Weak forms and finite element method in 1d
Problems & solutions

Simona Domesová

April 23, 2019

1 Weak forms in 1d

Problem 1. Consider the following Dirichlet boundary value problem:

$$\begin{cases} -\left(\left(2+\sin x\right)\cdot u'\left(x\right)\right)'+3\cdot u'\left(x\right) &= x^2 \qquad \text{in } \Omega=\left(K,L\right) \\ u\left(K\right) &= 4 \\ u\left(L\right) &= 5 \end{cases},$$

where $K\in\mathbb{R}$, $L\in\mathbb{R}$, $K<L$. Derive the weak form.

Solution 1. Consider the space $U=H^1\left(\Omega\right)$.
For all test functions $v\in V=\left\{w\in U: w\left(K\right)=w\left(L\right)=0\right\}$:

$$\underbrace{\int_K^L -\left(\left(2+\sin x\right)\cdot u'\left(x\right)\right)'\cdot v\left(x\right)\mathrm{d}x}_{\downarrow\ \text{per partes}\ \uparrow}+\int_K^L 3\cdot u'\left(x\right)\cdot v\left(x\right)\mathrm{d}x = \int_K^L x^2\cdot v\left(x\right)\mathrm{d}x$$

$$\overbrace{\int_K^L \left(2+\sin x\right)\cdot u'\left(x\right)\cdot v'\left(x\right)\mathrm{d}x-\left[\left(2+\sin x\right)\cdot u'\left(x\right)\cdot v\left(x\right)\right]_K^L}+\int_K^L 3\cdot u'\left(x\right)\cdot v\left(x\right)\mathrm{d}x = \int_K^L x^2\cdot v\left(x\right)\mathrm{d}x$$

Since $v\in V$, $v\left(K\right)=v\left(L\right)=0$ holds.

$$\underbrace{\int_K^L \left(2+\sin x\right)\cdot u'\left(x\right)\cdot v'\left(x\right)\mathrm{d}x+\int_K^L 3\cdot u'\left(x\right)\cdot v\left(x\right)\mathrm{d}x}_{a(u,v)} = \underbrace{\int_K^L x^2\cdot v\left(x\right)\mathrm{d}x}_{b(v)}$$

The weak solution of the boundary value problem is a function u that fulfills:

$$\begin{cases} u\in U_D=\left\{w\in U: w\left(K\right)=4\wedge w\left(L\right)=5\right\} \\ a\left(u,v\right)=b\left(v\right)\quad\forall v\in V \end{cases}.$$

Problem 2. Consider the following Neumann boundary value problem:

$$\begin{cases} -u''\left(x\right)+u\left(x\right) &= 0 \qquad \text{in } \Omega=\left(3,8\right) \\ -u'\left(3\right) &= 5 \\ u\left(8\right) &= 0 \end{cases}.$$

Derive the weak form.

Solution 2. Consider the space $U=H^1\left(\Omega\right)$.

For all test functions $v \in V = \{w \in U : w(8) = 0\}$:

$$\underbrace{\int_3^8 -u''(x) \cdot v(x)\,\mathrm{d}x}_{\downarrow \text{per partes} \uparrow} + \int_3^8 u(x) \cdot v(x)\,\mathrm{d}x \quad = \quad \int_3^8 0 \cdot v(x)\,\mathrm{d}x$$

$$\overbrace{\int_3^8 u'(x) \cdot v'(x)\,\mathrm{d}x - [u'(x) \cdot v(x)]_3^8} + \int_3^8 u(x) \cdot v(x)\,\mathrm{d}x \quad = \quad 0$$

$$\int_3^8 u'(x) \cdot v'(x)\,\mathrm{d}x - \left(u'(8) \cdot \underbrace{v(8)}_{=0} \underbrace{-u'(3)}_{=5} \cdot v(3) \right) + \int_3^8 u(x) \cdot v(x)\,\mathrm{d}x \quad = \quad 0$$

$$\underbrace{\int_3^8 u'(x) \cdot v'(x)\,\mathrm{d}x + \int_3^8 u(x) \cdot v(x)\,\mathrm{d}x}_{a(u,v)} \quad = \quad \underbrace{5 \cdot v(3)}_{b(v)}$$

The weak solution of the boundary value problem is a function u that fulfills:

$$\begin{cases} u \in U_D = V \\ a(u,v) = b(v) \quad \forall v \in V \end{cases} .$$

Problem 3. Consider the following Newton boundary value problem:

$$\begin{cases} -3 \cdot u''(x) & = & f(x) & \text{v } \Omega = (0,L) \\ u(0) & = & 1 & \\ -3 \cdot u'(L) & = & \alpha(u(L) - 7) & \end{cases} ,$$

where $L \in \mathbb{R}^+$ a $f \in L_2(\Omega)$. Derive the weak form.

Solution 3. Consider the space $U = H^1(\Omega)$.

For all test functions $v \in V = \{w \in U : w(0) = 0\}$:

$$\underbrace{\int_0^L -3 \cdot u''(x) \cdot v(x)\,\mathrm{d}x}_{\downarrow \text{per partes} \uparrow} \quad = \quad \int_0^L f(x) \cdot v(x)\,\mathrm{d}x$$

$$\overbrace{\int_0^L 3 \cdot u'(x) \cdot v'(x)\,\mathrm{d}x - [3u'(x) \cdot v(x)]_0^L} \quad = \quad \int_0^L f(x) \cdot v(x)\,\mathrm{d}x$$

$$\int_0^L 3 \cdot u'(x) \cdot v'(x)\,\mathrm{d}x - \left(\underbrace{3 \cdot u'(L)}_{=-\alpha(u(L)-7)} \cdot v(L) - u'(0) \cdot \underbrace{v(0)}_{=0} \right) \quad = \quad \int_0^L f(x) \cdot v(x)\,\mathrm{d}x$$

$$\int_0^L 3 \cdot u'(x) \cdot v'(x)\,\mathrm{d}x + (\alpha(u(L) - 7) \cdot v(L)) \quad = \quad \int_0^L f(x) \cdot v(x)\,\mathrm{d}x$$

$$\underbrace{\int_0^L 3 \cdot u'(x) \cdot v'(x)\,\mathrm{d}x + \alpha \cdot u(L) \cdot v(L)}_{a(u,v)} \quad = \quad \underbrace{\int_0^L f(x) \cdot v(x)\,\mathrm{d}x + 7\alpha \cdot v(L)}_{b(v)}$$

The weak solution of the boundary value problem is a function u that fulfills:

$$\begin{cases} u \in U_D = \{w \in U : w(0) = 1\} \\ a(u,v) = b(v) \quad \forall v \in V \end{cases} .$$

2 Finite element method in 1d

Problem 4 (Continuation of problem 1)**.** Consider the space $U = H^1(\Omega)$, where $\Omega = (K, L)$, test space $V = \{w \in U : w(K) = w(L) = 0\}$ and the following weak form:

$$\begin{cases} u \in U_D = \{w \in U : w(K) = 4 \wedge w(L) = 5\} \\ \underbrace{\int_K^L (2 + \sin x) \cdot u'(x) \cdot v'(x)\,\mathrm{d}x + \int_K^L 3 \cdot u'(x) \cdot v(x)\,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_K^L x^2 \cdot v(x)\,\mathrm{d}x}_{b(v)} \quad \forall v \in V \end{cases}.$$

Calculate the local FEM matrix A_E and the local right-hand-side vector $\mathbf{b}_E$ for the segment $E = \langle \pi, 2\pi \rangle$. Use linear finite elements. Assume $K < \pi$ and $L > 2\pi$.

Solution 4. We can simply derive that linear finite elements have the following forms and derivatives at the segment E (i.e. we consider restriction to E):

$$\begin{aligned} \phi_1(x) &= \frac{x - 2\pi}{-\pi},\ \phi_1'(x) = -\frac{1}{\pi} \\ \phi_2(x) &= \frac{x - \pi}{\pi},\ \phi_2'(x) = \frac{1}{\pi} \end{aligned}$$

The local matrix has the form:

$$A_E = \begin{bmatrix} a_E(\phi_1, \phi_1) & a_E(\phi_2, \phi_1) \\ a_E(\phi_1, \phi_2) & a_E(\phi_2, \phi_2) \end{bmatrix},$$

where

$$\begin{aligned} a_E(\phi_1, \phi_1) &= \int_\pi^{2\pi} (2 + \sin x) \cdot (\phi_1'(x))^2\,\mathrm{d}x + \int_\pi^{2\pi} 3 \cdot \phi_1'(x) \cdot \phi_1(x)\,\mathrm{d}x = \\ &= \frac{1}{\pi^2} \int_\pi^{2\pi} (2 + \sin x)\,\mathrm{d}x - \frac{3}{\pi} \underbrace{\int_\pi^{2\pi} \phi_1(x)\,\mathrm{d}x}_{=\pi/2\,(\text{area of}\triangle)} = \frac{1}{\pi^2} [2x - \cos x]_\pi^{2\pi} - \frac{3}{2} = \\ &= \frac{1}{\pi^2} \underbrace{(4\pi - \cos(2\pi) - 2\pi + \cos(\pi))}_{=2\pi-2} - \frac{3}{2} \\ a_E(\phi_1, \phi_2) &= \int_\pi^{2\pi} (2 + \sin x) \cdot \phi_1'(x) \cdot \phi_2'(x)\,\mathrm{d}x + \int_\pi^{2\pi} 3 \cdot \phi_1'(x) \cdot \phi_2(x)\,\mathrm{d}x = \\ &= -\frac{1}{\pi^2}(2\pi - 2) - \frac{3}{2} \\ a_E(\phi_2, \phi_1) &= \int_\pi^{2\pi} (2 + \sin x) \cdot \phi_2'(x) \cdot \phi_1'(x)\,\mathrm{d}x + \int_\pi^{2\pi} 3 \cdot \phi_2'(x) \cdot \phi_1(x)\,\mathrm{d}x = \\ &= -\frac{1}{\pi^2}(2\pi - 2) + \frac{3}{2} \\ a_E(\phi_2, \phi_2) &= \int_\pi^{2\pi} (2 + \sin x) \cdot (\phi_2'(x))^2\,\mathrm{d}x + \int_\pi^{2\pi} 3 \cdot \phi_2'(x) \cdot \phi_2(x)\,\mathrm{d}x = \\ &= \frac{1}{\pi^2}(2\pi - 2) + \frac{3}{2} \end{aligned}$$

The local right-hand-side vector has the form

$$\mathbf{b}_E = \begin{bmatrix} b_E(\phi_1) \\ b_E(\phi_2) \end{bmatrix},$$

where

$$
\begin{aligned}
b_E(\phi_1) &= \int_{\pi}^{2\pi} x^2 \cdot \phi_1(x)\,\mathrm{d}x = \int_{\pi}^{2\pi} x^2 \cdot \left(\frac{x-2\pi}{-\pi}\right)\mathrm{d}x = \int_{\pi}^{2\pi} -\frac{x^3}{\pi} + 2x^2\mathrm{d}x = \\
&= \left[-\frac{x^4}{4\pi} + 2\frac{x^3}{3}\right]_{\pi}^{2\pi} = -\frac{16\pi^4}{4\pi} + \frac{16\pi^3}{3} + \frac{\pi^4}{4\pi} - \frac{2\pi^3}{3} = \frac{11}{12}\pi^3 \\
b_E(\phi_2) &= \int_{\pi}^{2\pi} x^2 \cdot \phi_2(x)\,\mathrm{d}x = \int_{\pi}^{2\pi} x^2 \cdot \left(\frac{x-\pi}{\pi}\right)\mathrm{d}x = \int_{\pi}^{2\pi} \frac{x^3}{\pi} - x^2\mathrm{d}x = \\
&= \left[\frac{x^4}{4\pi} - \frac{x^3}{3}\right]_{\pi}^{2\pi} = \frac{16\pi^4}{4\pi} - \frac{8\pi^3}{3} - \frac{\pi^4}{4\pi} + \frac{\pi^3}{3} = \frac{17}{12}\pi^3
\end{aligned}
$$

Problem 5 (Continuation of problem 2)**.** Consider the space $U = H^1(\Omega)$, where $\Omega = (3,8)$, test space $V = \{w \in U : w(8) = 0\}$ and the following weak form:

$$
\begin{cases}
u \in V \\
\underbrace{\int_3^8 u'(x) \cdot v'(x)\,\mathrm{d}x + \int_3^8 u(x) \cdot v(x)\,\mathrm{d}x}_{a(u,v)} = \underbrace{5 \cdot v(3)}_{b(v)} \quad \forall v \in V
\end{cases}.
$$

Calculate the extended FEM matrix A and the extended right-hand-side vector $\mathbf{b}$ corresponding to the discretization $\mathcal{T} = \{\langle 3,5\rangle, \langle 5,6\rangle, \langle 6,8\rangle\}$. (Extended = contains also rows/columns corresponding to nodes with Dirichlet b. c.) Use linear finite elements.

Solution 5. First, we can find $a_E(\phi_i, \phi_j)$ for a general segment E. Note that in the case of this bilinear form, we can use substitution to the segment $E = \langle 0, h\rangle$ (see the geometric interpretation of the integration). At this segment, linear finite elements have the following forms and derivatives:

$$
\begin{aligned}
\phi_1(x) &= 1 - \frac{x}{h},\ \phi_1'(x) = -\frac{1}{h} \\
\phi_2(x) &= \frac{x}{h},\ \phi_2'(x) = \frac{1}{h}
\end{aligned}
$$

The calculations can be further simplified by noting that $a_E(\phi_1, \phi_1) = a_E(\phi_2, \phi_2)$ (again use the geometric interpretation of the integration).

$$
\begin{aligned}
a_E(\phi_1,\phi_1) = a_E(\phi_2,\phi_2) &= \int_0^h (\phi_2'(x))^2\,\mathrm{d}x + \int_0^h (\phi_2(x))^2\,\mathrm{d}x = \\
&= \int_0^h \frac{1}{h^2}\mathrm{d}x + \int_0^h \frac{x^2}{h^2}\mathrm{d}x = \frac{1}{h} + \left[\frac{x^3}{3h^2}\right]_0^h = \frac{1}{h} + \frac{h}{3} \\
a_E(\phi_1,\phi_2) = a_E(\phi_2,\phi_1) &= \int_0^h \phi_1'(x) \cdot \phi_2'(x)\,\mathrm{d}x + \int_0^h \phi_1(x) \cdot \phi_2(x)\,\mathrm{d}x = \\
&= \int_0^h -\frac{1}{h^2}\mathrm{d}x + \int_0^h \frac{x}{h} - \frac{x^2}{h^2}\mathrm{d}x = -\frac{1}{h} + \left[\frac{x^2}{2h} - \frac{x^3}{3h^2}\right]_0^h = -\frac{1}{h} + \frac{h}{6}
\end{aligned}
$$

The local matrices are obtained by substituting the lengths of the corresponding segments:

$$
\begin{aligned}
A_{\langle 3,5\rangle} = A_{\langle 6,8\rangle} &= \begin{bmatrix} \frac{1}{2} + \frac{2}{3} & -\frac{1}{2} + \frac{2}{6} \\ -\frac{1}{2} + \frac{2}{6} & \frac{1}{2} + \frac{2}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \\
A_{\langle 5,6\rangle} &= \begin{bmatrix} \frac{1}{1} + \frac{1}{3} & -\frac{1}{1} + \frac{1}{6} \\ -\frac{1}{1} + \frac{1}{6} & \frac{1}{1} + \frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 8 & -5 \\ -5 & 8 \end{bmatrix}
\end{aligned}
$$

The global FEM matrix is constructed using these local matrices:

$$A = \frac{1}{6}\begin{bmatrix} 7 & -1 & 0 & 0 \\ -1 & 7+8 & -5 & 0 \\ 0 & -5 & 8+7 & -1 \\ 0 & 0 & -1 & 7 \end{bmatrix}$$

The construction of local right-hand-side vectors is simple in this case (the only non-zero contribution is given by the value of the Neumann b. c.):

$$\mathbf{b}_{\langle 3,5\rangle} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \mathbf{b}_{\langle 5,6\rangle} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_{\langle 6,8\rangle} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, the global extended right-hand-side vector $\mathbf{b}$ is

$$\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 6 (Continuation of problem 3)**.** Consider the bilinear form

$$a(u,v) = \int_0^L 3 \cdot u'(x) \cdot v'(x) \, \mathrm{d}x + \alpha \cdot u(L) \cdot v(L),$$

where $u, v \in H^1((0,L))$. Calculate the local FEM matrix A_E for the segment $E = \langle K, L\rangle$, $0 < K < L$. Use linear finite elements.

Solution 6. We can simply derive that linear finite elements have the following forms and derivatives at the segment E (i.e. we consider restriction to E):

$$\begin{aligned} \phi_1(x) &= \frac{x-L}{K-L}, \ \phi_1'(x) = \frac{1}{K-L} \\ \phi_2(x) &= \frac{x-K}{L-K}, \ \phi_2'(x) = \frac{1}{L-K} \end{aligned}$$

The local matrix has the form:

$$A_E = \begin{bmatrix} a_E(\phi_1,\phi_1) & a_E(\phi_2,\phi_1) \\ a_E(\phi_1,\phi_2) & a_E(\phi_2,\phi_2) \end{bmatrix},$$

where

$$\begin{aligned} a_E(\phi_1,\phi_1) &= \int_K^L 3 \cdot (\phi_1'(x))^2 \, \mathrm{d}x + \alpha \cdot \underbrace{(\phi_1(L))^2}_{=0} = 3\left(\frac{1}{K-L}\right)^2 (L-K) = \frac{3}{L-K} \\ a_E(\phi_1,\phi_2) = a_E(\phi_2,\phi_1) &= \int_K^L 3 \cdot \phi_1'(x) \cdot \phi_2'(x) \, \mathrm{d}x + \alpha \cdot \underbrace{\phi_1(L)}_{=0} \cdot \underbrace{\phi_2(L)}_{=1} = \frac{3}{K-L} \\ a_E(\phi_2,\phi_2) &= \int_K^L 3 \cdot (\phi_2'(x))^2 \, \mathrm{d}x + \alpha \cdot (\phi_2(L))^2 = \frac{3}{L-K} + \alpha \end{aligned}$$

Problem 7 (Continuation of problem 3)**.** Consider the linear functional

$$b(v) = \int_0^L f(x) \cdot v(x) \, \mathrm{d}x + 7\alpha \cdot v(L),$$

where $v \in H^1((0,L))$ a $f(x) = \cos(x)$. Calculate the local right-hand-side vector $\mathbf{b}_E$ for the segment $E = \langle 0, K\rangle$, $0 < K < L$. Use linear finite elements.

Solution 7. We can simply derive that linear finite elements have the following forms and derivatives at the segment E (i.e. we consider restriction to E):

$$\begin{aligned}\phi_1(x) &= -\frac{x}{K}+1,\ \phi_1'(x)=-\frac{1}{K}\\ \phi_2(x) &= \frac{x}{K},\ \phi_2'(x)=\frac{1}{K}\end{aligned}$$

The local right-hand-side vector has the form:

$$\mathbf{b}_E=\left[\begin{array}{c} b_E(\phi_1)\\ b_E(\phi_2)\end{array}\right],$$

where

$$\begin{aligned}b_E(\phi_1) &= \int_0^K \cos(x)\cdot\phi_1(x)\,\mathrm{d}x+7\alpha\cdot\overbrace{\phi_1(L)}^{=0\,(L\notin E)}=-\frac{1}{K}\overbrace{\int_0^K x\cdot\cos(x)\,\mathrm{d}x}^{[x\cdot\sin(x)]_0^K-\int_0^K 1\cdot\sin(x)\mathrm{d}x}+\int_0^K\cos(x)\,\mathrm{d}x=\\ &= -\frac{1}{K}[x\cdot\sin(x)]_0^K+\frac{1}{K}[-\cos(x)]_0^K+[\sin(x)]_0^K=-\frac{\cos(K)}{K}+\frac{1}{K}\\ b_E(\phi_2) &= \int_0^K\cos(x)\cdot\phi_2(x)\,\mathrm{d}x+7\alpha\cdot\overbrace{\phi_2(L)}^{=0\,(L\notin E)}=\frac{1}{K}\int_0^K x\cdot\cos(x)\,\mathrm{d}x=\\ &= \frac{1}{K}[x\cdot\sin(x)]_0^K-\frac{1}{K}[-\cos(x)]_0^K=\sin(K)+\frac{\cos(K)}{K}-\frac{1}{K}\end{aligned}$$

References

[1] Blaheta, R. Matematické modelování a metoda konečných prvků. 2012. URL: http://mi21.vsb.cz/modul/matematicke-modelovani-metoda-konecnych-prvku-numericke-metody-2