Mathematical modeling
and
finite element method

Principles of mathematical modeling

mathematical model = mathematical description of a physical process

How to build a mathematical model to a physical process?

- determine physical quantities that characterize this process (state variable, flow variable)
- describe relations between these quantities (simplification of physical rules)

Solution

- analytical
- numerical (approximate)
  - solved in finite precision arithmetic (inaccurate)

State variable and flow variable in several physical processes

Stationary heat conduction in 1 dimension

- physical process = heat conduction in a domain $\Omega = (0, L)$
  - state variable = temperature $u(x)$
  - flow variable = temperature flow $\tau(x)$
- input variables:
  - material properties = thermal conductivity $k(x)$
  - density of heat sources $f(x)$
- relationships:
  - from the conservation law: $\tau'(x) = f(x)$
  - Fourier law: $\tau(x) = -k(x)\, u'(x)$

Flow in porous media (stationary)

- physical process = saturated flow of a fluid through porous media
  - state variable = pressure $p(x)$
  - flow variable = velocity $v(x)$
- input variables:
  - material properties = hydraulic conductivity $k(x)$
  - density of sources $f(x)$
- relationships:
  - from the mass conservation law: $v'(x) = f(x)$
  - Darcy law: $v(x) = -k(x)\,p'(x)$

String deformation

- physical process:
  - state variable = deformation $u(x)$
  - flow variable = stress $v(x)$
- input variables:
  - material properties (including Young's elastic modulus) $k(x)$
  - density of forces $f(x)$
- relationships:
  - from the conservation law: $v'(x) = f(x)$
  - Hooke's law: $v(x) = -k(x)\,u'(x)$

Conservation law

"Difference between the total amount of some physical variable in the domain at time t_1 and at time t_2

equals

total inflow/outflow through boundary plus total sources in the domain."

- state function $u(x,t)$
- total amount of u in the domain at time t_1 ... A
- total amount of u in the domain at time t_2 ... B
- total flow through the boundary ... C
- total sources in the domain ... D

$$B - A = C + D$$

Conservation law in 1 dimension

- domain in 1d: $\langle a, b\rangle \subset \mathbb{R}$
- boundary in 1d: $\{a, b\}$
- state function: $u\,(x, t)$
- flow function: $\phi\,(x, t)$

Derive:

- total amount of u in the domain at time t_1 ... ?
- total amount of u in the domain at time t_2 ... ?
- total flow through the boundary ... ?
- total sources in the domain ... ?

Conservation law in 1 dimension

(integral formulation)

$$\int_a^b u(x,t_2)\,\mathrm{d}x - \int_a^b u(x,t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} \left(\phi(a,t) - \phi(b,t)\right)\mathrm{d}t + \int_{t_1}^{t_2}\int_a^b f(x,t)\,\mathrm{d}x\mathrm{d}t$$

$$\downarrow$$

which requirements?

$$\downarrow$$

$$u_t(x,t) + \phi_x(x,t) = f(x,t)$$

(differential formulation)

Stationary problem:

$$\phi_x(x) = f(x)$$

Analytical solution of several boundary value problems

- these problems will be used as model exaples in numerical experiments
- prepare their analytical solutions, we will compare them with numerical solutions

Problem 1 (diffusion)

- constants: f, k, L, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) = f & \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

- analytical solution?

Problem 2 (material interface)

- constants: f, k_1, k_2, L, M, U, T
- boundary value problem:

$$\begin{cases} -k_1 \cdot u_1''(x) = f & \forall x \in (0, M) \\ -k_2 \cdot u_2''(x) = f & \forall x \in (M, L) \\ u_1(0) = U \\ -k_2 \cdot u_2'(L) = T \\ u_1(M) = u_2(M) \\ k_1 \cdot u_1'(M_+) = k_2 \cdot u_2'(M_-) \end{cases}$$

- analytical solution?

Problem 3 (with reaction)

- constants: k, k_0, L, g, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) + k_0 u(x) = g \quad \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

- analytical solution?

Problem 4 (with convection)

- constants: f, k, L, D, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) + D \cdot u'(x) = f & \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

- analytical solution?

Solution of several boundary value problems using finite differences method

- we will focus on numerical solutions using the finite elements method (FEM); however, numerical solution can also be obtained using a simpler method - finite differences method (FDM)

Problem 1

- constants: f, k, L, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) = f & \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

Solve this boundary value problem using finite differences method (for arbitrary parameters f, k, L, U, T).

Compare the numerical solution and the analytical solution, using different lengths of discretization steps.

Problem 2

- constants: f, k_1, k_2, L, M, U, T
- boundary value problem:

$$\begin{cases} -k_1 \cdot u_1''(x) = f & \forall x \in (0, M) \\ -k_2 \cdot u_2''(x) = f & \forall x \in (M, L) \\ u_1(0) = U \\ -k_2 \cdot u_2'(L) = T \\ u_1(M) = u_2(M) \\ k_1 \cdot u_1'(M_+) = k_2 \cdot u_2'(M_-) \end{cases}$$

Solve this boundary value problem using finite differences method (for arbitrary parameters f, k_1, k_2, L, M, U, T).

Compare the numerical solution and the analytical solution.

Problem 3

- constants: k, k_0, L, g, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) + k_0 u(x) = g & \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

Solve this boundary value problem using finite differences method (for arbitrary parameters k, k_0, L, g, U, T).

Compare the numerical solution and the analytical solution.

Problem 4

- constants: f, k, L, D, U, T
- boundary value problem:

$$\begin{cases} -k \cdot u''(x) + D \cdot u'(x) = f \quad \forall x \in (0, L) \\ u(0) = U \\ -k \cdot u'(L) = T \end{cases}$$

Solve this boundary value problem using finite differences method (for arbitrary boundary conditions and arbitrary parameters f, k, L, D, U, T):

1. using central differences
2. using up-wind (up-stream) differences
3. using down-wind (down-stream) differences