Two-Task Cobb–Douglas Team with Comparative Advantage: Equilibrium Taxonomy, Impossibility Results, and Parameter Regions

1 Model

Two players $i \in \{1,2\}$ supply nonnegative efforts $x_i, y_i \geq 0$ on tasks X and Y. Aggregate outputs are linear in efforts with productivity (benefit-side comparative advantage)

$$X = p_{1x}x_1 + p_{2x}x_2, \qquad Y = p_{1y}y_1 + p_{2y}y_2, \qquad p_{ix}, p_{iy} > 0.$$

Preferences are Cobb–Douglas; costs are quadratic in *own total effort*:

$$U_i(x_i, y_i; x_{-i}, y_{-i}) \; = \; X^{\,1-a_i}\, Y^{\,a_i} \; - \; \frac{c_i}{2}\,(x_i + y_i)^2, \qquad a_i \in (0,1),\ c_i > 0.$$

Let

$$r \;\equiv\; \frac{Y}{X} > 0, \qquad r_i \;\equiv\; \frac{a_i\, p_{iy}}{(1-a_i)\, p_{ix}} \quad (i = 1,2).$$

When player i is interior ($x_i > 0$ and $y_i > 0$), first-order conditions imply $r = r_i$.

For later use, define the total effort if player i acts as:

$$\text{X-only:} \quad s_i^X(r) = \frac{(1-a_i)\, p_{ix}}{c_i}\, r^{a_i}, \qquad\qquad \text{Y-only:} \quad s_i^Y(r) = \frac{a_i\, p_{iy}}{c_i}\, r^{-(1-a_i)}. \tag{1}$$

2 KKT (complementarity) conditions

Write $s_i = x_i + y_i$. For each choice variable we have the complementary slackness:

$$\begin{aligned} &x_i \geq 0, \quad c_1 s_1 - (1-a_1)p_{1x}r^{a_1} \;\geq 0, \quad x_1 \cdot \left[c_1 s_1 - (1-a_1)p_{1x}r^{a_1}\right] = 0,\\ &y_i \geq 0, \quad c_1 s_1 - a_1 p_{1y} r^{a_1 - 1} \;\geq 0, \quad y_1 \cdot \left[c_1 s_1 - a_1 p_{1y} r^{a_1-1}\right] = 0, \end{aligned}$$

and analogously for player 2 with c_2, p_{2x}, p_{2y}, a_2. Cobb–Douglas requires $X > 0$ and $Y > 0$ (since $a_i \in (0,1)$), so any equilibrium must deliver strictly positive X, Y.

3 From 16 support masks to 7 feasible patterns

A *support mask* is $m = (m_{x_1}, m_{y_1}, m_{x_2}, m_{y_2}) \in \{0,1\}^4$, where $m_{x_1} = 1$ means $x_1 > 0$, etc. There are $2^4 = 16$ masks.

Lemma 1 (Immediate impossibilities)**.** *Fix $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$. At any equilibrium:*

(a) *For each player i, it is impossible that $x_i = y_i = 0$.* (Because for any finite $r > 0$, both marginal benefits are strictly positive, violating complementary slackness.)

(b) *It is impossible that both players choose only X ($m_{y_1} = m_{y_2} = 0$), because $Y = 0$ contradicts Cobb–Douglas.*

(c) *It is impossible that both players choose only Y ($m_{x_1} = m_{x_2} = 0$), because $X = 0$.*

As a result, the only admissible *economic* patterns are obtained by classifying each player as:

$$\text{B (interior): } (x_i > 0, y_i > 0), \qquad \text{X-only: } (x_i > 0, y_i = 0), \qquad \text{Y-only: } (x_i = 0, y_i > 0),$$

which yields $3 \times 3 = 9$ patterns. By Lemma 1(b,c) two are infeasible (X,X) and (Y,Y) at the *pair* level, leaving the following **seven feasible patterns**:

$$(B, B),\ (X, Y),\ (Y, X),\ (B, X),\ (X, B),\ (B, Y),\ (Y, B).$$

4 Closed-form ratios and efforts for the seven feasible patterns

Define the two useful constants

$$K_{XY} \equiv \frac{a_2\, p_{2y}^2\, c_1}{(1 - a_1)\, p_{1x}^2\, c_2}, \qquad K_{YX} \equiv \frac{a_1\, p_{1y}^2\, c_2}{(1 - a_2)\, p_{2x}^2\, c_1}.$$

The exponents $2 + a_1 - a_2$ and $2 + a_2 - a_1$ lie in $(1, 3)$ since $a_i \in (0, 1)$.

(A) Both interior: (B,B)

Interior FOCs for both players imply $r = r_1 = r_2$. Given any r with $r_1 = r_2$, totals are

$$s_1 = \frac{(1 - a_1)p_{1x}}{c_1}\, r^{a_1} = \frac{a_1 p_{1y}}{c_1}\, r^{-(1-a_1)}, \qquad s_2 = \frac{(1 - a_2)p_{2x}}{c_2}\, r^{a_2} = \frac{a_2 p_{2y}}{c_2}\, r^{-(1-a_2)}.$$

Feasibility $Y = rX$ determines only a *line* of splits (y_1, y_2) within the box $0 \le y_i \le s_i$, hence a continuum of equilibria.

(B) Full specialization: (X,Y)

Player 1 is X-only, player 2 is Y-only. Aggregation and FOCs imply the scalar equation

$$r^{\,2+a_1-a_2} = K_{XY} \quad \Rightarrow \quad \boxed{r \;=\; K_{XY}^{\frac{1}{2+a_1-a_2}}}.$$

Totals are $s_1 = s_1^X(r)$, $s_2 = s_2^Y(r)$ from (1). The realized allocation is

$$(x_1, y_1; x_2, y_2) = (s_1, 0;\ 0, s_2).$$

(C) Full specialization: (Y,X)

Symmetrically,

$$r^{\,2+a_2-a_1} = K_{YX} \quad \Rightarrow \quad \boxed{r \;=\; K_{YX}^{\frac{1}{2+a_2-a_1}}}, \qquad (x_1, y_1; x_2, y_2) = (0, s_1;\ s_2, 0),$$

with $s_1 = s_1^Y(r)$, $s_2 = s_2^X(r)$.

(D) One interior, one specialist: (B,Y)

Here $r = r_1$ (player 1 interior). Player 2 is Y-only, so X initially comes only from player 1 and Y from both. Let

$$X_{\text{base}} = 0, \quad Y_{\text{base}} = p_{2y}\, s_2 \quad \text{with } s_2 = s_2^Y(r).$$

Player 1 total is $s_1 = s_1^X(r) = s_1^Y(r)$ at $r = r_1$. Feasibility $Y = rX$ pins player 1's split by solving

$$p_{1y} y_1 \; = \; r \cdot p_{1x}(s_1 - y_1) - Y_{\text{base}} \quad \Rightarrow \quad \boxed{y_1 \; = \; \frac{r\, p_{1x} s_1 - Y_{\text{base}}}{p_{1y} + r\, p_{1x}}, \qquad x_1 = s_1 - y_1.}$$

This split is feasible iff $0 \le y_1 \le s_1$. That inequality reduces to the single *capacity* condition

$$\boxed{r_1^{2+a_1-a_2} \; \ge \; K_{XY}} \qquad \text{(together with the consistency } r_1 \le r_2\text{).}$$

(E) One interior, one specialist: (Y,B)

Here $r = r_2$, player 1 is Y-only with $s_1 = s_1^Y(r)$, and player 2 is interior with $s_2 = s_2^X(r) = s_2^Y(r)$. Solving $Y = rX$ for y_2 gives

$$\boxed{y_2 \; = \; \frac{r\, p_{2x} s_2 - Y_{\text{base}}}{p_{2y} + r\, p_{2x}}, \qquad x_2 = s_2 - y_2, \quad Y_{\text{base}} = p_{1y} s_1.}$$

Feasibility $0 \le y_2 \le s_2$ reduces to

$$\boxed{r_2^{2+a_2-a_1} \; \ge \; K_{YX}} \qquad \text{(with consistency } r_2 \le r_1\text{).}$$

(F) One interior, one specialist: (B,X)

Here $r = r_1$, player 2 is X-only with $s_2 = s_2^X(r)$. Feasibility determines y_1 as in (B,Y), and $0 \le y_1 \le s_1$ reduces to the *dual* capacity bound

$$\boxed{r_1^{1+a_2-a_1} \; \le \; \frac{(1-a_1)\, p_{1y}\, p_{1x}\, c_2}{(1-a_2)\, p_{2x}^2\, c_1}} \qquad \text{(with consistency } r_1 \ge r_2\text{).}$$

(G) One interior, one specialist: (X,B)

Here $r = r_2$, player 1 is X-only with $s_1 = s_1^X(r)$; solving gives the dual bound

$$\boxed{r_2^{1+a_1-a_2} \; \le \; \frac{(1-a_2)\, p_{2y}\, p_{2x}\, c_1}{(1-a_1)\, p_{1x}^2\, c_2}} \qquad \text{(with consistency } r_2 \ge r_1\text{).}$$

Notes. (i) The (B,B) case produces a *continuum* of equilibria (a line segment of splits) whenever $r_1 = r_2$. (ii) The (X,Y), (Y,X) cases deliver a unique r and unique efforts. (iii) In the mixed (B,$\cdot$) cases, the total s_i is pinned and the split is uniquely pinned by feasibility; the capacity inequalities above are exactly the algebraic form of $0 \le y_i \le s_i$.

5 Parameter regions and uniqueness

Collect the consistency conditions and capacity inequalities:

$$
\begin{aligned}
&\textbf{(X,Y):}\ r = K_{XY}^{1/(2+a_1-a_2)} \quad \text{and} \quad r \in [r_1, r_2].\\
&\textbf{(Y,X):}\ r = K_{YX}^{1/(2+a_2-a_1)} \quad \text{and} \quad r \in [r_2, r_1].\\
&\textbf{(B,Y):}\ r = r_1, \quad r_1 \le r_2, \quad r_1^{2+a_1-a_2} \ge K_{XY}.\\
&\textbf{(Y,B):}\ r = r_2, \quad r_2 \le r_1, \quad r_2^{2+a_2-a_1} \ge K_{YX}.\\
&\textbf{(B,X):}\ r = r_1, \quad r_1 \ge r_2, \quad r_1^{1+a_2-a_1} \le \frac{(1-a_1)p_{1y}p_{1x}c_2}{(1-a_2)p_{2x}^2 c_1}.\\
&\textbf{(X,B):}\ r = r_2, \quad r_2 \ge r_1, \quad r_2^{1+a_1-a_2} \le \frac{(1-a_2)p_{2y}p_{2x}c_1}{(1-a_1)p_{1x}^2 c_2}.\\
&\textbf{(B,B):}\ r_1 = r_2 \quad \text{(knife-edge); a continuum of splits.}
\end{aligned}
$$

Proposition 1 (Generic uniqueness)**.** *For almost every parameter vector* $(a_1, a_2, p_{1x}, p_{1y}, p_{2x}, p_{2y}, c_1, c_2)$ *(i.e. away from the equalities that define the region boundaries), exactly* one *of the seven sets above holds, and the resulting equilibrium effort vector is unique. Multiple patterns can be feasible only on knife-edges (e.g.* $r_1 = r_2$ *or when the specialized ratio equals a threshold), and there the allocations coincide or form a one-dimensional continuum as in (B,B).*

6 Feasibility of the 16 support masks (exhaustive)

Let $m = (m_{x_1}, m_{y_1}, m_{x_2}, m_{y_2}) \in \{0,1\}^4$ indicate positivity of (x_1, y_1, x_2, y_2). Table 1 lists all masks and their status. "Type" maps each player's support to $\{X, Y, B\}$. Impossibilities follow from Lemma 1; all feasible masks fall into one of the seven patterns.

`mask`	P1 type	P2 type	feasible?	reason / pattern
`0000`	–	–	no	player 1 violates KKT; player 2 violates KKT
`0001`	Y	–	no	player 2 violates KKT ($x_2 = y_2 = 0$)
`0010`	X	–	no	player 2 violates KKT ($x_2 = y_2 = 0$)
`0011`	X	Y	yes	(X, Y)
`0100`	Y	–	no	player 2 violates KKT
`0101`	Y	Y	no	$X = 0$ impossible
`0110`	B	X	yes	(B, X)
`0111`	B	B	yes	(B, B) if $r_1 = r_2$; else no
`1000`	X	–	no	player 2 violates KKT
`1001`	X	Y	yes	(X, Y)
`1010`	X	X	no	$Y = 0$ impossible
`1011`	X	B	yes	(X, B)
`1100`	B	–	no	player 2 violates KKT
`1101`	B	Y	yes	(B, Y)
`1110`	B	X	yes	(B, X)
`1111`	B	B	yes	(B, B) if $r_1 = r_2$; else no

Table 1: All 16 support masks. Only the seven listed patterns are ever feasible.

7 Summary: which equation is used when (complete chart)

Pattern	Equilibrium ratio r	Consistency (threshold order)	Extra (capacity) condition
(X,Y)	$r = K_{XY}^{1/(2+a_1-a_2)}$	$r \in [r_1, r_2]$	none
(Y,X)	$r = K_{YX}^{1/(2+a_2-a_1)}$	$r \in [r_2, r_1]$	none
(B,Y)	$r = r_1$	$r_1 \le r_2$	$r_1^{2+a_1-a_2} \ge K_{XY}$
(Y,B)	$r = r_2$	$r_2 \le r_1$	$r_2^{2+a_2-a_1} \ge K_{YX}$
(B,X)	$r = r_1$	$r_1 \ge r_2$	$r_1^{1+a_2-a_1} \le \dfrac{(1-a_1)p_{1y}p_{1x}c_2}{(1-a_2)p_{2x}^2 c_1}$
(X,B)	$r = r_2$	$r_2 \ge r_1$	$r_2^{1+a_1-a_2} \le \dfrac{(1-a_2)p_{2y}p_{2x}c_1}{(1-a_1)p_{1x}^2 c_2}$
(B,B)	$r = r_1 = r_2$	equality	continuum of splits

Table 2: Complete mapping from parameters to the equation used. Away from the boundary equalities, exactly one row applies and the equilibrium effort vector is unique.

Derivation notes (capacity inequalities). In the mixed (B,·) cases the interior player has total s_i fixed by r; feasibility $Y = rX$ pins their split via $(p_{iy} + rp_{ix})\, y_i = r\big(X_{\text{base}} + p_{ix}s_i\big) - Y_{\text{base}}$. Requiring $0 \le y_i \le s_i$ reduces to the power inequalities reported above after substituting s_i^X, s_i^Y from (1) and simplifying.