AST1430 Assignment 1

Jessica Campbell

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1 Textbook Questions (not to be handed in)

Work through all problems in Chapter 1 of RL. Consult the solutions (end of book) if you get stuck. Make sure that you are proficient at these.

2 Photosphere

The temperature inside a star goes from a few million kelvins (interior) to practically zero (outside). On Earth ($\tau = 0$), we observe a radiation spectrum that is characterized by a temperature called the effective temperature and is measured at an optical depth of order unity (photosphere).

Part 1

Show that the bolometric flux radiated outward per unit area by a blackbody with temperature T is $F = \pi \int S_\nu d_\nu$ and is σT^4.

Solution

$$\begin{aligned}
F &= \int F_\nu d\nu = \int I_\nu \cos\theta d\Omega = \int_0^\infty I_\nu d\nu \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= \int_0^\infty I_\nu d\nu \int_0^{\pi/2} \frac{1}{2} \sin 2\theta d\theta \int_0^{2\pi} d\phi = \int_0^\infty I_\nu d\nu \left[-\frac{1}{2}\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} [\phi]_0^{2\pi} \\
&= -\frac{1}{4}\int_0^\infty I_\nu d\nu [\cos\pi - \cos 0][2\pi] = \frac{1}{2}\int_0^\infty I_\nu d\nu [1+1][\pi] = \pi \int_0^\infty I_\nu d\nu
\end{aligned}$$

For optically thick media ($\tau \gg 1$), $I_\nu \to S_\nu$. Thus,

$$F = \pi \int_0^\infty S_\nu d\nu.$$

For thermal radiation, $S_\nu \to B_\nu(T)$, so:

$$
\begin{aligned}
F &= \pi \int_0^\infty S_\nu d\nu = \pi \int_0^\infty B_\nu(T) d\nu = \pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_B T) - 1} d\nu \\
&= \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{\exp(h\nu/K_B T) - 1} d\nu.
\end{aligned}
$$

Let $x \equiv (h\nu/k_B T)$ such that $\nu = x(k_B T/h)$. Then,

$$
\begin{aligned}
F &= \frac{2\pi h}{c^2} \int_0^\infty \frac{(x k_B T/h)^3}{\exp(x) - 1} dx = \frac{2\pi h}{c^2} \left(\frac{k_B T}{h} \right)^4 \int_0^\infty \frac{x}{\exp(x) - 1} dx \\
&= \frac{2\pi k_B T^4}{c^2 h^3} \int_0^\infty \frac{x}{\exp(x) - 1} dx = \frac{2\pi k_B T^4}{c^2 h^3} \left[\frac{\pi^4}{15} \right] = \left(\frac{2\pi^5 k_B}{15 c^2 h^3} \right) T^4.
\end{aligned}
$$

Therefore,
$F = \sigma T^4$ where $\sigma \equiv \left(\frac{2\pi^5 k_B}{15c^2 h^3} \right)$ is Boltzmann's constant.

Part 2

In the case of a stellar atmosphere, show that the observed flux $F_\nu(\tau_\nu = 0) = \pi S_\nu(\tau_\nu = 2/3)$ (also called the Eddington-Barbier approximation), namely, the photospheric optical depth is 2/3. We suppress all ν subscript from now on. To accomplish this (in a cheap way), start from the equation of radiative transfer in plane-parallel atmosphere (RL eq. [1.116]). Taylor expand the source function S_τ around small τ, and iteratively solve the above equation to obtain $I(\tau = 0, \mu) = S(\tau = \mu)$. Obtain the corresponding $F(\tau = 0)$.

Solution

The equation of radiative transfer in a plane-parallel atmosphere is as follows:

$$
\mu \frac{\partial I}{\partial \tau} = I - S
$$

which can be rewritten as:

$$
I = S + \mu \frac{\partial I}{\partial \tau}.
$$

Taylor expanding $S(\tau)$ around $\tau \ll 1$:

$$
S(\tau) \approx S_0 + S_1 \tau + S_2 \tau^2.
$$

Using the Taylor expansion of $S(\tau)$ for our initial guess I_0:

$$I_0 \approx S(\tau) = S_0 + S_1\tau + S_2\tau^2.$$

$$\frac{\partial I_0}{\partial \tau} = \frac{\partial}{\partial \tau}(S_0 + S_1\tau + S_2\tau^2) = S_1 + 2S_2\tau.$$

Plugging this back into our radiative transfer equation for our refined guess I_1...

$$I_1 = S(\tau) + \mu\frac{\partial I_0}{\partial \tau} = (S_0 + S_1\tau + S_2\tau^2) + \mu(S_1 + 2S_2\tau) = (S_0 + \mu S_1) + (S_1 + 2\mu S_2)\tau + S_2\tau^2.$$

$$\frac{\partial I_1}{\partial \tau} = \frac{\partial}{\partial \tau}\left[(S_0 + S_1\tau + S_2\tau^2) + \mu(S_1 + 2S_2\tau)\right] = (S_1 + 2\mu S_2) + 2S_2\tau.$$

Plugging this back into our radiative transfer equation for our refined guess I_2...

$$\begin{aligned} I_2 &= S(\tau) + \mu\frac{\partial I_1}{\partial \tau} = (S_0 + S_1\tau + S_2\tau^2) + \mu\left[(S_1 + 2\mu S_2) + 2S_2\tau\right] \\ &= (S_0 + S_1\mu + 2\mu^2 S_2) + (S_1 + 2\mu S_2)\tau + S_2\tau^2. \end{aligned}$$

$$\frac{\partial I_2}{\partial \tau} = \frac{\partial}{\partial \tau}\left[(S_0 + S_1\mu + 2\mu^2 S_2) + (S_1 + 2\mu S_2)\tau + S_2\tau^2\right] = (S_1 + 2\mu S_2) + 2S_2\tau.$$

Plugging this back into our radiative transfer equation for our refined guess I_3...

$$\begin{aligned} I_3 &= S(\tau) + \mu\frac{\partial I_2}{\partial \tau} = (S_0 + S_1\tau + S_2\tau^2) + \mu\left[(S_1 + 2\mu S_2) + 2S_2\tau\right] \\ &= (S_0 + S_1\mu + 2\mu^2 S_2) + (S_1 + 2\mu S_2)\tau + S_2\tau^2. \end{aligned}$$

This is the same as I_2 so we have reached our solution:

$$I = (S_0 + S_1\mu + 2\mu^2 S_2) + (S_1 + 2\mu S_2)\tau + S_2\tau^2.$$

Given this solution, $I(\tau = 0, \mu) = S_0 + \mu S_1 + 2\mu^2 S_2$ is consistent with $(\tau = \mu) = S_0 + \mu S_1 + \mu^2 S_2$.

Part 3

Use result in 2) to explain Limb Darkening. Under what condition would you get Limb Brightening instead?

Solution

When observing the specific intensity I_ν across the solar disk, we are seeing different depths into the atmosphere (and therefore different temperatures) because the optical depth τ varies across the solar disk. Near the disk centre, the optical depth is higher ($\tau \sim 1$) so we are seeing deeper into the iterior where the temperature is hotter and brightness is greater. Towards the disk edges, the optical depth decreases so we are seeing higher elevations in the solar atmosphere where the temperature is cooler and brightness is lower. As we csn dee from Part 2, the specific intensity depends on the optical depth in such a way that decreasing $\tau \rightarrow$ decreasing I towards the solar edges. This is "limb darkening." In order to get "limb brightening," the temperature would need to increase with elevation such that the brightness would increase with decreasing optical depths. Such a scenario is believed to occur in the solar corona where the temperature drastically increases beyond the photosphere.

3 Photon Pressure

If photon momentum is related to energy as $p = \frac{3}{4}h\nu/c$ (an extra factor 3/4 compared to the real one), rederive the expressions for radiation pressure (RL eq. [1.10]) and the Stefan-Boltzman law (RL eq. [1.41]).

Solution

Radiation pressure:

$$p_\nu[\text{dynes} \cdot \text{cm}^{-2}] = \frac{3}{4}\frac{1}{c}\int I_\nu \cos^2\theta d\Omega.$$

For a reflecting enclosure containing an isotropic radiation field, each photon transfers twice its normal component of momentum upon reflection:

$$p_\nu = (2)\frac{3}{4}\frac{1}{c}\int I_\nu \cos^2\theta d\Omega = \frac{3}{2c}\int I_\nu \cos^2\theta d\Omega.$$

For isotropic radiation, $I_\nu = J_\nu (\equiv \frac{1}{4\pi}\int I_\nu d\nu)$, so:

$$\frac{3}{2c}\int_0^\infty J_\nu \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi.$$

Since the radiation energy density is $u = \int_0^\infty u_\nu d\nu = \frac{4\pi}{c}\int_0^\infty J_\nu d\nu$, we can make the substitution $\int_0^\infty J_\nu d\nu = \left(\frac{c}{4\pi}\right) u$:

$$p = \frac{3}{2c}\left[\frac{c}{4\pi}u\right]\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2}[2\pi]_0^{2\pi} = \frac{u}{4}.$$

Therefore $p = u/4$ is the radiation pressure.

Stefan-Boltzmann Law:

The first law of thermodynamics is $dQ = dU + pdV$ while the second law of thermodynamics: $dS = dQ/T$, where $U = uV$, $p = u/4$, and $u = (4\pi/c)\int J_\nu d\nu$; S is entropy, Q is heat, T is temperature, U is total energy, p is radiation pressure, and V is volume. Starting with the second law of thermodynamics:

$$\begin{aligned} dS = \frac{dQ}{T} &= \frac{(dU + pdV)}{T} = \frac{d(uV) + pdV}{T} = \frac{Vdu + udV + pdV}{T} \\ &= \frac{V}{T}\frac{dU}{dT}dT + \frac{u}{T}dV + \left(\frac{1}{4}\frac{u}{T}\right)dV = \frac{V}{T}\frac{dU}{dT}dT + \frac{5}{4}\frac{u}{T}dV. \end{aligned}$$

The entropy S therefore has the following partial derivatives:

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT}, \left(\frac{\partial S}{\partial V}\right)_T = \frac{5u}{4T}.$$

Using Clairaut's Theorem, we can now write the second partial derivatives of entropy as:

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V.$$

Substituting the partial derivatives from above, we can solve the partial differential equation for the new Stefan-Boltzmann relation:

$$\begin{aligned} \frac{\partial}{\partial T}\left(\frac{5u}{4T}\right) &= \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{du}{dT}\right) \\ \frac{5}{4}\frac{\partial}{\partial T}(uT^{-1}) &= \frac{1}{T}\frac{du}{dT}\frac{\partial}{\partial V}(V) \\ \frac{5}{4}\frac{\partial}{\partial T}(\frac{1}{T}\frac{\partial}{\partial T}u + u\frac{\partial}{\partial T}T^{-1}) &= \frac{1}{T}\frac{du}{dT} \\ \frac{5}{4}\left(\frac{1}{T}\frac{du}{dT} - \frac{u}{T^2}\right) &= \frac{1}{T}\frac{du}{dT} \\ \frac{5}{4}\left(\frac{du}{dT} - \frac{u}{T}\right) &= \frac{du}{dT} \\ \frac{du}{dT}\left(\frac{5}{4} - 1\right) &= \frac{5}{4}\frac{u}{T} \\ \frac{du}{u} &= 5\frac{dT}{T} \\ \int\frac{du}{u} &= 5\int\frac{dT}{T} \\ ln(u) &= 5ln(T) + ln(c) \\ e^{ln(u)} &= e^{5ln(T)}e^{ln(c)} \\ u &= cT^5, \end{aligned}$$

where c is an integration constant.

4 The Greenhouse Effect

The Sun heats the Earth. One can derive an equilibrium temperature for an air-less Earth T_p assuming energy conservation and a black-body surface. This has to be modified when an atmosphere exists. The earth atmosphere is optically thin in the visible wavelengths so the solar radiation hits the ground directly. However, it is optically thick in the infra-red wavelengths and intercepts radiation from the ground. The heat is eventually lost to the vacuum as the atmosphere radiates with an effective temperature T_{eff}. Necessarily, $T_{eff} = T_p$ (think why).

Part 1

Imagine the atmosphere as a single opaque layer with a uniform temperature T_p. It is receiving heat from the ground (T_g) and radiates as much luminosity toward the ground as towards the vacuum outside. Use energy conservation to show that $T_g = 2^{1/4} T_p$. So the ground has to be hotter.

Solution

Figure 4 depicts the flow of radiative flux between the Earth, atmosphere and space:

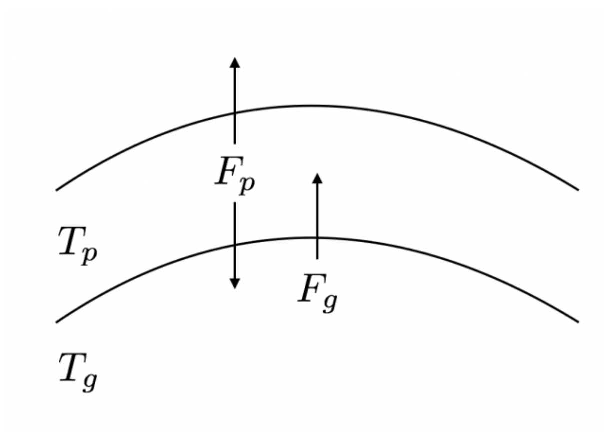

Figure 1: Radiative conservation for the Earth's atmosphere.

In thermal equilibrium, the radiative flux entering the atmosphere must equal that exiting the atmosphere:

$$\begin{aligned} F_{in} &= F_{out} \\ F_g &= 2F_p \\ (\sigma T_g^4) &= 2(\sigma T_p^4) \\ T_g &= 2^{1/4} T_p. \end{aligned}$$

Part 2

A more sophisticated approach is to allow different layers in the atmosphere to have different temperatures. Let τ_g be the optical depth (to outgoing radiation) measured at the ground, while $\tau = 2/3$ at the photosphere (problem 2). Use the radiative diffusion equation (RL eq. [1.111]) to show that

$$T_g^4 = T_p^4 = \left[1 + \frac{3}{4}\left(\tau_g - \frac{2}{3}\right)\right].$$

This reduces to the result in Part 1 when $\tau_g = 2$.

Solution

The one-dimensional radiative diffusion equation reads:

$$F(z) = \frac{16\sigma T^3}{3\alpha_R}\frac{dT}{dz},$$

where since $d\tau = \alpha_R dz$, we can use the fact that $dz = d\tau/\alpha_R$:

$$F_{rad} = \frac{16\sigma T^3}{3\alpha_R}\frac{dT}{(d\tau/\alpha_R)}$$
$$F_{rad} = \frac{16\sigma T^3}{3\alpha_R}\frac{dT}{(d\tau/\alpha_R)}$$
$$F_{rad} = \frac{16\sigma T^3}{3}\frac{dT}{d\tau}.$$

Since this F_{rad} must equal $F_p = \sigma T_p^4$:

$$\sigma T_p^4 = \frac{16\sigma T^3}{3}\frac{dT}{d\tau}$$
$$T_p^4 d\tau = \frac{16}{3}T^3 dT$$
$$T_p^4 \int_{\tau_g}^{2/3} d\tau = \frac{16}{3}\int_{T_g}^{T_p} T^3 dT$$
$$T_p^4[\tau]_{\tau_g}^{2/3} = \frac{16}{3}\left[\frac{T^4}{4}\right]_{T_g}^{T_p}$$
$$T_p^4(\frac{2}{3} - \tau_g) = \frac{4}{3}(T_p^4 - T_g^4)$$
$$T_g^4 = T_p^4 - \frac{3}{4}T_p^4(\frac{2}{3} - \tau_g)$$
$$T_g^4 = T_p^4\left[1 + \frac{3}{4}(\tau_g - \frac{2}{3})\right].$$

When $\tau_g = 2$,

$$T_g^4 = T_p^4 \left[1 + \frac{3}{4}(2 - \frac{2}{3})\right]$$
$$T_g^4 = 2T_p^4$$
$$T_g = 2^{1/4} T_p.$$

Part 3

The opacity used above is the so-called Rosseland mean opacity (RL eq. [1.110]). What chemical species (atoms, molecules, aerosols...) are most relevant for determining this opacity? In contrast, what is the dominant opacity source in the visible wavelengths?

Solution

The dominant chemical species for determining the Rosseland mean opacity for the Earth's atmosphere are: water (H_2O), carbon monoxide (CO), and methane (CH_4). In the visible wavelengths, there is water vapour (i.e., clouds) and Rayleigh scattering.

5 Equivalent Width and Curve of Growth Analysis

This problem is based on the data of Roberge et al (2004, Nature, 441, 72, also comes with supplementary material). The aim is to determine the range of uncertainty in the values for the atomic column densities obtained from absorption spectra.

Part 1

First extract from the NIST database (http://physics.nist.gov/PhysRefData/ASD, choose lines) Einstein A coefficient and statistical weights relevant for the following transitions: CII $\lambda = 1036.34$ Å line, CII* $\lambda = 1037.02$ Å line, and OI $\lambda = 976.45$ Å line.

Solution

	λ (Å)	A_{ki} (s^{-1})	g_i	g_k
CII	1036.34	7.38×10^8	2	2
CII*	1037.02	1.46×10^9	4	2
OI	976.45	3.86×10^7	5	3

Figure 2: NIST results.

Part 2

Starting from eq. (1.78) in RL, obtain the following absorption coefficient α_ν,

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2} n_a A_{21} \frac{g1}{g2}\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu) = \frac{c^2}{8\pi\nu^2} n_a A_{21} \frac{g1}{g2}\left[1 - \exp\left(-\frac{\Delta E}{k_B T}\right)\right]\phi(\nu),$$

where ΔE is the transition energy, T_s is the excitation temperature of atoms (see Problem 6) and for our problem here, the limit $k_B T_s \ll \Delta E$ applies.

Solution

The equation for the absorption coefficient is:

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu).$$

We therefore need to start by finding B_{12}. Starting with the following detailed balance equations:

$$\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}, \frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2}$$

we can solve for B_{12} via

$$B_{12} = B_{21}\left(\frac{g_2}{g_1}\right)$$

by plugging in the following for B_{21}:

$$B_{21} = A_{21}\left(\frac{c^2}{2h\nu^3}\right).$$

Doing this, we obtain the following:

$$B_{12} = \left[A_{21}\frac{c^2}{2h\nu^3}\right]\left(\frac{g_2}{g_1}\right).$$

Plugging this back into α_ν, we obtain:

$$\begin{aligned}\alpha_\nu &= \frac{h\nu}{4\pi} n_1 \left[A_{21}\left(\frac{c^2}{2h\nu^3}\right)\left(\frac{g_2}{g_1}\right)\right]\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu)\alpha_\nu \\ &= \frac{c^2 n_1}{8\pi\nu^2} A_{21}\left(\frac{g_2}{g_1}\right)\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu).\end{aligned}$$

We can now use the Boltzmann equation

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu}{k_B T}\right)$$

to solve for $(g_1 n_1/g_2 n_2)$:

$$\frac{g_1 h_1}{g_2 n_2} = \exp\left(-\frac{h\nu}{k_B T}\right).$$

Substituting this back into our equation for α_ν gives us the final form for the absorption coefficient:

$$\begin{aligned} \alpha_\nu &= \frac{c^2 n_1}{8\pi\nu^2} A_{21} \left(\frac{g_2}{g_1}\right) \left[1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right] \phi(\nu) \\ &= \frac{c^2 n_1}{8\pi\nu^2} A_{21} \left(\frac{g_2}{g_1}\right) \left[1 - \exp\left(-\frac{\Delta E}{k_B T_s}\right)\right] \phi(\nu) \end{aligned}$$

where ΔE is the transition energy and T_s is the excitation temperature.

In the limit where $k_B T_s \ll \Delta E$, this reduces to:

$$\alpha_\nu = \frac{c^2 n_1}{8\pi\nu^2} A_{21} \left(\frac{g_2}{g_1}\right) \phi(\nu).$$

Part 3

Show that for all three transitions (ignore the broad, redshifted components for carbon), the sort of column density values as listed in their Table 1 would imply that optical depths at the line centres ($\tau_{\nu=\nu_0}$) are greater than or comparable to 1. Or, these lines are saturated. In the author's opinion, Doppler broadening is important for the line-width,

$$\phi(\nu) \frac{1}{\sqrt{\pi}\Delta\nu_D} \exp\left[-\left(\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right]$$

where $\Delta\nu_D$ is the line width and is related to the Doppler parameter b (matter sound speed) as $\Delta\nu_D \approx (b/c)\nu_0$. Use the values for b as listed in Table 1.

Solution

Now that we have the equation for the absorption coefficient, let's start with the relation between optical depth and the absorption coefficient:

$$d\tau = \alpha_\nu ds d\tau = \frac{c^2 n_1}{8\pi\nu^2} A_{21} \left(\frac{g_2}{g_1}\right) \phi(\nu) ds \int d\tau = \frac{c^2}{8\pi\nu^2} A_{21} \left(\frac{g_2}{g_1}\right) \phi(\nu) \int n_1 ds.$$

Integrating and using the definition of column density, $N \equiv \int n ds$,

$$\tau = \frac{c^2}{8\pi\nu^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \phi(\nu).$$

Now, substituting the equation for a Doppler broadened dominated line profile into $\phi(\nu)$:

$$\tau = \frac{c^2}{8\pi\nu^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \left[\frac{1}{\sqrt{\pi}\Delta\nu_D} \exp\left(-\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right].$$

Using the fact that $\Delta\nu_D \approx (b/c)\nu_0$ and letting $\nu \equiv \nu_0$:

$$\tau = \frac{c^2}{8\pi\nu_0^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \left[\frac{1}{\sqrt{\pi}(b/c)\nu_0} \exp(0)^2\right] = \frac{c^2}{8\pi^{3/2} b \nu_0^3} A_{21} N_1 \left(\frac{g_2}{g_1}\right).$$

Solving for the optical depth for each of the three transitions, we obtain:

$$\tau(\text{CII}) = 1843.9$$
$$\tau(\text{CII}^*) = 1827.5$$
$$\tau(\text{CI}) = 1.2$$

Part 4

The authors use line-profile fitting to obtain column densities. This is highly ricky when a line is saturated and under-resolved. We adopt the equivalent width (EW) method here. Argue that the following definition is independent of spectral resolution:

$$EW_\nu = \int \frac{I_c - I_\nu}{I_c} d\nu,$$

where I_c is the specific intensity at continuum (the unabsorbed part). EW_ν has the unit of Hz.

Solution

Rewriting the integral as a sum:

$$EW_\nu = \int \frac{I_c - I_\nu}{I_c} d\nu = \sum_{i-1}^{N} \left(1 - \frac{I_{\nu i}}{I_c}\right) \delta\nu_i.$$

Assuming a uniform spectral resolution $\Delta\nu_i \equiv \Delta\nu/N$:

$$\begin{aligned} W_i &= \sum_{i-1}^{N} \left(1 - \frac{I_{vi}}{I_c}\right) \frac{\Delta\nu}{N} = \sum_{i-1}^{N} \left(1 - \frac{I_{vi}/N}{I_c}\right) \Delta\nu \\ &= \left(1 - \frac{\sum_{i-1}^{N} I_{vi}/N}{I_c}\right) \Delta\nu = \left(1 - \frac{< I >}{I_c}\right) \Delta\nu. \end{aligned}$$

Thus, the EW should be independent of $\Delta\nu_i$ and N and therefore of spectral resolution.

Part 5

When the line centre is optically thick, $EW_\nu \sim 2\Delta\nu$ where $\Delta\nu$ is the frequency displacement from the line centre at which $\tau_{\nu=\nu_0\pm1} = 1$. Show that for a Doppler broadened line, $EW_\nu \propto \sqrt{lnN}$, where N is th column density. (*Hint: Consult Spitzer p. 51 if confused.*)

Solution

From Part 3 above:

$$\tau_\nu = \frac{c^2}{8\pi^{3/2}\nu^2\Delta\nu_D} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \exp\left[-\left(\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right].$$

Letting $\tau = 1$ and $(\nu - \nu_0) = (EW_\nu/2)$, and solving for the equivalent width EW_ν:

$$\begin{aligned} (1) &= \frac{c^2}{8\pi^{3/2}\nu^2\Delta\nu_D} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \exp\left[-\left(\frac{EW_\nu}{2\Delta\nu_D}\right)^2\right] \\ \exp\left[-\left(\frac{EW_\nu}{2\Delta\nu_D}\right)^2\right] &= \frac{8\pi^{3/2}\nu^2\Delta\nu_D}{c^2 A_{21} N_1} \left(\frac{g_1}{g_2}\right) \\ \exp\left[\left(\frac{EW_\nu}{2\Delta\nu_D}\right)^2\right] &= \frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2\Delta\nu_D} \left(\frac{g_2}{g_1}\right) \\ \left(\frac{EW_\nu}{2\Delta\nu_D}\right)^2 &= \ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2\Delta\nu_D} \left(\frac{g_2}{g_1}\right)\right] \\ EW_\nu &= 2\Delta\nu_D \sqrt{\ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2\Delta\nu_D} \left(\frac{g_2}{g_1}\right)\right]}. \end{aligned}$$

Thus, $EW_\nu \propto ln\sqrt{N_1}$.

Part 6

For the following exercise, we introduce a more commonly used definition of EW,

$$EW_\lambda = \int \frac{I_c - I_\lambda}{I_c} d\lambda,$$

where the integration is over wavelength. Necessarily, $EW_\lambda/\lambda = EW_\nu/\nu$. By eyeball estimate, we assign an $EW_\lambda = 0.05$ Angstroms for the CII* line. Calculate the corresponding column densities when $b = 0.1$ km.s, 1.0 km/s, and 5.7 km/s (the 1σ range for b). What do your results say about the errorbars on the column densities listed in Table 1?

Solution

Let's start with our previously derived expression for the equivalent width:

$$EW_\nu = 2\Delta\nu_D \sqrt{\ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2 \Delta\nu_D}\left(\frac{g_2}{g_1}\right)\right]}.$$

Letting $\Delta\nu_d \approx (b/c)\nu_0$ and using the fact that $\frac{EW_\lambda}{\lambda} = \frac{EW\nu}{\nu}$, we can substitute $EW_\nu = (\nu/\lambda)EW_\lambda$ into the above equation for EW_ν:

$$\begin{aligned}
\left(\frac{\nu}{\lambda}EW_\lambda\right) &= 2\left(\frac{b}{c}\right)\sqrt{\ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2 (b/c)\nu_0}\left(\frac{g_2}{g_1}\right)\right]} \\
\left(\frac{\nu^2}{2b\nu_0}\right) EW_\lambda &= \sqrt{\ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2 (b/c)\nu_0}\left(\frac{g_2}{g_1}\right)\right]} \\
\left(\frac{\nu^2}{2b\nu_0}EW_\lambda\right)^2 &= \ln\left[\frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2 (b/c)\nu_0}\left(\frac{g_2}{g_1}\right)\right] \\
\exp\left(\frac{\nu^2}{2b\nu_0}EW_\lambda\right)^2 &= \frac{c^2 A_{21} N_1}{8\pi^{3/2}\nu^2 (b/c)\nu_0}\left(\frac{g_2}{g_1}\right) \\
N_1 &= \frac{8\pi^{3/2}\nu^2 (b/c)\nu_0}{c^2 A_{21}}\left(\frac{g_1}{g_2}\right)\exp\left(\frac{\nu^2}{2b\nu_0}EW_\lambda\right)^2 \\
N_1 &= \left(\frac{8\pi^{3/2}\nu^2 b}{c^2 A_{21}}\right)\left(\frac{g_1}{g_2}\right)(\nu_0\nu^2)\exp\left(\frac{\nu^2}{2b\nu_0}EW_\lambda\right)^2.
\end{aligned}$$

Solving for the column density for CII* for each of the three conditions, we obtain:

$$\begin{aligned}
N(\mathrm{CII}^*, b = 0.1\,\mathrm{km/s}) &= \infty \\
N(\mathrm{CII}^*, b = 1.0\,\mathrm{km/s}) &= 2.7\times 10^{35}\,\mathrm{cm}^{-2} \\
N(\mathrm{CII}^*, b = 5.7\,\mathrm{km/s}) &= 1.6\times 10^{14}\,\mathrm{cm}^{-2}
\end{aligned}$$

Part 7

You can obtain a lower limit to the CII column density by ignoring Doppler broadening ($b = 0$) but taking into accout the natural broadening (Lorentz profile, RL Section 10.6). In this case, one can show that $EW \propto \sqrt{N}$. Estimate the column density in the CII* level taking $\gamma = A_{21}$ and $EW_\lambda = 0.05$ Angstroms.

Similar exercises can be performed on the other lines and highlights the risk of abundance analysis for saturated lines.

Solution

The line profile equation for natural broadening is as follows:

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

where $\gamma \equiv \sum_{n'} A_{nn'}$.
Recalling that

$$\tau_\nu = \frac{c^2}{8\pi\nu^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \phi(\nu),$$

and substituting the above equation for $\phi(\nu)$ and allowing $\tau_\nu = 1$, $\gamma = A_{21}$, $\Delta\nu = (\nu - \nu_0)$, and $EW_\nu = 2\Delta\nu$:

$$\begin{aligned}
\tau_\nu &= \frac{c^2}{8\pi\nu^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \left[\frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}\right] \\
(1) &= \frac{c^2}{8\pi\nu^2} A_{21} N_1 \left(\frac{g_2}{g_1}\right) \left[\frac{A_{21}/4\pi^2}{(EW_\nu/2)^2 + (A_{21}/4\pi)^2}\right] \\
N_1 &= \frac{8\pi^3\nu^2}{A_{21}^2 c^2} \left(\frac{g_1}{g_2}\right) \left[EW_\nu^2 + \left(\frac{A_{21}}{4\pi}\right)^2\right].
\end{aligned}$$

Therefore $N_1 \propto EW_\nu^2$ and $EW_\nu \propto \sqrt{N_1}$.

Estimating the column density, we find that $N_1(\text{CII}^*, \gamma = A_{21}, EW_\lambda = 0.05\ \text{Å}) = 4.2 \times 10^{16}$ cm^{-2}.

We can thus see that the Lorentzian line profile is much more suitable for saturated lines than the Doppler line profile.