# Problem set I (Radiation Processes, Spring 2018)

*Hand in to me in person or slip in under my office door (MP1210) by Tuesday 2PM, Jan 20th*

*Please split yourselves equally into 3 groups of* $\sim 6$ *persons each, for purpose of collaborations. The solution write-up must be independent. Grading will be done by the other groups.*

**1**. (**NOT** to be handed in.) Work through all problems in Chapter 1 of RL. Consult the solutions (end of book) if you get stuck. Make sure that you are proficient at these.

**2. Photosphere** The temperature inside a star goes from a few million kelvins (interior) to practically zero (outside). On Earth ($\tau_\nu = 0$), we observe a radiation spectrum that is characterized by a temperature called the 'effective temperature' and is measured at an optical depth of order unity (photosphere).

1) Show that the bolometric flux radiated outward per unit area by a blackbody with temperature $T$ is $F = \pi \int S_\nu d\nu$ and is $\sigma T^4$.

2) In the case of a stellar atmosphere, show that the observed flux $F_\nu(\tau_\nu = 0) = \pi S_\nu(\tau_\nu = 2/3)$ (also called the Eddington-Barbier approximation), namely, the photospheric optical depth is 2/3. We suppress all $\nu$ subscript from now on. To accomplish this (in a cheap way), start from the equation of radiative transfer in plane-parallel atmosphere (RL eq. [1.116]). Taylor expand the source function $S(\tau)$ around small $\tau$, and iteratively solve the above equation to obtain $I(\tau = 0, \mu) \approx S(\tau = \mu)$. Obtain the corresponding $F(\tau = 0)$.

3) Use result in 2) to explain 'Limb Darkening'. Under what condition would you get 'Limb Brightening' instead?

**3**. **Photon pressure** If photon momentum is related to energy as $p = \frac{3}{4}\, h\nu/c$ (an extra factor 3/4 compared to the real one), rederive the expressions for radiation pressure (RL eq. [1.10]) and the Stefan-Boltzman law (RL eq. [1.41]).

**4**. **The greenhouse Effect** The Sun heats the Earth. One can derive an equilibrium temperature for an air-less Earth ($T_p$) assuming energy conservation and a black-body surface. This has to be modified when an atmosphere exists. The earth atmosphere is optically thin in the visible wavelengths so the solar radiation hits the ground directly. However, it is optically thick in the infra-red wavelengths and intercepts radiation from the ground. The heat is eventually lost to the vacuum as the atmosphere radiates with an effective temperature $T_{\rm eff}$. Necessarily, $T_{\rm eff} = T_p$ (think why).

1) Imagine the atmosphere as a single opaque layer with a uniform temperature $T_p$. It is receiving heat from the ground ($T_g$) and radiates as much luminosity toward the ground as towards the vacuum outside. Use energy conservation to show that $T_g = 2^{1/4} T_p$. So the ground has to be hotter.

2) A more sophisticated approach is to allow different layers in the atmosphere to have different temperatures. Let $\tau_g$ be the optical depth (to outgoing radiation) measured at the ground, while $\tau = 2/3$ at the photosphere (problem 2). Use the radiative diffusion equation (RL eq. [1.111]) to show that

$$T_g^4 = T_p^4 \left[1 + \frac{3}{4}\left(\tau_g - \frac{2}{3}\right)\right]. \qquad (1)$$

(This reduces to the result in part 1 when $\tau_g = 2$).

3) The opacity used above is the so-called Rosseland mean opacity (RL eq. [1.110]). What chemical species (atoms, molecules, aerosols...) are most relevant for determining this opacity? In contrast, what is the dominant opacity source in the visible wavelengths?

5. **Equivalent Width and Curve of Growth analysis** This problem is based on the data of Roberge et al (2004, Nature, 441, 72, also comes with supplementary material). The aim is to determine the range of uncertainty in the values for the atomic column densities obtained from absorption spectra.

1) First extract from the NIST database (http://physics.nist.gov/PhysRefData/ASD, choose "lines") Einstein A coefficient and statistical weights relevant for the following transitions: CII $\lambda = 1036.34$Å line, CII* $\lambda = 1037.02$Å line, and OI $\lambda = 976.45$Å line.

2) Starting from eq. (1.78) in RL, obtain the following absorption coefficient $\alpha_\nu$,

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2} n_1 A_{21} \frac{g_2}{g_1} \left(1 - \frac{g_1 n_2}{g_2 n_1}\right) \phi(\nu) = \frac{c^2}{8\pi\nu^2} n_1 A_{21} \frac{g_2}{g_1} \left[1 - \exp\left(-\frac{\Delta E}{k_B T_s}\right)\right] \phi(\nu), \tag{2}$$

where $\Delta E$ is the transition energy, $T_s$ is the excitation temperature of atoms (see Problem 6) and for our problem here, the limit $k_B T_s \ll \Delta E$ applies.

3) Show that for all three transitions (ignore the broad, redshifted components for carbon), the sort of column density values as listed in their Table 1 would imply that optical depths at the line centres ($\tau_{\nu=\nu_0}$) are greater than or comparable to 1. Or, these lines are saturated. In the author's opinion, Doppler broadening is important for the line-width,

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} \exp\left[-\left(\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right]. \tag{3}$$

where $\Delta\nu_D$ is the line width and is related to the Doppler parameter $b$ (matter sound speed) as $\Delta\nu_D \approx (b/c)\nu_0$. Use the values for $b$ as listed in Table 1.[1]

4) The authors use line-profile fitting to obtain column densities. This is highly risky when a line is saturated and under-resolved. We adopt the equivalent width (EW) method here. Argue that the following definition for the EW is independent of spectral resolution:

$$EW_\nu = \int \frac{I_c - I_\nu}{I_c} d\nu, \tag{4}$$

here $I_c$ is the specific intensity at continuum (the unabsorbed part). $EW_\nu$ has the unit of Hz.

5) When the line centre is optically thick, $EW_\nu \sim 2\Delta\nu$ where $\Delta\nu$ is the frequency displacement from the line centre at which $\tau_{\nu=\nu_0\pm\Delta\nu} = 1$. Show that for a Doppler broadened line, $EW_\nu \propto \sqrt{\ln N}$, where $N$ is the column density. *(Hint: consult Spitzer p.51 if you are confused.)*

6) For the following exercise we introduce a more commonly used definition of EW,

$$EW_\lambda = \int \frac{I_c - I_\lambda}{I_c} d\lambda, \tag{5}$$

[1] The fitted $b$ for OI is $\sim 10$ times greater than that for CII (and for other elements obtained from optical spectroscopy). This raises suspicion. But let us go on pretending it is correct.

where the integration is over wavelength. Necessarily, $EW_\lambda/\lambda = EW_\nu/\nu$. By eye-ball estimate, we assign a $EW_\lambda = 0.05$Å for the CII* line. Calculate the corresponding column densities when $b = 0.1\,\mathrm{km/s}$, $1.0\,\mathrm{km/s}$ and $5.7\,\mathrm{km/s}$ (the $1-\sigma$ range for $b$). What do your results say about the error-bars on the column densities listed in Table 1?

7) You can obtain a lower limit to the CII column density by ignoring Doppler broadening ($b = 0$) but taking into account the natural broadening (Lorentz profile, RL §10.6). In this case, one can show that EW $\propto \sqrt{N}$. Estimate the column density in the CII* level taking $\gamma = A_{21}$ and $EW_\lambda = 0.05$Å.

Similar exercise can be performed on the other lines and hightlights the risk of abundance analysis for saturated lines.

**6. Radiation Pressure** Consider a blackhole of mass $M$ and accreting at a maximal rate that is called the Eddington rate, $\dot{M}_{\rm Edd}$. Let matter radiates a fraction ($\epsilon$) of its gravitational energy at the Schwarzschild radius ($R_s = 2GM/c^2$) before it disappears into the blackhole. This is usually unavoidable as the accreting material likely has some angular momentum. It will orbit around the hole in an accretion disk. The inner material in the disk has to rub against the outer part to lose angular momentum and fall inward. This process generates heat and radiation. The radiation is intercepted by new infalling material, providing radiation pressure. The smallest cross-section for interception is usually the Thompson cross section.

1) The Eddington rate is approached when the outward pointing radiation pressure on the infalling plasma balances the inward pointing gravity. Derive an expression for this accretion rate.

2) Now assume the blackhole grows purely by accreting at the Eddington rate and with a constant $\epsilon$. What is the mass doubling time (the so-called Salpeter time) for the blackhole, if you take $\epsilon = 0.1$? Starting from a seed of $100M_\odot$ blackhole, and feeding it nonstop, how long before it can grow to a typical supermassive blackhole at, say, $10^9M_\odot$? For comparison, look up redshift record (and associatedly, the age of the universe) of the most distant quasars (which typically have a few billion solar masses) and check if they could have grown via the above pathway.

3) A variant of the Salpeter time. Now consider a massive star that burns nuclear fusion furiously. But the star can at best burn so fast that it radiates at the so-called Eddington luminosity, at which point the radiation pressure on matter (again, use Thompson scattering) balances the star's self-gravity. Let the fraction of rest mass that can be converted to radiation is 0.007, when hydrogen is fused into helium. What is the shortest lifetime of a massive star?