Theoretical considerations regarding the time state and $\omega^2 t$

Lukas Dobler

March 21, 2025

Let us calculate the integral I_T of $\omega^2 t$ for the time T, which can be approximated as sum assuming a time discretizations.

$$I_T = \int_{t=0}^{T} \omega_t^2 \, dt \approx \sum_{t=0}^{T} \omega_t^2 \Delta t \tag{1}$$

ω_t is the angular velocity at a given time t and can be described as following

$$\omega_t = \begin{cases} a \cdot t, & 0 \leq t \leq T_1 \\ \omega_c, & T_1 < t \leq T \end{cases} \tag{2}$$

T_1 is the point in time, when the acceleration phase ends and constant speed w_c is reached. Therefore Eq. (1) can be rewritten by separating the time of constant rotor speed into its own sum.

$$\begin{aligned} I_T &= \int_{t=0}^{T_1} \omega_t^2 \, dt + \int_{t=T_1}^{T} \omega_c^2 \, dt \\ &\approx \sum_{t=0}^{T_1} \omega_t^2 \Delta t + \sum_{t=T_1+\Delta t}^{T} \omega_c^2 \Delta t \end{aligned} \tag{3}$$

By evaluating the respective summand for the constant speed phase. It can be shown that they are equal.

$$\int_{t=T_1}^{T} \omega_c^2 \, dt = \omega_c \int_{t=T_1}^{T} dt = \omega_c^2 (T - T_1) \tag{4}$$

This applies especially for the case, that T is equal to the time of the first scan T_2.

Eq. (4) is for special importance as it matches the calculation of `UltraScan`. The exact solution is equal to the approximation and therefore the problem has to be in the first summand of Eq. (3).

The integral form evaluates as following

$$\int_{t=0}^{T_1} \omega_t^2 \, dt = \int_{t=0}^{T_1} (a \cdot t)^2 \, dt = \frac{a^2 T_1^3}{3} \overset{a \cdot T_1 = \omega_c}{=} \frac{\omega_c^2 T_1}{3} \tag{5}$$

This results differs from the prior March 2025 used equations in the `US_AstfemMath::low_acceleration`, which used $I_1 = \frac{\omega_c T_1}{4}$. Using the debug output of `us_astfem_sim` numbers with Eq. (5) results in the calculation of the correct acceleration rate.

$$T_1 = \frac{3}{2} \left(T_2 - \frac{I_2}{w_c^2} \right) = \frac{3}{2} \left(13 - \frac{1.10978\text{E}{+}06}{175459} \right) = 10.012 \text{ s}$$

Given the constant speed as 4000 rpm the acceleration rate is 400 $\frac{\text{rpm}}{\text{s}}$.

Using `US_AstfemMath::low_acceleration` the end of acceleration is calculated as

$$T_1 = \frac{4}{3} \left(T_2 - \frac{I_2}{w_c^2} \right) = \frac{4}{3} \left(13 - \frac{1.10978\text{E}{+}06}{175459} \right) = 8.900 \text{ s}$$

Resulting in an acceleration rate of 449 $\frac{\text{rpm}}{\text{s}}$, but the acceleration time is rounded to integer and therefore the acceleration rate is returned as 444 $\frac{\text{rpm}}{\text{s}}$, which matches the speed value in the time state file. The calculation was performed with actual experimental data for validation.

In order to calculate I_T or $\omega^2 t$ for a given time $T \geq 0$ for a given acceleration rate $a > 0$ and target speed $w_c > 0$

$$I_T = \frac{a^2}{3} \cdot (\min(T, \, w_c/a))^3 + \omega_c^2 \cdot \max(0, \, T - w_c/a) \tag{6}$$