Two-Task Cobb–Douglas Team: Model, Symmetry-Guided Regimes, and Equilibrium Shell

1 Model

Two players $i \in \{1,2\}$ choose efforts $x_i, y_i \geq 0$ on tasks X and Y. Aggregate outputs are linear in efforts:

$$X = p_{1x}x_1 + p_{2x}x_2, \qquad Y = p_{1y}y_1 + p_{2y}y_2, \qquad p_{ix}, p_{iy} > 0. \tag{1}$$

Preferences are Cobb–Douglas with quadratic cost in total own effort:

$$U_i(x_i, y_i; x_{-i}, y_{-i}) = X^{\,1-a_i}\, Y^{\,a_i} - \frac{c_i}{2}\,(x_i + y_i)^2, \qquad a_i \in (0,1),\ c_i > 0. \tag{2}$$

Parameter restrictions

- Productivities: $p_{ix} > 0,\ p_{iy} > 0$ for $i = 1, 2$.
- Preferences: $a_i \in (0,1)$ for $i = 1, 2$.
- Costs: $c_i > 0$ for $i = 1, 2$.

Derived notation (from the model, step by step)

Shorthands. Define the total-effort shorthand $s_i \coloneqq x_i + y_i$ and the output ratio

$$r \coloneqq \frac{Y}{X}.$$

(We will only use r in regimes where this ratio is well-defined.)

Useful identities. Whenever r is used, the following equalities are immediate:

$$X^{-a_i}Y^{a_i} = r^{a_i} \qquad \text{and} \qquad X^{1-a_i}Y^{a_i-1} = r^{a_i-1}.$$

2 First-order building blocks (four partial derivatives)

With $s_i = x_i + y_i$, the partial derivatives are

$$\frac{\partial U_1}{\partial x_1} = (1-a_1)\,X^{-a_1}\,Y^{a_1}\,p_{1x} \;-\; c_1\,s_1, \qquad \frac{\partial U_1}{\partial y_1} = a_1\,X^{1-a_1}\,Y^{a_1-1}\,p_{1y} \;-\; c_1\,s_1, \tag{3}$$

$$\frac{\partial U_2}{\partial x_2} = (1-a_2)\,X^{-a_2}\,Y^{a_2}\,p_{2x} \;-\; c_2\,s_2, \qquad \frac{\partial U_2}{\partial y_2} = a_2\,X^{1-a_2}\,Y^{a_2-1}\,p_{2y} \;-\; c_2\,s_2. \tag{4}$$

Using the identities above, these can be r-*reduced* to

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)\,p_{ix}\,r^{a_i} - c_i\,s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i\,p_{iy}\,r^{a_i-1} - c_i\,s_i, \qquad i \in \{1,2\}. \tag{5}$$

Interior benchmarks (used only when a choice is active)

For later algebra, record the marginal-equality totals that would hold when a given choice is strictly positive:

$$s_i^X(r) := \frac{(1-a_i)\,p_{ix}}{c_i}\,r^{a_i}, \qquad s_i^Y(r) := \frac{a_i\,p_{iy}}{c_i}\,r^{-(1-a_i)}. \tag{6}$$

Define the individual threshold ratio at which these coincide:

$$r_i := \frac{a_i\,p_{iy}}{(1-a_i)\,p_{ix}}. \tag{7}$$

3 Goal and symmetry plan

G1. Exhaustive masks. There are $4 \times 4 = 16$ ordered masks: for each player, `B` (both > 0), `X` ($x > 0, y = 0$), `Y` ($x = 0, y > 0$), `0` ($x = y = 0$).

G2. For each mask: (i) decide feasible vs. impossible; (ii) if feasible, record parameter-only bounds; (iii) give equilibrium efforts $(x_1^*, y_1^*, x_2^*, y_2^*)$ and (if used) the scalar equation for r.

G3. Use player-index symmetry to reduce work. Swapping labels $1 \leftrightarrow 2$ maps each ordered mask to a symmetric partner while keeping tasks (X, Y) and $r = Y/X$ fixed. Under this convention, the 16 ordered masks collapse to 10 unordered shapes: *four self-symmetric* `(B,B)`, `(X,X)`, `(Y,Y)`, `(0,0)` and *six symmetric pairs*:

$$(\mathtt{B},\mathtt{X}) \sim (\mathtt{X},\mathtt{B}),\ (\mathtt{B},\mathtt{Y}) \sim (\mathtt{Y},\mathtt{B}),\ (\mathtt{B},\mathtt{0}) \sim (\mathtt{0},\mathtt{B}),\ (\mathtt{X},\mathtt{Y}) \sim (\mathtt{Y},\mathtt{X}),\ (\mathtt{X},\mathtt{0}) \sim (\mathtt{0},\mathtt{X}),\ (\mathtt{Y},\mathtt{0}) \sim (\mathtt{0},\mathtt{Y}).$$

Flip rule (index swap, used throughout)

To obtain a symmetric partner from any derived case, swap indices $1 \leftrightarrow 2$ everywhere:

$$a_1 \leftrightarrow a_2, \quad c_1 \leftrightarrow c_2, \quad p_{1x} \leftrightarrow p_{2x}, \quad p_{1y} \leftrightarrow p_{2y}, \quad r_1 \leftrightarrow r_2.$$

Keep $r = Y/X$ fixed (do not invert it).

4 Pair P1: One interior + X-only `(B,X)` and `(X,B)`

Derive once: `(B,X)`. Setup. Player 1 is *interior* ($x_1 > 0$ and $y_1 > 0$). Player 2 is X*-only* ($x_2 > 0$ and $y_2 = 0$). We derive: (i) the ratio $r = Y/X$ (if pinned), (ii) the totals $s_1 = x_1 + y_1$ and $s_2 = x_2 + y_2$, (iii) the split (x_1, y_1), and (iv) parameter-only bounds ensuring this mask is feasible.

Step 1 (FOCs when a choice is active): from the reduced derivatives. From the four partials in the previous section, when a choice is *strictly positive* at optimum, its marginal equals zero. Using the r-reduced forms,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)p_{ix}r^{a_i} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i p_{iy} r^{a_i - 1} - c_i s_i.$$

Step 1a (Player 1 interior $\Rightarrow$ both equalities bind). Because $x_1 > 0$ *and* $y_1 > 0$, set both derivatives to zero:

$$(1-a_1)p_{1x}r^{a_1} - c_1 s_1 = 0, \tag{8}$$
$$a_1 p_{1y} r^{a_1-1} - c_1 s_1 = 0. \tag{9}$$

Both right-hand sides equal $c_1 s_1$, so equate the left-hand sides of (8) and (9):

$$(1-a_1)p_{1x}r^{a_1} \;=\; a_1 p_{1y} r^{a_1-1}.$$

Divide both sides by $r^{a_1-1} > 0$:

$$(1-a_1)p_{1x}\, r \;=\; a_1 p_{1y} \quad\Longrightarrow\quad \boxed{r \;=\; r_1 \;:=\; \frac{a_1 p_{1y}}{(1-a_1)p_{1x}}}.$$

Conclusion: in `(B,X)` the ratio is pinned to $r = r_1$.

Step 1b (Totals for Player 1). Either (8) or (9) gives the same s_1 once $r = r_1$:

$$\boxed{s_1 \;=\; \frac{(1-a_1)p_{1x}}{c_1}\, r_1^{a_1} \;=\; \frac{a_1 p_{1y}}{c_1}\, r_1^{-(1-a_1)}}.$$

(Equality of the two forms is guaranteed by the algebra that produced r_1.)

Step 1c (Totals for Player 2 with X-only). Here $x_2 > 0$ and $y_2 = 0$. Thus set the x_2-marginal to zero and only require the y_2-marginal to be ≤ 0:

$$(1-a_2)p_{2x}r^{a_2} - c_2 s_2 = 0 \quad\Longrightarrow\quad \boxed{s_2 \;=\; \frac{(1-a_2)p_{2x}}{c_2}\, r^{a_2}}.$$

Since $r = r_1$, substitute to get

$$\boxed{s_2 \;=\; \frac{(1-a_2)p_{2x}}{c_2}\, r_1^{a_2}}.$$

Because $y_2 = 0$, we also have $x_2 = s_2$.

Step 2 (Split (x_1, y_1) using $Y = rX$). By definition,

$$X \;=\; p_{1x}x_1 + p_{2x}x_2 \;=\; p_{1x}(s_1 - y_1) + p_{2x}s_2, \qquad Y \;=\; p_{1y}y_1.$$

Impose $Y = rX$ with $r = r_1$:

$$\begin{aligned} p_{1y}y_1 \;&=\; r_1\big[p_{1x}(s_1 - y_1) + p_{2x}s_2\big] \\ &=\; r_1 p_{1x}s_1 \;-\; r_1 p_{1x}y_1 \;+\; r_1 p_{2x}s_2. \end{aligned}$$

Collect the y_1 terms on the left:

$$(p_{1y} + r_1 p_{1x})\, y_1 \;=\; r_1\big(p_{1x}s_1 + p_{2x}s_2\big).$$

Divide by the positive denominator $p_{1y} + r_1 p_{1x} > 0$:

$$\boxed{y_1 \;=\; \frac{r_1\big(p_{1x}s_1 + p_{2x}s_2\big)}{p_{1y} + r_1 p_{1x}}}, \qquad \boxed{x_1 \;=\; s_1 - y_1}.$$

Step 3 (Feasibility/bounds as parameter-only inequalities). Two conditions must hold:
(A) Keep $y_2 = 0$ optimal. Require $\dfrac{\partial U_2}{\partial y_2} \le 0$:

$$\frac{\partial U_2}{\partial y_2} = a_2 p_{2y} r^{a_2-1} - c_2 s_2 \ \le\ 0,$$

$$\text{substitute } r = r_1,\ s_2 = \frac{(1-a_2)p_{2x}}{c_2} r_1^{a_2} \quad \Rightarrow \quad a_2 p_{2y} r_1^{a_2-1} \ \le\ c_2 \cdot \frac{(1-a_2)p_{2x}}{c_2} r_1^{a_2}$$

$$a_2 p_{2y} \ \le\ (1-a_2) p_{2x}\, r_1$$

$$\boxed{r_1 \ \ge\ r_2 := \frac{a_2 p_{2y}}{(1-a_2)p_{2x}}}.$$

(B) Ensure the interior split is feasible: $0 \le y_1 \le s_1$**.** - *Lower bound* $y_1 \ge 0$*:* numerator $r_1(p_{1x}s_1 + p_{2x}s_2)$ and denominator $p_{1y} + r_1 p_{1x}$ are both strictly positive, so $y_1 \ge 0$ holds automatically. - *Upper bound* $y_1 \le s_1$*:* start from the explicit y_1 formula:

$$\begin{aligned}
\frac{r_1(p_{1x}s_1 + p_{2x}s_2)}{p_{1y} + r_1 p_{1x}} \ \le\ s_1 \quad &\iff \quad r_1(p_{1x}s_1 + p_{2x}s_2) \ \le\ (p_{1y} + r_1 p_{1x}) s_1 \\
&\iff \quad r_1 p_{2x} s_2 \ \le\ p_{1y} s_1 \qquad \text{(cancel } r_1 p_{1x} s_1 \text{ on both sides)} \\
&\iff \quad r_1 p_{2x} \cdot \frac{(1-a_2)p_{2x}}{c_2} r_1^{a_2} \ \le\ p_{1y} \cdot \frac{(1-a_1)p_{1x}}{c_1} r_1^{a_1} \\
&\iff \quad \frac{(1-a_2)p_{2x}^2}{c_2} r_1^{a_2+1} \ \le\ \frac{(1-a_1)p_{1y}p_{1x}}{c_1} r_1^{a_1} \\
&\iff \quad r_1^{1+a_2-a_1} \ \le\ \boxed{\frac{(1-a_1)p_{1y}p_{1x}c_2}{(1-a_2)p_{2x}^2 c_1}}.
\end{aligned}$$

This is the *capacity inequality* for `(B,X)`. (Strict inequality gives $x_1 > 0$; equality is a knife-edge where a neighboring mask meets.)

Summary for `(B,X)`.

$$\boxed{r = r_1 = \frac{a_1 p_{1y}}{(1-a_1)p_{1x}}}, \qquad \boxed{s_1 = \frac{(1-a_1)p_{1x}}{c_1} r_1^{a_1}}, \qquad \boxed{s_2 = \frac{(1-a_2)p_{2x}}{c_2} r_1^{a_2}};$$

$$\boxed{y_1 = \frac{r_1(p_{1x}s_1 + p_{2x}s_2)}{p_{1y} + r_1 p_{1x}}, \quad x_1 = s_1 - y_1}, \qquad \boxed{x_2 = s_2,\ y_2 = 0};$$

$$\boxed{r_1 \ \ge\ r_2 = \frac{a_2 p_{2y}}{(1-a_2)p_{2x}}}, \qquad \boxed{r_1^{1+a_2-a_1} \ \le\ \frac{(1-a_1)p_{1y}p_{1x}c_2}{(1-a_2)p_{2x}^2 c_1}}.$$

Step 4 (Double-checks). (i) Substituting the y_1 solution into X and Y gives $\dfrac{Y}{X} = r_1$ by construction. (ii) With the two bounds satisfied, $\dfrac{\partial U_2}{\partial y_2} \le 0$ at $y_2 = 0$ and $0 < y_1 \le s_1$, so the mask is internally consistent.

Obtain by flip: `(X,B)`. Apply the index swap $1 \leftrightarrow 2$ (keep $r = Y/X$ unchanged). The results are the symmetric images:

$$\boxed{r = r_2 = \frac{a_2 p_{2y}}{(1-a_2)p_{2x}}}, \qquad \boxed{s_2 = \frac{(1-a_2)p_{2x}}{c_2} r_2^{a_2}}, \qquad \boxed{s_1 = \frac{(1-a_1)p_{1x}}{c_1} r_2^{a_1}};$$

$$\boxed{y_2 = \frac{r_2(p_{1x}s_1 + p_{2x}s_2)}{p_{2y} + r_2 p_{2x}}, \quad x_2 = s_2 - y_2}, \qquad \boxed{x_1 = s_1,\ y_1 = 0};$$

$$\boxed{r_2 \ \geq \ r_1}, \qquad \boxed{r_2^{1+a_1-a_2} \ \leq \ \frac{(1-a_2)p_{2y}p_{2x}c_1}{(1-a_1)p_{1x}^2 c_2}}.$$

5 Pair P2: One interior + Y-only `(B,Y)` and `(Y,B)`

Derive once: `(B,Y)`. **Setup.** Player 1 is *interior* ($x_1 > 0$ and $y_1 > 0$). Player 2 is Y*-only* ($y_2 > 0$ and $x_2 = 0$). We derive: (i) the ratio $r = Y/X$, (ii) totals $s_1 = x_1 + y_1$ and $s_2 = x_2 + y_2$, (iii) the split (x_1, y_1), and (iv) parameter-only bounds ensuring this mask is feasible.

Step 1 (FOCs when a choice is active): from the reduced derivatives. When a choice is *strictly positive* at optimum, its marginal equals zero. Using the r-reduced forms,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)p_{ix}r^{a_i} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i p_{iy} r^{a_i - 1} - c_i s_i.$$

Step 1a (Player 1 interior $\Rightarrow$ both equalities bind). Because $x_1 > 0$ *and* $y_1 > 0$, set both derivatives to zero:

$$(1-a_1)p_{1x}r^{a_1} - c_1 s_1 = 0, \tag{10}$$
$$a_1 p_{1y} r^{a_1 - 1} - c_1 s_1 = 0. \tag{11}$$

Equate the left-hand sides of (10)–(11) (both equal $c_1 s_1$):

$$(1-a_1)p_{1x}r^{a_1} \ = \ a_1 p_{1y} r^{a_1-1}.$$

Divide by $r^{a_1-1} > 0$ (since we use r only where it is well-defined/positive):

$$(1-a_1)p_{1x}\, r \ = \ a_1 p_{1y} \quad \Longrightarrow \quad \boxed{r \ = \ r_1 \ := \ \frac{a_1 p_{1y}}{(1-a_1)p_{1x}}}.$$

Step 1b (Totals for Player 1). Either (10) or (11) gives the same s_1 once $r = r_1$:

$$\boxed{s_1 \ = \ \frac{(1-a_1)p_{1x}}{c_1} r_1^{a_1} \ = \ \frac{a_1 p_{1y}}{c_1} r_1^{-(1-a_1)}}.$$

Step 1c (Totals for Player 2 with Y-only). Here $y_2 > 0$ and $x_2 = 0$. Thus set the y_2-marginal to zero and only require the x_2-marginal to be ≤ 0:

$$a_2 p_{2y} r^{a_2-1} - c_2 s_2 = 0 \quad \Longrightarrow \quad \boxed{s_2 \;=\; \frac{a_2 p_{2y}}{c_2}\, r^{a_2-1}}.$$

Since $r = r_1$, substitute to get

$$\boxed{s_2 \;=\; \frac{a_2 p_{2y}}{c_2}\, r_1^{a_2-1}}.$$

Because $x_2 = 0$, we also have $y_2 = s_2$.

Step 2 (Split (x_1, y_1) using $Y = rX$). By definition,

$$X \;=\; p_{1x}x_1 + p_{2x}x_2 \;=\; p_{1x}(s_1 - y_1) + \underbrace{p_{2x}\cdot 0}_{=0} = p_{1x}(s_1 - y_1), \qquad Y \;=\; p_{1y}y_1 + p_{2y}y_2 \;=\; p_{1y}y_1 + p_{2y}s_2.$$

Impose $Y = rX$ with $r = r_1$:

$$p_{1y}y_1 + p_{2y}s_2 \;=\; r_1\big[p_{1x}(s_1 - y_1)\big] \;=\; r_1 p_{1x} s_1 \;-\; r_1 p_{1x} y_1.$$

Collect the y_1 terms on the left:

$$(p_{1y} + r_1 p_{1x})\, y_1 \;=\; r_1 p_{1x} s_1 \;-\; p_{2y} s_2.$$

Divide by $p_{1y} + r_1 p_{1x} > 0$:

$$\boxed{y_1 \;=\; \frac{r_1 p_{1x} s_1 - p_{2y} s_2}{p_{1y} + r_1 p_{1x}}}, \qquad \boxed{x_1 \;=\; s_1 - y_1}.$$

Step 3 (Feasibility/bounds as parameter-only inequalities). Two conditions must hold: **(A) Keep $x_2 = 0$ optimal.** Require $\dfrac{\partial U_2}{\partial x_2} \leq 0$:

$$\begin{aligned}
&& \frac{\partial U_2}{\partial x_2} = (1 - a_2) p_{2x} r^{a_2} - c_2 s_2 \;\leq\; 0,\\
\text{substitute } r = r_1,\; s_2 = \frac{a_2 p_{2y}}{c_2} r_1^{a_2-1} & \quad\Rightarrow & (1 - a_2) p_{2x} r_1^{a_2} \;\leq\; a_2 p_{2y} r_1^{a_2-1}\\
&& (1 - a_2) p_{2x}\, r_1 \;\leq\; a_2 p_{2y}\\
&& \boxed{r_1 \;\leq\; r_2 := \frac{a_2 p_{2y}}{(1 - a_2) p_{2x}}}.
\end{aligned}$$

(B) Ensure the interior split is feasible: $0 \leq y_1 \leq s_1$**.** - *Upper bound $y_1 \leq s_1$ is automatic:* from $y_1 \leq s_1 \iff r_1 p_{1x} s_1 - p_{2y} s_2 \leq (p_{1y} + r_1 p_{1x}) s_1 \iff -p_{2y} s_2 \leq p_{1y} s_1$, which always holds because the left side is ≤ 0 and the right side is ≥ 0.

- *Lower bound $y_1 \geq 0$ gives the nontrivial constraint:*

$$\frac{r_1 p_{1x} s_1 - p_{2y} s_2}{p_{1y} + r_1 p_{1x}} \;\geq\; 0 \quad \Longleftrightarrow \quad r_1 p_{1x} s_1 \;\geq\; p_{2y} s_2.$$

Insert $s_1 = \dfrac{(1-a_1)p_{1x}}{c_1} r_1^{a_1}$ and $s_2 = \dfrac{a_2 p_{2y}}{c_2} r_1^{a_2-1}$:

$$\begin{aligned}
r_1 p_{1x} \cdot \frac{(1-a_1)p_{1x}}{c_1} r_1^{a_1} &\geq p_{2y} \cdot \frac{a_2 p_{2y}}{c_2} r_1^{a_2-1} \\
\frac{(1-a_1)p_{1x}^2}{c_1} r_1^{a_1+1} &\geq \frac{a_2 p_{2y}^2}{c_2} r_1^{a_2-1} \\
r_1^{2+a_1-a_2} &\geq \frac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2} =: \boxed{K_{XY}}.
\end{aligned}$$

Thus the capacity inequality is

$$\boxed{r_1^{2+a_1-a_2} \geq K_{XY} \quad \text{with} \quad K_{XY} = \frac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2}}.$$

Summary for `(B,Y)`.

$$\boxed{r = r_1 = \frac{a_1 p_{1y}}{(1-a_1)p_{1x}}}, \qquad \boxed{s_1 = \frac{(1-a_1)p_{1x}}{c_1} r_1^{a_1}}, \qquad \boxed{s_2 = \frac{a_2 p_{2y}}{c_2} r_1^{a_2-1}};$$

$$\boxed{y_1 = \frac{r_1 p_{1x} s_1 - p_{2y} s_2}{p_{1y} + r_1 p_{1x}}, \quad x_1 = s_1 - y_1}, \qquad \boxed{x_2 = 0,\ y_2 = s_2};$$

$$\boxed{r_1 \leq r_2 = \frac{a_2 p_{2y}}{(1-a_2)p_{2x}}}, \qquad \boxed{r_1^{2+a_1-a_2} \geq K_{XY} = \frac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2}}.$$

Step 4 (Double-checks). (i) Substituting the y_1 solution into X and Y returns $\dfrac{Y}{X} = r_1$ by construction. (ii) With the two bounds satisfied, $\dfrac{\partial U_2}{\partial x_2} \leq 0$ at $x_2 = 0$ and $0 \leq y_1 \leq s_1$, so the mask is internally consistent. (iii) $x_1 > 0$ holds strictly whenever the capacity inequality is strict, since $x_1 = s_1 - y_1 = \dfrac{p_{1y} s_1 + p_{2y} s_2}{p_{1y} + r_1 p_{1x}} > 0$.

Obtain by flip: `(Y,B)`. Apply the index swap $1 \leftrightarrow 2$ (keep $r = Y/X$ unchanged). The results are the symmetric images:

$$\boxed{r = r_2 = \frac{a_2 p_{2y}}{(1-a_2)p_{2x}}}, \qquad \boxed{s_2 = \frac{(1-a_2)p_{2x}}{c_2} r_2^{a_2} = \frac{a_2 p_{2y}}{c_2} r_2^{-(1-a_2)}}, \qquad \boxed{s_1 = \frac{a_1 p_{1y}}{c_1} r_2^{a_1-1}};$$

$$\boxed{y_2 = \frac{r_2 p_{2x} s_2 - p_{1y} s_1}{p_{2y} + r_2 p_{2x}}, \quad x_2 = s_2 - y_2}, \qquad \boxed{x_1 = 0,\ y_1 = s_1};$$

$$\boxed{r_2 \leq r_1}, \qquad \boxed{r_2^{2+a_2-a_1} \geq K_{YX} = \frac{a_1 c_2 p_{1y}^2}{(1-a_2)c_1 p_{2x}^2}}.$$

6 Pair P3: One interior + Zero-effort (B,0) and (0,B)

Derive once: `(B,0)`. **Setup.** Player 1 is *interior* ($x_1 > 0$ and $y_1 > 0$). Player 2 exerts *zero effort* ($x_2 = 0$ and $y_2 = 0$). We ask: (i) is $r = Y/X$ used and what is it? (ii) what are s_1 and the split (x_1, y_1)? (iii) under what *parameter-only* bounds (if any) can player 2 optimally choose zero? We will see this mask is *impossible* under our standing restrictions.

Step 1 (Player 1 interior pins r and s_1). From the reduced marginal conditions, when a choice is strictly positive its marginal equals zero:

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)p_{ix}r^{a_i} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i p_{iy} r^{a_i - 1} - c_i s_i.$$

Because $x_1 > 0$ and $y_1 > 0$ (player 1 interior), set both to zero:

$$(1-a_1)p_{1x}r^{a_1} - c_1 s_1 = 0, \tag{12}$$
$$a_1 p_{1y} r^{a_1-1} - c_1 s_1 = 0. \tag{13}$$

Equate the left-hand sides of (12)–(13) (both equal $c_1 s_1$):

$$(1-a_1)p_{1x}r^{a_1} = a_1 p_{1y} r^{a_1-1} \quad\Longrightarrow\quad (1-a_1)p_{1x}\, r = a_1 p_{1y} \quad\Longrightarrow\quad \boxed{r = r_1 = \frac{a_1 p_{1y}}{(1-a_1)p_{1x}}}.$$

Is r used/defined? Yes. Since player 1 is interior, $X = p_{1x}x_1 > 0$ and $Y = p_{1y}y_1 > 0$, so $r = Y/X$ is well-defined and strictly positive. Either (12) or (13) yields the same total s_1:

$$\boxed{s_1 = \frac{(1-a_1)p_{1x}}{c_1}\, r_1^{a_1} = \frac{a_1 p_{1y}}{c_1}\, r_1^{-(1-a_1)}}.$$

Step 2 (Split (x_1, y_1) using $Y = rX$ with $x_2 = y_2 = 0$). Here $X = p_{1x}x_1 = p_{1x}(s_1 - y_1)$ and $Y = p_{1y}y_1$. Impose $Y = rX$ with $r = r_1$:

$$p_{1y}y_1 = r_1\, p_{1x}(s_1 - y_1) = r_1 p_{1x} s_1 - r_1 p_{1x} y_1.$$

Collect y_1-terms on the left and divide by $p_{1y} + r_1 p_{1x} > 0$:

$$\boxed{y_1 = \frac{r_1 p_{1x} s_1}{p_{1y} + r_1 p_{1x}}, \qquad x_1 = s_1 - y_1}.$$

(These produce $Y/X = r_1$ by construction.)

Step 3 (Can player 2 optimally choose zero? KKT at $x_2 = y_2 = 0$). For player 2 to play $(0,0)$ optimally, both one-sided KKT inequalities at zero must hold:

$$\left.\frac{\partial U_2}{\partial x_2}\right|_{(0,0)} \le 0 \qquad \text{and} \qquad \left.\frac{\partial U_2}{\partial y_2}\right|_{(0,0)} \le 0.$$

Evaluate them at $s_2 = 0$ and $r = r_1 > 0$:

$$\left.\frac{\partial U_2}{\partial x_2}\right|_{(0,0)} = (1-a_2)p_{2x}\, r^{a_2} - c_2\, s_2 = (1-a_2)p_{2x}\, r_1^{a_2} > 0,$$

$$\left.\frac{\partial U_2}{\partial y_2}\right|_{(0,0)} = a_2 p_{2y}\, r^{a_2-1} - c_2\, s_2 = a_2 p_{2y}\, r_1^{a_2-1} > 0.$$

Each strict positivity follows from:

$$(1-a_2) > 0,\ p_{2x} > 0,\ a_2 \in (0,1),\ p_{2y} > 0,\ r_1 > 0 \implies r_1^{a_2} > 0,\ r_1^{a_2-1} > 0.$$

Therefore both inequalities *fail* at $(0,0)$; player 2 has strictly positive marginal gains in both directions while the marginal cost term $c_2 s_2$ equals 0 at $s_2 = 0$.

Conclusion for `(B,0)`. Under our standing restrictions ($a_i \in (0,1)$, $p_{ix}, p_{iy}, c_i > 0$), the mask `(B,0)` is *impossible for any parameter values.* The only ways to make one of those marginals nonpositive at $(0,0)$ would be to violate a restriction (e.g., $p_{2x} = 0$ or $p_{2y} = 0$) or drive r to 0 or ∞ (which cannot happen here because player 1 is interior and pins $r = r_1 \in (0,\infty)$).

Summary for `(B,0)`.

$$\boxed{\text{Mask impossible (no parameter region).}}$$

(If one nonetheless writes the would-be Player 1 split, it is given above; but it cannot be an equilibrium because Player 2's best response is not $(0,0)$.)

Obtain by flip: `(0,B)`. Swap $1 \leftrightarrow 2$ (keep $r = Y/X$). The same logic applies symmetrically:

$$\boxed{\texttt{(0,B)}\text{ is impossible for all admissible parameters.}}$$

Indeed, with Player 2 interior one pins $r = r_2 > 0$; at $(x_1, y_1) = (0,0)$, player 1's one-sided marginals are $(1-a_1)p_{1x} r_2^{a_1} > 0$ and $a_1 p_{1y} r_2^{a_1-1} > 0$, so $(0,0)$ cannot be optimal for player 1.

7 Pair P4: Full specialization `(X,Y)` and `(Y,X)`

Derive once: `(X,Y)`. Setup. Player 1 is X-only ($x_1 > 0$, $y_1 = 0$). Player 2 is Y-only ($y_2 > 0$, $x_2 = 0$). We derive: (i) the ratio $r = Y/X$, (ii) totals $s_1 = x_1$ and $s_2 = y_2$, (iii) parameter-only bounds that keep the inactive choices at zero, and (iv) the equilibrium efforts.

Step 1 (FOCs when a choice is active): use the reduced derivatives. From the r-reduced forms,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)p_{ix} r^{a_i} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i p_{iy} r^{a_i-1} - c_i s_i.$$

Active choices bind:

$$x_1 > 0 \Rightarrow (1-a_1)p_{1x} r^{a_1} = c_1 s_1, \qquad y_2 > 0 \Rightarrow a_2 p_{2y} r^{a_2-1} = c_2 s_2.$$

Thus the *totals* at a given r are

$$\boxed{s_1 = s_1^X(r) = \frac{(1-a_1)p_{1x}}{c_1} r^{a_1}}, \qquad \boxed{s_2 = s_2^Y(r) = \frac{a_2 p_{2y}}{c_2} r^{a_2-1}}.$$

Step 2 (Pin r from the identity $r = Y/X$). Under `(X,Y)` we have

$$X = p_{1x}x_1 = p_{1x}s_1, \qquad Y = p_{2y}y_2 = p_{2y}s_2,$$

since $y_1 = 0$ and $x_2 = 0$. Therefore

$$r = \frac{Y}{X} = \frac{p_{2y}s_2}{p_{1x}s_1} = \frac{\dfrac{a_2p_{2y}^2}{c_2}r^{a_2-1}}{\dfrac{(1-a_1)p_{1x}^2}{c_1}r^{a_1}} = \frac{a_2c_1p_{2y}^2}{(1-a_1)c_2p_{1x}^2}\, r^{\,a_2-1-a_1}.$$

Multiply both sides by $r^{\,1+a_1-a_2} > 0$:

$$\boxed{r^{\,2+a_1-a_2} = K_{XY}}, \qquad \boxed{K_{XY} := \frac{a_2c_1p_{2y}^2}{(1-a_1)c_2p_{1x}^2}}.$$

Hence

$$\boxed{r = K_{XY}^{\,1/(2+a_1-a_2)}}.$$

Uniqueness. Since $a_1, a_2 \in (0,1)$, the exponent $2 + a_1 - a_2 \in (1,3)$, so this positive root is unique.

Step 3 (Feasibility/bounds: keep inactive choices at zero). We must ensure the *inactive* choices are indeed optimal at zero:

(A) $y_1 = 0$ optimal for player 1. Require $\dfrac{\partial U_1}{\partial y_1} \le 0$ at $y_1 = 0$. Using $c_1s_1 = (1-a_1)p_{1x}r^{a_1}$ from $x_1 > 0$,

$$\frac{\partial U_1}{\partial y_1} = a_1p_{1y}r^{a_1-1} - c_1s_1 = a_1p_{1y}r^{a_1-1} - (1-a_1)p_{1x}r^{a_1} \le 0$$

$$\iff \quad a_1p_{1y} \le (1-a_1)p_{1x}\,r \quad \iff \quad \boxed{r \;\ge\; r_1 := \frac{a_1p_{1y}}{(1-a_1)p_{1x}}}.$$

(B) $x_2 = 0$ optimal for player 2. Require $\dfrac{\partial U_2}{\partial x_2} \le 0$ at $x_2 = 0$. Using $c_2s_2 = a_2p_{2y}r^{a_2-1}$ from $y_2 > 0$,

$$\frac{\partial U_2}{\partial x_2} = (1-a_2)p_{2x}r^{a_2} - c_2s_2 = (1-a_2)p_{2x}r^{a_2} - a_2p_{2y}r^{a_2-1} \le 0$$

$$\iff \quad (1-a_2)p_{2x}\,r \le a_2p_{2y} \quad \iff \quad \boxed{r \;\le\; r_2 := \frac{a_2p_{2y}}{(1-a_2)p_{2x}}}.$$

Bounds summary: the specialized equilibrium is feasible iff

$$\boxed{r \in [r_1, r_2]}.$$

(At the endpoints the inactive FOC is exactly binding; strict inequalities give a strict corner.)

Step 4 (Equilibrium efforts). Plug the pinned r into the active totals; the efforts are

$$\boxed{x_1^* = s_1 = \frac{(1-a_1)p_{1x}}{c_1}\, r^{a_1}, \qquad y_1^* = 0}, \qquad \boxed{x_2^* = 0, \qquad y_2^* = s_2 = \frac{a_2 p_{2y}}{c_2}\, r^{a_2-1}}.$$

(Optionally: $X = \dfrac{(1-a_1)p_{1x}^2}{c_1} r^{a_1}$ and $Y = \dfrac{a_2 p_{2y}^2}{c_2} r^{a_2-1}$, which satisfy $Y/X = r$ by construction.)

Summary for `(X,Y)`.

$$\boxed{r^{\,2+a_1-a_2} = K_{XY} = \frac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2}, \quad r \in [r_1, r_2]\,;}$$

$$\boxed{(x_1^*, y_1^*; x_2^*, y_2^*) = \Big(\frac{(1-a_1)p_{1x}}{c_1} r^{a_1},\ 0;\ 0,\ \frac{a_2 p_{2y}}{c_2} r^{a_2-1}\Big)}.$$

Obtain by flip: `(Y,X)`. Apply the index swap $1 \leftrightarrow 2$ (keep $r = Y/X$). Active equalities are now $s_1 = s_1^Y(r)$ and $s_2 = s_2^X(r)$, and the same steps give

$$\boxed{r^{\,2+a_2-a_1} = K_{YX} := \frac{a_1 c_2 p_{1y}^2}{(1-a_2)c_1 p_{2x}^2}, \quad r \in [r_2, r_1]\,;}$$

$$\boxed{(x_1^*, y_1^*; x_2^*, y_2^*) = \Big(0,\ \frac{a_1 p_{1y}}{c_1} r^{a_1-1};\ \frac{(1-a_2)p_{2x}}{c_2} r^{a_2},\ 0\Big)}.$$

8 Pair P5: One X-only + Zero-effort `(X,0)` and `(0,X)`

Derive once: `(X,0)`. Setup. Player 1 is X-only ($x_1 > 0$, $y_1 = 0$). Player 2 exerts zero effort ($x_2 = 0$, $y_2 = 0$). Under this mask,

$$X \;=\; p_{1x}x_1 \;>\; 0, \qquad Y \;=\; p_{1y}y_1 + p_{2y}y_2 \;=\; 0.$$

Important: here $r = Y/X$ is *not used* (the ratio is 0), so we work directly with the X, Y-form derivatives.

Step 1 (FOCs/KKT evaluated at the mask). From the original (non-r-reduced) partial derivatives,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)X^{-a_i}Y^{a_i}p_{ix} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i X^{1-a_i}Y^{a_i-1}p_{iy} - c_i s_i,$$

with $a_i \in (0,1)$, $p_{ix}, p_{iy}, c_i > 0$.

Player 1 ($x_1 > 0, y_1 = 0$). Here $s_1 = x_1 > 0$ and $X > 0$, $Y = 0$.

$$\frac{\partial U_1}{\partial x_1} = (1-a_1)X^{-a_1}\underbrace{Y^{a_1}}_{=0}p_{1x} - c_1 s_1 = -\,c_1 s_1 \;<\; 0.$$

Since $x_1 > 0$ (an active choice) requires $\frac{\partial U_1}{\partial x_1} = 0$ at optimum, this is a *contradiction* (it says "decrease x_1"). Moreover, for the *inactive* choice $y_1 = 0$, KKT demands $\frac{\partial U_1}{\partial y_1} \le 0$; but

$$\frac{\partial U_1}{\partial y_1} = a_1 X^{1-a_1} \underbrace{Y^{a_1-1}}_{\text{diverges to } +\infty} p_{1y} \;-\; c_1 s_1 = +\infty \;-\; c_1 s_1 \;=\; +\infty \;>\; 0,$$

because $a_1 - 1 < 0$ and $Y = 0$ imply $Y^{a_1-1} = 1/Y^{1-a_1} \to +\infty$. So $y_1 = 0$ also violates its KKT inequality: player 1 strictly wants to increase y_1.

Player 2 ($x_2 = y_2 = 0$). Here $s_2 = 0$, $X > 0$, $Y = 0$.

$$\frac{\partial U_2}{\partial x_2} = (1-a_2) X^{-a_2} \underbrace{Y^{a_2}}_{=0} p_{2x} - c_2 s_2 = 0 \quad (\le 0 \text{ holds as equality}),$$

but

$$\frac{\partial U_2}{\partial y_2} = a_2 X^{1-a_2} \underbrace{Y^{a_2-1}}_{\to +\infty} p_{2y} - c_2 s_2 = +\infty \;>\; 0.$$

Thus player 2 also strictly wants to deviate to $y_2 > 0$.

Step 2 (Feasibility verdict). Both players' KKT conditions are violated under the mask: - An *active* $x_1 > 0$ cannot satisfy $\partial U_1/\partial x_1 = 0$ when $Y = 0$ (it is strictly negative). - The *inactive* $y_1 = 0$ and $y_2 = 0$ fail their ≤ 0 inequalities because $\partial U/\partial y$ blows up $+\infty$ at $Y = 0$ for $a \in (0, 1)$.

Therefore the mask `(X,0)` is *impossible for all admissible parameters* ($a_i \in (0, 1)$, $p_{ix}, p_{iy}, c_i > 0$).

Feasibility/bounds (parameter-only).

$$\boxed{\text{Mask impossible (no parameter region).}}$$

Equilibrium efforts.

$$\boxed{\text{Not applicable (mask cannot be an equilibrium).}}$$

Obtain by flip: `(0,X)`. Swap $1 \leftrightarrow 2$. The symmetric argument applies verbatim: with player 2 X-only and player 1 at zero, you again have $Y = 0$ and the same derivative contradictions.

$$\boxed{\texttt{(0,X)} \text{ is impossible for all admissible parameters.}}$$

9 Pair P6: One Y-only + Zero-effort `(Y,0)` and `(0,Y)`

Derive once: `(Y,0)`. **Setup.** Player 1 is Y-only ($x_1 = 0$, $y_1 > 0$). Player 2 exerts zero effort ($x_2 = 0$, $y_2 = 0$). Under this mask,

$$X \;=\; p_{1x}x_1 + p_{2x}x_2 \;=\; 0, \qquad Y \;=\; p_{1y}y_1 + p_{2y}y_2 \;=\; p_{1y}y_1 \;>\; 0.$$

Important: here $r = Y/X$ is *not used* (the ratio is $+\infty$), so we work directly with the X, Y-form derivatives.

Step 1 (FOCs/KKT evaluated at the mask). From the original (non-r-reduced) partial derivatives,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)\,X^{-a_i}\,Y^{a_i}\,p_{ix} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i\,X^{1-a_i}\,Y^{a_i-1}\,p_{iy} - c_i s_i,$$

with $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$.

Player 1 ($x_1 = 0,\ y_1 > 0$; $s_1 = y_1 > 0$; $X = 0$, $Y > 0$).

$$\frac{\partial U_1}{\partial y_1} = a_1\,X^{1-a_1}\,Y^{a_1-1}\,p_{1y} - c_1 s_1 = a_1\,\underbrace{X^{\,1-a_1}}_{0^{\,1-a_1}=0}\,Y^{a_1-1}\,p_{1y} - c_1 s_1 = -\,c_1 s_1 \;<\; 0.$$

Because $y_1 > 0$ is *active*, optimality would require $\partial U_1/\partial y_1 = 0$, which is impossible here (strictly negative). For the *inactive* choice $x_1 = 0$, KKT requires $\partial U_1/\partial x_1 \le 0$, but

$$\frac{\partial U_1}{\partial x_1} = (1-a_1)\,X^{-a_1}\,Y^{a_1}\,p_{1x} - c_1 s_1 = (1-a_1)\,\underbrace{X^{-a_1}}_{0^{-a_1}=+\infty}\,Y^{a_1}\,p_{1x} - c_1 s_1 = \;+\infty \;>\; 0,$$

since $a_1 \in (0,1) \Rightarrow -a_1 < 0$ and $0^{-a_1} = +\infty$. Thus $x_1 = 0$ also violates its KKT inequality.

Player 2 ($x_2 = y_2 = 0$; $s_2 = 0$; $X = 0$, $Y > 0$).

$$\frac{\partial U_2}{\partial x_2} = (1-a_2)\,X^{-a_2}\,Y^{a_2}\,p_{2x} - c_2 s_2 = \;+\infty \;>\; 0,$$

so $x_2 = 0$ is *not* optimal; player 2 strictly wants to increase x_2. For completeness,

$$\frac{\partial U_2}{\partial y_2} = a_2\,X^{1-a_2}\,Y^{a_2-1}\,p_{2y} - c_2 s_2 = 0 - 0 = 0 \;(\le 0),$$

so $y_2 = 0$ passes its inequality, but the failure for $x_2 = 0$ already invalidates the mask.

Step 2 (Feasibility verdict). Under $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$, the mask `(Y,0)` violates KKT for both players: - Player 1's *active* $y_1 > 0$ cannot satisfy $\partial U_1/\partial y_1 = 0$ at $X = 0$ (it is strictly negative), and the *inactive* $x_1 = 0$ also fails since $\partial U_1/\partial x_1 = +\infty > 0$. - Player 2's $x_2 = 0$ fails because $\partial U_2/\partial x_2 = +\infty > 0$. Therefore `(Y,0)` is *impossible for all admissible parameters.*

Feasibility/bounds (parameter-only).

$$\boxed{\text{Mask impossible (no parameter region).}}$$

Equilibrium efforts.

$$\boxed{\text{Not applicable (mask cannot be an equilibrium).}}$$

Obtain by flip: `(0,Y)`. Swap $1 \leftrightarrow 2$. The same logic applies with roles reversed: player 2 Y-only and player 1 zero effort again implies $X = 0$, $Y > 0$ and the same derivative contradictions.

$$\boxed{\texttt{(0,Y)}\text{ is impossible for all admissible parameters.}}$$

10 Self-symmetric S1: Fully interior `(B,B)`

Setup and feasibility. Both players are interior: $x_i > 0$ and $y_i > 0$ for $i \in \{1, 2\}$. Thus both marginal equalities bind:

$$(1 - a_i)p_{ix}r^{a_i} = c_i s_i, \qquad a_i p_{iy} r^{a_i - 1} = c_i s_i, \qquad s_i := x_i + y_i.$$

Equating the two left–hand sides gives, for each i,

$$(1 - a_i)p_{ix}\, r = a_i p_{iy} \quad \Longrightarrow \quad r = r_i := \frac{a_i p_{iy}}{(1 - a_i)p_{ix}}.$$

Hence a necessary condition for `(B,B)` is

$$\boxed{r_1 = r_2} \quad \text{(knife-edge in parameters).}$$

When $r_1 = r_2$ holds, set $r := r_1 = r_2$. Then the interior totals are pinned for each player (either equality gives the same value):

$$\boxed{s_i = \frac{(1 - a_i)p_{ix}}{c_i}\, r^{a_i} = \frac{a_i p_{iy}}{c_i}\, r^{-(1-a_i)}}, \qquad i \in \{1, 2\}.$$

Since $x_i, y_i > 0$ and $p_{ix}, p_{iy} > 0$, both $X = p_{1x}x_1 + p_{2x}x_2$ and $Y = p_{1y}y_1 + p_{2y}y_2$ are strictly positive and $r = Y/X$ is well-defined.

Feasibility region (parameter-only).

$$\boxed{\texttt{(B,B)} \text{ is feasible if and only if } r_1 = r_2.}$$

(Generically, this is a measure-zero "boundary" in parameter space; off this equality, at least one player must be corner.)

Equilibrium efforts: continuum of splits + a canonical selection. The marginal equalities pin the *totals* s_1, s_2 and the common ratio r, but they do not pin each player's split (x_i, y_i) separately because cost depends only on $s_i = x_i + y_i$. The remaining equilibrium condition is the accounting identity $Y = rX$:

$$X = p_{1x}x_1 + p_{2x}x_2, \qquad Y = p_{1y}y_1 + p_{2y}y_2 = p_{1y}(s_1 - x_1) + p_{2y}(s_2 - x_2).$$

Impose $Y = rX$ and rearrange to a linear constraint in (x_1, x_2):

$$\boxed{(p_{1y} + rp_{1x})\, x_1 + (p_{2y} + rp_{2x})\, x_2 \;=\; p_{1y}s_1 + p_{2y}s_2.}$$

Together with the box constraints $0 < x_1 < s_1$, $0 < x_2 < s_2$, this defines a *line segment* of equilibria (a continuum). An explicit parameterization is convenient:

$$x_2(x_1) = \frac{p_{1y}s_1 + p_{2y}s_2 - (p_{1y} + rp_{1x})x_1}{p_{2y} + rp_{2x}}, \qquad y_i = s_i - x_i,\ i \in \{1, 2\}.$$

The feasible interval of x_1 that keeps $0 \le x_2 \le s_2$ is

$$\boxed{x_1 \in I := \Big[\ \max\Big\{0,\ \frac{p_{1y}s_1 - rp_{2x}s_2}{p_{1y} + rp_{1x}}\Big\},\ \min\Big\{s_1,\ \frac{p_{1y}s_1 + p_{2y}s_2}{p_{1y} + rp_{1x}}\Big\}\Big]},$$

which is nonempty under $p_{ix}, p_{iy}, s_i > 0$. Any choice $x_1 \in I$ yields $0 < x_1 < s_1$ and $0 < x_2 < s_2$ and hence an equilibrium effort vector with $x_i > 0$, $y_i > 0$.

Canonical selection (equal fraction to X). For a single-valued "selected interior equilibrium," choose a common fraction $\lambda \in (0,1)$ so that both players allocate the *same share* of their totals to task X:

$$x_1 = \lambda s_1, \qquad x_2 = \lambda s_2, \qquad y_i = (1-\lambda)s_i.$$

Plugging into $r = \dfrac{Y}{X}$ with $Y = p_{1y}y_1 + p_{2y}y_2$ and $X = p_{1x}x_1 + p_{2x}x_2$ gives

$$\frac{Y}{X} = \frac{(1-\lambda)(p_{1y}s_1 + p_{2y}s_2)}{\lambda(p_{1x}s_1 + p_{2x}s_2)} = r \quad \Longrightarrow \quad \boxed{\lambda = \frac{p_{1y}s_1 + p_{2y}s_2}{p_{1y}s_1 + p_{2y}s_2 + r\,(p_{1x}s_1 + p_{2x}s_2)}}.$$

At the knife-edge $r = r_1 = r_2$ with s_i as above, this produces a unique canonical point on the equilibrium line segment:

$$\boxed{x_i^{\text{can}} = \lambda s_i, \qquad y_i^{\text{can}} = (1-\lambda)s_i, \qquad i \in \{1,2\}.}$$

Summary (ready for the master table).

$$\boxed{\text{Feasible iff } r_1 = r_2. \quad s_i = \frac{(1-a_i)p_{ix}}{c_i}\, r^{a_i} = \frac{a_i p_{iy}}{c_i}\, r^{-(1-a_i)}.}$$

$$\boxed{\text{Efforts: any } (x_1,x_2) \text{ on } (p_{1y}+rp_{1x})x_1 + (p_{2y}+rp_{2x})x_2 = p_{1y}s_1 + p_{2y}s_2,\ 0 < x_i < s_i;\ y_i = s_i - x_i.}$$

$$\boxed{\text{Canonical selection: } \lambda = \frac{p_{1y}s_1 + p_{2y}s_2}{p_{1y}s_1 + p_{2y}s_2 + r(p_{1x}s_1 + p_{2x}s_2)},\ x_i = \lambda s_i,\ y_i = (1-\lambda)s_i.}$$

11 Self-symmetric S2: Both X-only `(X,X)`

Setup and feasibility. Both players choose X only: $x_i > 0$ and $y_i = 0$ for $i \in \{1,2\}$. Then

$$X = p_{1x}x_1 + p_{2x}x_2 \ > \ 0, \qquad Y = p_{1y}y_1 + p_{2y}y_2 = 0.$$

Important: here $r = Y/X$ is *not used* (the ratio is 0). We work directly with the original X, Y-form partial derivatives:

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)\, X^{-a_i}\, Y^{a_i}\, p_{ix} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i\, X^{1-a_i}\, Y^{a_i-1}\, p_{iy} - c_i s_i,$$

with $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$.

KKT at the mask. Since $x_i > 0$ is *active*, optimality would require $\frac{\partial U_i}{\partial x_i} = 0$. But with $Y = 0$:

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)X^{-a_i}\underbrace{Y^{a_i}}_{=0}p_{ix} - c_i s_i = -\,c_i s_i \ < \ 0,$$

a contradiction (it says "decrease x_i"). For the *inactive* choice $y_i = 0$, KKT requires $\frac{\partial U_i}{\partial y_i} \le 0$. Yet with $Y = 0$ and $a_i - 1 < 0$,

$$\frac{\partial U_i}{\partial y_i} = a_i X^{1-a_i} \underbrace{Y^{a_i-1}}_{=1/Y^{1-a_i}\to+\infty} p_{iy} - c_i s_i = \ +\infty \ > \ 0.$$

Thus $y_i = 0$ is also not optimal. These two violations occur for *both* players.

Bounds (parameter-only). Under the standing restrictions $a_i \in (0,1)$, $p_{ix}, p_{iy}, c_i > 0$, there are *no* parameters that make `(X,X)` satisfy KKT:

$$\boxed{\text{Mask impossible (no parameter region).}}$$

(Only by leaving the admissible class—e.g., setting some $a_i = 0$ or $p_{iy} = 0$—could one remove the blow-up at $Y = 0$, but such cases are excluded here.)

Equilibrium efforts $(x_1^*, y_1^*; x_2^*, y_2^*)$**.**

$$\boxed{\text{Not applicable (mask cannot be an equilibrium).}}$$

12 Self-symmetric S3: Both Y-only `(Y,Y)`

Setup and feasibility. Both players choose Y only: $y_i > 0$ and $x_i = 0$ for $i \in \{1,2\}$. Then

$$X \;=\; p_{1x}x_1 + p_{2x}x_2 \;=\; 0, \qquad Y \;=\; p_{1y}y_1 + p_{2y}y_2 \;>\; 0.$$

Important: here $r = Y/X$ is *not used* (the ratio is $+\infty$). We work directly with the original X, Y-form partial derivatives:

$$\frac{\partial U_i}{\partial x_i} = (1-a_i)\, X^{-a_i}\, Y^{a_i}\, p_{ix} - c_i s_i, \qquad \frac{\partial U_i}{\partial y_i} = a_i\, X^{1-a_i}\, Y^{a_i-1}\, p_{iy} - c_i s_i,$$

with $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$.

KKT at the mask (each player). Because $y_i > 0$ is *active*, optimality would require $\frac{\partial U_i}{\partial y_i} = 0$. But with $X = 0$:

$$\frac{\partial U_i}{\partial y_i} = a_i \underbrace{X^{1-a_i}}_{0^{1-a_i}=0} Y^{a_i-1}\, p_{iy} - c_i s_i = -\,c_i s_i \;<\; 0,$$

a contradiction (it says "decrease y_i"). For the *inactive* choice $x_i = 0$, KKT requires $\frac{\partial U_i}{\partial x_i} \le 0$; but with $X = 0$ and $a_i \in (0,1)$,

$$\frac{\partial U_i}{\partial x_i} = (1-a_i) \underbrace{X^{-a_i}}_{0^{-a_i}=+\infty} Y^{a_i}\, p_{ix} - c_i s_i = \;+\infty \;>\; 0.$$

Thus $x_i = 0$ is also not optimal. Both violations occur for *both* players.

Bounds (parameter-only). Under the standing restrictions $a_i \in (0,1)$, $p_{ix}, p_{iy}, c_i > 0$, there are *no* parameters that make `(Y,Y)` satisfy KKT:

$$\boxed{\text{Mask impossible (no parameter region).}}$$

(Only by leaving the admissible class—e.g., setting some $a_i = 1$ or $p_{ix} = 0$—could one remove the blow-up at $X = 0$, but such cases are excluded here.)

Equilibrium efforts $(x_1^*, y_1^*; x_2^*, y_2^*)$**.**

$$\boxed{\text{Not applicable (mask cannot be an equilibrium).}}$$

13 Self-symmetric S4: Both zero effort `(0,0)`

Setup and feasibility. Both players choose zero effort: $x_i = y_i = 0$ for $i \in \{1,2\}$. Then

$$X \; = \; p_{1x}x_1 + p_{2x}x_2 \; = \; 0, \qquad Y \; = \; p_{1y}y_1 + p_{2y}y_2 \; = \; 0.$$

With $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$, Cobb–Douglas utility for player i is

$$U_i = X^{\,1-a_i} Y^{\,a_i} - \frac{c_i}{2}(x_i + y_i)^2.$$

At $(X,Y) = (0,0)$ the product term is not well-defined as a differentiable function; rather than rely on derivatives, we use a *direct deviation* argument to test whether $(0,0)$ can be a best reply.

Direct deviation test (constructive). Fix the opponent at zero effort. Consider for player i a small deviation that makes *both* outputs positive:

$$x_i = t, \qquad y_i = t, \qquad t > 0 \text{ small} \quad \text{(others stay at 0)}.$$

Then $X = p_{ix}t > 0$ and $Y = p_{iy}t > 0$, and the resulting payoff is

$$U_i(t) = (p_{ix}t)^{1-a_i}(p_{iy}t)^{a_i} - \frac{c_i}{2}(2t)^2 = \underbrace{p_{ix}^{1-a_i}p_{iy}^{a_i}}_{=:B_i}\, t \; - \; 2c_i t^2.$$

For sufficiently small t, the linear term dominates the quadratic cost:

$$U_i(t) - U_i(0) \;\; = \;\; B_i\, t - 2c_i t^2 \;\; > \;\; 0 \quad \text{whenever} \quad 0 < t < \frac{B_i}{2c_i}.$$

Hence $t = 0$ is *not* a best reply to the opponent's zero effort. This holds for each player $i = 1, 2$.

Feasibility verdict. Because each player has a profitable unilateral deviation from zero (for all admissible parameters), the mask `(0,0)` cannot be a Nash equilibrium.

$$\boxed{\text{Mask impossible (no parameter region).}}$$

Bounds (parameter-only). None: under $a_i \in (0,1)$ and $p_{ix}, p_{iy}, c_i > 0$, there are no parameters for which `(0,0)` is an equilibrium.

Equilibrium efforts $(x_1^*, y_1^*; x_2^*, y_2^*)$.

$$\boxed{\text{Not applicable (mask cannot be an equilibrium).}}$$

14 Equilibrium Tables

Table 1: Symmetric feasible pairs placed side-by-side. Feasibility/bounds (with how r is determined) and stacked equilibrium efforts.

Pair	**Left mask (feasibility/bounds & efforts)**	**Right mask (feasibility/bounds & efforts)**
P1: `(B,X)` $\leftrightarrow$ `(X,B)`	`(B,X)` (Player 1 interior; Player 2 X-only) *Bounds:* $r_1 \geq r_2$, $r_1^{1+a_2-a_1} \leq \dfrac{(1-a_1)p_{1y}p_{1x}c_2}{(1-a_2)p_{2x}^2 c_1}$. *Efforts (evaluate at* r_1*):* $s_1 = \dfrac{(1-a_1)p_{1x}}{c_1}\, r_1^{a_1}$ $s_2 = \dfrac{(1-a_2)p_{2x}}{c_2}\, r_1^{a_2}$ $y_1 = \dfrac{r_1(p_{1x}s_1 + p_{2x}s_2)}{p_{1y} + r_1 p_{1x}}$ $x_1 = s_1 - y_1$ $x_2 = s_2$ $y_2 = 0$	`(X,B)` (Player 1 X-only; Player 2 interior) *Bounds:* $r_2 \geq r_1$, $r_2^{1+a_1-a_2} \leq \dfrac{(1-a_2)p_{2y}p_{2x}c_1}{(1-a_1)p_{1x}^2 c_2}$. *Efforts (evaluate at* r_2*):* $s_2 = \dfrac{(1-a_2)p_{2x}}{c_2}\, r_2^{a_2}$ $s_1 = \dfrac{(1-a_1)p_{1x}}{c_1}\, r_2^{a_1}$ $y_2 = \dfrac{r_2(p_{1x}s_1 + p_{2x}s_2)}{p_{2y} + r_2 p_{2x}}$ $x_2 = s_2 - y_2$ $x_1 = s_1$ $y_1 = 0$
P2: `(B,Y)` $\leftrightarrow$ `(Y,B)`	`(B,Y)` (Player 1 interior; Player 2 Y-only) *Bounds:* $r_1 \leq r_2$, $\quad r_1^{2+a_1-a_2} \geq K_{XY}$, $K_{XY} = \dfrac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2}$. *Efforts (evaluate at* r_1*):* $s_1 = \dfrac{(1-a_1)p_{1x}}{c_1}\, r_1^{a_1}$ $s_2 = \dfrac{a_2 p_{2y}}{c_2}\, r_1^{a_2-1}$ $y_1 = \dfrac{r_1 p_{1x} s_1 - p_{2y} s_2}{p_{1y} + r_1 p_{1x}}$ $x_1 = s_1 - y_1$ $x_2 = 0$ $y_2 = s_2$	`(Y,B)` (Player 1 Y-only; Player 2 interior) *Bounds:* $r_2 \leq r_1$, $\quad r_2^{2+a_2-a_1} \geq K_{YX}$, $K_{YX} = \dfrac{a_1 c_2 p_{1y}^2}{(1-a_2)c_1 p_{2x}^2}$. *Efforts (evaluate at* r_2*):* $s_2 = \dfrac{(1-a_2)p_{2x}}{c_2}\, r_2^{a_2} = \dfrac{a_2 p_{2y}}{c_2}\, r_2^{-(1-a_2)}$ $s_1 = \dfrac{a_1 p_{1y}}{c_1}\, r_2^{a_1-1}$ $y_2 = \dfrac{r_2 p_{2x} s_2 - p_{1y} s_1}{p_{2y} + r_2 p_{2x}}$ $x_2 = s_2 - y_2$ $x_1 = 0$ $y_1 = s_1$

(table continues on next page)

Table 1 (continued).

Pair	Left mask (feasibility/bounds & efforts)	Right mask (feasibility/bounds & efforts)
P3 (specialization): `(X,Y)` $\leftrightarrow$ `(Y,X)`	`(X,Y)` (1: X-only; 2: Y-only) Let $r_* := K_{XY}^{1/(2+a_1-a_2)}$ with $K_{XY} = \dfrac{a_2 c_1 p_{2y}^2}{(1-a_1)c_2 p_{1x}^2}$; require $r_* \in [r_1, r_2]$. *Efforts (evaluate at r_*):* $x_1 = \dfrac{(1-a_1)p_{1x}}{c_1}\, r_*^{a_1}$ $y_1 = 0$ $x_2 = 0$ $y_2 = \dfrac{a_2 p_{2y}}{c_2}\, r_*^{a_2-1}$	`(Y,X)` (1: Y-only; 2: X-only) Let $\tilde{r}_* := K_{YX}^{1/(2+a_2-a_1)}$ with $K_{YX} = \dfrac{a_1 c_2 p_{1y}^2}{(1-a_2)c_1 p_{2x}^2}$; require $\tilde{r}_* \in [r_2, r_1]$. *Efforts (evaluate at $\tilde{r}_*$):* $x_1 = 0$ $y_1 = \dfrac{a_1 p_{1y}}{c_1}\, \tilde{r}_*^{a_1-1}$ $x_2 = \dfrac{(1-a_2)p_{2x}}{c_2}\, \tilde{r}_*^{a_2}$ $y_2 = 0$

Table 2: Self-symmetric feasible case.

Mask	Feasibility/bounds & equilibrium efforts (stacked)
`(B,B)`	Feasible *iff* $r_1 = r_2$ (knife-edge). Set $r = r_1 = r_2$ and $s_i = \dfrac{(1-a_i)p_{ix}}{c_i} r^{a_i} = \dfrac{a_i p_{iy}}{c_i} r^{-(1-a_i)}$. Splits lie on $(p_{1y} + rp_{1x})x_1 + (p_{2y} + rp_{2x})x_2 = p_{1y}s_1 + p_{2y}s_2$ with $0 < x_i < s_i$. Canonical selection (equal fraction to X): $\lambda = \dfrac{p_{1y}s_1 + p_{2y}s_2}{p_{1y}s_1 + p_{2y}s_2 + r(p_{1x}s_1 + p_{2x}s_2)}$; $x_1 = \lambda s_1$, $y_1 = (1-\lambda)s_1$, $x_2 = \lambda s_2$, $y_2 = (1-\lambda)s_2$.

Table 3: Masks that are impossible for all admissible parameters ($a_i \in (0,1)$, $p_{ix}, p_{iy}, c_i > 0$).

Mask	Why impossible (brief)
`(B,0)`	With Player 1 interior (so $r = r_1 > 0$), at $(x_2, y_2) = (0,0)$ both one-sided marginals for Player 2 are > 0 (violates KKT at zero).
`(0,B)`	Symmetric of `(B,0)`: with Player 2 interior ($r = r_2 > 0$), Player 1's one-sided marginals at $(0,0)$ are > 0.
`(X,0)`	Forces $Y = 0$; $\partial U/\partial y$ blows up $+\infty$ at $Y = 0$ (for $a \in (0,1)$), so $y = 0$ cannot be optimal.
`(0,X)`	Symmetric of `(X,0)`.
`(Y,0)`	Forces $X = 0$; $\partial U/\partial x$ blows up $+\infty$ at $X = 0$, so $x = 0$ cannot be optimal.
`(0,Y)`	Symmetric of `(Y,0)`.

Table 3 (continued).

Mask	**Why impossible (brief)**
`(X,X)`	Forces $Y = 0$; for each player, active $x > 0$ cannot satisfy $\partial U/\partial x = 0$ and inactive $y = 0$ violates $\partial U/\partial y \leq 0$.
`(Y,Y)`	Forces $X = 0$; symmetric argument to `(X,X)`.
`(0,0)`	Each player has profitable small deviations (t, t): $U_i(t) - U_i(0) = p_{ix}^{1-a_i} p_{iy}^{a_i} t - 2c_i t^2 > 0$ for small $t > 0$.