DMFM and Tensor 3PRF: A Nowcasting Framework for Mixed-Frequency Data in the Euro Area

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1 Vector Models

1.1 Data Preparation

1.1.1 Data

- **Source:** mixed–frequency *EA–MD–QD* dataset from Barigozzi and Lissona (2024).
- **Transformations:** all series are pre–transformed to ensure stationarity.[1]
- **Time span:** April 2000 – October 2025.[2]
- **Crises covered:**
  - Global Financial Crisis (2008–2009): no special treatment for any class of variables.
  - COVID pandemic (March 2020 – July 2021): masking of real variables (those most affected by the crisis).
- **Types of missingness:**
  - Regular missingness from quarterly variables.
  - COVID–induced blocks of missing values for real variables during the masked period.
  - Ragged edge from heterogeneous publication calendars, handled in the pseudo real–time forecasting exercise.

1.1.2 Factor-Number Selection under Missing Data

Given the arbitrary missingness pattern in our large dataset, selecting the number of factors on a different subset (e.g. only monthly or only quarterly data, or excluding the COVID period) would be inconsistent with the final specification. We therefore follow an alternative approach.

All-purpose Covariance Estimator and Eigenvalue–Ratio

- Let $X \in \mathbb{R}^{T\times N}$ be the standardized panel with missing entries.
- Define the missingness matrix and the masked data as

$$w_{t,i} = \begin{cases} 1, & X_{t,i} \text{ observed}, \\ 0, & X_{t,i} \text{ missing}, \end{cases} \qquad \widetilde{X}_{t,i} = w_{t,i} X_{t,i},$$

  i.e. missing values are imputed by zero in $\widetilde{X}$.

[1] Each quarterly variable is placed in the third month of the corresponding quarter, regardless of its original transformation.

[2] The sample starts in April 2000 to avoid missing values in the annual growth rate at the beginning of the time span, when the EA aggregate did not yet exist in the previous year.

- Following Xiong and Pelger (2023), estimate the covariance as

$$\widetilde{\Sigma}_{ij} = \frac{1}{\Theta_{ij}} \sum_{t: w_{t,i} w_{t,j} = 1} X_{t,i} X_{t,j}, \qquad \Theta_{ij} = \sum_{t=1}^{T} w_{t,i} w_{t,j}.$$

- Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the eigenvalues of $\widetilde{\Sigma}$. For a given $K_{\max}$, select

$$\widehat{r} = \arg \max_{1 \leq j \leq K_{\max}} \frac{\lambda_j}{\lambda_{j+1}},$$

as in Ahn and Horenstein (2013).[3]

Otherwise, as an alternative, we could reconstruct the dataset by first estimating factors using the method of Xiong and Pelger (2023), substituting the estimated common component for the missing entries, and then apply the classical Bai–Ng information criteria (Bai and Ng 2002), which assume a clean, balanced, and exogenous panel.

1.1.3 Predictor Selection Based on GDP Correlation

Preprocessing

- Let y_τ denote quarterly GDP.
- Let $X_{m,t}$ and $X_{q,\tau}$ be monthly and quarterly predictors.
- Monthly series are aggregated to quarterly frequency using an operator $\omega(L)$ (sum or average depending on the variable type):

$$X_{\tau,j}^{(m \to q)} = \omega(L) X_{m,t,j}.$$

Selection criteria (country-by-country) Selection is performed separately for each country to reflect heterogeneity. Thus, for each country c and each predictor x_j:

1. **Fixed-number criterion**
   - Compute absolute correlation

$$\rho_j = \left| \operatorname{corr}(X_{\cdot,j}^{(m \to q)}, y) \right|.$$

   - Keep the n_m monthly and n_q quarterly predictors with largest $|\rho_j|$ according to the user parameter's specification.
2. **Correlation-threshold criterion**

[3] This factor–number selection is parallel to the one used in the matrix formulation, via the tensor extensions of Cen and Lam (2025) and Yu et al. (2022) applied to Xiong and Pelger (2023) and Ahn and Horenstein (2013), respectively.

- Select all predictors with

$$\left|\mathrm{corr}(X_{\cdot,j}^{(m\to q)}, y)\right| \geq \tau_m, \qquad \left|\mathrm{corr}(X_{\cdot,j}^{(q)}, y)\right| \geq \tau_q.$$

3. **F-test significance criterion**

- For each predictor, run a simple regression of $X_{\tau,j}^{(m\to q)}$ on y_τ.
- Compute the F-statistic for $H_0 : \beta_j = 0$ and retain j if

$$p\text{-value}(F_j) \leq \tau_F.$$

1.2 Mixed-Frequency TPRF (MF–TPRF)

Overview and motivation

- Kelly and Pruitt (2015) propose the Three-Pass Regression Filter (3PRF) as a simple, OLS-based method to extract *targeted factors* for forecasting a specific variable using a large set of predictors.
- The procedure is:
  - computationally inexpensive,
  - easy to implement (only linear regressions),
  - designed to focus on the predictive content of high-dimensional predictors.
- However, the standard 3PRF relies on assumptions usually violated in macroeconomic applications:
  - a single frequency,
  - a balanced panel without ragged edges,
  - no temporary data unavailability.
- To address these issues, Hepenstrick and Marcellino (2016) extend 3PRF to the mixed-frequency framework (MF–3PRF), allowing:
  - a quarterly target y_τ to coexist with higher-frequency predictors x_t,
  - explicit handling of ragged edges and unbalanced panels in x_t.

1.2.1 Standard Three-Pass Regression Filter

Model setup

- Let F_t denote latent factors, and consider the model:

$$y_{t+1} = \beta' F_t + \eta_{t+1}, \qquad z_t = \Lambda F_t + \omega_t, \qquad x_t = \Phi F_t + \varepsilon_t,$$

  where:

  - y_{t+1} is the target variable to be forecast [4],
  - z_t is a low-dimensional proxy (e.g. asset returns, macro indicator, or tipically the target itself),
  - x_t is a large-dimensional set of predictors exhibiting balancedness.

Pass 1: Targeting step

- For each predictor $x_{i,t}$, run the regression

$$x_{i,t} = z_t' \phi_i + u_{i,t},$$

  and stack the OLS estimates into

$$\hat{\Phi}_z' = (Z'Z)^{-1} Z'X,$$

  where Z collects the proxies z_t and X the predictors x_t.

- Interpretation: $\hat{\Phi}_z$ measures how strongly each predictor responds to the proxy z_t and "targets" the part of X that is most relevant for forecasting y.

Pass 2: Factor extraction

- Using $\hat{\Phi}_z$, extract targeted factors from the high-dimensional predictors via

$$\hat{F}_t = (\hat{\Phi}_z' \hat{\Phi}_z)^{-1} \hat{\Phi}_z' x_t,$$

  or, in matrix form,

$$\hat{F} = X \hat{\Phi}_z (\hat{\Phi}_z' \hat{\Phi}_z)^{-1}.$$

- Interpretation: $\hat{F}_t$ summarizes the portion of X_t that is most correlated with the proxy z_t and therefore most relevant for forecasting y.

[4] the target is always excluded from the matrix of predictors X.

Pass 3: Forecast regression

- Regress the target on the extracted factors:

$$y_{t+1} = \beta' \hat{F}_t + \eta_{t+1}, \qquad \hat{\beta} = (\hat{F}'\hat{F})^{-1}\hat{F}'Y.$$

- The forecast is then given by

$$\hat{y}_{t+1} = \hat{\beta}' \hat{F}_t.$$

- Interpretation: the final regression uses a small number of targeted factors, rather than the full set of predictors, delivering a low-dimensional but information-rich forecasting model.

1.2.2 MF Three-Pass Regression Filter: Preparation

- Let y_τ and z_τ be observed at quarterly frequency, and let x_t be observed monthly.
- Following Hepenstrick and Marcellino (2016), quarters are indexed by τ and months by t.
- Temporal aggregation from monthly to quarterly is defined by [5]:

$$\omega(L) = \omega_0 + \omega_1 L + \cdots + \omega_{k-1} L^{k-1}, \qquad k = 3,$$

1.2.3 Pass 1: Quarterly Regressions

Aggregation and regressions

- Monthly predictors and proxies are first aggregated to quarterly frequency [6]:

$$x_{i,\tau} = \omega(L) x_{i,t}, \qquad z_\tau = \omega(L) z_t.$$

- Here,

$$x_{i,\tau} \in \mathbb{R}, \qquad z_\tau \in \mathbb{R}^L$$

  collects the quarterly aggregated values.

- For each predictor $i = 1, \dots, N$, estimate the quarterly regression

$$x_{i,\tau} = z_\tau' \alpha_i + u_{i,\tau}, \qquad \alpha_i \in \mathbb{R}^L.$$

Stacked representation and OLS

- Stacking across predictors:

$$x_\tau = \mathbf{A}_z z_\tau + u_\tau, \qquad \mathbf{A}_z = \begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_N' \end{pmatrix} \in \mathbb{R}^{N \times L}.$$

- In matrix form, with $Z \in \mathbb{R}^{T_Q \times L}$ and $X \in \mathbb{R}^{T_Q \times N}$ collecting quarterly observations,

$$\widehat{\mathbf{A}}_z' = (Z'Z)^{-1} Z'X$$

  is the OLS estimator of the loading matrix $\mathbf{A}_z$.

[5] e.g. quarterly growth is given by the simple sum $\omega(L) = 1 + L + L^2$.

[6] In the empirical exercise we treat stocks as avarages over the quarter and flows as the sum

1.2.4 Pass 2: Monthly Factor Extraction

Projection on targeted loadings

- At the monthly frequency, the cross-section is projected onto the estimated loadings:

$$x_t = \widehat{\mathbf{A}}_z F_t + \varepsilon_t, \qquad x_t \in \mathbb{R}^N, \quad F_t \in \mathbb{R}^L.$$

- Cross-sectional OLS yields the targeted factors

$$\widehat{F}_t = (\widehat{\mathbf{A}}_z' \widehat{\mathbf{A}}_z)^{-1} \widehat{\mathbf{A}}_z' x_t.$$

Stacked factors

- Stacking over months t gives

$$\widehat{F} = X_M \widehat{\mathbf{A}}_z (\widehat{\mathbf{A}}_z' \widehat{\mathbf{A}}_z)^{-1}, \qquad \widehat{F} \in \mathbb{R}^{T_M \times L},$$

where X_M collects the monthly predictors.

1.2.5 Pass 3: Quarterly U–MIDAS Regression

Within-quarter factor positions

- Monthly factors are grouped into their positions within each quarter:

$$F_t \longrightarrow F_\tau^Q = \begin{pmatrix} F_\tau^{(1)} \\ F_\tau^{(2)} \\ F_\tau^{(3)} \end{pmatrix} \in \mathbb{R}^{3L},$$

where $F_\tau^{(1)}, F_\tau^{(2)}, F_\tau^{(3)}$ denote the factors in the first, second, and third month of quarter τ.

Quarterly regression and estimation

- The quarterly target is modeled as a U–MIDAS regression on lagged within-quarter factors:

$$y_\tau = \beta_0 + B' F_{\tau-1}^Q + \eta_\tau, \qquad B \in \mathbb{R}^{3L}.$$

- In matrix form, with $\widehat{F}^Q$ stacking $F_{\tau-1}^Q$ and Y collecting y_τ,

$$\widehat{B} = \left[(\widehat{F}^Q)' \widehat{F}^Q \right]^{-1} (\widehat{F}^Q)' Y, \qquad Y \in \mathbb{R}^{T_Q}.$$

1.2.6 Handling Mixed Frequency and Ragged Edges

As already mentioned, y_τ and z_τ are observed at quarterly frequency, but nothing prevents some $x_{i,t}$ be observed at the same lower frequency.

Selection matrices

- Let A_i be a *selection matrix* for series i, mapping the full time series into the vector of observed entries:
$$x_{i,\text{obs}} = A_i x_i, \qquad A_i \in \{0,1\}^{m_i \times T_M},$$
where m_i is the number of observed monthly values for series i.
- Missing or lower-frequency data correspond to rows of A_i that skip certain months.

EM-style imputation

- At iteration j, missing values of series i are updated using the current factor estimates:
$$\widehat{x}_{i,t}^{(j)} = \widehat{F}^{(j-1)}\widehat{\Phi}_i^{(j-1)} + A_i\left(A_i'A_i\right)^{-1}\left(x_{i,\text{obs}} - A_i\widehat{F}^{(j-1)}\widehat{\Phi}_i^{(j-1)}\right).$$
- Intuitively, the first term is the model-implied common component, and the second term corrects it to exactly match the observed entries.
- Factors are then re-estimated (e.g. via PCA) on the completed dataset, and the procedure is iterated until convergence.
- To implement succesfully this step the following procedures needs to be followed:
  - standardize the set of predictors,
  - extarct the number of factors,
  - Run the EM,
  - aggregate at quarterly level the reconstructed dataset,
  - center the data again to run the MF-TPRF or add a constant to the regressions in Step 1

1.2.7 Nowcasting Structure

In the empirical exercise the real time forecastimg exercise has been computed considering the delays in each indicator release. For any time in the evaluation sample (Spanning from January 2017 to October 2025) we always recomputed the number of factors and the lag in the U-MIDAS regression.

Within-quarter nowcasts

- Given monthly factors $(F_\tau^{(1)}, F_\tau^{(2)}, F_\tau^{(3)})$, the nowcast of y_τ can be updated as new monthly information arrives:

$$\hat{y}_\tau^{(1)} = \hat{\beta}_0 + \hat{\beta}_1 F_\tau^{(1)},$$
$$\hat{y}_\tau^{(2)} = \hat{\beta}_0 + \hat{\beta}_1 F_\tau^{(1)} + \hat{\beta}_2 F_\tau^{(2)},$$
$$\hat{y}_\tau^{(3)} = \hat{\beta}_0 + \hat{\beta}_1 F_\tau^{(1)} + \hat{\beta}_2 F_\tau^{(2)} + \hat{\beta}_3 F_\tau^{(3)}.$$

- These correspond to nowcasts after the first, second, and third month of the quarter, respectively.

1.2.8 Lag-Length Selection in the MF–TPRF

MF–TPRF requires choosing the number of quarterly lags L in the U–MIDAS target equation. We first run Pass 1–2 of the MF–TPRF as described in the previous subsection to obtain the quarterly factors $F_\tau^{(1)}, F_\tau^{(2)}, F_\tau^{(3)}$. In Pass 3, we select L as follows:

- For each candidate $L \leq L_{\max}$, estimate the quarterly regression

$$y_\tau = \beta_0 + \sum_{\ell=1}^{L} \left(F_{\tau-\ell}^{(1)} \gamma_\ell^{(1)} + F_{\tau-\ell}^{(2)} \gamma_\ell^{(2)} + F_{\tau-\ell}^{(3)} \gamma_\ell^{(3)} \right) + \varepsilon_\tau.$$

- Compute information criteria for each L:

$$\hat{L}_{\text{AIC}} = \arg\min_L \text{AIC}(L), \qquad \hat{L}_{\text{BIC}} = \arg\min_L \text{BIC}(L).$$

- The selected $\hat{L}$ provides a fully data–driven lag choice, consistent with the mixed-frequency setting and the projected TPRF structure.

1.3 Vector Models - Results

1.3.1 MF-TPRF

2 Matrix Models

3 Tensor-Based TPRF

Here we extend the classical Three-Pass Regression Filter to the case where predictors are *matrices* (e.g. country $\times$ variable) observed at mixed frequencies over time.[7] This approach preserves the proxy-based targeting idea while exploiting the tensor structure of the data.

- **Step 0: Completion of matrix-valued predictors**
  - As discussed above, monthly predictors are matrices (e.g. countries $\times$ indicators) affected by:
    - ragged edges,
    - publication delays,
    - blocks of missing values due to COVID masking.
  - We vectorize each matrix and apply a static-factor EM algorithm in order to:
    - estimate common factors and loadings,
    - impute missing entries in a model-consistent way,
    - obtain a completed monthly panel of matrix-valued predictors.
  - After reconstruction, we reshape the completed data back into its original tensor form ($T \times P_1 \times P_2$).
- **Step 1: Proxy-based targeting in the matrix domain**
  - Given a scalar proxy (e.g. GDP averaged across countries), we construct proxy-weighted moment matrices between this proxy and the completed predictors.
  - For each proxy, we compute a truncated SVD of these moment matrices to extract a low-rank *bilinear* pattern:
    - left singular vectors $\Rightarrow$ loadings over the P_1 dimension (e.g. countries),
    - right singular vectors $\Rightarrow$ loadings over the P_2 dimension (e.g. indicators).
  - This is the matrix analogue of the targeting step in the standard TPRF.
- **Step 2: Tensor factors preserving matrix structure**
  - Instead of stacking everything into a single long vector, we keep the data in matrix form and extract factors directly as *matrix factors*. This preserves:
    - the cross-country structure,
    - the cross-indicator structure.
  - This tensor formulation avoids losing information about the two-dimensional geometry of the predictors.
- **Step 3: Mixed-frequency forecasting via U–MIDAS**

[7] This structure can be represented as a tensor of dimension $T \times P_1 \times P_2$.

– Tensor factors in Step 2 are naturally estimated at monthly frequency directly from the matrices. At this stage, the tensor can be vectorized for convenience in the regression step.

– For each quarter, we organise the monthly factors into three components

$$(F_\tau^{(1)}, F_\tau^{(2)}, F_\tau^{(3)}),$$

corresponding to the three months of quarter τ.

– These monthly tensor factors enter a U–MIDAS regression for the quarterly target (e.g. GDP), preserving the mixed-frequency nature:

  * the regressors retain their monthly information content,
  * the target is quarterly and linked through MIDAS lags.

Through this approach, we respect the tensor (matrix) structure of the data, handle missingness from ragged edges and COVID masking in a unified way, and, most importantly, retain the proxy-driven targeting philosophy of the original TPRF.

3.0.1 Step 1 of the Tensor 3PRF: Single Proxy and Multiple Proxies

Step 1 of the Tensor Mixed-Frequency Three-Pass Regression Filter (T-3PRF) is formalised in this section, beginning with the simplest case of a single proxy ($L = 1$) and then extend the formulation to an arbitrary number of proxies ($L > 1$). Throughout, quarterly observations are indexed by $\tau = 1, \dots, T_q$, and each quarterly observation of the mixed-frequency dataset is a matrix

$$X_\tau \in \mathbb{R}^{P_1 \times P_2},$$

where P_1 denotes the number of cross-sectional units (e.g. countries) and P_2 the number of variables per unit.

Case $L = 1$: Regression on a single proxy

Let the proxy be a scalar time series $Z_\tau \in \mathbb{R}$. Step 1 of the T–3PRF specifies the matrix regression model

$$X_\tau = Z_\tau AB^\top + U_\tau, \qquad A \in \mathbb{R}^{P_1 \times r}, \quad B \in \mathbb{R}^{P_2 \times r}, \tag{1}$$

where $AB^\top$ is a rank-r coefficient matrix. The parameters (A, B) are obtained by solving the least–squares problem

$$(A, B) = \arg \min_{\mathrm{rank}(AB^\top) \le r} \sum_{\tau=1}^{T_q} \|X_\tau - Z_\tau AB^\top\|_F^2. \tag{2}$$

The objective (2) expands as

$$\sum_\tau \|X_\tau\|_F^2 - 2\sum_\tau Z_\tau \langle X_\tau, AB^\top \rangle + \sum_\tau Z_\tau^2 \|AB^\top\|_F^2.$$

Differentiating with respect to $AB^\top$ yields the normal equation

$$\sum_{\tau=1}^{T_q} Z_\tau X_\tau = \Big(\sum_{\tau=1}^{T_q} Z_\tau^2\Big) AB^\top. \tag{3}$$

Define the weighted moments

$$M \;\equiv\; \sum_{\tau=1}^{T_q} Z_\tau X_\tau, \qquad S \;\equiv\; \sum_{\tau=1}^{T_q} Z_\tau^2.$$

Then the least–squares solution satisfies

$$AB^\top = S^{-1} M. \tag{4}$$

Since multiplying a matrix by a scalar does not affect its rank-r structure, the entire information about the targeted component $AB^\top$ is contained in

$$M = \sum_{\tau=1}^{T_q} Z_\tau X_\tau \in \mathbb{R}^{P_1 \times P_2}. \tag{5}$$

Thus, *M is a sufficient statistic for estimating the rank-r targeted pattern.*

We now have a connection with low-rank approximation since the constrained problem (2) is equivalent to

$$\min_{\operatorname{rank}(C) \le r} \|M - C\|_F^2, \qquad C = AB^\top S, \tag{6}$$

because (1) implies that the least–squares fit depends on X_τ only through their aggregated moment M. Therefore, Step 1 reduces to finding the *best rank-r approximation* of M.

By the Eckart–Young–Mirsky theorem, the solution is given by the truncated SVD:

$$M = U \Sigma V^\top, \qquad A = U_{[:,1:r]}, \qquad B = V_{[:,1:r]}. \tag{7}$$

The SVD provides the unique (up to rotation) minimiser of (6) and hence of the original regression problem (2). In other words, performing least squares under a rank constraint is *mathematically equivalent to computing the best rank-r SVD approximation of M*, from where we can finally estimate $AB^\top$

Case $L > 1$: Regression on multiple proxies

Let the proxy vector at quarter τ be

$$z_\tau = \begin{pmatrix} Z_\tau^{(0)} \\ \vdots \\ Z_\tau^{(L-1)} \end{pmatrix} \in \mathbb{R}^L,$$

and collect all proxies across time into

$$Z = \begin{pmatrix} z_1^\top \\ \vdots \\ z_{T_q}^\top \end{pmatrix} \in \mathbb{R}^{T_q \times L}.$$

The Step 1 regression generalises to

$$X_\tau = \sum_{\ell=0}^{L-1} Z_\tau^{(\ell)} A^{(\ell)} B^{(\ell)\top} + U_\tau, \qquad A^{(\ell)} \in \mathbb{R}^{P_1 \times r},\ B^{(\ell)} \in \mathbb{R}^{P_2 \times r}. \tag{1}$$

Each proxy $Z^{(\ell)}$ targets its own rank-r bilinear pattern $A^{(\ell)} B^{(\ell)\top}$, in direct analogy with the multiple-proxy structure of the classical 3PRF.

The least-squares criterion is

$$\min_{\{A^{(\ell)}, B^{(\ell)}\}} \sum_{\tau=1}^{T_q} \left\| X_\tau - \sum_{\ell=0}^{L-1} Z_\tau^{(\ell)} A^{(\ell)} B^{(\ell)\top} \right\|_F^2. \tag{2}$$

First we need to specify the partial residual equation. When the parameters $\{A^{(j)}, B^{(j)}\}_{j \neq \ell}$ are held fixed, the subproblem for proxy ℓ reduces to a single-regressor matrix regression. Define the partial residual

$$R_\tau^{(\ell)} = X_\tau - \sum_{j \neq \ell} Z_\tau^{(j)} A^{(j)} B^{(j)\top}.$$

Then the subproblem satisfies

$$R_\tau^{(\ell)} = Z_\tau^{(\ell)} A^{(\ell)} B^{(\ell)\top} + U_\tau^{(\ell)}. \tag{*}$$

The normal equation is identical to the previous case. By solving $(*)$ using least squares we gain the normal equation

$$\sum_{\tau=1}^{T_q} Z_\tau^{(\ell)} R_\tau^{(\ell)} = \left(\sum_{\tau=1}^{T_q} \big(Z_\tau^{(\ell)}\big)^2 \right) A^{(\ell)} B^{(\ell)\top}.$$

Define the proxy-specific weighted moment

$$M^{(\ell)} \equiv \sum_{\tau=1}^{T_q} Z_\tau^{(\ell)} R_\tau^{(\ell)} \in \mathbb{R}^{P_1 \times P_2}, \tag{3}$$

and the scalar weight

$$S^{(\ell)} \equiv \sum_{\tau=1}^{T_q} \big(Z_\tau^{(\ell)}\big)^2.$$

Then the least-squares solution must satisfy

$$A^{(\ell)} B^{(\ell)\top} = \big(S^{(\ell)}\big)^{-1} M^{(\ell)}.$$

Since multiplication by the scalar $(S^{(\ell)})^{-1}$ does not affect the rank-r structure, the matrix $M^{(\ell)}$ is a *sufficient statistic* for the bilinear pattern targeted by proxy ℓ.

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Again, the connection with low-rank approximation is straightforward. The constrained least-squares problem for proxy ℓ is equivalent to

$$\min_{\text{rank}(C)\leq r} \|M^{(\ell)} - C\|_F^2, \qquad C = S^{(\ell)}A^{(\ell)}B^{(\ell)\top}.$$

Thus, for each proxy, Step 1 reduces to computing the best rank-r approximation of the weighted moment matrix $M^{(\ell)}$. By the Eckart–Young–Mirsky theorem, the optimal solution is its truncated SVD:

$$M^{(\ell)} = U^{(\ell)}\Sigma^{(\ell)}V^{(\ell)\top}, \qquad A^{(\ell)} = U^{(\ell)}_{[:,1:r]}, \qquad B^{(\ell)} = V^{(\ell)}_{[:,1:r]}.$$

Interpretation. Each proxy $Z^{(\ell)}$ extracts a different "targeted" direction in the space of matrix-valued observations. The matrices $A^{(\ell)}$ contains cross-sectional patterns across the P_1 units, while $B^{(\ell)}$ encode cross-variable patterns across the P_2 features. Despite the matrix structure of X_τ, Step 1 remains a *time-series regression*, where the truncated SVD provides the best rank-r bilinear approximation to the proxy-weighted moment matrix $M^{(\ell)}$.

3.0.2 Step 2: Estimation of Monthly Tensor Factors

Let $X_t \in \mathbb{R}^{P_1\times P_2}$ denote the high-frequency (monthly) observation of the matrix-valued dataset at time t, and let $\{A^{(\ell)}, B^{(\ell)}\}_{\ell=1}^L$ be the loading matrices obtained in Step 1 from the L targeted proxies. For each proxy ℓ, the tensor regression model takes the form

$$X_t = A^{(\ell)}F_t^{(\ell)}B^{(\ell)\top} + U_t^{(\ell)}, \qquad A^{(\ell)} \in \mathbb{R}^{P_1\times r_1}, \quad B^{(\ell)} \in \mathbb{R}^{P_2\times r_2}, \tag{4}$$

where $F_t^{(\ell)} \in \mathbb{R}^{r_1\times r_2}$ is the tensor factor associated with proxy ℓ at time t, and $U_t^{(\ell)}$ is an idiosyncratic error matrix.

Pass 2 is a cross-sectional tensor regression. In analogy with the classical TPRF of Hepenstrick and Marcellino (2016), the monthly factors are obtained by estimating, for each time period t and for each proxy ℓ, the coefficient matrix $F_t^{(\ell)}$ that minimizes the Frobenius-norm loss:

$$\widehat{F}_t^{(\ell)} = \arg\min_{F\in\mathbb{R}^{r_1\times r_2}} \left\|X_t - A^{(\ell)}FB^{(\ell)\top}\right\|_F^2. \tag{5}$$

Problem (5) admits a closed-form solution, given by a doubly-projected bilinear least-squares estimator:

$$\widehat{F}_t^{(\ell)} = \left(A^{(\ell)\top}A^{(\ell)}\right)^{-1}A^{(\ell)\top}X_tB^{(\ell)}\left(B^{(\ell)\top}B^{(\ell)}\right)^{-1}. \tag{6}$$

This expression is the tensor analogue of the cross-sectional regression used in the vector-valued TPRF. The estimator performs a left-projection of X_t onto the row-space of $A^{(\ell)}$ and a right-projection onto the column-space of $B^{(\ell)}$, thus producing an $r_1 \times r_2$ matrix of monthly latent factors.

For each time t, the full set of tensor factors is composed by Stacking across proxies and it is represented by:

$$\widehat{F}_t = \{\widehat{F}_t^{(1)}, \widehat{F}_t^{(2)}, \ldots, \widehat{F}_t^{(L)}\}.$$

In subsequent steps of the Mixed-Frequency TPRF, these matrices can be vectorised and concatenated as $\mathrm{vec}(\widehat{F}_t^{(\ell)}) \in \mathbb{R}^{r_1 r_2}$ for use in the MIDAS regression of quarterly GDP. Indeed, after this step we have achieved the goal of extracting factors and loadings from a matrix, that was our purpose from the beginning.

3.0.3 Step 3: Quarterly U-MIDAS Regression and Real-Time Nowcasting

Let y_τ denote the quarterly target variable (e.g., GDP growth), observed at quarterly dates $\tau = 1, \ldots, T_q$. From Step 2, for each proxy $\ell = 1, \ldots, L$ and for each month $m = 1, 2, 3$ within quarter τ, we obtain a matrix-valued monthly factor $F_{m,\tau}^{(\ell)} \in \mathbb{R}^{k_1 \times k_2}$. In order to use these components in a linear quarterly regression, we apply the vectorisation operator:

$$f_{m,\tau}^{(\ell)} \equiv \mathrm{vec}\left(F_{m,\tau}^{(\ell)}\right) \in \mathbb{R}^{k_1 k_2}.$$

To onstruct the MIDAS regressors, let K denote the maximum number of quarterly MIDAS lags. For each quarter τ, the full regressor vector is formed by stacking all vectorised monthly tensor factors and their lags:

$$X_\tau = \left[f_{1,\tau}^{(1)}, f_{2,\tau}^{(1)}, f_{3,\tau}^{(1)}, \ldots, f_{1,\tau}^{(L)}, f_{2,\tau}^{(L)}, f_{3,\tau}^{(L)}, \; f_{1,\tau-1}^{(1)}, \ldots, f_{3,\tau-K+1}^{(L)}\right].$$

Hence, $X_\tau \in \mathbb{R}^{3LK \cdot (k_1 k_2)}$. [8]

3.0.4 Nowcasting Structure

From this point onwards it is possivble to run two practical exercises: forecasting and nowcasting.

U-MIDAS forecasting regression. Following the classical TPRF logic of Hepenstrick and Marcellino (2016), the quarterly forecasting step is performed using the *one-step-ahead* regression

$$y_{\tau+1} = \beta_0 + X_\tau^\top \beta + \varepsilon_{\tau+1}, \qquad \tau = K, \ldots, T_q - 1, \tag{7}$$

where $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^{3LK(k_1 k_2)}$ are estimated by ordinary least squares. Equation (7) is the tensor-valued analogue of the U-MIDAS regression used in the vector-based Mixed-Frequency TPRF.

Real-time nowcasting In real time, the quarter $\tau + 1$ is observed only partially. Let $M_\tau \in \{1, 2, 3\}$ denote the number of monthly observations currently available for quarter $\tau + 1$. For each proxy ℓ, the

[8] Although the dimension increases with the number of proxies, lags and tensor modes, macroeconomic datasets typically exhibit very low intrinsic rank. In practice, L is small, quarterly lags rarely exceed $K = 1$ or $K = 2$, and tensor ranks $(k_1 k_2)$ are extremely modest, preventing any explosion in the size of the regressor.

corresponding available factors are $F^{(\ell)}_{m,\tau+1}$ for $m = 1,\dots,M_\tau$. We construct the truncated regressor vector

$$X^{(\mathrm{rt})}_{\tau+1} = \left[f^{(1)}_{1,\tau+1},\dots,f^{(1)}_{M_\tau,\tau+1},\ \dots,\ f^{(L)}_{1,\tau+1},\dots,f^{(L)}_{M_\tau,\tau+1}\right],$$

which includes only the information available in real time.

The real-time nowcast of $y_{\tau+1}$ is then given by

$$\widehat{y}^{(\mathrm{rt})}_{\tau+1} = \beta_0 + \left(X^{(\mathrm{rt})}_{\tau+1}\right)^\top \beta, \tag{8}$$

where the unavailable factor blocks (months $m > M_\tau$) are omitted from the regressor. Equation (8) yields three nested nowcasts:

- **first-month nowcast** ($M_\tau = 1$): uses only $f^{(\ell)}_{1,\tau+1}$;
- **second-month nowcast** ($M_\tau = 2$): uses $f^{(\ell)}_{1,\tau+1}$ and $f^{(\ell)}_{2,\tau+1}$;
- **full-quarter nowcast** ($M_\tau = 3$): uses all three monthly factor blocks.

To summarise, Step 3 extends the classical U-MIDAS forecasting regression to the tensor-valued factor representation obtained in Steps 1 and 2, while maintaining the real-time structure of the Mixed-Frequency TPRF.

3.0.5 EM Algorithm under a Tensor-Valued Predictor Structure

In the mixed-frequency setting, each monthly predictor is a matrix

$$X_t \in \mathbb{R}^{P_1 \times P_2}, \qquad t = 1,\dots,T,$$

typically featuring arbitrary missing entries induced by ragged edges, different publication calendars, or variables masked during the COVID period. Although the Tensor-TPRF model preserves a bilinear structure for the *targeted regression* step, the EM algorithm used to impute the missing predictor data is applied in the *vectorized* domain, following the standard procedure for static-factor models.

Vectorization. We define the vector representation

$$x_t = \mathrm{vec}(X_t) \in \mathbb{R}^N, \qquad N = P_1 P_2,$$

obtained by stacking the columns of X_t. Collecting all observations yields the $T \times N$ panel

$$X_{\mathrm{vec}} = \begin{pmatrix} x_1^\top \\ \vdots \\ x_T^\top \end{pmatrix}.$$

Vectorization does not impose any structural constraint nor does it interfere with the bilinear representation used later in the T-TPRF. Its purpose is simply to map matrix-valued predictors into the N-dimensional setting required by the classical EM algorithm for static-factor models.

Selection matrices. For each series $i = 1, \dots, N$, let A_i denote the binary selection matrix that extracts the observed entries of $x_{i,t}$:

$$x_{i,\mathrm{obs}} = A_i x_i, \qquad A_i \in \{0,1\}^{m_i \times T}.$$

This representation accommodates arbitrary missing patterns, including mixed-frequency observations and ragged edges.

E-step. Given factor estimates $\widehat{F}^{(j-1)}$ and loadings $\widehat{\Phi}^{(j-1)}$, the completed value of series i at iteration j is

$$\widehat{x}_i^{(j)} = \widehat{F}^{(j-1)} \widehat{\Phi}_i^{(j-1)} + A_i (A_i' A_i)^{-1} \left(x_{i,\mathrm{obs}} - A_i \widehat{F}^{(j-1)} \widehat{\Phi}_i^{(j-1)} \right).$$

This formula replaces missing entries with their projection onto the factor space, adjusted to match exactly the observed data.

M-step (PCA update). Once the completed dataset

$$\widehat{X}^{(j)} = (\widehat{x}_1^{(j)}, \dots, \widehat{x}_N^{(j)})$$

is available, the factors are updated via principal components:

$$\widehat{F}^{(j)} \;=\; \mathrm{PC}_r\left(\widehat{X}^{(j)}\right), \qquad \widehat{\Phi}^{(j)} \;=\; \left(\widehat{F}^{(j)}\right) \widehat{X}^{(j)},$$

where $\mathrm{PC}_r(\cdot)$ denotes the rank-r principal components.

Reconstruction of tensor-valued predictors. Finally, the completed predictors are reshaped back to their tensor form:

$$\widehat{X}_t = \mathrm{mat}(\widehat{x}_t,\, P_1,\, P_2), \qquad t = 1, \dots, T.$$

The Tensor-TPRF estimation then operates on the reconstructed matrices $\widehat{X}_t$ in Step 1, where the bilinear low-rank constraints are imposed through the truncated SVD of the proxy-weighted moment matrices $M^{(\ell)}$.

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