AST1430 Assignment 2

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1. LTE

This is preparation for Problem 2. An atom of two energy levels is immersed in a sea of electrons (of number density n_e and kinetic temperature T_K), and bathed in a black-body background radiation (of radiation temperature T_R). This atom is excited both collisionally and radiatively. Define an excitation temperature (T_s, sometimes called the spin temperature when the transition concerns spin) by

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{\Delta E}{k_B T_s}\right)$$

where $\Delta E = E_2 - E_1 > 0$ is the transition energy and k_B the Stefan-Boltzman constant.

Part 1

First write down the equation that describes the statistical equilibrium at which excitation and de-excitation of the upper level balances, and manipulate it to find

$$\exp\left(-\frac{\Delta E}{k_B T_s}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \frac{\xi}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}}{1 + \frac{\xi \exp\left(\frac{\Delta E}{k_B T_R}\right)}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}}$$

where the dimensionless number $\xi = A_{21}/(n_e q_{21}) = A_{21}/(n_e \sigma_{21} v_e) = n_{crit}/n_e$. Here, n_{crit} is the critical density for matter LTE.

Solution

In statistical equilibrium, excitation balances de-excitation:

$$n_1 n_e q_{12} + n_1 \bar{J} B_{12} = n_2 n_e q_{21} + n_2 B_{21} \bar{J} + n_2 A_{21}$$

where n_1 and n_2 are the number densities of atoms in levels 1 and 2 respectively. In thermodynamic equilibrium, the ratio of these number densities follows the Boltzmann distribution:

$$\frac{n_2}{n_1} = \left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_S}\right).$$

So, let's re-write the balance equation in a form that gives us n_2/n_1:

$$\begin{aligned}
n_1 n_e q_{12} + n_1 \bar{J} B_{12} &= n_2 n_e q_{21} + n_2 B_{21} \bar{J} + n_2 A_{21} \\
n_1(n_e q_{12} + \bar{J} B_{12}) &= n_2(n_e q_{21} + B_{21} \bar{J} + A_{21}) \\
\frac{n_2}{n_1} &= \frac{n_e q_{12} + \bar{J} B_{12}}{n_e q_{21} + B_{21} \bar{J} + A_{21}}
\end{aligned}$$

and now set this equal to the above Boltzmann equation:

$$\left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{n_e q_{12} + \bar{J} B_{12}}{n_e q_{21} + B_{21} \bar{J} + A_{21}}.$$

Dividing the top and bottom of the RHS by $(n_e q_{21})$,

$$\begin{aligned}
\left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_S}\right) &= \frac{(n_e q_{12} + \bar{J} B_{12})/(n_e q_{21})}{(n_e q_{21} + B_{21} \bar{J} + A_{21})/(n_e q_{21})} \\
\left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_S}\right) &= \frac{\frac{q_{12}}{q_{21}} + \frac{\bar{J} B_{12}}{n_e q_{21}}}{1 + \frac{B_{21} \bar{J}}{n_e q_{21}} + \frac{A_{21}}{n_e q_{21}}},
\end{aligned}$$

we can now use the fact that the ratio of collisional coefficients is given by:

$$\frac{q_{12}}{q_{21}} = \left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_K}\right).$$

Substituting this for q_{12}/q_{21} into our running balance equation:

$$\left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right) \exp\left(-\frac{\Delta E}{k_B T_K}\right) + \frac{\bar{J} B_{12}}{n_e q_{21}}}{1 + \frac{B_{21} \bar{J}}{n_e q_{21}} + \frac{A_{21}}{n_e q_{21}}}.$$

In thermodynamic equilibrium, $\bar{J} \approx B_\nu(T)$ which can be written as:

$$\bar{J} = \frac{A_{21}/B_{21}}{\left(\frac{g_1 B_{12}}{g_2 B_{21}}\right) \exp\left(\frac{\Delta E}{k_B T_R}\right) - 1}.$$

Making this substitution for $\bar{J}$:

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}/B_{21}}{\left(\frac{g_1 B_{12}}{g_2 B_{21}}\right)\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)\frac{B_{12}}{n_e q_{21}}}{1+\left(\frac{A_{21}/B_{21}}{\left(\frac{g_1 B_{12}}{g_2 B_{21}}\right)\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)\frac{B_{21}}{n_e q_{21}}+\frac{A_{21}}{n_e q_{21}}}$$

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{B_{12}}{n_e q_{21}}\frac{A_{21}}{B_{21}}\frac{g_2 B_{21}}{g_1 B_{12}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{B_{21}}{n_e q_{21}}\frac{A_{21}}{B_{21}}\frac{g_2 B_{21}}{g_1 B_{12}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}$$

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}}{n_e q_{21}}\frac{g_2}{g_1}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{A_{21}}{n_e q_{21}}\frac{B_{21} g_2}{B_{12} g_1}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}.$$

From equations of detailed balance, we can use the fact that

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2}$$

to substitute this for (B_{21}/B_{12}) into our equation:

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}}{n_e q_{21}}\frac{g_2}{g_1}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{A_{21}}{n_e q_{21}}\frac{g_1 g_2}{g_2 g_1}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}$$

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}}{n_e q_{21}}\frac{g_2}{g_1}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{A_{21}}{n_e q_{21}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}$$

$$\left(\frac{g_2}{g_1}\right)\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \left(\frac{g_2}{g_1}\right)\frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}}{n_e q_{21}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{A_{21}}{n_e q_{21}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{A_{21}}{n_e q_{21}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\left(\frac{A_{21}}{n_e q_{21}}\right)\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\frac{A_{21}}{n_e q_{21}}}.$$

Now, using the definition $\xi \equiv A_{21}/n_e q_{21}$ and rearranging terms:

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \xi\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\xi\left(\frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)+\xi}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{\xi}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\xi\left(\frac{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1+1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{\xi}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\xi\left(\frac{\exp\left(\frac{\Delta E}{k_B T_R}\right)}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}.$$

And now to take a moment to cry because I can't believe I typed that out.

Part 2

Simplify the above expression to obtain that in the limits of $\xi \gg 1$, $T_s \approx T_R$; and when $\xi \ll 1$, $T_s \approx T_K$. Contemplate the meaning of these results.

Solution

Starting with the equation we derived above:

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \left(\frac{\xi}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}{1+\xi\left(\frac{\exp\left(\frac{\Delta E}{k_B T_R}\right)}{\exp\left(\frac{\Delta E}{k_B T_R}\right)-1}\right)}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) = \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right) + \xi\left(\exp\left(\frac{\Delta E}{k_B T_R}\right)-1\right)^{-1}}{1+\xi\exp\left(\frac{\Delta E}{k_B T_R}\right)\left(\exp\left(\frac{\Delta E}{k_B T_R}\right)-1\right)^{-1}}.$$

If $\xi \gg 1$, the non-ξ terms are negligible and can be ignored:

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) \approx \frac{\xi\left(\exp\left(\frac{\Delta E}{k_B T_R}\right)-1\right)^{-1}}{\xi\exp\left(\frac{\Delta E}{k_B T_R}\right)\left(\exp\left(\frac{\Delta E}{k_B T_R}\right)-1\right)^{-1}}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) \approx \frac{1}{\exp\left(\frac{\Delta E}{k_B T_R}\right)}$$

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) \approx \exp\left(-\frac{\Delta E}{k_B T_R}\right).$$

Therefore,

$$T_S \approx T_R.$$

Now, if $\xi \ll 1$ the ξ terms now become negligible:

$$\exp\left(-\frac{\Delta E}{k_B T_S}\right) \approx \frac{\exp\left(-\frac{\Delta E}{k_B T_K}\right)}{1} \exp\left(-\frac{\Delta E}{k_B T_S}\right) \approx \exp\left(-\frac{\Delta E}{k_B T_K}\right).$$

Therefore,

$$T_S \approx T_K.$$

Part 3

Prove that if $T_K > T_R$, independent of the value of ξ, we have $T_s > T_R$. In other words, there cannot be absorption of the background radiation.

Solution

I don't even know.

2. 21 cm Emission From the Early Universe

We apply results from the above problem to investigate the 21cm emission from the epoch of reionization (when all hydrogen in the intergalactic space is ionized by UV photons from, presumably, massive stars). The 21 cm transition is between two energy levels (one being the hydrogen ground state) split by the nucleus spin (so called hyper-fine transitions). ΔE when expressed in temperature equals 0.068 K. This is much smaller than T_K or T_R. We ignore redshift effect in this problem.

Part 1

For this transition, $A_{21} = 2.87 \times 10^{-15}$ s^{-1}, $g_2 = 3$ and $g_1 = 1$. Assume that σ_{21} is of order (1 A)^2 and that $T_K = 100$ K. What is your estimated value of n_{crit}? How does this compare with the mean baryon density of the universe $n_{baryon} \sim 10^6(1+z)^3$ at re-ionization (currently thought to occur at $z \sim 10$)?

Solution

The critical density can be found via:

$$n_{crit} \equiv \frac{A_{21}}{q_{21}} \equiv \frac{A_{21}}{\sigma_{21} v_e},$$

so we just need to solve for v_e by setting the kinetic energy equal to the thermal energy:

$$\begin{aligned}
E_K &= E_{th} \\
\frac{1}{2} m_e v_e^2 &= \frac{3}{2} k_B T_K \\
v_e &= \sqrt{\frac{3 k_B T_K}{m_e}} \\
v_e &= \sqrt{\frac{3(1.38 \times 10^{-23}\,\mathrm{J/K})(100\,\mathrm{K})}{9.11 \times 10^{-31}\,\mathrm{kg}}} \\
v_e &= 67430.70\,\mathrm{m/s}.
\end{aligned}$$

We can now use this to calculate the critical density:

$$\begin{aligned}
n_{crit} &\equiv \frac{A_{21}}{\sigma_{21} v_e} \\
n_{crit} &= \frac{2.87 \times 10^{-15}\,\mathrm{s}^{-1}}{(10^{-10}\,\mathrm{m})^2\, 67430.70\,\mathrm{m/s}} \\
n_{crit} &= 4.26 \times 10^{-6}\,\mathrm{cm}^{-3}.
\end{aligned}$$

To compare it to the mean baryon density of the universe, let's first calculate n_{baryon}:

$$\begin{aligned}
n_{baryon} &\sim 10^{-6}(1+z)^3 \\
n_{baryon} &\sim 10^{-6}(11)^3 \\
n_{baryon} &\sim 1.33 \times 10^{-3}\,\mathrm{cm}^{-3}.
\end{aligned}$$

The critical density of 21 cm emission is therefore much less than mean baryonic density of the universe, so it is in matter LTE.

Part 2

Derive the following absorption coefficient for a clump of neutral hydrogen (number density n_{HI}) in the 21 cm wavelength,

$$\alpha_\nu = \frac{3c^2 A_{21} n_1}{8\pi\nu^2} \frac{0.068\,\mathrm{K}}{T_s} \phi(\nu) \approx \frac{3c^2 A_{21} n_{\mathrm{HI}}}{32\pi\nu^2} \frac{0.068\,\mathrm{K}}{T_s} \phi(\nu),$$

where $\phi(\nu)$ is the line profile function normalized as $\int_0^\infty \phi(\nu) d\nu = 1$. The optical depth is the absorption coefficient integrated through the cloud, $\tau_\nu = \int \alpha_\nu ds$. Typically, $\tau_\nu \ll 1$.

Solution

Starting from the expression we derived for the absorption coefficient in the previous assignment:

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2} n_1 A_{21} \left(\frac{g_2}{g_1}\right)\left(1 - \exp\left(-\frac{\Delta E}{k_B T_s}\right)\right)\phi(\nu)$$
$$\alpha_\nu = \frac{c^2}{8\pi\nu^2} n_1 A_{21} \left(\frac{3}{1}\right)\left(1 - \exp\left(-\frac{0.068\,\mathrm{K}}{T_s}\right)\right)\phi(\nu)$$
$$\alpha_\nu = \frac{3c^2}{8\pi\nu^2} n_1 A_{21} \left(1 - \exp\left(-\frac{0.068\,\mathrm{K}}{T_s}\right)\right)\phi(\nu).$$

Assuming $0.068 \ll T_s$, we can do a Taylor expansion and disregard higher-order terms:

$$\alpha_\nu = \frac{3c^2}{8\pi\nu^2} n_1 A_{21} \left(1 - \left(1 - \frac{0.068\,\mathrm{K}}{T_s}\right)\right)\phi(\nu)$$
$$\alpha_\nu = \frac{3c^2}{8\pi\nu^2} n_1 A_{21} \left(\frac{0.068\,\mathrm{K}}{T_s}\right)\phi(\nu).$$

To convert n_1 into n_{HI}, we can use the following as a substitution:

$$n_1 = \frac{n_{\mathrm{HI}}}{g_1 + g_2} = \frac{n_{\mathrm{HI}}}{1+3} = \frac{n_{\mathrm{HI}}}{4}.$$

Therefore,

$$\alpha_\nu = \frac{3c^2}{8\pi\nu^2} \left(\frac{n_{\mathrm{HI}}}{4}\right) A_{21} \left(\frac{0.068\,\mathrm{K}}{T_s}\right)\phi(\nu)$$
$$\alpha_\nu = \frac{3c^2}{32\pi\nu^2} n_{\mathrm{HI}} A_{21} \left(\frac{0.068\,\mathrm{K}}{T_s}\right)\phi(\nu).$$

Part 3

Define a brightness temperature for the 21 cm emission, $T_b = T_b(\nu) = c^2/(2\nu^2 k_B)\, I_\nu$. We are in the Rayleigh-Jeans limit ($h\nu = \Delta E k_B T$, where T can be T_s, T_b, T_R or T_K). Show that the source function $S_\nu \approx 2\nu^2/c^2 k_B T_s$ and that this allows the equation of radiative transport to be simplified to the following form (eq. [1.61] in RL):

$$\frac{dT_b}{d\tau_\nu} = (T_s - T_b).$$

Write down the solution for T_b in terms of variables defined above. (*Hint: the initial specific intensity originates from the radiation background.*)

Solution

Since the specific intensity originates from the radiation background, $S_\nu = B_\nu(T)$ so:

$$S_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{\Delta E}{k_B T_s}\right) - 1}.$$

Now, since we're in the Rayleigh-Jeans limit, $\Delta E \ll k_B T_s$ we can use a Taylor expansion to obtain the following simplification:

$$\begin{aligned}
S_\nu &\approx \frac{2h\nu^3}{c^2}\frac{1}{\left(1 + \frac{\Delta E}{k_B T_s}\right) - 1} \\
S_\nu &\approx \frac{2h\nu^3}{c^2}\frac{1}{\left(\frac{\Delta E}{k_B T_s}\right)} \\
S_\nu &\approx \frac{2h\nu^3}{c^2}\left(\frac{k_B T_s}{\Delta E}\right).
\end{aligned}$$

Substituting $\Delta E = h\nu$ and simplifying for our final form:

$$\begin{aligned}
S_\nu &\approx \frac{2h\nu^3}{c^2}\left(\frac{k_B T_s}{h\nu}\right) \\
S_\nu &\approx \frac{2\nu^2}{c^2}\left(\frac{k_B T_s}{1}\right) \\
S_\nu &\approx \left(\frac{2\nu^2 k_B}{c^2}\right) T_s.
\end{aligned}$$

Turning to the canonical equation of radiative transfer,

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu$$

we just derived an expression for S_ν and we can use the definition of brightness temperature to obtain one for I_ν:

$$\begin{aligned}
T_b \equiv T_b(\nu) &= \frac{c^2}{2\nu^2 k_B} I_\nu \\
I_\nu &= \left(\frac{2\nu^2 k_B}{c^2}\right) T_b.
\end{aligned}$$

Substituting these into the equation of radiative transfer we obtain the following simplification:

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{2\nu^2 k_B}{c^2}T_b\right) &= \left(\frac{2\nu^2 k_B}{c^2}\right)T_s - \left(\frac{2\nu^2 k_B}{c^2}\right)T_b \\
\left(\frac{2\nu^2 k_B}{c^2}\right)\frac{d}{d\tau}\left(T_b\right) &= \left(\frac{2\nu^2 k_B}{c^2}\right)T_s - \left(\frac{2\nu^2 k_B}{c^2}\right)T_b \\
\frac{dT_b}{d\tau} &= T_s - T_b.
\end{aligned}
$$

To find the solution of T_b, we simply need to solve the differential equation:

$$
\begin{aligned}
\frac{dT_b}{d\tau} &= T_s - T_b \\
\frac{dT_b}{T_s - T_b} &= d\tau \\
\int_{T_{b_0}}^{T_b} \frac{dT_b}{T_s - T_b} &= \int_0^{\tau} d\tau.
\end{aligned}
$$

Introducing a change of variables where $x \equiv T_s - T_b$ such that $dx = -dT_b$ and solving for T_b:

$$
\begin{aligned}
\int_{T_s - T_{b_0}}^{T_s - T_b} -\frac{dx}{x} &= \int_0^{\tau} d\tau \\
\int_{T_s - T_{b_0}}^{T_s - T_b} \frac{dx}{x} &= -\int_0^{\tau} d\tau \\
\ln(x)\big|_{T_s - T_{b_0}}^{T_s - T_b} &= -\tau\big|_0^{\tau} \\
\ln(T_s - T_{b_0}) - \ln(T_s - T_b) &= -\tau \\
\ln\left(\frac{T_s - T_b}{T_s - T_{b_0}}\right) &= -\tau \\
\exp\left[\ln\left(\frac{T_s - T_b}{T_s - T_{b_0}}\right)\right] &= \exp(-\tau) \\
\left(\frac{T_s - T_b}{T_s - T_{b_0}}\right) &= \exp(-\tau) \\
T_s - T_b &= (T_s - T_{b_0})\exp(-\tau) \\
T_b &= (T_{b_0} - T_s)\exp(-\tau) + T_s \\
T_b &= T_s(1 - \exp(-\tau)) + T_{b_0}\exp(-\tau).
\end{aligned}
$$

Part 4

Radio telescopes will be able to measure 21cm emission from the early universe by detecting excess temperature over the CMB,

$$\delta T_b = T_b - T_R.$$

Show that this excess is independent of the actual spin temperature as long as it lies sufficiently far above T_R (currently at 2.7 K). This measurement allows one to obtain a good estimate of the HI column density as a function of redshift, thereby probing the processes occurring during reionization.

References for Problems 1 2: Field, 1958, Proc. IRE, 46, 240 (original, no electronic version available) Field, 1959, ApJ, 129,525 (visionary work) Zaldarriaga et al 2004, ApJ, 608, 622 (recent analytical development)

Solution

Starting with the solution that we just found for T_b:

$$T_b = T_s(1 - \exp(-\tau)) + T_{b_0} \exp(-\tau),$$

we can calculate δT_b as follows:

$$\delta T_b = T_b - T_R \delta T_b = T_s(1 - \exp(-\tau)) + T_{b_0} \exp(-\tau) - T_R.$$

Since we know that $\tau \equiv \int \alpha_\nu ds$, this implies that $\tau_\nu \propto \alpha_\nu \propto 1/T_s$, so:

$$\delta T_b \sim T_s \left(1 - \exp\left(-\frac{1}{T_s}\right)\right) + T_{b_0} \exp\left(-\frac{1}{T_s}\right) - T_R.$$

If the spin temperature lies sufficiently far above the background radiation temperature, $T_s \gg T_R$, then $1/T_s \ll 1$ so we can perform a Taylor expansion:

$$\begin{aligned}
\delta T_b &\approx T_s \left(1 - \left[1 - \frac{1}{T_s}\right]\right) + T_{b_0} \left[1 - \frac{1}{T_s}\right] - T_R \\
\delta T_b &\approx T_s \left(\frac{1}{T_s}\right) + T_{b_0} \left[1 - \frac{1}{T_s}\right] - T_R \\
\delta T_b &\approx 1 + T_{b_0} \left[1 - \frac{1}{T_s}\right] - T_R \\
\delta T_b &\approx 1 + T_{b_0}(1) - T_R \\
\delta T_b &\approx 1 + T_{b_0} - T_R.
\end{aligned}$$

Therefore, δT_b is independent of the spin temperature T_s.

4. Emission Line Fluxes

Atomic and molecular infrared transitions are important for cooling the interstellar gas. One example for this is the cooling of metallic gas in the debris disk around stars like β Pictoris and others. The gas is largely ionized (except for OI which has a first ionization potential 16 eV) and is hydrogen-depleted. The gas is mostly composed of oxygen and carbon and has a mean ion/electron density of $\sim$ 10 cm^{-3}. Ignore carbon here.

Part 1

The ground-state of OI (which has a configuration of $2s^2 2p^4$) is split by spin-orbit coupling to three. Transitions between these produce the OI 44 μm, 63.2 μm and 145.6 μm lines. Assuming LTE, which transition is most important for cooling the gas when gas temperature $T \sim 300$ K? What about when $T \sim 50$ K? For an OI number density of 1 cm^{-3} and LTE, a gas temperature of 300 K, calculate the cooling luminosity from individual lines radiated by a cubic centimeter of material.

Solution

Figure summarizes the three OI transitions: Earth, atmosphere and space:

	$\lambda\,(\mu\text{m})$	$\nu\,(\text{GHz})$	$\Delta E\,(\text{J})$	$\Delta E\,(\text{eV})$
OI($0 \to 2$)	44	6813.46	4.52×10^{-21}	0.0282
OI($0 \to 1$)	63.2	4743.55	3.14×10^{-21}	0.0196
OI($1 \to 2$)	145.6	2059.01	1.36×10^{-21}	0.0085

Figure 1: Summary of transitions.

Gas Cooling

To determine which transition dominates gas cooling, we must simply find the transitional energy that is closest to the value of $k_B T$ for the gas. For a gas temperature of $T \sim 300\,\text{K}$, $(k_B T) = 4.14 \times 10^{-21}\,\text{J} = 0.0259\,\text{eV}$ so OI($0 \to 1$) dominates the gas cooling. Similarly, for a gas temperature of $T \sim 50\,\text{K}$, $(k_B T) = 6.903 \times 10^{-22}\,\text{J} = 0.0043\,\text{eV}$ so OI($1 \to 2$) dominates the gas cooling.

Cooling Luminosity

For radiative cooling via emission lines such as the three OI transitions, atoms occupy two different energy levels i and j separated by an energy of ΔE_{ji}. The Einstein coefficient A_{ji} characterizes the rate of spontaneous emission from state $j \to i$ which have respective number densities of n_j and n_i.

The cooling luminosity of a transition from state $j \to i$ in a volume V is given by:

$$L_{ji} = n_j A_{ji} \Delta E_{ji} V.$$

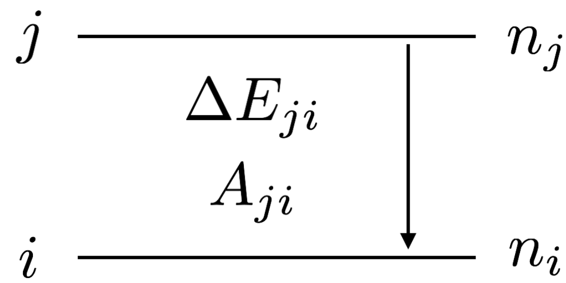

Figure 2: Schematic of energy transitions.

We're given that the total OI number density is $1\,\mathrm{cm}^{-3}$, but in order to determine a cooling luminosity for each of the transitions independently, we must solve for the independent number densities of the three states. Here, we can use the Boltzmann equation which relates two states i and j:

$$\frac{n_j}{n_i} = \left(\frac{g_j}{g_i}\right) \exp\left(\frac{\Delta E_{ji}}{k_B T}\right).$$

This allows us to relate the two number densities n_1 and n_2 to n_0 as such:

$$\begin{aligned} n_1 &= n_0 \left(\frac{g_1}{g_0}\right) \exp\left(\frac{\Delta E_{10}}{k_B T}\right) \\ n_2 &= n_0 \left(\frac{g_2}{g_0}\right) \exp\left(\frac{\Delta E_{20}}{k_B T}\right) \end{aligned}$$

which allows to solve for n_0:

$$\begin{aligned} n_0 + n_1 + n_2 &= 1\,\mathrm{cm}^{-3} \\ n_0 + n_0 \left(\frac{g_1}{g_0}\right) \exp\left(\frac{\Delta E_{10}}{k_B T}\right) + n_0 \left(\frac{g_2}{g_0}\right) \exp\left(\frac{\Delta E_{20}}{k_B T}\right) &= 1\,\mathrm{cm}^{-3} \\ n_0 \left[1 + \left(\frac{g_1}{g_0}\right) \exp\left(\frac{\Delta E_{10}}{k_B T}\right) + \left(\frac{g_2}{g_0}\right) \exp\left(\frac{\Delta E_{20}}{k_B T}\right)\right] &= 1\,\mathrm{cm}^{-3} \\ n_0 = \frac{1\,\mathrm{cm}^{-3}}{1 + \left(\frac{g_1}{g_0}\right) \exp\left(\frac{\Delta E_{10}}{k_B T}\right) + \left(\frac{g_2}{g_0}\right) \exp\left(\frac{\Delta E_{20}}{k_B T}\right)}&. \end{aligned}$$

To compute n_0, we must first obtain the transitional information from NIST.

The following table summarizes the transitional information from NIST:

	$\lambda\,(\mu\mathrm{m})$	$A_{21}\,(\mathrm{s}^{-1})$	g_i	g_j
$\mathrm{OI}(0 \to 2)$	44	1×10^{-10}	5	1
$\mathrm{OI}(0 \to 1)$	63.2	9×10^{-5}	5	3
$\mathrm{OI}(1 \to 2)$	145.6	1.7×10^{-5}	3	1

Figure 3: NIST results.

Using this transitional information along with $T = 300\,\mathrm{K}$, we can now solve for n_0:

$$
\begin{aligned}
n_0 &= \frac{1\,\mathrm{cm}^{-3}}{1 + \left(\frac{g_1}{g_0}\right)\exp\left(\frac{\Delta E_{10}}{k_B T}\right) + \left(\frac{g_2}{g_0}\right)\exp\left(\frac{\Delta E_{20}}{k_B T}\right)} \\
n_0 &= \frac{1\,\mathrm{cm}^{-3}}{1 + \left(\frac{3}{5}\right)\exp\left(\frac{0.0196\,\mathrm{eV}}{k_B(300\,\mathrm{K})}\right) + \left(\frac{1}{5}\right)\exp\left(\frac{0.0282\,\mathrm{eV}}{k_B(300\,\mathrm{K})}\right)} \\
n_0 &= 0.35\,\mathrm{cm}^{-3}.
\end{aligned}
$$

We can now return to the Boltzmann equations to determine the remaining number densities:

$$
\begin{aligned}
n_1 &= n_0\left(\frac{g_1}{g_0}\right)\exp\left(\frac{\Delta E_{10}}{k_B T}\right) = 0.35\,\mathrm{cm}^{-3}\left(\frac{3}{5}\right)\exp\left(\frac{0.019\,\mathrm{eV}}{k_B(300\,\mathrm{K})}\right) = 0.45\,\mathrm{cm}^{-3} \\
n_2 &= n_0\left(\frac{g_2}{g_0}\right)\exp\left(\frac{\Delta E_{20}}{k_B T}\right) = 0.35\,\mathrm{cm}^{-3}\left(\frac{1}{5}\right)\exp\left(\frac{0.0282\,\mathrm{eV}}{k_B(300\,\mathrm{K})}\right) = 0.21\,\mathrm{cm}^{-3}
\end{aligned}
$$

We can now use these densities to calculate the cooling luminosity of each transition:

$$
\begin{aligned}
L_{20} &= n_2 A_{20}\Delta E_{20} V = (0.21\,\mathrm{cm}^{-3})(1 \times 10^{-10}\,\mathrm{s}^{-1})(0.0282\,\mathrm{eV})(1\,\mathrm{cm}^3) = 5.84 \times 10^{-13}\,\mathrm{eV/s} \\
L_{10} &= n_1 A_{10}\Delta E_{10} V = (0.45\,\mathrm{cm}^{-3})(9 \times 10^{-5}\,\mathrm{s}^{-1})(0.0196\,\mathrm{eV})(1\,\mathrm{cm}^3) = 7.85 \times 10^{-7}\,\mathrm{eV/s} \\
L_{21} &= n_2 A_{21}\Delta E_{21} V = (0.21\,\mathrm{cm}^{-3})(1.7 \times 10^{-5}\,\mathrm{s}^{-1})(0.0085\,\mathrm{eV})(1\,\mathrm{cm}^3) = 2.99 \times 10^{-8}\,\mathrm{eV/s}
\end{aligned}
$$

Part 2

Perform the same calculations but without assuming LTE. Ignore radiative pumping. Table 8 of Hollenbach McKee (1989, ApJ, 342, 30) lists relevant numbers for all three transitions (Oxygen is an important ISM coolant). (*Hint: for this question, you may not have to solve for the population occupations. Cross off the unimportant terms.*)

Solution

Do I have to.

Part 3

Simplify the debris disk to a homogeneous sphere of radius 100 AU, lying at a distance of 20 pc. Will the line fluxes you obtain be measurable by the current Herschel mission (a typical sensitivity of 3×10^{-18} Wm^{-2} over 1 hour of integration)?

Solution

Returning to our equation of luminosity while adjusting the volume term V to reflect the spherical shape of the debris disk:

$$L_{20} = n_2 A_{20} \Delta E_{20} V = (0.21\,\mathrm{cm}^{-3})(1 \times 10^{-10}\,\mathrm{s}^{-1})(0.0282\,\mathrm{eV})(\frac{4}{3}\pi(100\,\mathrm{AU})^3) = 1.31 \times 10^{15}\,\mathrm{W}$$
$$L_{10} = n_1 A_{10} \Delta E_{10} V = (0.45\,\mathrm{cm}^{-3})(9 \times 10^{-5}\,\mathrm{s}^{-1})(0.0196\,\mathrm{eV})(\frac{4}{3}\pi(100\,\mathrm{AU})^3) = 1.76 \times 10^{21}\,\mathrm{W}$$
$$L_{21} = n_2 A_{21} \Delta E_{21} V = (0.21\,\mathrm{cm}^{-3})(1.7 \times 10^{-5}\,\mathrm{s}^{-1})(0.0085\,\mathrm{eV})(\frac{4}{3}\pi(100\,\mathrm{AU})^3) = 6.72 \times 10^{19}\,\mathrm{W}$$

and calculating the line fluxes via $F = L/(4\pi a^2)$:

$$F_{20} = \frac{L_{20}}{4\pi(a)^2} = \frac{1.31 \times 10^{15}\,\mathrm{W}}{4\pi(20\,\mathrm{pc})^2} = 2.74 \times 10^{-22}\,\mathrm{Wm}^{-2}$$
$$F_{10} = \frac{L_{10}}{4\pi(a)^2} = \frac{1.76 \times 10^{21}\,\mathrm{W}}{4\pi(20\,\mathrm{pc})^2} = 3.69 \times 10^{-16}\,\mathrm{Wm}^{-2}$$
$$F_{21} = \frac{L_{21}}{4\pi(a)^2} = \frac{6.72 \times 10^{19}\,\mathrm{W}}{4\pi(20\,\mathrm{pc})^2} = 1.40 \times 10^{-17}\,\mathrm{Wm}^{-2}.$$

Therefore, Herschel could detect the OI($1 \to 0$) and OI($2 \to 1$) transitions with a one-hour integration.

Part 4

Assuming that the two strongest lines are detectable, what can we learn about disk density and temperature?

Solution

Based on discussions in class, we can learn about whether the gas is in LTE or NLTE.

Part 5

Riviere-Marichalar et al (2012, AA, 546, 8) reported detection of OI 63 μm line from star HD 172555, using the Herschel telescope (http://arxiv.org/abs/1210.0089). They used the observed flux to derive the mass of oxygen around that star (appendix B). Can you spot any inappropriate assumption? How would it change their result on the oxygen mass?

Solution

At least I'm making the marking easier.

5. Photoevaporating a Close-in Planet

Stellar XUV ($100 - 1000$ A) photons can ionize hydrogen in the upper atmosphere of a planet. The dissociated electrons typically have an energy of $\sim$ eV and heats the ionized gas to a temperature of $T \sim 10^4\,\mathrm{K}$. The vertical pressure scale height associated with such a temperature is so large (of order the planet radius) that the gas readily gets lost from the planet. Estimate the mass loss rate from a Jupiter-like planet at 0.05 AU, based on the following simplifying assumptions:

1) If the XUV photon can ionize the atmosphere down to a layer with base density n (cm^{-3}), we will obtain a mass loss (Parker wind, a hydrodynamical outflow) rate of $\dot{M} \sim 4\pi R_p^2 n m_{\mathrm{H}} v_{\mathrm{th}}$, where m_{H} is the proto mass, R_p is the planet radius, and v_{th} is the thermal velocity of the gas.

2) Recombination is important in determining the extent of the ionized region.

3) A star similar to the Sun in mass and age has a XUV luminosity of $L_{XUV} \sim 10^{-6} L_{\odot}$.

Compare your results with papers calculating precisely this process. Lammer et al (2003, ApJ, 598, L121) produces $\dot{M} \sim 10^{12}\,\mathrm{gs}^{-1}$, Yelle (2004, Icarus, 170, 167) gives $\dot{M} \sim 10^{10}\,\mathrm{gs}^{-1}$. The only observation concerns the transiting planet HD 209468b and claims a detection of mass-loss with a rate $> 10^{10}\,\mathrm{gs}^{-1}$ (Vidal-Madjar et al 2003, Nature, 422, 143, but see Holmstrom et al 2008, Nature 451, 970).

Solution

Solution:

To calculate $\dot{M}$, we will first need to determine the thermal gas velocity v_{th}. We can do this by setting the kinetic energy equal to the thermal energy for a mostly ionized gas:

$$\frac{1}{2} m_e v_{th}^2 = k_B T v_{th} = \sqrt{\frac{2k_B T}{m_e}}.$$

This can be substituted into the mass-loss equation:

$$\dot{M} \sim 4\pi R_p^2 n m_{\mathrm{H}} \sqrt{\frac{2k_B T}{m_e}}.$$

We'll now need to determine n which can be done by setting the photoionization rate equal to the recombination rate:

$$\frac{4\pi}{h\nu} n_n \sigma_{bf} \left(1 - \exp\left(\frac{-h\nu}{k_B T}\right)\right) B_\nu(T) d\nu = n_+ n_e \sigma_{fb} f(\nu) \nu d\nu,$$

where n_n, n_+ and n_e are number densities of the neutral species, cations and electrons, respectively, σ_{bf} and σ_{fb} are the ionization and recombination cross-sections, respectively, $B_\nu(T)$ is the Planck function, and $f(\nu)$ is the Maxwell velocity distribution. Ignoring the exponential term which subtracts stimulated recombination,

$$\frac{4\pi}{h\nu} n_n \sigma_{bf} B_\nu(T) d\nu = n_+ n_e \sigma_{fb} f(\nu) \nu d\nu,$$

and integrating the result:

$$\frac{4\pi}{h\nu} n_n \sigma_{bf} \int B_\nu(T) d\nu = n_+ n_e \int \sigma_{fb} f(\nu) \nu d\nu.$$

We have that $B(T) \equiv \int B_\nu(T)$ is the integrated Planck function and for an isotropic radiation field, $F = \pi B(T)$:

$$\begin{aligned}
\frac{4\pi}{h\nu} n_n \sigma_{bf} B(T) d\nu &= n_+ n_e \int \sigma_{fb} f(\nu) \nu d\nu \\
\frac{4\pi}{h\nu} n_n \sigma_{bf} \left(\frac{F}{\pi}\right) &= n_+ n_e \int \sigma_{fb} f(\nu) \nu d\nu \\
\frac{4}{h\nu} n_n \sigma_{bf} F &= n_+ n_e \int \sigma_{fb} f(\nu) \nu d\nu.
\end{aligned}$$

Now, the RHS can be simplified by noting that $\int \sigma_{fb} f(\nu) \nu d\nu \equiv \alpha_{rec}$ where α_{rec} is the recombination coefficient:

$$\frac{4}{h\nu} n_n \sigma_{bf} F = n_+ n_e \alpha_{rec}.$$

If we assume that $n \equiv n_+ \equiv n_e$ and use the fact that $n_n \sim 1/(\sigma bf H)$ where H is the scale height:

$$\begin{aligned}
\frac{4}{h\nu} \left(\frac{1}{\sigma_{bf} H}\right) \sigma_{bf} F &\sim n^2 \alpha_{rec} \\
\frac{4}{h\nu} \left(\frac{F}{H}\right) &\sim n^2 \alpha_{rec} \\
n &\sim \sqrt{\frac{4F}{h\nu H \alpha_{rec}}}.
\end{aligned}$$

Lastly, substituting $F = L/(4\pi a^2)$:

$$\begin{aligned}
n &\sim \sqrt{\frac{4\left(\frac{L}{4\pi a^2}\right)}{h\nu H \alpha_{rec}}} \\
n &\sim \sqrt{\frac{L}{h\nu \pi a^2 H \alpha_{rec}}}.
\end{aligned}$$

We can now plug this expression back into the mass-loss rate equation for its final form:

$$\dot{M} \sim 4\pi R_p^2 m_{\mathrm{H}} \sqrt{\frac{L}{h\nu\pi a^2 H\alpha_{rec}}}\sqrt{\frac{2k_B T}{m_e}}.$$

In estimating the mass-loss rate:

$$\dot{M} \sim 4\pi (R_J)^2 m_{\mathrm{H}} \sqrt{\frac{10^{-6} L_\odot}{h\nu\pi(0.05\,\mathrm{AU})^2(1.0)\cdot 2.7\times 10^{-13}\left(\frac{T}{10^4\,\mathrm{K}}\right)^{-0.9}\,\mathrm{cm}^3\mathrm{s}^{-1}}}\sqrt{\frac{2k_B(1\times 10^4\,\mathrm{K})}{m_e}}.$$

I didn't quite have time to figure out the issue with my units, so my value is currently off but it's just a matter of finding the bug in my code.