Finite differences method (FDM)

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Finite differences in 1d and resulting linear system

(e.g. linear elasticity, heat conduction, etc.)

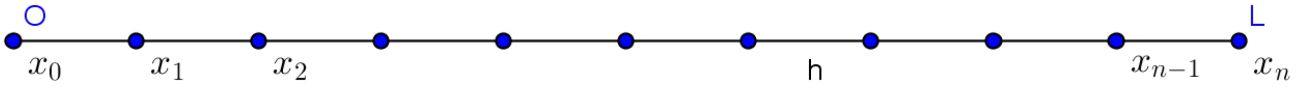
Boundary value problem:

$$\begin{cases} -u''(x) = f(x) & x \in (0, L) \\ u(0) = u(L) = 0 & \text{(Dirichlet boundary conditions)} \end{cases}$$

Approximation of second derivative (can be derived using Taylor series):

$$u''(x) \approx \frac{u(x-h) - 2 \cdot u(x) + u(x+h)}{h^2}$$

Discretization of $(0, L)$ into n sub-intervals of length h (equidistant):



$$f(x_i) \overset{ozn}{=} f_i, \quad u(x_i) \overset{def}{=} u_i$$

$\rightarrow$ linear system ($n-1$ equations, unknowns $u_1, \ldots, u_{n-1}$):

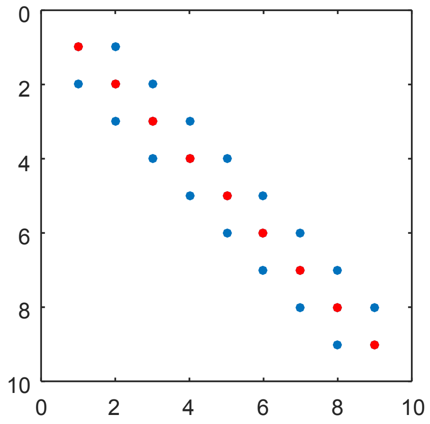
$$-(u_{i-1} - 2 \cdot u_i + u_{i+1}) = h^2 f_i$$

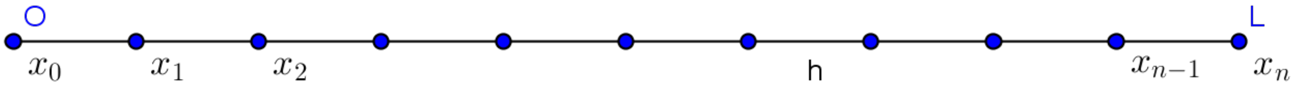
$$-\left(u_{i-1} - 2 \cdot u_i + u_{i+1}\right) = h^2 f_i$$

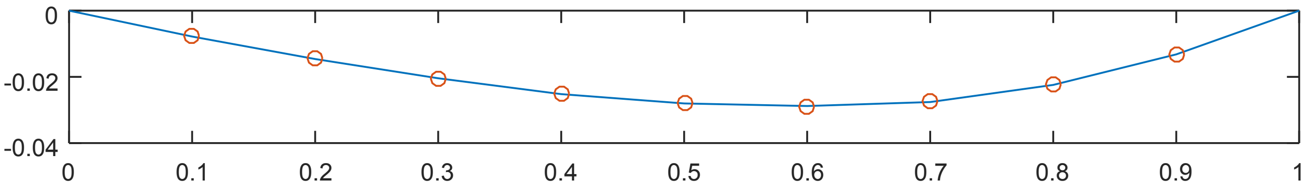
linear system with SPD (symmetric, positive definite) matrix

$$A \cdot u = b$$

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 f_1 + u_0 \\ h^2 f_2 \\ \vdots \\ h^2 f_{n-2} \\ h^2 f_{n-1} + u_n \end{pmatrix}$$







Finite differences in 2d and resulting linear system

Boundary value problem:

$$\begin{cases} -\triangle u\left(x\right) = f\left(x\right) & x \in \Omega = (0, L) \times (0, L) \\ u\left(x, y\right) = 0 & x \in \partial\Omega \end{cases}$$

Approximation of second partial derivatives

$$\frac{\partial^2}{\partial x^2} u\left(x, y\right) \approx \frac{u\left(x-h, y\right) - 2 \cdot u\left(x, y\right) + u\left(x+h, y\right)}{h^2}$$

$$\frac{\partial^2}{\partial y^2} u\left(x, y\right) \approx \frac{u\left(x, y-h\right) - 2 \cdot u\left(x, y\right) + u\left(x, y+h\right)}{h^2}$$

Discretization of $(0, L) \times (0, L)$ into $n \times n$ squares (side h):

$$f\left(x_i, y_j\right) \overset{ozn}{=} f_{i,j}, \quad u\left(x_i, y_j\right) \overset{def}{=} u_{i,j}$$

$\rightarrow$ linear system, $(n-1) \times (n-1)$ equations:

$$-u_{i-1,j} - u_{i,j-1} + 4 \cdot u_{i,j} - u_{i+1,j} - u_{i,j+1} = h^2 f_{i,j}$$

$$A = \begin{pmatrix} B & I & 0 & \cdots & 0 \\ I & B & I & \ddots & \vdots \\ 0 & I & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & 0 & I & B \end{pmatrix}, \text{ where } B = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & -1 & 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix}$$