\alpha Y_t^i dt + \frac{1}{2} dB_t^i, \quad \sum_{i=1}^n (Y_0^i)^2 = V_0',$$

with independent Brownian motions B_t^i.

Since the Ornstein-Uhlenbeck processes Y_t^i are Gaussian, for arbitrary constant $c > 1$ and all $t \in [0, T]$ there exists small $\varepsilon > 0$ such that

$$\mathrm{E}\, e^{\frac{\varepsilon}{2}(Y_t^i)^2} < c.$$

Therefore, $\mathrm{E}\, e^{\frac{\varepsilon}{2}V_t} \leq c'$ for some constant c'. □

2 The formula for the price for a call option

2.1 Representation of the option price through the exercise probability

Assume P is a martingale measure. Define the new measure

$$dP = \frac{F_T}{F_0} dP.$$

The undiscounted price of a call is given by

$$f(t,x,v) = e^x \tilde{\Pi}(t,x,v) - K\Pi(t,x,v),$$

where Π and $\tilde{\Pi}$ are the conditional exercise probabilities:

$$\Pi(t,x,v) = P_{t,x,v}(X_T \geq \ln K), \quad \tilde{\Pi}(t,x,v) = \tilde{P}_{t,x,v}(X_T \geq \ln K).$$

Proof. We have

$$f(t,x,v) = E^P_{t,x,v}(F_T - K)^+ = E^P_{t,x,v} F_T I(F_T \geq K) - KP_{t,x,v}(F_T \geq K).$$

Let

$$\Pi(t,x,v) = P_{t,x,v}(F_T \geq K),$$

and define

$$\widetilde{\Pi}(t,x,v) = \frac{E^P_{t,x,v}(F_T I(F_T \geq K))}{e^x} = \widetilde{P}_{t,x,v}(F_T \geq K),$$

where the second equality follows from the formula for conditional expectations under an equivalent measure. With these Π and $\widetilde{\Pi}$, we get the claim of the lemma. □

2.2 Computation of the characteristic functions

Define the conditional characteristic functions

$$\varphi(t,x,v;u) = \mathrm{E}^{\mathrm{P}}_{t,x,v} e^{iuX_T}, \quad \widetilde{\varphi}(t,x,v;u) = \mathrm{E}^{\tilde{\mathrm{P}}}_{t,x,v} e^{iuX_T}.$$

It holds that

$$\widetilde{\varphi}(t,x,v;u) = \frac{\varphi(t,v,x;u-i)}{\varphi(t,v,x;-i)}$$

and the function φ satisfies the equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \frac{v}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \varphi}{\partial v^2} + \rho\sigma v \frac{\partial^2 \varphi}{\partial x \partial v} - \frac{v}{2}\frac{\partial \varphi}{\partial x} + \kappa(\theta - v)\frac{\partial \varphi}{\partial v} = 0, \\ \varphi(T,x,v;u) = e^{iux}. \end{cases}$$

Proof. Fix t, x, v. Then

$$\tilde{\varphi}(u) = E^{\tilde{P}}_{t,x,v} e^{iuX_T} = \frac{E^P_{t,x,v} F_T e^{iuX_T}}{e^x} = \frac{E^P_{t,x,v} e^{i(u-i)X_T}}{e^x} = \frac{\varphi(u-i)}{\varphi(-i)},$$

since $\varphi(-i) = E_{t,x,v} e^{X_T} = E_{t,x,v} F_T = e^x$. The differential equation follows from the Feynman-Kac formula. □

Let $\tau = T - t$ denote the time until expiration.

It holds that

$$\varphi(t, x, v; u) = \exp(C(\tau, u) + D(\tau, u)v + iux),$$

where the functions C and D satisfy the equations

$$\frac{\partial D}{\partial \tau} = \rho\sigma u D - \frac{1}{2}u^2 + \frac{1}{2}\sigma^2 D^2 - \frac{1}{2}iu - \kappa D, \quad D(0, u) = 0,$$

$$\frac{\partial C}{\partial \tau} = \kappa\theta D, \quad C(0, u) = 0.$$

The equation for D is a Riccati equation.

Proof. Substituting φ of the above form in the equation from the previous lemma, we get

$$v\left(-\frac{\partial D}{\partial \tau} + \rho\sigma iuD - \frac{1}{2}u^2 + \frac{1}{2}\sigma^2 D^2 - \frac{1}{2}iu - \kappa D\right) + \left(-\frac{\partial C}{\partial \tau} + \kappa\theta D\right) = 0.$$

If we equate each bracket to zero, we get the required equations. □

Fix u. Then we have the following equation for $D(\tau, u)$ ($'$ denotes the derivative with respect to τ)

$$D' = \alpha + \beta D + \gamma D^2, \quad D(0, u) = 0,$$

where

$$\alpha = -\frac{iu + u^2}{2}, \quad \beta = \rho\sigma iu - \kappa, \quad \gamma = \frac{\sigma^2}{2}.$$

The roots of the corresponding characteristic equation are

$$r_\pm = \frac{\beta \pm d}{2}, \quad d = \sqrt{\beta^2 - 4\alpha\gamma} = \sqrt{(\rho\sigma iu - \kappa)^2 + \sigma^2(iu + u^2)}.$$

This implies

$$D(\tau) = -\frac{1}{\gamma}\left(\frac{K_1 r_+ e^{r_+\tau} + K_2 r_- e^{r_-\tau}}{K_1 e^{r_+\tau} + K_2 e^{r_-\tau}}\right).$$

In order to satisfy the condition $D(0, \phi) = 0$, we can take

$$K_1 = 1, \quad K_2 = -\frac{r_+}{r_-}.$$

Then we get the following formula for D and integrating it get the formula for C:

$$D(\tau, u) = \frac{\kappa - \rho\sigma iu - d}{\sigma^2}\left(\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}\right), \quad \text{where } g = \frac{r_+}{r_-},$$

$$C(\tau, u) = \frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma iu - d)\tau - 2\ln\left(\frac{1 - ge^{-d\tau}}{1 - g}\right)\right).$$

2.3 Inversion of the characteristic functions

[Gil-Pelaez, 1951] Let $\varphi(u)$ be a characteristic function of a distribution $F(x)$, i.e. $\varphi(u) = \int_{\mathbb{R}} e^{iux} dF(x)$. Then

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-iux}\varphi(u)}{iu} \right) du.$$

This implies

$$\Pi(t,x,v) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\ln K}\varphi(t,x,v;u)}{iu} \right) du,$$

$$\widetilde{\Pi}(t,x,v) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\ln K}\widetilde{\varphi}(t,x,v;u)}{iu} \right) du.$$

2.4 The final result

Assume the model is given by equations (1)-(2). Define the functions

$$\varphi(t,x,v;u) = \exp(C(T-t,u) + D(T-t,u)v + iux),$$

$$\widetilde{\varphi}(t,x,v;u) = \frac{\varphi(t,x,v;u-i)}{\varphi(t,x,v;-i)},$$

where

$$C(\tau,u) = \frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma iu - d(u))\tau - 2\ln\left(\frac{1-g(u)e^{-d(u)\tau}}{1-g(u)} \right) \right),$$

$$D(\tau,u) = \frac{\kappa - \rho\sigma iu - d(u)}{\sigma^2}\left(\frac{1-e^{-d(u)\tau}}{1-g(u)e^{-d(u)\tau}} \right),$$

$$d(u) = \sqrt{(\rho\sigma iu - \kappa)^2 + \sigma^2(iu+u^2)}, \quad g(u) = \frac{\rho\sigma iu - \kappa + d(u)}{\rho\sigma iu - \kappa - d(u)}.$$

For a call option with strike K and expiration time T, the price at time t is given by $C_t = B(t,T)f(t, \ln F_t, V_t)$, where $B(t,T) = B_t/B_T$ and

$$f(t,x,v) = e^x\tilde{\Pi}(t,x,v) - K\Pi(t,x,v)$$

with the functions $\tilde{\Pi}, \Pi$ given by

$$\Pi(t,x,v) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\ln K}\varphi(t,x,v;u)}{iu} \right) du,$$

$$\tilde{\Pi}(t,x,v) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\ln K}\tilde{\varphi}(t,x,v;u)}{iu} \right) du.$$

3 Qualitative role of the model parameters

On the following plots, the blue line shows the volatility smile for v0 = 0.1, = 2, = 0.1, = 0.7, = 0.3, T = 0.5, and the two other smiles are obtained by varying one of the parameters.

4 Appendix: Riccati equations

A Riccati differential equation with constant coefficients is an equation of the form

$$y'(x) = \alpha + \beta y(x) + \gamma y(x)^2 \quad (\gamma \neq 0). \tag{*}$$

Let $\beta^2 - 4\alpha\gamma \neq 0$. Then the solution of (*) is given by

$$y(x) = -\frac{1}{\gamma}\left(\frac{K_1 r_+ e^{r_+ x} + K_2 r_- e^{r_- x}}{K_1 e^{r_+ x} + K_2 e^{r_- x}}\right),$$

where K_1, K_2 are constants, and $r_\pm \in \mathbb{C}$ are the roots of the equation $r^2 + \beta r + \alpha\gamma = 0$:

$$r_\pm = \frac{\beta \pm d}{2}, \quad d = \sqrt{\beta^2 - 4\alpha\gamma}.$$

Proof. It is easy to check that if $z(x)$ solves the equation

$$z'' - \beta z' + \alpha\gamma z = 0,$$

then

$$y(x) = -\frac{z'(x)}{\gamma z(x)}$$

solves the Riccati equation.

The solution $z(x)$ is given by

$$z(x) = K_1 e^{r_+ x} + K_2 e^{r_- x}.$$

□